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Existence Results for the Higher-Order Weighted Caputo-Fabrizio Fractional Derivative*

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Abstract By the definition of the higher-order fractional derivative, we explore the central properties of the higher-order Caputo-Fabrizio fractional derivative and integral with a weighted term. Furthermore, by dint of Schaefer's fixed point theorem, α-ψ-Contraction theorem, etc., we establish the existence of solutions for nonlinear equations. We also give three examples to make our main conclusion clear.

Keywords Higher-order weighted fractional derivative, Caputo-Fabrizio derivative, existence

MSC(2010) 26A33, 35A01, 47H10.

1. Introduction

During the past decades, the Caputo fractional derivative (CFD) has been investigated by many scholars (see [13,18]). In the last few years, a large number of essays about a novel fractional derivative, Caputo-Fabrizio fractional derivative (CFFD), have emerged, and this kind of derivative has a better nature than the usual fractional derivative (see [1–3,5,7,9,12,15,16,19]). For instance, in 2020, Eiman et al., dealt with the nether class of fractional differential equations involving the CFFD and obtained the existence theory

$$\begin{cases} {}^{CF}_{0}D^{\theta}_{x}u(x) = f(x, u(x), {}^{CF}_{0}D^{\theta}_{x}u(x)), \quad x \in [0, T] = \mathbb{J}, \\ u(0) = u_0, \quad u_0 \in \mathbb{R}, \end{cases}$$

where $\theta \in (0, 1]$, f: $\mathbb{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ (see [9]). In 2021, Abbas et al., investigated the existence of solutions for the following Cauchy problem of Caputo-Fabrzio impulsive fractional differential equations

$$\begin{cases} ({}^{CF}D^{r}_{t_k}u)(t) = f(t, u(t)); t \in I_k, k = 0, \cdots, m, \\ u(t_k^+) = u(t_k^-) + L_k(u(t_k^-)); k = 1, \cdots, m, \\ u(0) = u_0, \end{cases}$$

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where $I_0 = [0, t_1]$, $I_k = (t_k, t_{k+1}]$, $k = 1, \cdots, m$; $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $u_0 \in \mathbb{R}$, and $f : I_k \times \mathbb{R} \to \mathbb{R}$, $k = 0, \cdots, m$, $L_k : \mathbb{R} \to \mathbb{R}$; $k = 1, \cdots, m$ are given continuous functions, ${}^{CF}D^r_{t_k}$ is the Caputo-Fabrizio fractional derivative of order $r \in (0, 1)$ (see [2]). In 2022, Abbas et al., investigated the existence of solutions for the Cauchy problem of Caputo-Fabrzio fractional differential equations without instantaneous impulses

$$\begin{cases} ({}^{CF}D^r_{s_k}u)(t) = f(t, u(t)); t \in I_k, k = 0, \cdots, m, \\ u(t) = g_k(t, u(t_k^-)); \text{if } t \in J_k, k = 1, \cdots, m, \\ u(0) = u_0 \in \mathbb{R}, \end{cases}$$

where $I_0 := [0, t_1]$, $J_k := (t_k, s_k]$, $I_k := (s_k, t_{k+1}]$; $k = 1, \cdots, m$, and $f : I_k \times \mathbb{R} \to \mathbb{R}$, $g_k : J_k \times \mathbb{R} \to \mathbb{R}$ are given continuous functions, $0 = s_0 < t_1 \le s_1 < t_2 \le s_2 < \cdots \le s_{m-1} < t_m \le s_m < t_{m+1} = T$ (see [3]).

Abreast of the times, in 2022, Fernandez et al., conducted a formal study of weighted fractional calculus, and emphasized the importance of the conjugation relationships with the classical Riemann-Liouville fractional calculus (see [11]). For the study of Caputo-Fabrizio fractional derivative (CFFD) in the weighted field, in 2019, Al-Refai and Jarrah first proposed the weighted Caputo-Fabrizio fractional derivative (WCFFD) of order 0 to 1, and demonstrated the existence and uniqueness of the nonlinear fractional initial value problem

$$\begin{cases} (D^\alpha_{a,[z,w]}f)(t) = g(t, f), t > a, 0 < \alpha < 1, \\ f(a) = f_0 \in \mathbb{R}, \end{cases}$$

where $D^\alpha_{a,[z,w]}$ is the WCFFD (see [4]). In 2020, Wu, Chen and Deng studied the existence and stability of solutions for the WCFFD type differential equations of order 0 to 1 (see [20]). However, fewer papers are on the higher-order WCFFD.

In this paper, we are concerned with the existence of solutions for the following nonlinear equations

$$\begin{cases} (D^r_{a,[z,w]}y)(t) = \xi(t, y(t)), \\ y^{(k)}(a) = 0, \qquad k = 0, 1, 2, \cdots, n-1, \\ y^{(n)}(a) = 1, \end{cases} \tag{1.1}$$

where $1 \le n < r < n+1$, $D^r_{a,[z,w]}$ is the higher order WCFFD, and $y \in AC^n([a, T], \mathbb{R})$, ξ are binary continuous functions

$$\begin{cases} D^r_{a,[z,w]}(y(t) - \varpi(t, y(t)) = \xi(t, y(t)), \\ (y - \varpi)^{(k)}(a) = 0, \qquad k = 0, 1, 2, \cdots, n-1, \\ (y - \varpi)^{(n)}(T) = 0, \end{cases} \tag{1.2}$$

where $y - \varpi \in AC^n([a, T], \mathbb{R})$, and ϖ are binary continuous functions

$$\begin{cases} D^r_{a,[z,w]}\frac{y(t)}{\varphi(t, y(t))} = \xi(t, y(t)), \\ y^{(k)}(a) = 0, \qquad k = 0, 1, 2, \cdots, n-1, \\ y^{(n)}(a) = 1, \end{cases} \tag{1.3}$$

where $\frac{y}{\varphi} \in AC^n([a, T], \mathbb{R})$, and φ are binary continuous functions. Here, $AC([a, T], \mathbb{R})$ is Banach space, which contains all absolutely continuous functions from $[a, T]$ into $\mathbb{R}$, provided with the usual maximum norm. $AC^n([a, T], \mathbb{R}) = \{x : [a, T] \to$

$\mathbb{R}, and\ x^{(n-1)} \in AC([a,T],\mathbb{R})\}$.

The main components of this article are as follows. First, the definitions and properties of the higher-order WCFFD are introduced. Then, the existence results of the nonlinear equations are obtained. Finally, we give three examples to make our main conclusion clear.

2. Preliminary results

In this segment, we introduce preliminary results related to this dissertation.

Definition 2.1 ([4]). Let $0 < r < 1$, and $y \in AC([a,T],\mathbb{R})$. The weighted Caputo-Fabrizio fractional derivative (WCFFD) of y of order r is defined by

$$(D^r_{a,[z,w]}y)(t) = \frac{M(r)}{1-r}\frac{1}{w(t)}\int_a^t e^{-\mu_r(z(t)-z(s))}\frac{d}{ds}(wy)(s)ds, a < t < T.$$

Here, $\mu_r = \frac{r}{1-r}$, $M(r)$ is a normalization function, which satisfies $M(0) = M(1) = 1$, $w, z \in AC^1[a,T]$, and $w, w', z' > 0$ on $[a,T]$.

Definition 2.2 ([4]). For $0 < r < 1$, the weighted Caputo-Fabrizio fractional integral (WCFFI) of y of order r is defined by

$$(I^r_{a,[z,w]}y)(t) = \frac{1}{M(r)}\left((1-r)y(t) + \frac{r}{w(t)}\int_a^t z'(s)w(s)y(s)ds\right).$$

Definition 2.3. Let $n < r < n+1$, and $y \in AC^n([a,T],\mathbb{R})$, we define the WCFFD of y of order r as follows:

$$\begin{aligned}(D^r_{a,[z,w]}y)(t) &= (D^{r-n}_{a,[z,w]}y^{(n)})(t)\\ &= \frac{M(r-n)}{1-r+n}\frac{1}{w(t)}\int_a^t e^{-\mu_{r-n}(z(t)-z(s))}\frac{d}{ds}(wy^{(n)})(s)ds, \qquad (2.1)\end{aligned}$$

where $\mu_{r-n} = \frac{r-n}{1-r+n}$.

Definition 2.4. For $n < r < n+1$, the WCFFI of y of order r as follows:

$$\begin{aligned}&(I^r_{a,[z,w]}y)(t) = (I^n I^{r-n}_{a,[z,w]}y)(t)\\ =& \frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{M(r-n)}\left((1-r+n)y(s) + \frac{r-n}{w(s)}\int_a^s z'(u)w(u)y(u)du\right)ds\\ =& \frac{n+1-r}{\Gamma(n)M(r-n)}\int_a^t (t-s)^{n-1}y(s)ds\\ &+\frac{r-n}{\Gamma(n)M(r-n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)y(u)duds.\end{aligned}$$

Definition 2.5 ([6]). For $n < r < n+1$, we call the usual Caputo-Fabrizio fractional derivative (CFFD) as follows:

$$D^r_a y(t) = \frac{M(r-n)}{1-r+n}\int_a^t e^{-\mu_{r-n}(t-s)}y^{(n+1)}ds. \qquad (2.2)$$

Let us consider the difference between the WCFFD and the usual CFFD in the interval $[-25, 25]$.

(i) As $y(t) = \sin t$, we choose $z(t) = w(t) = t$, $a = -25$, and $M = 1$. We observe the following simulations of the WCFFD and the usual CFFD with $r = 0.8$ (see Figures 1-2):

$$(D^r_{a,[z,w]}y)(t) = \frac{1}{0.2}\frac{1}{t}\int_{-25}^{t} e^{4(s-t)}(\sin s + s\cdot\cos s)ds, \tag{2.3}$$

$$(D^r_a y)(t) = \frac{1}{0.2}\int_{-25}^{t} e^{4(s-t)}\cos s ds. \tag{2.4}$$

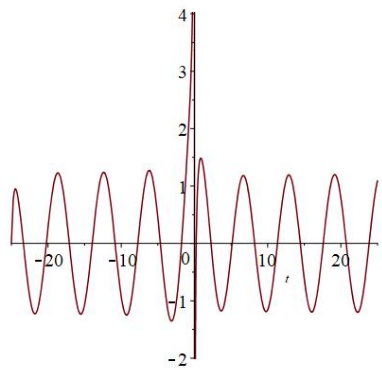

Figure 1. Simulation of WCFFD (2.3)

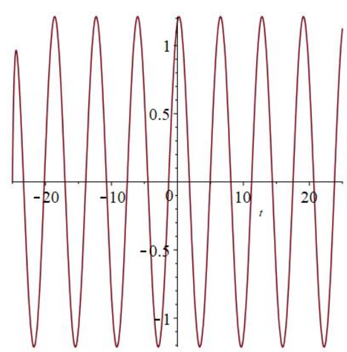

Figure 2. Simulation of the usual CFFD (2.4)

(ii) As $y(t) = \sin t$, we choose $z(t) = w(t) = t$, $a = -25$, $M = 1$. We observe the following simulations of the WCFFD and the usual CFFD with $r = 1.8$ (see Figures 3-4):

$$(D^r_{a,[z,w]}y)(t) = \frac{1}{0.2}\frac{1}{t}\int_{-25}^{t} e^{4(s-t)}(\cos s - s\cdot\sin s)ds, \tag{2.5}$$

$$(D_a^r y)(t) = \frac{1}{0.2}\int_{-25}^{t} e^{4(s-t)}(-\sin s)ds. \tag{2.6}$$

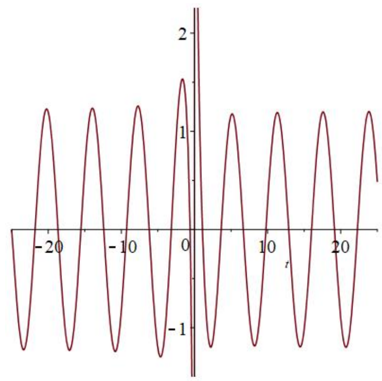

Figure 3. Simulation of WCFFD(2.5)

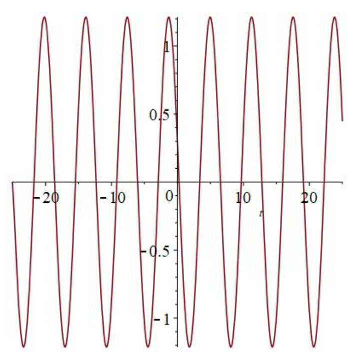

Figure 4. Simulation of the usual CFFD (2.6)

From the above simulations we can observe different actions between the WCFFD and the usual CFFD. There is a difference between Figure 1 and Figure 2. Otherwise, it appears less different in the other two images (see Figures 3-4).

Now, we consider the relations between the differential and integral operators.

Theorem 2.1. *Letting $n < r < n+1$, and $y \in AC^n([a,T],\mathbb{R})$, then*
(i) $(D^r_{a,[z,w]} I^r_{a,[z,w]} y)(t) = y(t) - \frac{e^{\mu_{r-n}(z(a)-z(t))} w(a)y(a)}{w(t)}$.
(ii) $(I^r_{a,[z,w]} D^r_{a,[z,w]} y)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(t-a)^k - \frac{w(a)y^{(n)}(a)}{\Gamma(n)} \int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds.$

Proof. Letting $\beta = r-n$, then $\beta \in (0,1)$.
(i) Since

$$\begin{aligned} &(I^r_{a,[z,w]} y)^{(n)}(t) \\ = &\left[I^n \left(\frac{1}{M(\beta)} \left((1-\beta) y(t) + \frac{\beta}{w(t)} \int_a^t z'(s) w(s) y(s) ds \right) \right) \right]^{(n)} \end{aligned}$$

$$= \frac{1}{M(\beta)} \left((1-\beta) y(t) + \frac{\beta}{w(t)} \int_a^t z'(s) w(s) y(s) ds \right),$$

we have

$$\frac{d}{dt} (w I^r_{a,[z,w]} y^{(n)})(t) = \frac{1}{M(\beta)} \left((1-\beta) \frac{d}{dt} (wy)(t) + \beta (z'wy)(t) \right).$$

Thus,

$$\begin{aligned}
&(D^r_{a,[z,w]} I^r_{a,[z,w]} y)(t) \\
&= \frac{1}{1-\beta} \frac{e^{-\mu_\beta z(t)}}{w(t)} \int_a^t e^{\mu_\beta z(s)} \left((1-\beta) \frac{d}{ds} (wy)(s) + \beta (z'wy)(s) \right) ds \\
&= \frac{1}{1-\beta} \frac{e^{-\mu_\beta z(t)}}{w(t)} \Big((1-\beta) \int_a^t e^{\mu_\beta z(s)} \frac{d}{ds} (wy)(s) ds \\
&\quad + \beta \int_a^t e^{\mu_\beta z(s)} (z'wy)(s) ds \Big). \qquad (2.7)
\end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
&(1-\beta) \int_a^t e^{\mu_\beta z(s)} \frac{d}{ds} (wy)(s) ds \\
&= (1-\beta) \left(e^{\mu_\beta z(t)} (wy)(t) - e^{\mu_\beta z(a)} (wy)(a) - \mu_\beta \int_a^t e^{\mu_\beta z(s)} (z'wy)(s) ds \right) \\
&= (1-\beta) e^{\mu_\beta z(t)} w(t) y(t) - (1-\beta) e^{\mu_\beta z(a)} w(a) y(a) \\
&\quad - \beta \int_a^t e^{\mu_\beta z(s)} (z'wy)(s) ds. \qquad (2.8)
\end{aligned}$$

Substituting the result of (2.8) into (2.7),

$$\begin{aligned}
&(D^r_{a,[z,w]} I^r_{a,[z,w]} y)(t) \\
&= \frac{1}{1-\beta} \frac{e^{-\mu_\beta z(t)}}{w(t)} \Big((1-\beta) e^{\mu_\beta z(t)} (wy)(t) - (1-\beta) e^{\mu_\beta z(a)} (wy)(a) \Big) \\
&= y(t) - \frac{e^{\mu_\beta (z(a) - z(t))} w(a) y(a)}{w(t)}.
\end{aligned}$$

If we consider $y(a) = 0$, we get $(D^r_{a,[z,w]} I^r_{a,[z,w]} y)(t) = y(t)$.
(ii)

$$\begin{aligned}
(I^r_{a,[z,w]} D^r_{a,[z,w]} y)(t) = &\frac{1-\beta}{\Gamma(n) M(\beta)} \int_a^t (t-s)^{n-1} (D^r_{a,[z,w]} y)(s) ds \\
&+ \frac{\beta}{\Gamma(n) M(\beta)} \int_a^t (t-s)^{n-1} \frac{1}{w(s)} \int_a^s z'(u) w(u) (D^r_{a,[z,w]} y)(u) du ds. \qquad (2.9)
\end{aligned}$$

Let $k_y(t) = \int_a^t e^{\mu_\beta z(s)} (w y^{(n)}(s))' ds$. Then

$$k_y'(t) = e^{\mu_\beta z(t)} \left(w y^{(n)}(t) \right)'$$

and

$$(D^r_{a,[z,w]}y)(t) = \frac{M(\beta)}{1-\beta}\frac{e^{-\mu_\beta z(t)}}{w(t)}k_y(t).$$

Hence,

$$(z'wD^r_{a,[z,w]}y)(t) = \frac{M(\beta)}{1-\beta}z'e^{-\mu_\beta z(t)}k_y(t).$$

Integrating by parts, we have

$$\begin{aligned}
&\int_a^t (z'wD^r_{a,[z,w]}y)(s)ds \\
&= \frac{M(\beta)}{1-\beta}\int_a^t z'(s)e^{-\mu_\beta z(s)}k_y(s)ds \\
&= \frac{M(\beta)}{1-\beta}\int_a^t k_y(s)d(-\frac{1}{\mu_\beta}e^{-\mu_\beta z(s)}) \\
&= \frac{M(\beta)}{1-\beta}\left[-\frac{1}{\mu_\beta}e^{-\mu_\beta z(s)}k_y(s)\mid_a^t + \int_a^t \frac{1}{\mu_\beta}e^{-\mu_\beta z(s)}\frac{d}{ds}k_y(s)\right] \\
&= \frac{M(\beta)}{1-\beta}\Big[-\frac{1}{\mu_\beta}e^{-\mu_\beta z(t)}k_y(t) + \frac{1}{\mu_\beta}e^{-\mu_\beta z(a)}k_y(a) \\
&\quad + \int_a^t \frac{1}{\mu_\beta}e^{-\mu_\beta z(s)}e^{\mu_\beta z(s)}\frac{d}{ds}(wy^{(n)})ds\Big] \\
&= \frac{M(\beta)}{1-\beta}\left[-\frac{1}{\mu_\beta}e^{-\mu_\beta z(t)}k_y(t) + \frac{1}{\mu_\beta}\int_a^t \frac{d}{ds}(wy^{(n)})(s)ds\right] \\
&= -\frac{M(\beta)}{\mu_\beta(1-\beta)}\left[e^{-\mu_\beta z(t)}k_y(t) - (wy^{(n)})(t) + (wy^{(n)})(a)\right] \\
&= -\frac{M(\beta)}{\beta}\left[e^{-\mu_\beta z(t)}k_y(t) - (wy^{(n)})(t) + (wy^{(n)})(a)\right].
\end{aligned}$$

Substituting the result into (2.9), we have

$$\begin{aligned}
&(I^r_{a,[z,w]}D^r_{a,[z,w]}y)(t) \\
&= \frac{1-\beta}{\Gamma(n)M(\beta)}\int_a^t (t-s)^{n-1}\frac{M(\beta)}{1-\beta}\frac{e^{-\mu_\beta z(s)}}{w(s)}k_y(s)ds \\
&\quad -\frac{\beta}{\Gamma(n)M(\beta)}\int_a^t \frac{(t-s)^{n-1}}{w(s)}\frac{M(\beta)}{\beta}\left[e^{-\mu_\beta z(s)}k_y(s) - (wy^{(n)})(s) + (wy^{(n)})(a)\right]ds \\
&= \frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{e^{-\mu_\beta z(s)}}{w(s)}k_y(s)ds \\
&\quad -\frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\left[e^{-\mu_\beta z(s)}k_y(s) - (wy^{(n)})(s) + (wy^{(n)})(a)\right]ds \\
&= \frac{1}{\Gamma(n)}\int_a^t (t-s)^{n-1}y^{(n)}(s)ds - \frac{w(a)y^{(n)}(a)}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds \\
&= y(t) - \sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k - \frac{w(a)y^{(n)}(a)}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds.
\end{aligned}$$

If we consider $y^{(n)}(a)=0$, we get $(I^r_{a,[z,w]}D^r_{a,[z,w]}y)(t)=y(t)-\sum\limits_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^k$.

□

3. Existence results for the nonlinear equation

In the following segment, we will investigate the existence results for nonlinear equations in Section 1. Several lemmas related are given first.

Definition 3.1 ([14, 17]). Let Ψ be the family of nondecreasing function $\psi : [0,\infty)\to[0,\infty)$ satisfying $\sum\limits_{n=1}^{\infty}\psi^n(t)<\infty$ for $t>0$. Let (X,d) be a metric space, and $\alpha : X\times X\to[0,\infty)$ be a map and $\psi\in\Psi$. A mapping $T : X\to X$ is called an α-ψ-contraction, if $\alpha(x,y)d(Tx,Ty)\le\psi(d(x,y))$ for all $x,y\in X$.

Definition 3.2 ([14,17]). $T : X\to X$ is said to be α-admissible, if $\alpha(x,y)\ge1\Rightarrow\alpha(Tx,Ty)\ge1$, for $x,y\in X$, where $\alpha : X\times X\to[0,\infty)$.

Lemma 3.1 ([10]). *Let M be a Banach space, and $P : M\to M$ be completely continuous, if $A(P)=\{y\in M : y=\lambda Py,\ for\ some\ \lambda\in[0,1]\}$ is bounded. Then, P has a fixed point.*

Lemma 3.2 ([14,17]). *(α-ψ-Contraction theorem)*

Let (M,d) be a complete metric space and $T : M\to M$ be an α-ψ contraction mapping. Further,
(i) T is α-admissible;
(ii) there exists $x_0\in M$ such that $\alpha(x_0,Tx_0)\ge1$;
(iii) if x_n is a sequence in M such that $\alpha(x_n,x_{n+1})\ge1$ for all n and $x_n\to x\in M$ as $n\to\infty$, then $\alpha(x_n,x)\ge1$ for all n.

Then, there exists $y\in M$ such that Ty=y.

Lemma 3.3 ([8]). *Let S be a non-empty, bounded and closed convex subset of Banach algebra Ω. $F_1 : \Omega\to\Omega$ and $F_2 : S\to\Omega$ satisfy*
(i) F_1 is Lipschitzian, and the lipschitz constant is written as α;
(ii) F_2 is completely continuous;
(iii) $y_1=F_1y_1F_2y_2\Rightarrow y_1\in S$ for all $y_2\in S$;
(iv) $\alpha M<1$, where $M=\sup\{\|F_2(y_1)\| : y_1\in S\}$,

then $F_1y_1F_2y_1=y_1$ has a solution in S.

By means of Theorem 2.1, the following conclusion can be reached.

Lemma 3.4. *Let $y\in AC^n[a,T]$, ξ be a binary continuous function, and y be a solution to the nonlinear fractional equation (1.1), if it satisfies the integral equation*

$$y(t)-\frac{w(a)}{\Gamma(n)}\int_a^t(t-s)^{n-1}\frac{1}{w(s)}ds$$
$$=a_r\int_a^t(t-s)^{n-1}\xi(s,y(s))ds+b_r\int_a^t(t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds,$$

where $a_r=\frac{1-r+n}{\Gamma(n)M(r-n)}$, $b_r=\frac{r-n}{\Gamma(n)M(r-n)}$.

Theorem 3.1. *Let $1\le n<r<n+1$ and $a\le t\le T$. ξ is a binary continuous function which satisfies $|\xi(t,y(t))|\le L_1(1+|y(t)|)$, and here $L_1>0$. If $(\theta_1+\theta_2)L_1<$*

1, *then boundary value problem* (1.1) *has at least one solution, where* $\theta_1 = \frac{a_r(T-a)^n}{n}$, *and* $\theta_2 = \frac{b_r(T-a)^n w(T)(z(T)-z(a))}{nw(a)}$.

Proof. Define $P : AC^n([a,T],\mathbb{R}) \to AC^n([a,T],\mathbb{R})$ as follows.

$$\begin{aligned}(Py)(t)=&\frac{w(a)}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds + a_r\int_a^t (t-s)^{n-1}\xi(s,y(s))ds\\ &+ b_r\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds.\end{aligned}$$

Letting $y_n \to y$ in $[a,T]$, for all $t \in [a,T]$,

$$\begin{aligned}&|(Py_n)(t)-(Py)(t)|\\ \leq\ & a_r\int_a^t (t-s)^{n-1}\big|\xi\left(s,y_n(s)\right)-\xi\left(s,y(s)\right)\big|ds\\ &+b_r\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\big|\xi\left(u,y_n(u)\right)-\xi\left(u,y(u)\right)\big|duds\\ \leq\ & a_r \parallel \xi(\cdot,y_n(\cdot))-\xi(\cdot,y(\cdot))\parallel \int_a^t (t-s)^{n-1}ds\\ &+b_r \parallel \xi(\cdot,y_n(\cdot))-\xi(\cdot,y(\cdot))\parallel \int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)duds.\end{aligned}$$

From $z', w' > 0$ and the mean value theorem for integrals, for some $a < \sigma < T$, we have

$$\int_a^s z'(u)w(u)du = w(\sigma)(z(s)-z(a)) \leq w(T)(z(T)-z(a)).$$

Thus,

$$\begin{aligned}&|(Py_n)(t)-(Py)(t)|\\ \leq\ &\frac{a_r(T-a)^n}{n}\parallel \xi(\cdot,y_n(\cdot))-\xi(\cdot,y(\cdot))\parallel\\ &+\frac{b_r w(T)(z(T)-z(a))(T-a)^n}{nw(a)}\parallel \xi(\cdot,y_n(\cdot))-\xi(\cdot,y(\cdot))\parallel.\end{aligned}$$

Since ξ is continuous, we can derive that P is continuous.

In the following, we will testify that P is a bounded operator. For

$$y \in B_\rho = \{y \in AC^n([a,T],\mathbb{R}) : \sup_{t\in[a,T]}|y(t)| \leq \rho\},$$

we get

$$\begin{aligned}&|Py(t)|\\ =&\Big|\frac{w(a)}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds + a_r\int_a^t (t-s)^{n-1}\xi(s,y(s))ds\\ &+b_r\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds\Big|\\ \leq\ &\frac{w(a)}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds + a_r\int_a^t (t-s)^{n-1}L_1(1+|y(s)|)ds\end{aligned}$$

$$
\begin{aligned}
&+b_r\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)L_1(1+|y(u)|)duds\\
&\le \frac{(T-a)^n}{n\Gamma(n)}+\frac{a_r(T-a)^nL_1(1+\rho)}{n}+\frac{b_r(T-a)^nw(T)(z(T)-z(a))L_1(1+\rho)}{nw(a)}\\
&=\frac{(T-a)^n}{\Gamma(n+1)}+\theta_1L_1(1+\rho)+\theta_2L_1(1+\rho)\\
&=\frac{(T-a)^n}{\Gamma(n+1)}+L_1(1+\rho)(\theta_1+\theta_2):=l.
\end{aligned}
$$

Thus,

$$
\sup_{t\in[a,T]}|Py(t)|\le l.
$$

Afterwards, the equicontinuity will be demonstrated. Let $t_1,t_2\in[a,T]$ be with $a\le t_1\le t_2\le T$, $y\in B_\rho$. We have

$$
\begin{aligned}
&|Py(t_1)-Py(t_2)|\\
=&\left|\frac{w(a)}{\Gamma(n)}\left[\int_a^{t_2}(t_2-s)^{n-1}\frac{1}{w(s)}ds-\int_a^{t_1}(t_1-s)^{n-1}\frac{1}{w(s)}ds\right]\right.\\
&+a_r\left[\int_a^{t_2}(t_2-s)^{n-1}\xi(s,y(s))ds-\int_a^{t_1}(t_1-s)^{n-1}\xi(s,y(s))ds\right]\\
&+b_r\left[\int_a^{t_2}(t_2-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds\right.\\
&\left.\left.-\int_a^{t_1}(t_1-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds\right]\right|\\
\le&\frac{(t_2-a)^n-(t_1-a)^n}{n\Gamma(n)}+\frac{a_rL_1(1+\rho)\big[(t_2-a)^n-(t_1-a)^n\big]}{n}\\
&+\frac{b_rL_1(1+\rho)w(T)(z(T)-z(a))\big[(t_2-a)^n-(t_1-a)^n\big]}{nw(a)}\\
\le&\left[\frac{1}{n\Gamma(n)}+\frac{a_rL_1(1+\rho)}{n}+\frac{b_rL_1(1+\rho)w(T)(z(T)-z(a))}{nw(a)}\right]\big[(t_2-a)^n-(t_1-a)^n\big].
\end{aligned}
$$

Applying the Lagrange mean value theorem, there exists $\zeta\in[t_1,t_2]$ such that

$$
(t_2-a)^k-(t_1-a)^k=k(\zeta-a)^{k-1}(t_2-t_1).
$$

Thus,

$$
\begin{aligned}
&|Py(t_1)-Py(t_2)|\\
&\le\left[\frac{1}{\Gamma(n)}+a_rL_1(1+\rho)+\frac{b_rL_1(1+\rho)w(T)(z(T)-z(a))}{w(a)}\right](\zeta-a)^{n-1}(t_2-t_1).
\end{aligned}
$$

Then, P is equicontinuous.

Combining the above steps with the Arzela-Ascoli theorem, we can conclude that P is completely continuous.

Eventually, we consider the boundedness of the set $A(P)=\{y\in AC^n([a,T],\mathbb{R}):$

$y = \lambda Py$, for some $\lambda \in [0,1]\}$. Letting $y \in A(P)$, for every $t \in [a,T]$, we are able to derive that

$$\begin{aligned}|y(t)| &= |\lambda Py(t)| \\ &\le \frac{(T-a)^n}{\Gamma(n+1)} + L_1(1+\|y\|)(\theta_1+\theta_2) \\ &\le \frac{(T-a)^n}{\Gamma(n+1)} + L_1(\theta_1+\theta_2) + L_1\|y\|(\theta_1+\theta_2).\end{aligned}$$

Using the condition $(\theta_1+\theta_2)L_1 < 1$, we obtain

$$\|y\| \le \frac{\frac{(T-a)^n}{\Gamma(n+1)} + L_1(\theta_1+\theta_2)}{1 - L_1(\theta_1+\theta_2)},$$

which means the set $A(P)$ is bounded.

By dint of Lemma 3.1, we derive that P has a fixed point, which is a solution to (1.1). The proof is completed. □

Now, we consider the nonlinear boundary value problem (1.2).

Lemma 3.5. *Let $y \in AC^n([a,T],\mathbb{R})$. ξ and ϖ are binary continuous functions, $y - \varpi \in AC^n([a,T],\mathbb{R})$ and y is a solution to the nonlinear fractional boundary value problem* (1.2)*, if it satisfies the equation*

$$\begin{aligned}&y(t) - \varpi(t,y(t)) + a_r w(T)\xi(T,y(T)) \int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds \\ &+b_r \int_a^T z'(u)w(u)\xi(u,y(u))du \int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds \\ &= a_r \int_a^t (t-s)^{n-1}\xi(s,y(s))ds + b_r \int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds.\end{aligned}$$

Proof. By means of Theorem 2.1, we have

$$\begin{aligned}y(t) - \varpi(t,y(t)) = &\sum_{k=0}^{n-1} c_k(t-a)^k + \frac{w(a)c_n}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds \\ &+ a_r \int_a^t (t-s)^{n-1}\xi(s,y(s))ds \\ &+ b_r \int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds.\end{aligned}$$

According to $(y-\varpi)^{(k)}(a) = 0$, we know that $c_k = 0$. That is,

$$\begin{aligned}&y(t) - \varpi(t,y(t)) - \frac{w(a)c_n}{\Gamma(n)}\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds \\ &= a_r \int_a^t (t-s)^{n-1}\xi(s,y(s))ds + b_r \int_a^t \frac{(t-s)^{n-1}}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds.\end{aligned}$$

Then,

$$(y-\varpi)^{(n)}(t) = \frac{w(a)c_n}{w(t)} + a_r\Gamma(n)\xi(t,y(t)) + b_r\Gamma(n)\frac{1}{w(t)}\int_a^t z'(s)w(s)\xi(s,y(s))ds,$$

$$(y-\varpi)^{(n)}(T)=\frac{w(a)c_n}{w(T)}+a_r\Gamma(n)\xi(T,y(T))+b_r\Gamma(n)\frac{1}{w(T)}\int_a^T z'(s)w(s)\xi(s,y(s))ds.$$

For $(y-\varpi)^{(n)}(T)=0$, we get

$$w(a)c_n=-a_rw(T)\Gamma(n)\xi(T,y(T))-b_r\Gamma(n)\int_a^T z'(s)w(s)\xi(s,y(s))ds.$$

Thus,

$$\begin{aligned}&y(t)-\varpi(t,y(t))+a_rw(T)\xi(T,y(T))\int_a^t(t-s)^{n-1}\frac{1}{w(s)}ds\\&+b_r\int_a^T z'(u)w(u)\xi(u,y(u))du\int_a^t(t-s)^{n-1}\frac{1}{w(s)}ds\\=&\,a_r\int_a^t(t-s)^{n-1}\xi(s,y(s))ds+b_r\int_a^t(t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds,\end{aligned}$$

which completes the proof. □

Denote $V=\{y:y\in AC^n([a,T],\mathbb{R})\}$, and $d(y_1,y_2)=\|y_1-y_2\|$. Obviously, (V,d) is a complete metric space.

Define operator $T:V\to V$,

$$\begin{aligned}&(Ty)(t)\\=&-a_rw(T)\xi(T,y(T))\int_a^t\frac{(t-s)^{n-1}}{w(s)}ds\\&-b_r\int_a^T z'(u)w(u)\xi(u,y(u))du\int_a^t(t-s)^{n-1}\frac{1}{w(s)}ds+a_r\int_a^t(t-s)^{n-1}\xi(s,y(s))ds\\&+b_r\int_a^t\frac{(t-s)^{n-1}}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds+\varpi(t,y(t)).\end{aligned}$$

By dint of Lemma 3.5, we derive that the boundary value problem (1.2) has solutions, if T has fixed points.

We define function $\varsigma:\mathbb{R}^2\to\mathbb{R}$, and make the following conditions hold.

(H_1) There exists a map $\psi\in\Psi$ and a constant $m>0$ satisfying

$$|\xi(t,y_1)-\xi(t,y_2)|\le\psi(|y_1-y_2|),|\varpi(t,y_1)-\varpi(t,y_2)|\le m\psi(|y_1-y_2|).$$

(H_2) There exists $\widetilde{x_0}\in V$ such that $\varsigma(\widetilde{x_0},T\widetilde{x_0}(t))\ge0$ for $a\le t\le T$.

(H_3) For $\forall t\in[a,T]$, $\varsigma(x(t),y(t))\ge0$ implies $\varsigma(Tx(t),Ty(t))\ge0$.

(H_4) For $\{x_n\}\subset V$, $x_n\to x\in V$, for each $t\in[a,T]$ and every n, $\varsigma(x_n(t),x_{n+1}(t))\ge0$, we have $\varsigma(x_n(t),x(t))\ge0$.

Theorem 3.2. *Assume that(H_1)-(H_4) are satisfied. If*

$$\frac{a_r(T-a)^n(w(T)+w(a))+2b_rw(T)(z(T)-z(a))(T-a)^n}{nw(a)}+m<1,$$

then equation (1.2) *has a solution.*

Proof. Let $\alpha:V\times V\to[0,\infty)$ by

$$\alpha(x,y)=\begin{cases}1,&\varsigma(x(t),y(t))\ge0,\\0,&else.\end{cases}\tag{3.1}$$

We explain that T is α-admissible. Choosing $x, y \in V$, for $\forall t \in [a,T]$, $\alpha(x,y) \geq 1$ implies $\varsigma(x(t), y(t)) \geq 0$, then $\varsigma(Tx(t), Ty(t)) \geq 0$. We have $\alpha(Tx, Ty) \geq 1$. Hence, T is α-admissible.

Next, according to hypothesis (H_2), there exists $\widetilde{x_0} \in V$ such that $\varsigma(\widetilde{x_0}, T\widetilde{x_0}(t)) \geq 0$. That is, $\alpha(\widetilde{x_0}, T\widetilde{x_0}) \geq 1$.

The following shows that T is an α-ψ-contraction.

Letting $y_1, y_2 \in V$, for each $t \in [a,T]$, we have

$$
\begin{aligned}
&|(Ty_1)(t) - (Ty_2)(t)| \\
= &\Big| a_r w(T)\big(\xi(T, y_1(T)) - \xi(T, y_2(T))\big) \int_a^t (t-s)^{n-1} \frac{1}{w(s)} ds \\
&+ b_r \int_a^T z'(u) w(u) \big(\xi(u, y_1(u)) - \xi(u, y_2(u))\big) du \int_a^t (t-s)^{n-1} \frac{1}{w(s)} ds \\
&+ a_r \int_a^t (t-s)^{n-1} \big(\xi(s, y_1(s)) - \xi(s, y_2(s))\big) ds \\
&+ b_r \int_a^t (t-s)^{n-1} \frac{1}{w(s)} \int_a^s z'(u) w(u) \big(\xi(u, y_1(u)) - \xi(u, y_2(u))\big) du ds \\
&+ \varpi(t, y_1(t)) - \varpi(t, y_2(t)) \Big| \\
\leq & a_r w(T) \Big| \xi\big(T, y_1(T)\big) - \xi\big(T, y_2(T)\big) \Big| \int_a^t (t-s)^{n-1} \frac{1}{w(s)} ds \\
&+ b_r \int_a^T z'(u) w(u) \Big| \xi(u, y_1(u)) - \xi(u, y_2(u)) \Big| du \int_a^t (t-s)^{n-1} \frac{1}{w(s)} ds \\
&+ a_r \int_a^t (t-s)^{n-1} \Big| \xi\big(s, y_1(s)\big) - \xi\big(s, y_2(s)\big) \Big| ds \\
&+ b_r \int_a^t (t-s)^{n-1} \frac{1}{w(s)} \int_a^s z'(u) w(u) \Big| \xi\big(u, y_1(u)\big) - \xi\big(u, y_2(u)\big) \Big| du ds \\
&+ |\varpi(t, y_1(t)) - \varpi(t, y_2(t))|.
\end{aligned}
$$

Applying the mean value theorem for integrals,

$$
\begin{aligned}
&\|Ty_1 - Ty_2\| \\
\leq & \frac{a_r w(T)(T-a)^n \psi(\|y_1 - y_2\|)}{n w(a)} + \frac{b_r w(T)(z(T) - z(a))(T-a)^n \psi(\|y_1 - y_2\|)}{n w(a)} \\
&+ \frac{a_r (T-a)^n \psi(\|y_1 - y_2\|)}{n} + \frac{b_r w(T)(z(T) - z(a))(T-a)^n \psi(\|y_1 - y_2\|)}{n w(a)} \\
&+ m\psi(\|y_1 - y_2\|) \\
\leq & \psi(\|y_1 - y_2\|).
\end{aligned}
$$

Thus, we get

$$d(Ty_1, Ty_2) \leq \psi(d(y_1, y_2)),$$

which implies

$$\alpha(y_1, y_2) d(Ty_1, Ty_2) \leq \psi(d(y_1, y_2)).$$

Then, We obtain that T is α-ψ-contraction.

Lastly, from hypothesis (H_4), letting $\{x_n\}$ be a sequence in V with $\varsigma(x_n(t), x(t))$

≥ 0, we are able to derive $\alpha(x_n, x) \geq 1$.

Based on Lemma 3.2, there exists u such that $u = Tu$, which completes the proof. □

In the following, we consider the nonlinear boundary value problem (1.3). Similar to the proof of Lemma 3.5, we have the conclusion as below.

Lemma 3.6. *Assume that* $y \in AC^n([a,T],\mathbb{R})$, ξ *and* φ *are binary continuous functions,* $\varphi \in C([a,T]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, *and* $\frac{y}{\varphi} \in AC^n([a,T],\mathbb{R})$. y *is a solution to the nonlinear equation* (1.3), *if it satisfies*

$$\frac{y(t)}{\varphi(t,y(t))} - \frac{w(a)}{\Gamma(n)}\Big(\frac{1}{\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)}{M(r-n)}\Big)\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds$$
$$= a_r\int_a^t (t-s)^{n-1}\xi(s,y(s))ds + b_r\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds.$$

We suppose that neither of the assumptions holds.
(H_5) For $\varphi \in C([a,T]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, the inequality $|\varphi(t,y_1)-\varphi(t,y_2)| \leq L_2|y_1-y_2|$ holds, where $L_2 > 0$.
(H_6) There exists $\eta \in AC^n([a,T],\mathbb{R}^+)$ satisfying $|\xi(t,y(t))| \leq \eta(t)$.

Theorem 3.3. *Assume that hypotheses* (H_5)*-*(H_6) *are satisfied. If*

$$L_2\bigg(\Big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\Big| + (\theta_1+\theta_2)\|\eta\|\bigg) < 1, \quad (3.2)$$

then the boundary value problem (1.3) *has a solution, where* $\theta_1 = \frac{a_r(T-a)^n}{n}$, *and* $\theta_2 = \frac{b_r(T-a)^n w(T)(z(T)-z(a))}{nw(a)}$.

Proof. Let $\Lambda = (AC^n([a,T],\mathbb{R}), \|\cdot\|)$, where $\|y\| = \sup_{t\in[a,T]}|y(t)|$. Then, Λ is a Banach algebra with multiplication defined by $(y_1y_2)(t) = y_1(t)y_2(t)$, $y_1,y_2 \in \Lambda$, $t \in [a,T]$. Define

$$Q = \frac{M_\varphi\Big(\big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\big| + (\theta_1+\theta_2)\|\eta\|\Big)}{1-L_2\Big(\big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\big| + (\theta_1+\theta_2)\|\eta\|\Big)},$$

where $M_\varphi = \sup_{t\in[a,T]}|\varphi(t,0)|$. From condition (3.2), we can derive $Q > 0$.

Considering the set $U = \{y \in \Lambda : \|y\| \leq Q\}$, we can easily obtain that U is a bounded subset of Λ, which is closed and convex.

Considering the operators $F_1 : \Lambda \to \Lambda$ and $F_2 : U \to \Lambda$:

$$(F_1y)(t) = \varphi(t,y(t)),$$

$$(F_2y)(t) = \frac{w(a)}{\Gamma(n)}\left(\frac{1}{\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)}{M(r-n)}\right)\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds$$
$$+ a_r\int_a^t (t-s)^{n-1}\xi(s,y(s))ds$$
$$+ b_r\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u,y(u))duds,$$

we can write the fractional integral equation of Lemma 3.6 as an equivalent operator equation $y = F_1yF_2y, y \in \Lambda$.

Now, we verify the conditions of Lemma 3.3.

(i) F_1 is Lipschitz.

For any $y_1, y_2 \in \Lambda$, $t \in [a, T]$,

$$|(F_1y)(t) - (F_2y)(t)| = |\varphi(t, y_1) - \varphi(t, y_2)| \le L_2|y_1 - y_2|.$$

We obtain

$$\|F_1y - F_2y\| \le L_2\|y_1 - y_2\|.$$

(ii) F_2 is completely continuous.

Letting $y_n \to y$ in $[a, T]$, for all $t \in [a, T]$, we get

$$\begin{aligned}
&|(F_2y_n)(t) - (F_2y)(t)| \\
=& \Big|a_r\int_a^t (t-s)^{n-1}\big(\xi(s, y_n(s)) - \xi(s, y(s))\big)ds \\
&+b_r\int_a^t (t-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\big(\xi(u, y_n(u)) - \xi(u, y(u))\big)duds\Big|.
\end{aligned}$$

Similar to the first step of Theorem 3.1, we can derive that F_2 is continuous.

$$\begin{aligned}
&|(F_2y)(t)| \\
\le& \Big|\frac{w(a)}{\Gamma(n)}\Big(\frac{1}{\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)}{M(r-n)}\Big)\int_a^t (t-s)^{n-1}\frac{1}{w(s)}ds\Big| \\
&+a_r\int_a^t (t-s)^{n-1}\big|\xi(s, y(s))\big|ds + b_r\int_a^t \frac{(t-s)^{n-1}}{w(s)}\int_a^s z'(u)w(u)\big|\xi(u, y(u))\big|duds \\
\le& \Big|\frac{(T-a)^n}{n\Gamma(n)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{n\Gamma(n)M(r-n)}\Big| + \frac{a_r(T-a)^n\|\eta\|}{n} \\
&+\frac{b_r(T-a)^n w(T)(z(T)-z(a))\|\eta\|}{nw(a)} \\
\le& \Big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\Big| + (\theta_1+\theta_2)\|\eta\|,
\end{aligned}$$

which shows that F_2 is uniformly bounded.

Choosing $t_1, t_2 \in [a, T]$ with $a \le t_1 \le t_2 \le T$, we get

$$\begin{aligned}
&|(F_2y)(t_2) - (F_2y)(t_1)| \\
\le& \frac{w(a)}{\Gamma(n)}\Big|\frac{1}{\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)}{M(r-n)}\Big|\Big[\int_a^{t_1}\big((t_2-s)^{n-1} - (t_1-s)^{n-1}\big)\frac{1}{w(s)}ds \\
&+\int_{t_1}^{t_2}(t_2-s)^{n-1}\frac{1}{w(s)}ds\Big] \\
&+a_r\Big[\int_a^{t_1}\big((t_2-s)^{n-1} - (t_1-s)^{n-1}\big)\xi(s, y(s))ds + \int_{t_1}^{t_2}(t_2-s)^{n-1}\xi(s, y(s))ds\Big] \\
&+b_r\Big[\int_a^{t_1}\big((t_2-s)^{n-1} - (t_1-s)^{n-1}\big)\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u, y(u))duds \\
&+\int_{t_1}^{t_2}(t_2-s)^{n-1}\frac{1}{w(s)}\int_a^s z'(u)w(u)\xi(u, y(u))duds\Big]
\end{aligned}$$

$$\leq \Big[\frac{1}{n\Gamma(n)}\Big|\frac{1}{\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)}{M(r-n)}\Big| + \frac{a_r\|\eta\|}{n} + \frac{b_r\|\eta\|w(T)(z(T)-z(a))}{nw(a)}\Big]$$
$$[(t_2-a)^n - (t_1-a)^n].$$

As t_1 approaches t_2, we have $|(F_2y)(t_2) - (F_2y)(t_1)| \leq 0$, then F_2 is equicontinuous. Combining the above steps with the Arzela-Ascoli theorem, we can conclude that F_2 is completely continuous.
(iii) Let any $y_2 \in U$. For $y_1 \in \Lambda$, we consider that the operator equation $y_1 = F_1y_1F_2y_2$.

Our aim is to prove that $y_1 \in U$,

$$\begin{aligned}
&|y_1(t)| \\
&\leq |(F_1y_1)(t)||(F_2y_2)(t)| \\
&\leq |\varphi(t,y_1(t)) - \varphi(t,0) + \varphi(t,0)||(F_2y_2)(t)| \\
&\leq (L_2|y_1(t)| + M_\varphi)\Big(\Big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\Big| \\
&+ (\theta_1+\theta_2)\|\eta\|\Big).
\end{aligned}$$

This gives

$$|y_1(t)| \leq \frac{M_\varphi\Big(\Big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\Big| + (\theta_1+\theta_2)\|\eta\|\Big)}{1 - L_2\Big(\Big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\Big| + (\theta_1+\theta_2)\|\eta\|\Big)}.$$

Therefore,

$$|y_1(t)| \leq Q,$$

which proves $y_1 \in U$.
(iv) Let

$$\alpha = L_2,\ M = \Big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\Big| + (\theta_1+\theta_2)\|\eta\|,$$

Thus, by condition (3.2),

$$\alpha M = L_2M < 1.$$

According to the above steps (i)-(iv), we are able to derive that all the conditions of Lemma 3.3 are satisfied. Consequently, the operator equation $y = F_1yF_2y$ has a fixed point in U, which is just a solution to boundary value problem (1.3). □

4. Examples

The results that we have obtained will be tested in this section.

For the sake of convenience, we suppose the normalization function $M(r) = 1$, $w(t) = e^t$, and $z(t) = t^2$.

Example 4.1. Consider

$$\begin{cases} (D^{\frac{3}{2}}_{0,[t^2,e^t]}z)(t) = \frac{e^{-2t}}{1+e^t}|z(t)|, & t \in [0, \frac{1}{4}], \\ z(0) = 0, \\ z'(0) = 1. \end{cases} \tag{4.1}$$

Setting $r = \frac{3}{2}$, $a = 0$, $T = \frac{1}{4}$, and $\xi(t, z(t)) = \frac{e^{-2t}}{1+e^t}|z(t)|$, then

$$a_r = \frac{1-r+n}{\Gamma(n)M(r-n)} = \frac{1}{2}, \quad b_r = \frac{r-n}{\Gamma(n)M(r-n)} = \frac{1}{2},$$

and

$$\theta_1 = \frac{a_r(T-a)^n}{n} = \frac{1}{8}, \quad \theta_2 = \frac{b_r(T-a)^n w(T)(z(T)-z(a))}{nw(a)} = \frac{1}{128}e^{\frac{1}{4}}.$$

We obtain

$$\xi(t, z(t)) = \frac{e^{-2t}}{1+e^t}|z(t)| \leq \frac{e^{-2t}}{2}|z| \leq \frac{1}{2}|z| \leq \frac{1}{2}(1+|z|).$$

Letting $L_1 = \frac{1}{2}$, then

$$(\theta_1 + \theta_2)L_1 = \frac{1}{16} + \frac{1}{256}e^{\frac{1}{4}} \approx 0.07253144857 < 1.$$

On the basis of Theorem 3.1, there exists a solution to (4.1).

Example 4.2. Consider

$$\begin{cases} D^{\frac{3}{2}}_{0,[t^2,e^t]}(z(t) - \frac{e^{-t}}{9+e^t}|z(t)|) = \frac{e^{-2t}}{1+e^t}|z(t)| + 1, \qquad t \in [0, \frac{1}{4}], \\ (z - \frac{e^{-t}}{9+e^t}|z|)(0) = 0, \\ (z - \frac{e^{-t}}{9+e^t}|z|)'(\frac{1}{4}) = 0. \end{cases} \tag{4.2}$$

Setting $r = \frac{3}{2}$, $a = 0$, $T = \frac{1}{4}$, $\varpi(t, z(t)) = \frac{e^{-t}}{9+e^t}|z(t)|$, and $\xi(t, z(t)) = \frac{e^{-2t}}{1+e^t}|z(t)| + 1$, then $a_r = \frac{1}{2}$, and $b_r = \frac{1}{2}$.

For every $t \in [0, \frac{1}{4}]$,

$$|\xi(t, z_1) - \xi(t, z_2)| \leq \left|\frac{e^{-2t}}{1+e^t}\right| |z_1 - z_2| \leq \left|\frac{e^{-2t}}{2}\right| |z_1 - z_2| \leq \frac{1}{2}|z_1 - z_2|,$$

$$|\varpi(t, z_1) - \varpi(t, z_2)| \leq \left|\frac{e^{-t}}{9+e^t}\right| |z_1 - z_2| \leq \left|\frac{e^{-t}}{10}\right| |z_1 - z_2| \leq \frac{1}{10}|z_1 - z_2|.$$

Letting $\psi(t) = \frac{1}{2}t$, $m = \frac{1}{5}$, and for $x, y \in V$, putting $\varsigma(x, y) = 1$, then $(H_1)-(H_4)$ are satisfied. Further, we are able to get

$$\begin{aligned} &\frac{a_r(T-a)^n(w(T)+w(a)) + 2b_r w(T)(z(T)-z(a))(T-a)^n}{nw(a)} + m \\ &= \frac{13}{40} + \frac{9}{64}e^{\frac{1}{4}} \approx 0.5055660743 < 1. \end{aligned}$$

On the basis of Theorem 3.2, there exists a solution of (4.2).

Example 4.3. Consider

$$\begin{cases} D^{\frac{3}{2}}_{0,[t^2,e^t]} \frac{y(t)}{\frac{e^{-t}}{1+e^{2t}}|y(t)|+1} = \frac{e^{-2t}}{1+e^t}|\sin y(t)|, \qquad t \in [0, \frac{1}{4}], \\ y(0) = 0, \\ y'(0) = 1. \end{cases} \tag{4.3}$$

Setting $r = \frac{3}{2}$, $a = 0$, $T = \frac{1}{4}$, $\varphi(t, y) = \frac{e^{-t}}{1+e^{2t}}|y| + 1$, and $\xi(t, y) = \frac{e^{-2t}}{1+e^{t}}|\sin y|$, then $a_r = \frac{1}{2}$, $b_r = \frac{1}{2}$, $\theta_1 = \frac{1}{8}$, and $\theta_2 = \frac{1}{128}e^{\frac{1}{4}}$.

For every $t \in [0, \frac{1}{4}]$,

$$|\varphi(t, y_1) - \varphi(t, y_2)| \le \left|\frac{e^{-t}}{1 + e^{2t}}\right||y_1 - y_2| \le \frac{1}{2}|y_1 - y_2|.$$

Thus, $L_2 = \frac{1}{2}$.

Letting $\eta(t) = \frac{1}{2}(t + 1)$, then

$$|\xi(t, y)| = \frac{e^{-2t}}{1 + e^{t}}|\sin y| \le \frac{e^{-2t}}{2}|\sin y| \le \frac{1}{2}|\sin y| \le \frac{1}{2} \le \eta(t).$$

The condition (H_6) is satisfied, and $\|\eta\| = \sup_{t\in[0,\frac{1}{4}]} |\eta(t)| = \frac{5}{8}$.

We can also easily get $\xi(0, 0) = 0$, $\varphi(0, 0) = 1$. Therefore,

$$\begin{aligned} &L_2\left(\Big|\frac{(T-a)^n}{\Gamma(n+1)\varphi(a,0)} - \frac{(1-r+n)\xi(a,0)(T-a)^n}{\Gamma(n+1)M(r-n)}\Big| + (\theta_1+\theta_2)\|\eta\|\right) \\ &= \frac{13}{128} + \frac{5}{2048}e^{\frac{1}{4}} \\ &\approx 0.1046973277 < 1. \end{aligned}$$

On the basis of Theorem 3.3, there exists a solution of (4.3).

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