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On Some SEIRS Epidemic Models*

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Abstract We discuss a few variations of the SEIRS epidemic model. How basic dynamical properties of the models can be derived by using some tools of the computer algebra system Mathematica is shown, and how invariant surfaces of the system can be found by using computer algebra system Singular is explained. Some numerical simulations are presented as well.

Keywords Epidemiology, SEIRS model, stability of solution, invariant surface

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1 Introduction

The first applications of mathematical methods to the analysis of epidemics are associated with the works of D. Bernoulli, I. Lambert and P. S. Laplace. It appears that modern mathematical models of epidemiology go back to the work of R. Ross, published in 1911, on the study of the spread of malaria [12], and to the SIR model proposed in 1927 by W. Kermack and A. McKendrick [10]. The SIR model is based on the division of the entire population into three groups of susceptible, infected and recovered individuals, and describes the transition of individuals from the group of susceptible to the group of infected and then recovered. Mathematically, it is given as a system of differential equations that describe the change in the size of each of these population groups over time. However, the SIR model does not take into account the presence of the incubation period of the disease, i.e., it is assumed that a person who has had contact with a sick person immediately falls ill. This shortcoming is overcome in the SEIR model, which incorporates a group of contacts (exposed) (see, for example, [8, 9]). Thus, in the process of infection, a person susceptible to the disease first becomes exposed, and only after some time becomes infected.

A further development of the model is the SEIRS (Susceptible–Exposed–Infectious–Recovered–Susceptible) model (see e.g., [1]). The model considers the population divided in four groups: susceptible (S), exposed (E), infectious (I) and recovered

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(R). However, it is assumed that recovered people may become susceptible again. Mathematically, the simplest SEIRS model is described by the following system of differential equations:

$$\begin{aligned}\dot{S} =& -\beta SI/N + \omega R,\\ \dot{E} =& \beta SI/N - \sigma E,\\ \dot{I} =& \sigma E - \gamma I,\\ \dot{R} =& \gamma I - \omega R.\end{aligned} \tag{1.1}$$

In the model, the infectious rate β is the rate of spread which represents the probability of transmitting the disease between a susceptible and an infectious individual. The incubation rate σ is the rate of latent individuals becoming infectious. Recovery rate $\gamma = 1/d$ is determined by the average duration d of the infection, and ω is the rate the recovered individuals return to the susceptible state due to a loss of immunity. $N = S + I + E + R$ is the total population and since in model (1.1), it is assumed that the population is closed with no births and deaths, and N is a constant.

Some recent extensions of the model taking into account vaccination or time-delay are presented (e.g., in [2, 6, 13]), and works are referenced there.

In this paper, we review some main dynamical properties of a few variations of the model, and show how they can be easily derived by using certain tools of the computer algebra system MATHEMATICA. We also clarify the behavior of the model with the vital dynamics comparing our computational results with the ones obtained in [1].

2 Singular points and invariant surfaces in the SEIRS model

Introducing the notation

$$S = x_1,\ E = x_2,\ I = x_3,\ R = x_4, \tag{2.1}$$

we write system (1.1) as

$$\begin{aligned}\dot{x}_1 =& -(\beta x_1 x_3)/N + x_4\omega,\\ \dot{x}_2 =& -\sigma x_2 + (\beta x_1 x_3)/N,\\ \dot{x}_3 =& \sigma x_2 - \gamma x_3,\\ \dot{x}_4 =& \gamma x_3 - x_4\omega.\end{aligned} \tag{2.2}$$

System (2.2) has a line filled with steady states

$$x_1 = \frac{\gamma N}{\beta},\quad x_2 = \frac{\gamma x_3}{\sigma},\quad x_4 = \frac{\gamma x_3}{\omega}$$

and the first integral

$$\Psi = x_1 + x_2 + x_3 + x_4. \tag{2.3}$$

After rescaling of the phase variables

$$x_1 = \frac{x_1}{N},\quad x_2 = \frac{x_2}{N},\quad x_3 = \frac{x_3}{N},\quad x_4 = \frac{x_4}{N},$$

we obtain a system of the same form as (2.2), but with $N = 1$, so now x_1, x_1, x_3, x_4 represent the fractions of susceptible, exposed, infectious and recovered people in the population.

Using the first integral (2.3) and the fact that

$$x_1 + x_2 + x_3 + x_4 = 1, \tag{2.4}$$

we can reduce the dimension of the system. Namely, after the change of variables

$$y = x_1 + x_2 + x_3 + x_4,$$

we obtain from (2.2) the system

$$\begin{aligned}
\dot{x}_1 &= -\beta x_1 x_3 - (x_1 + x_2 + x_3 - y)\omega,\\
\dot{x}_2 &= -\sigma x_2 + \beta x_1 x_3,\\
\dot{x}_3 &= \sigma x_2 - \gamma x_3,\\
\dot{y} &= 0.
\end{aligned}$$

Since by (2.4), $y = 1$, system (2.2) is reduced to the three dimensional system

$$\begin{aligned}
\dot{x}_1 &= -\beta x_1 x_3 - (x_1 + x_2 + x_3 - 1)\omega,\\
\dot{x}_2 &= -\sigma x_2 + \beta x_1 x_3,\\
\dot{x}_3 &= \sigma x_2 - \gamma x_3,
\end{aligned} \tag{2.5}$$

which has the steady states A with the coordinates

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$$

and B with the coordinates

$$\tilde{x}_1 = \frac{\gamma}{\beta}, \quad \tilde{x}_2 = \frac{\gamma\omega(\beta - \gamma)}{\beta(\gamma(\sigma + \omega) + \sigma\omega)}, \quad \tilde{x}_3 = \frac{\sigma\omega(\beta - \gamma)}{\beta(\gamma(\sigma + \omega) + \sigma\omega)}. \tag{2.6}$$

Point A is the so-called disease-free equilibrium, and B is the endemic equilibrium.

Computing the Jacobi matrix of (2.5) at B, we find that its characteristic polynomial is

$$p(y) = y^3 + a_2 y^2 + a_1 y + a_0 \tag{2.7}$$

with

$$\begin{aligned}
a_2 &= (\gamma^2(\sigma + \omega) + \gamma(\sigma + \omega)^2 + \sigma\omega(\beta + \sigma + \omega))/(\sigma\omega + \gamma(\sigma + \omega)),\\
a_1 &= (\omega((\gamma^2 + \gamma\sigma + \sigma^2)\omega + \beta\sigma(\gamma + \sigma + \omega)))/(\sigma\omega + \gamma(\sigma + \omega)),\\
a_0 &= (\beta - \gamma)\sigma\omega.
\end{aligned} \tag{2.8}$$

According to the Routh-Hurwitz criterion the polynomial $p(y)$ has all roots with negative real parts, if and only if

$$a_2 > 0 \ \wedge \ a_0 > 0 \ \wedge a_2 a_1 - a_0 > 0.$$

Solving the semialgebraic system

$$a_2 > 0 \wedge a_0 > 0 \wedge a_1 a_2 - a_0 > 0 \wedge \gamma > 0 \wedge \beta > 0 \wedge \sigma > 0 \wedge \omega > 0 \tag{2.9}$$

with the routine Cylindrical Decomposition (based on the algorithm of [3]) of MATHEMATICA, we obtain the result

$$\sigma > 0 \wedge \omega > 0 \wedge \beta > \gamma > 0. \tag{2.10}$$

Thus, observing from (2.6) that the coordinates of point B are non-negative only if $\beta \geq \gamma$, we conclude that point B is asymptotically stable, if and only if the condition (2.10) is fulfilled. If condition (2.10) not holding the system has only the disease-free equilibrium A in the first octant. From this observation, it follows that system (2.5) cannot have Hopf bifurcations at B.

From the point of view of applications, the first integrals of a system of autonomous differential equations represent strong conservation laws and invariant surfaces represent weak conservation laws. To finish this section, we describe how to find invariant surfaces in system (2.5) by using the computer algebra system SINGULAR.

For the differential system

$$\dot{x} = P(x_1, x_2, x_3), \quad \dot{y} = Q(x_1, x_2, x_3), \quad \dot{z} = R(x_1, x_2, x_3), \tag{2.11}$$

where P, Q, and R a polynomials of degree at most n, we denote by $\mathcal{X}$ the associated vector field. A polynomial $F(x_1, x_2, x_3)$ satisfying the equation

$$\mathcal{X}(F) = \frac{\partial F}{\partial x_1} P + \frac{\partial F}{\partial x_2} Q + \frac{\partial F}{\partial x_3} R = KF \tag{2.12}$$

for some polynomial K is called a Darboux polynomial of (2.11). In this case, K is called the cofactor of F, and has the degree at most $n-1$. It is not difficult to see that the equation $F = 0$ defines an invariant algebraic surface of system (2.11).

We look for invariant surfaces of degree two in system (2.5) in the form

$$F = \quad h_{000} + h_{100}x_1 + h_{200}x_1^2 + h_{010}x_2 + h_{110}x_1x_2 + \tag{2.13}$$

$$h_{020}x_2^2 + h_{001}x_3 + h_{101}x_1x_3 + h_{011}x_2x_3 + h_{002}x_3^2 \tag{2.14}$$

with the cofactor

$$K = k_0 + k_1x_1 + k_2x_2 + k_3x_3.$$

Substituting this expressions into (2.12) (where P, Q and R are polynomials on the right hand side of (2.5)) and equating coefficients of similar terms on both sides of the obtained identity, we obtain the polynomial system

$$\begin{aligned}
&0 = -h_{200}k_1 = -h_{020}k_2 = -h_{020}k_1 - h_{110}k_2 = -h_{110}k_1 - h_{200}k_2 = -h_{002}k_3 = \\
&-h_{002}k_2 - h_{011}k_3 = -h_{011}k_2 - h_{020}k_3 = \beta h_{011} - \beta h_{101} - h_{002}k_1 - h_{101}k_3 = \\
&2\beta h_{020} - \beta h_{110} - h_{011}k_1 - h_{101}k_2 - h_{110}k_3 = \beta h_{110} - 2\beta h_{200} - h_{101}k_1 - h_{200}k_3 = \\
&-h_{000}k_0 + h_{100}\omega = -2\gamma h_{002} - h_{002}k_0 - h_{001}k_3 - h_{101}\omega = -\gamma h_{001} - h_{001}k_0 \\
&-h_{000}k_3 - h_{100}\omega + h_{101}\omega = -h_{200}k_0 - h_{100}k_1 - 2h_{200}\omega = \beta h_{010} - \beta h_{100} - \\
&\gamma h_{101} - h_{101}k_0 - h_{001}k_1 - h_{100}k_3 - h_{101}\omega - 2h_{200}\omega = -h_{100}k_0 - h_{000}k_1 - h_{100}\omega + \\
&2h_{200}\omega = -h_{010}k_0 - h_{000}k_2 - h_{100}\omega + h_{110}\omega + h_{001}\sigma - h_{010}\sigma = -\gamma h_{011} - h_{011}k_0 - \\
&h_{001}k_2 - h_{010}k_3 - h_{101}\omega - h_{110}\omega + 2h_{002}\sigma - h_{011}\sigma = -h_{020}k_0 - h_{010}k_2 - h_{110}\omega + \\
&h_{011}\sigma - 2h_{020}\sigma = -h_{110}k_0 - h_{010}k_1 - h_{100}k_2 - h_{110}\omega - 2h_{200}\omega + h_{101}\sigma - h_{110}\sigma
\end{aligned} \tag{2.15}$$

To find Darboux polynomials of system (2.5) of degree two, we have to eliminate from this system the variables $h_{000}, h_{001}, h_{002}, h_{010}, h_{011}, h_{020}, h_{100}, h_{101}, h_{110}, h_{200}$, $k_0, k_1,\ k_2, k_3$. However, it is necessary to add to system (2.15) the condition, and one of the coefficients of monomials of degree two in (2.15) is not zero (otherwise, the elimination is always possible, since system (2.15) always has the trivial solution). We first set in (2.15) $h_{200} = 1$, and perform the elimination by using the routine eliminate of SINGULAR and then solve the obtained system with the routine minAssGTZ [5] (the routine performs irreducible decomposition of the variety of a polynomial ideal, and it is based on the algorithm of [7]) by obtaining the following five solutions:

1) $\gamma = 0$,
2) $\omega = 0$,
3) $\sigma - \omega + \gamma = \beta\omega - 2\omega^2 - \beta\gamma - \omega\gamma + \gamma^2 = 0$,
4) $\beta = 0$,
5) $\sigma = 0$.

Obviously, only the third solution is relevant to the epidemiological point of view. Performing similar computations with $h_{110} = 1, h_{101} = 1$, etc, we do not obtain any other relevant solutions (for some cases, it was impossible to compute decomposition over the field of rational numbers, so modular computations were used as it is described in [11]).

In the case when condition 3 is satisfied, the system has an invariant surface

$$F = \frac{2\gamma(x_2 + 3x_3)}{3(\gamma - 2\omega)} + \frac{4\gamma x_2}{3(\gamma + \omega)} + \frac{\gamma x_3^2}{\omega - \gamma} + (x_1 + x_2 + x_3 - 1)^2 = 0.$$

Simple computations show that both steady states A and B are on the surface.

3 SEIRS model with vital dynamics

In this section, we consider a generalization of the SEIRS model from the previous section, where the birth rate and mortality rate are taken into account. The model is described by the system of differential equations

$$\begin{aligned} \dot{S} =& \mu N - \nu S - \beta SI/N + \omega R, \\ \dot{E} =& \beta SI/N - \nu E - \sigma E, \\ \dot{I} =& \sigma E - \gamma I - (\nu + \alpha) I, \\ \dot{R} =& \gamma I - \omega R - \nu R, \end{aligned} \tag{3.1}$$

where μ is the birth rate, ν is the background mortality rate and α is the death rate due to the infection. In a stable population, the mortality rate is approximately the same as the birth rate $\mu = \nu$, and if the disease is not severe, then α can be assumed equal to zero.

3.1 The case of a stable population

In the case when $\mu = \nu$ and $\alpha = 0$ the total population $N = S + E + I + R$ is a constant. Therefore, we can rescale variables S, E, I, R by N, or equivalently, we

set $N = 1$ in (3.1) by obtaining (using notation (2.1)) the system

$$\begin{aligned}\dot{x}_1 =&\nu - \nu x_1 - \beta x_1 x_3 + \omega x_4,\\ \dot{x}_2 =&-\nu x_2 - \sigma x_2 + \beta x_1 x_3,\\ \dot{x}_3 =&\sigma x_2 - \gamma x_3 - \nu x_3,\\ \dot{x}_4 =&\gamma x_3 - \nu x_4 - \omega x_4.\end{aligned} \tag{3.2}$$

Since $H = x_1 + x_2 + x_3 + x_4 - 1$ is an invariant subspace of (3.2), using the substitution $z = 1 - x_1 - x_2 - x_3 - x_3$ on the invariant space $z = 0$, we obtain from system (3.2),

$$\begin{aligned}\dot{x}_1 =&\nu - \nu x_1 - \beta x_1 x_3 + \omega(1 - x_1 - x_2 - x_3),\\ \dot{x}_2 =&-\nu x_2 - \sigma x_2 + \beta x_1 x_3,\\ \dot{x}_3 =&\sigma x_2 - \gamma x_3 - \nu x_3.\end{aligned} \tag{3.3}$$

For system (3.3), the reproduction number is (see e.g., [1])

$$r_0 = \frac{\beta\sigma}{(\gamma + nu)(\nu + \sigma)}.$$

It is convenient to use r_0 instead of the parameter β. Straightforward computations show that system (3.3) has the disease-free steady state A with the coordinates

$$x_1 = 1, \quad x_2 = 0, \quad x_3 = 0$$

and the epidemic equilibrium $B(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$, where

$$\begin{aligned}\tilde{x}_1 =&\frac{1}{r_0},\\ \tilde{x}_2 =&\frac{(r_0 - 1)(\gamma + \nu)(\nu + \omega)}{r_0(\gamma(\nu + \omega + \sigma) + (\nu + \omega)(\nu + \sigma))},\\ \tilde{x}_3 =&\frac{(r_0 - 1)\sigma(\nu + \omega)}{r_0(\gamma(\nu + \omega + \sigma) + (\nu + \omega)(\nu + \sigma))}.\end{aligned} \tag{3.4}$$

Thus, we see that point B is in the first octant, if and only if $r_0 \geq 1$.

Since for system (3.2) $x_1 + x_2 + x_3 + x_4 = 1$, in the case of system (3.3), we have the condition

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \leq 1. \tag{3.5}$$

From (3.4), we have

$$\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 = \frac{\gamma(r_0(\nu + \omega) + \sigma) + r_0(\nu + \omega)(\nu + \sigma)}{\gamma r_0(\nu + \omega + \sigma) + r_0(\nu + \omega)(\nu + \sigma)}.$$

From this expression, we see that condition (3.5) is satisfied, if $r_0 \geq 1$.

For system (3.3) with condition (3.5) fulfills the endemic equilibrium, B is always asymptotically stable. To see this, one simply needs to compute the Jacobi matrix of system (3.3) at point B and then its characteristic polynomial

$$p(y) = y^3 + a_2 y^2 + a_1 y + a_0. \tag{3.6}$$

Solving with Cylindrical Decomposition, the semi-algebraic system

$$a_2 > 0 \wedge a_0 > 0 \wedge a_1 a_2 - a_0 > 0 \wedge r_0 \geq 0 \wedge \gamma > 0 \wedge \sigma > 0 \wedge \nu > 0 \wedge \omega > 0, \tag{3.7}$$

we obtain the result

$$r_0 > 1 \wedge \gamma > 0 \wedge \sigma > 0 \wedge \nu > 0 \wedge \omega > 0. \tag{3.8}$$

By the Routh-Hurwitz criterion, it means that all roots of the characteristic polynomial $p(y)$ have negative real parts, unless $r_0 \leq 1$. However, in the latter case, the system has only point A (B collides with A for $r_0 = 1$).

Thus, point B is always asymptotically stable. Observing that all eigenvalues of the Jacobian of system (3.3) at A are always real, we conclude that a Hopf bifurcation cannot occur in system (3.3).

As we have seen, if condition (3.8) holds, then all roots of the characteristic polynomial at point B have negative real parts. However, there are two possibilities:
(i) all roots of the characteristic polynomial are real;
(ii) there is a pair of complex conjugate roots.

In the second case, the trajectories exhibit damped oscillations approaching the steady state B. If solutions to (3.3) exhibit oscillatory behavior, we refer to them as epidemic waves (see e.g., [1]).

As it is known, a cubic polynomial has a pair of complex conjugate roots, if its discriminant is negative. Thus, to detect the values of parameters for which damping oscillations exists, one can add to system (3.7) the condition $D < 0$, where D is the discriminant of polynomial (3.6) (in MATHEMATICA, it can be computed with the command Discriminant) and apply the routine Cylindrical Decomposition to the obtained system. That is, we can use the routine in order to determine the values of parameters of system (3.3), for which the system has epidemic waves and for which the waves most likely do not exist. It is possible to perform computations, only if we fix a few parameters in system (3.3).

As an example, we consider the case, when

$$\gamma = \frac{1}{14},\ \sigma = \frac{1}{7},\ \omega = \frac{1}{365},\ \nu = \frac{1}{76 \times 365}. \tag{3.9}$$

Adding to system (3.7) the condition $D < 0$ and solving the obtained system with the Cylindrical Decomposition, we obtain the result

$$1.01489 < r_0 < 460.421.$$

Thus, for r_0, in this interval, the Jacobian of the linearized system has a pair of complex conjugate eigenvalues, and we can observe periodic waves (Figure 2). For $r_0 < 1.01489$, the trajectories fast approach the steady state (Figure 1).

3.2 The case of changing population

Now, we consider the case when in system (3.1) $\mu = \nu$ and $\alpha > 0$. That is, the mortality due to infection is not negligible. In this case,

$$N = S + E + I + R \tag{3.10}$$

is no longer a constant, but a function of time. From (3.1), we have

$$\dot{N} = -\nu(S + E + I + R - N) - \alpha I.$$

Taking into account (3.10), we can use instead of the above equation the equation

$$\dot{N} = -\alpha I$$

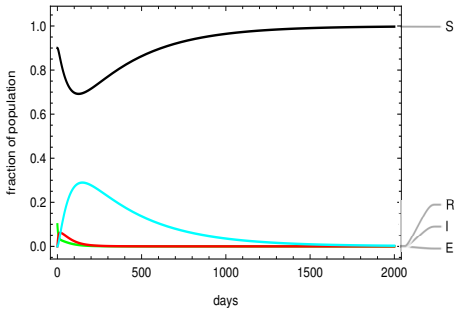

Figure 1: Solutions to (3.2) for $r_0 = 1.005$ and the initial values $x_1 = 0.9, x_2 = 0.1, x_3 = x_4 = 0$

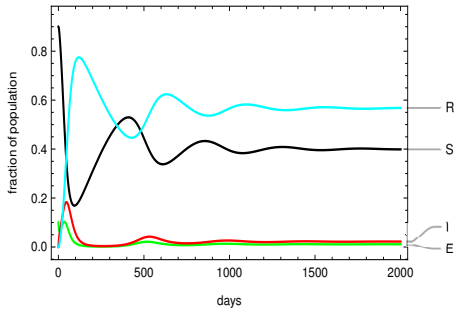

Figure 2: Solutions to (3.2) for $r_0 = 2.5$ and the initial values $x_1 = 0.9, x_2 = 0.1, x_3 = x_4 = 0$

by extending (3.1) to the five dimensional system

$$\begin{aligned}
\dot{S} =&\mu N - \nu S - \beta SI/N + \omega R,\\
\dot{E} =&\beta SI/N - \nu E - \sigma E,\\
\dot{I} =&\sigma E - \gamma I - (\nu + \alpha)I,\\
\dot{R} =&\gamma I - \omega R - \nu R,\\
\dot{N} =&-\alpha I.
\end{aligned} \tag{3.11}$$

Letting $x_1 = S$, $x_2 = E$, $x_3 = I$, $x_4 = R$, $x_5 = N$, performing the variable change $y_k = x_k/x_5$ $(k = 1, 2, 3, 4)$ and keeping the notation x_k instead of y_k, we obtain from system (3.11) that

$$\begin{aligned}
\dot{x}_1 =&\nu - \nu x_1 + \alpha x_1 x_3 - \beta x_1 x_3 + \omega x_4,\\
\dot{x}_2 =&-\nu x_2 - \sigma x_2 + \beta x_1 x_3 + \alpha x_2 x_3,\\
\dot{x}_3 =&\sigma x_2 - \alpha x_3 - \gamma x_3 - \nu x_3 + \alpha x_3^2,\\
\dot{x}_4 =&\gamma x_3 - \nu x_4 - \omega x_4 + \alpha x_3 x_4,\\
\dot{x}_5 =&-\alpha x_3 x_5.
\end{aligned} \tag{3.12}$$

Since the first four equations of the system do not depend on x_5, it is sufficient to consider instead of (3.13) the system

$$\begin{aligned}
\dot{x}_1 =&\nu - \nu x_1 + \alpha x_1 x_3 - \beta x_1 x_3 + \omega x_4,\\
\dot{x}_2 =&-\nu x_2 - \sigma x_2 + \beta x_1 x_3 + \alpha x_2 x_3,\\
\dot{x}_3 =&\sigma x_2 - \alpha x_3 - \gamma x_3 - \nu x_3 + \alpha x_3^2,\\
\dot{x}_4 =&\gamma x_3 - \nu x_4 - \omega x_4 + \alpha x_3 x_4.
\end{aligned} \tag{3.13}$$

The system has the invariant subspace (the Darboux factor)

$$H = x_1 + x_2 + x_3 + x_4 - 1$$

(which reflects the fact that the sum of the fractions of susceptible, exposed, infectious and recovered in the population is one), and on this subspace the flow of system (3.13) is defined by the equations

$$\begin{aligned}
\dot{x}_1 =&\nu + \omega - \nu x_1 + (\alpha - \beta)x_1 x_3 - \omega(x_1 + x_2 + x_3),\\
\dot{x}_2 =&-(nu + \sigma x_2) + (\beta x_1 + \alpha x_2)x_3,\\
\dot{x}_3 =&\sigma x_2 - (\alpha + \gamma + nu)x_3 + \alpha x_3^2.
\end{aligned} \tag{3.14}$$

In [1], the authors presented calculations for the case $\mu = \nu$ and α different from zero. For the values of parameters defined by (3.9) and $r_0 = 2.5, \alpha = 0.02$, the picture obtained in [1] looks as in Figure 3 (https://shiny.bcgsc.ca/posepi2/).

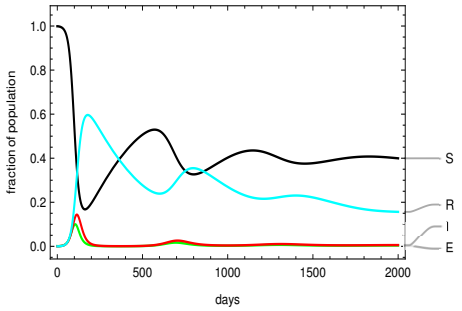

Figure 3: $\alpha = 0.02,\ r_0 = 2.5$

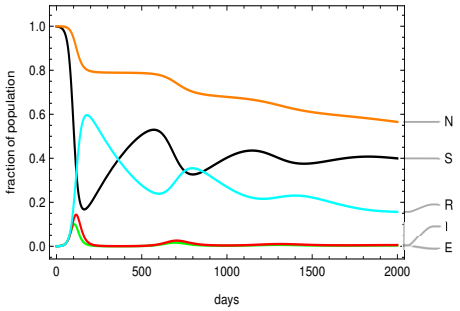

Figure 4: $\alpha = 0.02,\ r_0 = 2.5$

One can also obtain the graph given in Figure 3 by solving system (3.1) with $N = 1$, the parameter values given by (3.9) and the initial conditions $S = 0.999, E = 0.001, I = R = 0$. However, for such solutions, the sum $S + I + E + R = N$ is decreasing (see Figure 4), which contradicts the assumption $N = 1$.

As is shown above, if $\alpha \neq 0$, then instead of system (3.1) with $N = 1$ (equivalently, (3.12)) we have to use system (3.13). In this case, the picture is in Figure 5. Solving system (3.11) with the initial conditions $N(0) = 1000000$, $S(0) = 999000$, $E(0) = 1000$, $I(0) = R(0) = 0$, we obtain the picture in Figure 6. Plotting the ratios $S/N, E/N, I/N, R/N$, we obtain the same picture as shown in Figure 5. Thus, we see that although the total population decreases (Figure 6), the fractions of susceptible, exposed, infectious and recovered in the population stabilize (Figure 5).

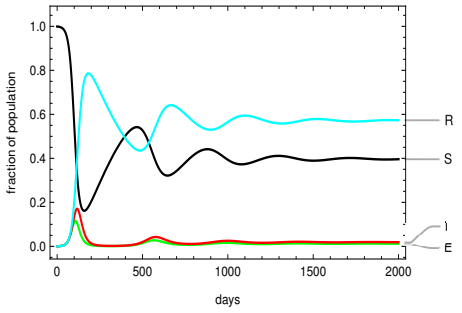

Figure 5: $\alpha = 0.02,\ r_0 = 2.5$

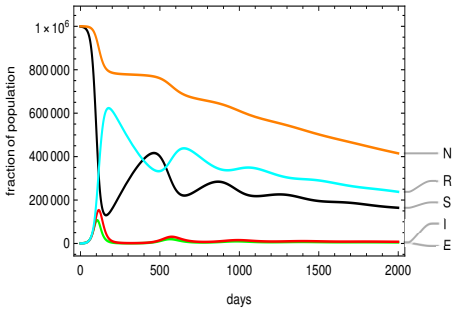

Figure 6: $\alpha = 0.02,\ r_0 = 2.5$

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