Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 1, March 2024, 107–132 DOI:10.12150/jnma.2024.107

Existence of Three Weak Solutions for a Class of Quasi-Linear Elliptic Operators with a Mixed Boundary Value Problem Containing $p(\cdot)$-Laplacian in a Variable Exponent Sobolev Space

Junichi Aramaki[1,†]

Abstract In this paper, we consider a mixed boundary value problem to a class of nonlinear operators containing $p(\cdot)$-Laplacian. More precisely, we are concerned with the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of at least three weak solutions under some hypotheses on given functions and the values of parameters.

Keywords $p(\cdot)$-Laplacian type equation, three weak solutions, mixed boundary value problem

MSC(2010) 35H30, 35D05, 35J60, 35J70.

1. Introduction

In this paper, we consider the following nonlinear problem

$$\begin{cases} -\mathrm{div}\,[S_t(x, |\boldsymbol{\nabla} u(x)|^2)\boldsymbol{\nabla} u(x)] = \lambda f_0(x, u(x)) + \mu f_1(x, u(x)) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma_1, \\ S_t(x, |\boldsymbol{\nabla} u(x)|^2)\frac{\partial u}{\partial \boldsymbol{n}}(x) = \lambda g_0(x, u(x)) + \mu g_1(x, u(x)) & \text{on } \Gamma_2, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ $(d \geq 2)$ is a bounded domain with a Lipschitz-continuous boundary Γ, Γ_1 and Γ_2 are disjoint open subsets of Γ such that $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma$, and $\boldsymbol{n}$ denotes the unit, outer and normal vector to Γ. Thus, we impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g_i : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ for $i = 0, 1$ are Carathéodory functions, and λ, μ are parameters. The function $S(x, t)$ is a Carathéodory function on $\Omega \times [0, \infty)$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$ and $S_t = \partial S/\partial t$. Then, the function $\mathrm{div}\,[S_t(x, |\boldsymbol{\nabla} u(x)|^2)\boldsymbol{\nabla} u(x)]$ is a more general operator than the $p(\cdot)$-Laplacian $\Delta_{p(x)} u(x) = \mathrm{div}\,[|\boldsymbol{\nabla} u(x)|^{p(x)-2}\boldsymbol{\nabla} u(x)]$, where $p(x) > 1$. This generality brings about difficulties and requires more conditions.

[†]The corresponding author.
Email address: aramaki@hctv.ne.jp (J. Aramaki)
[1]Division of Science, Tokyo Denki University, Hatoyama-machi, Saitama, 350-0394, Japan

The study of such a type of differential equations with $p(\cdot)$-growth conditions has been a very interesting topic recently. Studying such a problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [32]) and electrorheological fluids (Diening [12], Halsey [19], Mihăilescu and Rădulescu [24] as well as Růžička [26]).

Over the last two decades, there have been many articles on the existence of weak solutions for the Dirichlet boundary condition for the $p(\cdot)$-Laplacian type, that is,

$$\begin{cases} -\mathrm{div}\,[|\nabla u|^{p(x)-2}\nabla u] = f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \Gamma. \end{cases} \tag{1.2}$$

See, Fan [14], Ji [21,22], Fan and Zhang [16], Avci [7] and Yücedaĝ [28], for example. On the other hand, for the Steklov boundary condition, that is,

$$\begin{cases} -\mathrm{div}\,[|\nabla u|^{p(x)-2}\nabla u] = f(x,u) & \text{in } \Omega, \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \boldsymbol{n}} = 0 & \text{on } \Gamma. \end{cases} \tag{1.3}$$

See, Fan and Ji [15], Wei and Chen [27], Yücedaĝ [29], Allaoui et al., [1], Ayoujil [8] and Deng [11], for example.

However, since we can find only a few papers on the problem with the mixed boundary value condition in variable exponent Sobolev space as in (1.1) (cf. Aramaki [4–6]), we are convinced of the reason for existence of this paper.

Throughout this paper, we assume that Γ_1 and Γ_2 are disjoint open subsets of Γ such that

$$\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset. \tag{1.4}$$

When $p(x) = p = \text{const.}$, Zeidler [30] considered the following mixed boundary value problem

$$\begin{cases} \mathrm{div}\,\boldsymbol{j} = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ \boldsymbol{j}\cdot\boldsymbol{n} = h & \text{on } \Gamma_2, \end{cases} \tag{1.5}$$

where $\boldsymbol{j}$ is the current density, and $f(x), g(x)$ and $h(x)$ are given functions. If $\boldsymbol{j}$ is of the form

$$\boldsymbol{j} = -\alpha(|\nabla u|^2)\nabla u, \tag{1.6}$$

problem (1.5) corresponds to many physical problems, for example, hydrodynamics, gas dynamics, electrostatics, heat conduction, elasticity and plasticity.

If $\alpha \equiv 1$, then problem (1.5) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial \boldsymbol{n}} = h & \text{on } \Gamma_2. \end{cases} \tag{1.7}$$

From the mathematical point of view, this is a mixed boundary value problem for the Poisson equation. If $\alpha(|\nabla u|^2) = |\nabla u|^{p-2}$, problem (1.5) corresponds to the p-Laplacian equation. Definitely, if $\Gamma_2 = \emptyset$ (resp. $\Gamma_1 = \emptyset$), then system (1.5) becomes the first (resp. second) boundary value problem respectively. In order to have an intuitive understanding, let $d = 3$ and regard $u(x)$ as the temperature of a body Ω at the point x. Then, $\boldsymbol{j}$ in (1.6) is the current density vector of stationary heat flow in Ω, f describes outer heat source, and the boundary conditions means the prescription of the temperature on Γ_1 and heat flow through Γ_2. System (1.5) represents a constitutive law which depends on the specific properties of the material. If α is a positive constant, α represents the heat conductivity, and (1.5) is called heat conductivity.

Under some assumptions on f_i, g_i ($i = 0, 1$) and parameters λ and μ in (1.1), we show the existence of at least three weak solutions using at least three critical points theorem of Ricceri [25]. Here, functions f_1 and g_1 represent perturbation terms. We make efforts to be self-contained to this paper.

The paper is organized as follows. Section 2 consists of three subsections. In Subsection 2.1, we recall some results on variable exponent Lebesgue-Sobolev spaces. In Subsection 2.2, we introduce a Carathéodory function $S(x, t)$ satisfying the structure conditions and some properties. In Subsection 2.3, we consider the Nemyckii operator. Section 3 is devoted to the setting of problem (1.1) rigorously and giving a main theorem (Theorem 3.1) on the existence of at least three weak solutions. The proof of Theorem 3.1 and its corollary (Corollary 4.1) are given in Section 4.

2. Preliminaries

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d \geq 2$) with a Lipschitz-continuous (abbreviated as $C^{0,1}$) boundary Γ. Moreover, we assume that Γ satisfies (1.4).

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^d by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \ldots, a_d)$ and $\boldsymbol{b} = (b_1, \ldots, b_d)$ in $\mathbb{R}^d$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^{d} a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$. Furthermore, we denote the dual space of B by B^* and the duality bracket by $\langle \cdot, \cdot \rangle_{B^*, B}$.

2.1. Variable exponent Lebesgue and Sobolev spaces

In this subsection, we recall some well-known results on variable exponent Lebesgue-Sobolev spaces. See Diening et al., [13], [16], Kováčik and Rácosník [23] and the references therein for more details. Throughout this paper, let Ω be a bounded domain in $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ, and Ω is locally on the same side of Γ. Define $\mathcal{P}(\Omega) = \{p : \Omega \to [1, \infty); p \text{ is a measurable function}\}$. Define

$$p^+ = \operatorname*{ess\,sup}_{x \in \Omega} p(x) \text{ and } p^- = \operatorname*{ess\,inf}_{x \in \Omega} p(x). \tag{2.1}$$

For any measurable function u on Ω, a modular $\rho_{p(\cdot)} = \rho_{p(\cdot),\Omega}$ is defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx. \tag{2.2}$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \{u; u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \rho_{p(\cdot)}(u) < \infty\} \tag{2.3}$$

equipped with the Luxemburg norm

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0; \rho_{p(\cdot)}\left(\frac{u}{\lambda}\right) \le 1\right\}. \tag{2.4}$$

Then, $L^{p(\cdot)}(\Omega)$ is a Banach space. We also define, for any integer $m \ge 0$,

$$W^{m,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega); \partial^\alpha u \in L^{p(\cdot)}(\Omega) \text{ for } |\alpha| \le m\}, \tag{2.5}$$

where $\alpha = (\alpha_1, \dots, \alpha_d)$ is a multi-index, $|\alpha| = \sum_{i=1}^d \alpha_i$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ and $\partial_i = \partial/\partial x_i$, endowed with the norm

$$\|u\|_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le m} \|\partial^\alpha u\|_{L^{p(\cdot)}(\Omega)}. \tag{2.6}$$

Definitely, $W^{0,p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$. Define

$$W_0^{m,p(\cdot)}(\Omega) = \text{ the closure of the set of } W^{m,p(\cdot)}(\Omega)\text{-functions}$$

with compact supports in Ω.

The following three propositions are well-known (see Fan et al., [17], [27], Fan and Zhao [18], Zhao et al., [28,31]).

Proposition 2.1. *Let $p \in \mathcal{P}(\Omega)$ and let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \dots)$. Then, we have the following properties.*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1) \Longleftrightarrow \rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.

(ii) $\|u\|_{L^{p(\cdot)}(\Omega)} > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.

(iv) $\lim_{n\to\infty} \|u_n - u\|_{L^{p(\cdot)}(\Omega)} = 0 \Longleftrightarrow \lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty \Longleftrightarrow \rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

The following proposition is a generalized Hölder inequality.

Proposition 2.2. *Let $p \in \mathcal{P}_+(\Omega)$, where*

$$\mathcal{P}_+(\Omega) = \{p \in \mathcal{P}(\Omega); 1 < p^- \le p^+ < \infty\}. \tag{2.7}$$

(i) *For any $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that*

$$\left|\int_\Omega u(x)v(x)dx\right| \le \varepsilon\rho_{p(\cdot)}(u) + C(\varepsilon)\rho_{p'(\cdot)}(v) \text{ for all } u \in L^{p(\cdot)}(\Omega) \text{ and } v \in L^{p'(\cdot)}(\Omega). \tag{2.8}$$

(ii) *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\int_\Omega |u(x)v(x)|dx \le \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)} \le 2\|u\|_{L^{p(\cdot)}(\Omega)}\|v\|_{L^{p'(\cdot)}(\Omega)}. \tag{2.9}$$

Here and from now on, $p'(\cdot)$ is the conjugate exponent of $p(\cdot)$, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

For $p \in \mathcal{P}(\Omega)$, define

$$p^*(x) = \begin{cases} \frac{dp(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \ge d. \end{cases} \tag{2.10}$$

Proposition 2.3. *Let Ω be a bounded domain with $C^{0,1}$-boundary and let $p \in \mathcal{P}_+(\Omega)$ and $m \geq 0$ be an integer. Then, we have the following.*

(i) *The spaces $L^{p(\cdot)}(\Omega)$ and $W^{m,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If $q(\cdot) \in \mathcal{P}_+(\Omega)$ and satisfies $q(x) \leq p(x)$ for all $x \in \Omega$, then $W^{m,p(\cdot)}(\Omega) \hookrightarrow W^{m,q(\cdot)}(\Omega)$, where $\hookrightarrow$ means that the embedding is continuous.*

(iii) *If $q(x) \in \mathcal{P}_+(\Omega)$ satisfies $q(x) < p^*(x)$ for all $x \in \Omega$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact.*

We say that $p \in \mathcal{P}(\Omega)$ belongs to $\mathcal{P}^{\log}(\Omega)$, if p has the log-Hölder continuity in Ω. That is, $p : \Omega \to \mathbb{R}$ satisfies that there exists a constant $C_{\log}(p) > 0$ such that

$$|p(x) - p(y)| \leq \frac{C_{\log}(p)}{\log(e + 1/|x-y|)} \text{ for all } x, y \in \Omega. \tag{2.11}$$

We also write $\mathcal{P}^{\log}_+(\Omega) = \{p \in \mathcal{P}^{\log}(\Omega); 1 < p^- \leq p^+ < \infty\}$.

Proposition 2.4. *If $p \in \mathcal{P}^{\log}_+(\Omega)$ and $m \geq 0$ is an integer, then $\mathcal{D}(\Omega) := C_0^\infty(\Omega)$ is dense in $W_0^{m,p(\cdot)}(\Omega)$.*

For the proof, see [13, Corollary 11.2.4].

Next, we consider the trace. Let Ω be a domain of $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ and $p \in \mathcal{P}_+(\overline{\Omega})$. Since $W^{1,p(\cdot)}(\Omega) \subset W^{1,1}_{\text{loc}}(\Omega)$, the trace $\gamma(u) = u\big|_\Gamma$ to Γ of any function u in $W^{1,p(\cdot)}(\Omega)$ is well defined as a function in $L^1_{\text{loc}}(\Gamma)$. We define

$$\begin{aligned}[\mathrm{Tr}(W^{1,p(\cdot)}(\Omega)) &= (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma) \\ &= \{f; f \text{ is the trace to } \Gamma \text{ of a function } F \in W^{1,p(\cdot)}(\Omega)\}\end{aligned} \tag{2.12}$$

equipped with the norm

$$\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)} = \inf\{\|F\|_{W^{1,p(\cdot)}(\Omega)}; F \in W^{1,p(\cdot)}(\Omega) \text{ satisfying } F\big|_\Gamma = f\} \tag{2.13}$$

for $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, where the infimum can be achieved. Then, $(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$ is a Banach space. More precisely, see [13, Chapter 12]. Later, we also write $F\big|_\Gamma = g$ by $F = g$ on Γ. Moreover, we denote

$$(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i) = \{f\big|_{\Gamma_i}; f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)\} \text{ for } i = 1, 2 \tag{2.14}$$

equipped with the norm

$$\|g\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)} = \inf\{\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}; f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma) \text{ satisfying } f\big|_{\Gamma_i} = g\}, \tag{2.15}$$

where the infimum also can be achieved. Therefore, for any $g \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)$, there exists $F \in W^{1,p(\cdot)}(\Omega)$ such that $F\big|_{\Gamma_i} = g$ and $\|F\|_{W^{1,p(\cdot)}(\Omega)} = \|g\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma_i)}$.

Let $q \in \mathcal{P}_+(\Gamma) := \{q \in \mathcal{P}(\Gamma); q^- > 1\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma$. We define

$$L^{q(\cdot)}(\Gamma) = \Big\{u; u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma \text{ satisfying} \\ \int_\Gamma |u(x)|^{q(x)} d\sigma < \infty\Big\},$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0; \int_\Gamma \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma \le 1\right\}, \tag{2.16}$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma. \tag{2.17}$$

Proposition 2.5. *We have the following properties.*

(i) $\|u\|_{L^{q(\cdot)}(\Gamma)} \ge 1 \Longrightarrow \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(ii) $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1 \Longrightarrow \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \le \rho_{q(\cdot),\Gamma}(u) \le \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

Proposition 2.6. *Let Ω be a bounded domain with a $C^{0,1}$-boundary Γ satisfying (1.4) and let $p \in \mathcal{P}^{\log}_+(\overline{\Omega})$. If $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma)$ and there exists a constant $C > 0$ such that*

$$\|f\|_{L^{p(\cdot)}(\Gamma)} \le C\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}. \tag{2.18}$$

In particular, if $f \in (\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)$, then $f \in L^{p(\cdot)}(\Gamma_1)$ and

$$\|f\|_{L^{p(\cdot)}(\Gamma_1)} \le C\|f\|_{(\mathrm{Tr}W^{1,p(\cdot)})(\Gamma)}. \tag{2.19}$$

For $p \in \mathcal{P}_+(\overline{\Omega})$, define

$$p^\partial(x) = \begin{cases} \frac{(d-1)p(x)}{d-p(x)} & \text{if } p(x) < d, \\ \infty & \text{if } p(x) \ge d. \end{cases} \tag{2.20}$$

Proposition 2.7. *Let $p \in \mathcal{P}_+(\overline{\Omega})$. Then, if $q(x) \in \mathcal{P}_+(\overline{\Omega})$ satisfies $q(x) < p^\partial(x)$ for all $x \in \Gamma$, the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well-defined, compact and continuous. In particular, the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$ is compact and continuous, and there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Gamma)} \le C\|u\|_{W^{1,p(\cdot)}(\Omega)} \textit{ for } u \in W^{1,p(\cdot)}(\Omega). \tag{2.21}$$

Define a basic space of this paper by

$$X = \{v \in W^{1,p(\cdot)}(\Omega); v = 0 \text{ on } \Gamma_1\}. \tag{2.22}$$

Then, it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive, separable and uniformly convex Banach space. We show the following Poincaré type inequality (cf. Ciarlet and Dinca [10]).

Lemma 2.1. *Let $p \in \mathcal{P}^{\log}_+(\Omega)$. Then, there exists a constant $C = C(\Omega, d, p) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \le C\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \textit{ for all } u \in X. \tag{2.23}$$

In particular, $\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ is equivalent to $\|u\|_{W^{1,p(\cdot)}(\Omega)}$ for $u \in X$.

Proof. If the conclusion is false, then there exists a sequence $\{u_n\} \subset X$ such that $\|u_n\|_{L^{p(\cdot)}(\Omega)} = 1$ and $1 > n\|\boldsymbol{\nabla} u_n\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$. Since $\|u_n\|_{L^{p(\cdot)}(\Omega)} = 1$ and $\boldsymbol{\nabla} u_n \to \mathbf{0}$ strongly in $\boldsymbol{L}^{p(\cdot)}(\Omega)$, $\{u_n\}$ is bounded in $W^{1,p(\cdot)}(\Omega)$. Therefore, by the fact that X is a reflexive Banach space, there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u \in X$ such that $u_{n'} \to u$ weakly in $W^{1,p(\cdot)}(\Omega)$ and in $L^{p(\cdot)}(\Omega)$. Thus, $u_{n'} \to u$ in $\mathcal{D}'(\Omega)$ (the set of all distributions in Ω), so $\boldsymbol{\nabla} u_{n'} \to \boldsymbol{\nabla} u$ in $\boldsymbol{\mathcal{D}'}(\Omega)$. Since $\boldsymbol{\nabla} u_{n'} \to \mathbf{0}$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$, $\boldsymbol{\nabla} u = \mathbf{0}$ in $\boldsymbol{\mathcal{D}'}(\Omega)$. Therefore, $u =$ const. (cf. Boyer and Fabrie [9, Lemma II. 2.44]). As $u = 0$ on $\Gamma_1 (\neq \emptyset)$, we have $u = 0$. Thus, $u_{n'} \to 0$ weakly in $W^{1,p(\cdot)}(\Omega)$. Since $p(x) < p^*(x)$ for all $x \in \overline{\Omega}$, the embedding mapping $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact, so $u_{n'} \to 0$ strongly in $L^{p(\cdot)}(\Omega)$. This contradicts $\|u_{n'}\|_{L^{p(\cdot)}(\Omega)} = 1$. □

Thus, we can define the norm on X, so that

$$\|v\|_X = \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for } v \in X, \tag{2.24}$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$.

2.2. A Carathéodory function

Let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$ be fixed. Let $S(x,t)$ be a Carathéodory function on $\Omega \times [0,\infty)$ and assume that for a.e. $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C([0,\infty))$ satisfies the following structure conditions: there exist positive constants $0 < s_* \le s^* < \infty$ such that for a.e. $x \in \Omega$.

$$S(x,0) = 0 \text{ and } s_* t^{(p(x)-2)/2} \le S_t(x,t) \le s^* t^{(p(x)-2)/2} \text{ for } t > 0. \tag{2.25a}$$

$$s_* t^{(p(x)-2)/2} \le S_t(x,t) + 2tS_{tt}(x,t) \le s^* t^{(p(x)-2)/2} \text{ for } t > 0. \tag{2.25b}$$

$$\begin{aligned} &S_{tt}(x,t) < 0 \text{ when } 1 < p(x) < 2 \\ &\text{and } S_{tt}(x,t) \ge 0 \text{ when } p(x) \ge 2 \text{ for } t > 0, \end{aligned} \tag{2.25c}$$

where $S_t = \partial S/\partial t$ and $S_{tt} = \partial^2 S/\partial t^2$. We note that from (2.25a), we have

$$\frac{2}{p(x)} s_* t^{p(x)/2} \le S(x,t) \le \frac{2}{p(x)} s^* t^{p(x)/2} \text{ for } t \ge 0. \tag{2.26}$$

We introduce two examples.

Example 2.1. (i) When $S(x,t) = \nu(x)\frac{1}{p(x)} t^{p(x)/2}$, where ν is a measurable function in Ω satisfying $0 < \nu_* \le \nu(x) \le \nu^* < \infty$ for a.e. in Ω, the function $S(x,t)$ satisfies (2.25a)-(2.25c). We see that (1.1) is an extension of the problem (1.2) or (1.3).

(ii) As an another example, we can take

$$g(t) = \begin{cases} ae^{-1/t} + a & \text{for } t > 0, \\ a & \text{for } t = 0, \end{cases} \tag{2.27}$$

where $a > 0$ is a constant. Then, we can see that $S(x,t) = \nu(x) g(t) \frac{1}{p(x)} t^{p(x)/2}$ satisfies (2.25a)-(2.25c), if $p(x) \ge 2$ for all $x \in \overline{\Omega}$, (cf. Aramaki [2]).

We have the following estimate of S_t.

Lemma 2.2. *Under hypotheses* (2.25a)-(2.25c), *there exists a constant $c > 0$ depending only on s_* and p^+ such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$,*

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big)\cdot(\boldsymbol{a}-\boldsymbol{b})$$
$$\geq\begin{cases} c|\boldsymbol{a}-\boldsymbol{b}|^{p(x)} & \textit{when } p(x)\geq 2,\\ c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p(x)-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \textit{when } 1<p(x)<2.\end{cases} \tag{2.28}$$

For the proof, see Aramaki [3, Lemma 3.6].

Lemma 2.3. *Under hypotheses (2.25a)-(2.25b), the function* $T(x,t)=\frac{1}{2}S(x,t^2)$ *defined in* $\Omega\times[0,\infty)$ *is uniformly convex with respect to* $t\in[0,\infty)$*, that is, for any* $\varepsilon>0$*, there exists a constant* $\delta=\delta(\varepsilon)>0$ *independent of* x *such that*

$$|t-s|\leq\varepsilon\max\{t,s\}\quad \textit{or}\quad T\left(x,\frac{t+s}{2}\right)\leq(1-\delta)\frac{T(x,t)+T(x,s)}{2}$$

for a.e. $x\in\Omega$ *and all* $t,s\geq 0$*. Moreover, the function* $T(x,t)$ *is strictly monotonically increasing and strictly convex with respect to* $t\in[0,\infty)$*.*

For the proof, see [4, Lemma 2.8].
Lemma 2.3 is extended as follows.

Proposition 2.8. *For any* $\varepsilon_2>0$*, there exists a constant* $\delta_2=\delta_2(\varepsilon_2)>0$ *independent of* x *such that*

$$|\boldsymbol{a}-\boldsymbol{b}|\leq\varepsilon_2\max\{|\boldsymbol{a}|,|\boldsymbol{b}|\}\quad \textit{or}\quad T\left(x,\left|\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right|\right)\leq(1-\delta_2)\frac{T(x,|\boldsymbol{a}|)+T(x,|\boldsymbol{b}|)}{2}$$

for any $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^d$ *and a.e.* $x\in\Omega$*.*

Proof. Fix ε_2, so that $0<\varepsilon_2<\sqrt{16/3}$ and put $\varepsilon=\varepsilon_2/2$. Choose $\delta=\delta(\varepsilon)>0$ as in Lemma 2.3. Let $|\boldsymbol{a}-\boldsymbol{b}|>\varepsilon_2\max\{|\boldsymbol{a}|,|\boldsymbol{b}|\}$. If

$$||\boldsymbol{a}|-|\boldsymbol{b}||>\varepsilon\max\{|\boldsymbol{a}|,|\boldsymbol{b}|\}\left(\geq\varepsilon\frac{|\boldsymbol{a}|+|\boldsymbol{b}|}{2}\right), \tag{2.29}$$

then it follows from Lemma 2.3 that the conclusion holds with $\delta_2=\delta$. Thus, we assume $||\boldsymbol{a}|-|\boldsymbol{b}||\leq\varepsilon\max\{|\boldsymbol{a}|,|\boldsymbol{b}|\}$. Then,

$$|\boldsymbol{a}-\boldsymbol{b}|>\varepsilon_2\max\{|\boldsymbol{a}|,|\boldsymbol{b}|\}=2\varepsilon\max\{|\boldsymbol{a}|,|\boldsymbol{b}|\}\geq 2\,||\boldsymbol{a}|-|\boldsymbol{b}||\,. \tag{2.30}$$

Therefore, we have

$$\begin{aligned}\left|\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right|^2&=\frac{|\boldsymbol{a}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2}-\left|\frac{\boldsymbol{a}-\boldsymbol{b}}{2}\right|^2\\&=\frac{|\boldsymbol{a}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2}-\frac{3}{4}\left|\frac{\boldsymbol{a}-\boldsymbol{b}}{2}\right|^2-\frac{1}{4}\left|\frac{\boldsymbol{a}-\boldsymbol{b}}{2}\right|^2\\&\leq\frac{|\boldsymbol{a}|^2}{2}+\frac{|\boldsymbol{b}|^2}{2}-\frac{3}{4}\left|\frac{\boldsymbol{a}-\boldsymbol{b}}{2}\right|^2-\left(\frac{|\boldsymbol{a}|-|\boldsymbol{b}|}{2}\right)^2\\&=\left(\frac{|\boldsymbol{a}|+|\boldsymbol{b}|}{2}\right)^2-\frac{3}{4}\left|\frac{\boldsymbol{a}-\boldsymbol{b}}{2}\right|^2.\end{aligned}$$

Since $|\boldsymbol{a}-\boldsymbol{b}|>\varepsilon_2\max\{|\boldsymbol{a}|,|\boldsymbol{b}|\}\geq\varepsilon_2\frac{|\boldsymbol{a}|+|\boldsymbol{b}|}{2}$, we have

$$\left|\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right|^2\leq\left(1-\frac{3\varepsilon_2^2}{16}\right)\left(\frac{|\boldsymbol{a}|+|\boldsymbol{b}|}{2}\right)^2. \tag{2.31}$$

Let $\delta_2 = 1 - \sqrt{1 - 3\varepsilon_2^2/16} > 0$. Since $T(x,t)$ is monotonically increasing, convex with respect to t and $T(x,0) = 0$, we see

$$\begin{aligned}
T\left(x, \left|\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right|\right) &\leq T\left(x, (1-\delta_2)\frac{|\boldsymbol{a}|+|\boldsymbol{b}|}{2}\right) \\
&\leq (1-\delta_2) T\left(x, \frac{|\boldsymbol{a}|+|\boldsymbol{b}|}{2}\right) \\
&\leq (1-\delta_2)\frac{T(x,|\boldsymbol{a}|)+T(x,|\boldsymbol{b}|)}{2}.
\end{aligned}$$

□

Remark 2.1. This proposition is a slight extension of [13, Lemma 2.4.7].

2.3. The Nemytskii operator

In this subsection, we consider the Nemytskii operator. Let Ω be a open subset of $\mathbb{R}^d$ and let $\boldsymbol{f} : \Omega \times \mathbb{R}^m \to \mathbb{R}^l$ be a function, that is,

$$\boldsymbol{f}(x,\boldsymbol{p}) = (f_1(x,p_1,\dots,p_m),\dots,f_l(x,p_1,\dots,p_m)),\ x \in \Omega, \boldsymbol{p} = (p_1,\dots,p_m) \in \mathbb{R}^m.$$

Assume the following (H.1) and (H.2).

(H.1) $\boldsymbol{f}(x,\boldsymbol{p})$ is a Carathéodory function defined in $\Omega \times \mathbb{R}^m$, that is, for any $\boldsymbol{p} \in \mathbb{R}^m$, a function $x \mapsto \boldsymbol{f}(x,\boldsymbol{p})$ is measurable in Ω and for a.e. $x \in \Omega$, a function $\boldsymbol{p} \mapsto \boldsymbol{f}(x,\boldsymbol{p})$ is continuous in $\mathbb{R}^m$.

(H.2) The growth condition: for every $j = 1,\dots,l$, there exist a constant $b_j > 0$ and functions $p_i, q_j \in \mathcal{P}^{\log}(\Omega)$ such that $1 \leq p_i^- \leq p_i^+ < \infty$ for $i = 1,\dots,m$, $1 \leq q_j^- \leq q_j^+ < \infty$ and a non-negative function $a_j \in L^{q_j}(\Omega)$ such that

$$|f_j(x,\boldsymbol{p})| \leq a_j(x) + b_j \sum_{i=1}^{m} |p_i|^{p_i(x)/q_j(x)}. \tag{2.32}$$

For any function $\boldsymbol{u}(x) = (u_1(x),\dots,u_m(x))$, define the Nemytskii operator $\boldsymbol{F}$ by

$$\boldsymbol{F}(\boldsymbol{u})(x) = \boldsymbol{f}(x,\boldsymbol{u}(x)) \text{ for } x \in \Omega. \tag{2.33}$$

Then, we have the following proposition.

Proposition 2.9. *Under hypotheses (H.1) and (H.2), the Nemytskii operator*

$$\boldsymbol{F} : \prod_{i=1}^{m} L^{p_i(\cdot)}(\Omega) \to \prod_{j=1}^{l} L^{q_j(\cdot)}(\Omega) \tag{2.34}$$

is continuous and bounded with

$$\rho_{q_j(\cdot)}(F_j\boldsymbol{u}) \leq C_j \left(\rho_{q_j(\cdot)}(a_j) + \sum_{i=1}^{m} \rho_{p_i(\cdot)}(u_i)\right) \text{ for } j = 1,\dots,l. \tag{2.35}$$

Proof. It suffices to show the case $l = 1$. For the brevity of notations, we write $q = q_1, a = a_1, b = b_1, f = \boldsymbol{f}$ and $F = \boldsymbol{F}$.

First, we show the continuity of F. Let $\boldsymbol{u}^{(n)} \to \boldsymbol{u}$ in $\prod_{i=1}^{m} L^{p_i(\cdot)}(\Omega)$. By the convergent principle (cf. [5, Appendix]), there exist a subsequence $\{\boldsymbol{u}^{(n')}\}$ of $\{\boldsymbol{u}^{(n)}\}$

and $g_i \in L^{p_i(\cdot)}(\Omega)$ $(i = 1, \dots, m)$ such that $u_i^{(n')}(x) \to u_i(x)$ as $n' \to \infty$ for a.e. $x \in \Omega$ and $|u_i^{(n')}(x)| \le g_i(x)$ for all n', a.e. $x \in \Omega$ and $i = 1, \dots, m$. We show that $\|F\boldsymbol{u}^{(n')} - F\boldsymbol{u}\|_{L^{q(\cdot)}(\Omega)} \to 0$ as $n' \to \infty$. By Proposition 2.1 (iv), it suffices to show

$$\rho_{q(\cdot)}(F\boldsymbol{u}^{(n')} - F\boldsymbol{u}) = \int_\Omega |f(x, \boldsymbol{u}^{(n')}(x)) - f(x, \boldsymbol{u}(x))|^{q(x)} dx \to 0 \text{ as } n' \to \infty. \tag{2.36}$$

Since f is a Carathéodory function and $\boldsymbol{u}^{(n')}(x) \to \boldsymbol{u}(x)$ a.e. $x \in \Omega$ as $n' \to \infty$, we see

$$|f(x, \boldsymbol{u}^{(n')}(x)) - f(x, \boldsymbol{u}(x))|^{q(x)} \to 0 \text{ for a.e. } x \in \Omega \text{ as } n' \to \infty.$$

Moreover, from (H.2), we have

$$\begin{aligned}
&|f(x, \boldsymbol{u}^{(n')}(x)) - f(x, \boldsymbol{u}(x))|^{q(x)} \\
&\quad \le C(|f(x, \boldsymbol{u}^{(n')}(x))|^{q(x)} + |f(x, \boldsymbol{u}(x))|^{q(x)}) \\
&\quad \le C_1\left(a(x)^{q(x)} + b\sum_{i=1}^m |u_i^{(n')}(x)|^{p_i(x)} + b\sum_{i=1}^m |u_i(x)|^{p_i(x)}\right) \\
&\quad \le C_2\left(a(x)^{q(x)} + b\sum_{i=1}^m g_i(x)^{p_i(x)} + b\sum_{i=1}^m |u_i(x)|^{p_i(x)}\right)
\end{aligned}$$

for some constants C, C_1 and C_2. The last term is an integrable function in Ω which is independent of n'. By the Lebesgue dominated convergent theorem, (2.36) holds. By the convergent principle (cf. [30, Proposition 10.13 (i)], for the full sequence $\{\boldsymbol{u}^{(n)}\}$, we have $\rho_{q(\cdot)}(F\boldsymbol{u}^{(n)} - F\boldsymbol{u}) \to 0$, as $n \to \infty$.

Next, we consider the estimate (2.35) with $l = 1$,

$$\begin{aligned}
\rho_{q(\cdot)}(F\boldsymbol{u}) &= \int_\Omega |f(x, \boldsymbol{u}(x))|^{q(x)} dx \\
&\le \int_\Omega \left(a(x) + b\sum_{i=1}^m |u_i(x)|^{p_i(x)/q(x)}\right)^{q(x)} dx \\
&\le C_3 \int_\Omega \left(a(x)^{q(x)} + \sum_{i=1}^m |u_i(x)|^{p_i(x)}\right) dx \\
&= C_3\left(\rho_{q(\cdot)}(a) + \sum_{i=1}^m \rho_{p_i(\cdot)}(u_i)\right),
\end{aligned}$$

where C_3 is a constant. Thus, the estimate (2.35) with $l = 1$ holds. □

3. Setting of the problem and the main theorem

In this section, we consider system (1.1). From now on, we suppose the following conditions. For $i = 0, 1$,

(f_i) a Carathéodory function $f_i : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|f_i(x, t)| \le C_{1,i} + C_{2,i}|t|^{\alpha_i(x)-1} \text{ for a.e } x \in \Omega \text{ and all } t \in \mathbb{R}, \tag{3.1}$$

where $C_{1,i}$ and $C_{2,i}$ are non-negative constants and $\alpha_i \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies that $\alpha_i(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

(g_i) a Carathéodory function $g_i : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ satisfies

$$|g_i(x,t)| \leq D_{1,i} + D_{2,i}|t|^{\beta_i(x)-1} \text{ for a.e } x \in \Gamma_2 \text{ and all } t \in \mathbb{R}, \tag{3.2}$$

where $D_{1,i}$ and $D_{2,i}$ are non-negative constants and $\beta_i \in \mathcal{P}_+^{\log}(\overline{\Gamma_2})$ satisfies $\beta_i(x) < p^\partial(x)$ for all $x \in \overline{\Gamma_2}$.

We introduce the notion of weak solutions for problem (1.1).

Definition 3.1. We say $u \in X$ defined by (2.22) is a weak solution of (1.1), if

$$\begin{aligned} &\int_\Omega S_t(x, |\nabla u(x)|^2)\nabla u(x) \cdot \nabla v(x)dx \\ =&\lambda \left(\int_\Omega f_0(x,u(x))v(x)dx + \int_{\Gamma_2} g_0(x,u(x))v(x)d\sigma \right) \\ &+ \mu \left(\int_\Omega f_1(x,u(x))v(x)dx + \int_{\Gamma_2} g_1(x,u(x))v(x)d\sigma \right) \text{ for all } v \in X. \end{aligned} \tag{3.3}$$

We want to solve problem (1.1). For this purpose, we consider the functional on X defined by

$$I(u) = \Phi(u) - \lambda J(u) - \mu K(u) \text{ for } u \in X, \tag{3.4}$$

where

$$\Phi(u) = \frac{1}{2}\int_\Omega S(x, |\boldsymbol{\nabla} u(x)|^2)dx, \tag{3.5}$$

$$J(u) = \int_\Omega F_0(x,u(x))dx + \int_{\Gamma_2} G_0(x,u(x))d\sigma, \tag{3.6}$$

$$K(u) = \int_\Omega F_1(x,u(x))dx + \int_{\Gamma_2} G_1(x,u(x))d\sigma, \tag{3.7}$$

and for every $i = 0, 1$,

$$F_i(x,t) = \int_0^t f_i(x,s)ds \text{ for } (x,t) \in \Omega \times \mathbb{R}, \tag{3.8}$$

$$G_i(x,t) = \int_0^t g_i(x,s)ds \text{ for } (x,t) \in \Gamma_2 \times \mathbb{R}. \tag{3.9}$$

We are in a position to state the main theorem.

Theorem 3.1. *Let Ω be a bounded domain of $\mathbb{R}^d$ ($d \geq 2$) with a $C^{0,1}$-boundary Γ satisfying* (1.4) *and $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$ verifying*

$$p^+ - p^- < \frac{p^+p^- - p^-}{d} \text{ if } p^- < d. \tag{3.10}$$

Assume that the functions f_0 and g_0 satisfy (f_0) and (g_0). Moreover, suppose

$$\max\left\{ \limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F_0(x,t)}{|t|^{p^+}}, \limsup_{|t|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} F_0(x,t)}{|t|^{p^-}} \right\} \leq 0, \tag{3.11}$$

$$\max\left\{ \limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Gamma_2} G_0(x,t)}{|t|^{p^+}}, \limsup_{|t|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Gamma_2} G_0(x,t)}{|t|^{p^-}} \right\} \leq 0 \tag{3.12}$$

and

$$\sup_{u\in X}\left(\int_\Omega F_0(x,u(x))dx+\int_{\Gamma_2}G_0(x,u(x))d\sigma\right)>0. \tag{3.13}$$

Set

$$\theta=\inf\left\{\frac{\frac{1}{2}\int_\Omega S(x,|\nabla u(x)|^2)dx}{\int_\Omega F_0(x,u(x))dx+\int_{\Gamma_2}G_0(x,u(x))d\sigma};u\in X \text{ with } \int_\Omega F_0(x,u(x))dx+\int_{\Gamma_2}G_0(x,u(x))d\sigma>0\right\}. \tag{3.14}$$

Then, for each compact interval $[a,b]\subset(\theta,\infty)$, there exists $r>0$ with the following property: for every $\lambda\in[a,b]$ and any functions f_1 and g_1 satisfying (f_1) and (g_1), there exists $\delta>0$ such that for each $\mu\in[0,\delta]$, problem (1.1) has at least three weak solutions whose norms are less than r.

Remark 3.1. In Ji [20], the author considered the case $\Gamma_2=\emptyset$, and insisted that there exists $q\in\mathbb{R}$ such that $p^+<q<p^*(x)$ for all $x\in\Omega$. However, in general, this does not hold without the hypothesis $p^+-p^-<p^+p^-/d$.

Before the proof of Theorem 3.1, we consult the properties of the functionals Φ, J and K defined by (3.5), (3.6) and (3.7) respectively in the following subsections.

3.1. The property of Φ

Proposition 3.1. *The functional Φ on X defined by (3.5) is a positive and continuous modular, and uniformly convex, that is, for any $\varepsilon>0$, there exists $\delta>0$ such that*

$$\Phi\left(\frac{u-v}{2}\right)\le\varepsilon\frac{\Phi(u)+\Phi(v)}{2}\quad \text{or}\quad \Phi\left(\frac{u+v}{2}\right)\le(1-\delta)\frac{\Phi(u)+\Phi(v)}{2} \tag{3.15}$$

for all $u,v\in X$.

Proof. First, we note that $u_n\to u$ in X means that $\nabla u_n\to\nabla u$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$, and that the function $S(x,t)$ is a Carathéodory function on $\Omega\times[0,\infty)$. Since $S(x,|\nabla u(x)|^2)\le\frac{2s^*}{p^-}|\nabla u(x)|^{p(x)}$, by (2.26), the Nemytskii operator

$$F(\nabla u)(x)=T(x,|\nabla u(x)|)=\frac{1}{2}S(x,|\nabla u(x)|^2) \tag{3.16}$$

is continuous from $\boldsymbol{L}^{p(\cdot)}(\Omega)$ to $L^1(\Omega)$. Thus, if $u_n\to u$ in X, then

$$\int_\Omega T(x,|\nabla u_n(x)|dx\to\int_\Omega T(x,|\nabla u(x)|)dx \text{ as } n\to\infty. \tag{3.17}$$

Therefore, Φ is continuous on X. Since $u=0$ in X, if and only if $\nabla u=\mathbf{0}$ from the Poincaré type inequality Lemma 2.1, it is easy to see that Φ is a positive modular (cf. [13, Definition 2.1.1]). We derive that Φ is uniformly convex. Let $0<\varepsilon<1$. Put $\varepsilon_2=\varepsilon/2$ and choose $\delta_2>0$ as in Proposition 2.8. Assume

$$\Phi\left(\frac{u-v}{2}\right)>\varepsilon\frac{\Phi(u)+\Phi(v)}{2}. \tag{3.18}$$

We show

$$\Phi\left(\frac{u+v}{2}\right) \le \left(1-\frac{\delta_2\varepsilon}{2}\right)\frac{\Phi(u)+\Phi(v)}{2}. \tag{3.19}$$

Put $E = \left\{x \in \Omega; |\nabla u(x) - \nabla v(x)| > \frac{\varepsilon}{2}\max\{|\nabla u(x)|, |\nabla v(x)|\}\right\}$, for a.e $x \in \Omega \setminus E$,

$$|\nabla u(x) - \nabla v(x)| \le \varepsilon_2 \max\{|\nabla u(x)|, |\nabla v(x)|\} \le \varepsilon_2(|\nabla u(x)| + |\nabla v(x)|). \tag{3.20}$$

Then, we decompose Φ as $\Phi(u) = \Phi_E(u) + \Phi_{\Omega\setminus E}(u)$, where

$$\Phi_E(u) = \int_E T(x, |\nabla u(x)|)dx \text{ and } \Phi_{\Omega\setminus E}(u) = \int_{\Omega\setminus E} T(x, |\nabla u(x)|)dx. \tag{3.21}$$

Since $T(x,t)$ is a monotonically increasing and convex function on $\Omega \times [0,\infty)$ with respect to t-variable and $T(x,0) = 0$, we have

$$\begin{aligned}
\Phi_{\Omega\setminus E}\left(\frac{u-v}{2}\right) &= \int_{\Omega\setminus E} T\left(x, \left|\frac{\nabla u(x) - \nabla v(x)}{2}\right|\right) dx \\
&\le \int_{\Omega\setminus E} T\left(x, \varepsilon_2 \frac{|\nabla u(x)| + |\nabla v(x)|}{2}\right) dx \\
&\le \varepsilon_2 \int_{\Omega\setminus E} T\left(x, \frac{|\nabla u(x)| + |\nabla v(x)|}{2}\right) dx \\
&\le \varepsilon_2 \int_{\Omega\setminus E} \frac{T(x, |\nabla u(x)|) + T(x, |\nabla v(x)|)}{2} dx \\
&= \frac{\varepsilon}{2}\frac{\Phi_{\Omega\setminus E}(u) + \Phi_{\Omega\setminus E}(v)}{2} \\
&\le \frac{\varepsilon}{2}\frac{\Phi(u)+\Phi(v)}{2}.
\end{aligned}$$

From this inequality and (3.18), we have

$$\begin{aligned}
\Phi_E\left(\frac{u-v}{2}\right) &= \Phi\left(\frac{u-v}{2}\right) - \Phi_{\Omega\setminus E}\left(\frac{u-v}{2}\right) \\
&> \varepsilon\frac{\Phi(u)+\Phi(v)}{2} - \frac{\varepsilon}{2}\frac{\Phi(u)+\Phi(v)}{2} \\
&= \frac{\varepsilon}{2}\frac{\Phi(u)+\Phi(v)}{2}.
\end{aligned} \tag{3.22}$$

On the other hand, for a.e. $x \in E$, since

$$|\nabla u(x) - \nabla v(x)| > \varepsilon_2 \max\{|\nabla u(x)|, |\nabla v(x)|\}, \tag{3.23}$$

it follows from Proposition 2.8 that

$$T\left(x, \left|\frac{\nabla u(x) + \nabla v(x)}{2}\right|\right) \le (1-\delta_2)\frac{T(x, |\nabla u(x)|) + T(x, |\nabla v(x)|)}{2}. \tag{3.24}$$

Hence,

$$\Phi_E\left(\frac{u+v}{2}\right) = \int_E T\left(x, \left|\frac{\nabla u(x) + \nabla v(x)}{2}\right|\right) dx \le (1-\delta_2)\frac{\Phi_E(u)+\Phi_E(v)}{2}. \tag{3.25}$$

Since the function $\mathbb{R}^d \ni \boldsymbol{a} \mapsto T(x, |\boldsymbol{a}|)$ is convex, we see

$$\frac{T(x,|\boldsymbol{a}|)+T(x,|\boldsymbol{b}|)}{2} - T\left(x, \left|\frac{\boldsymbol{a}+\boldsymbol{b}}{2}\right|\right) \geq 0. \tag{3.26}$$

Therefore, we have

$$\begin{aligned}
&\frac{\Phi_{\Omega\setminus E}(u)+\Phi_{\Omega\setminus E}(v)}{2} - \Phi_{\Omega\setminus E}\left(\frac{u+v}{2}\right) \\
&= \int_{\Omega\setminus E}\left\{\frac{T(x,|\boldsymbol{\nabla} u(x)|)+T(x,|\boldsymbol{\nabla} v(x)|)}{2} - T\left(x,\left|\frac{\boldsymbol{\nabla} u(x)+\boldsymbol{\nabla} v(x)}{2}\right|\right)\right\} dx \geq 0.
\end{aligned} \tag{3.27}$$

Therefore, from (3.25) and (3.22), we have

$$\begin{aligned}
\frac{\Phi(u)+\Phi(v)}{2} - \Phi\left(\frac{u+v}{2}\right) &= \frac{\Phi_E(u)+\Phi_E(v)}{2} - \Phi_E\left(\frac{u+v}{2}\right) \\
&\quad + \frac{\Phi_{\Omega\setminus E}(u)+\Phi_{\Omega\setminus E}(v)}{2} - \Phi_{\Omega\setminus E}\left(\frac{u+v}{2}\right) \\
&\geq \frac{\Phi_E(u)+\Phi_E(v)}{2} - \Phi_E\left(\frac{u+v}{2}\right) \\
&\geq \frac{\Phi_E(u)+\Phi_E(v)}{2} - (1-\delta_2)\frac{\Phi_E(u)+\Phi_E(v)}{2} \\
&= \delta_2 \frac{\Phi_E(u)+\Phi_E(v)}{2} \\
&= \delta_2 \int_E \frac{T(x,|\boldsymbol{\nabla} u(x)|)+T(x,|\boldsymbol{\nabla} v(x)|)}{2} dx \\
&\geq \delta_2 \int_E T\left(x,\left|\frac{\boldsymbol{\nabla} u(x)-\boldsymbol{\nabla} v(x)}{2}\right|\right) dx \\
&= \delta_2 \Phi_E(u-v) \\
&\geq \frac{\delta_2\varepsilon}{2}\frac{\Phi(x)+\Phi(v)}{2}.
\end{aligned}$$

This means that (3.19) holds. □

Since Φ is a modular on X, the modular space and the Luxembourg norm associated with Φ are defined by

$$X_\Phi = \{u \in X; \lim_{\tau\to 0}\Phi(\tau u) = 0\} \tag{3.28}$$

and

$$\|u\|_\Phi = \inf\left\{\tau > 0; \Phi\left(\frac{u}{\tau}\right) \leq 1\right\} \text{ for } u \in X_\Phi. \tag{3.29}$$

Clearly, we see $X_\Phi = X$. By (2.26), we have

$$\begin{aligned}
\int_\Omega \frac{s_*}{p(x)}\left|\frac{\boldsymbol{\nabla} u(x)}{\tau}\right|^{p(x)} dx \leq \Phi\left(\frac{u}{\tau}\right) &= \frac{1}{2}\int_\Omega S\left(x,\left|\frac{\boldsymbol{\nabla} u(x)}{\tau}\right|^2\right) dx \\
&\leq \int_\Omega \frac{s^*}{p(x)}\left|\frac{\boldsymbol{\nabla} u(x)}{\tau}\right|^{p(x)} dx.
\end{aligned} \tag{3.30}$$

Hence,

$$\frac{s_*}{p^+}\int_\Omega \left|\frac{\nabla u(x)}{\tau}\right|^{p(x)} dx \le \Phi\left(\frac{u}{\tau}\right) \le \frac{s^*}{p^-}\int_\Omega \left|\frac{\nabla u(x)}{\tau}\right|^{p(x)} dx. \tag{3.31}$$

Therefore, there exist $0 < c < 1$ and $C > 1$ such that

$$c\int_\Omega \left|\frac{\nabla u(x)}{\tau}\right|^{p(x)} dx \le \Phi\left(\frac{u}{\tau}\right) \le C\int_\Omega \left|\frac{\nabla u(x)}{\tau}\right|^{p(x)} dx. \tag{3.32}$$

Since $p(x) > 1$, we have $c^{p(x)} \le c$ and $C \le C^{p(x)}$. Thus, we have

$$c\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \le \|u\|_\Phi \le C\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}. \tag{3.33}$$

Lemma 3.1. *If $u_n \to u$ weakly in X and $\Phi(u_n) \to \Phi(u)$ as $n \to \infty$, then $u_n \to u$ strongly in X.*

Proof. If $u_n \to u$ weakly in X, then clearly $u_n \to u$ weakly in X_Φ. From this and the hypothesis, it follows from [13, Lemma 2.4. 17] that

$$\Phi\left(\frac{u_n - u}{2}\right) \to 0 \text{ as } n \to \infty. \tag{3.34}$$

From (3.32) with $\tau = 1$, $\rho_{p(\cdot)}(\nabla(u_n - u)) \to 0$ as $n \to \infty$, so from Proposition 2.1 (iv) , $u_n \to u$ strongly in X as $n \to \infty$. □

Proposition 3.2. *Let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$. Then, we can see that the following properties are satisfied.*

(i) *We can see $\Phi \in C^1(X, \mathbb{R})$.*

(ii) *The functional Φ is sequentially weakly lower semi-continuous, coercive on X, that is,*

$$\lim_{\|u\|_X \to \infty} \frac{\Phi(u)}{\|u\|_X} = \infty, \tag{3.35}$$

and bounded on every bounded subset of X.

(iii) *$\Phi \in \mathcal{W}_X$, that is, if $u_n \to u$ weakly in X and $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$, then the sequence $\{u_n\}$ has a subsequence converging to u strongly in X.*

Proof. (i) Clearly, Φ is Gâteaux differentiable at every $u \in X$ and for any $v \in X$, the Gâteaux differential $d\Phi$ is written by

$$d\Phi(u)(v) = \int_\Omega S_t(x, |\nabla u(x)|^2)\nabla u(x) \cdot \nabla v(x) dx. \tag{3.36}$$

We show the continuity of $d\Phi$. Let $u_n \to u$ in X, so $\nabla u_n \to \nabla u$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$. By Proposition 2.2 (iv), we have

$$\begin{aligned} &|\big(d\Phi(u_n) - d\Phi(u)\big)(v)| \\ =&\left|\int_\Omega \big(S_t(x, |\nabla u_n(x)|^2)\nabla u_n(x) - S_t(x, |\nabla u(x)|^2)\nabla u(x)\big) \cdot \nabla v(x) dx\right| \\ \le& 2\|S_t(\cdot, |\nabla u_n(\cdot)|^2)\nabla u_n(\cdot) - S_t(\cdot, |\nabla u(\cdot)|^2)\nabla u(\cdot)\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}\|v\|_X \text{ for all } v \in X. \end{aligned} \tag{3.37}$$

Thus, we have

$$\|d\Phi(u_n) - d\Phi(u)\|_{X^*} \le 2\|S_t(\cdot, |\nabla u_n(\cdot)|^2)\nabla u_n(\cdot) - S_t(\cdot, |\nabla u(\cdot)|^2)\nabla u(\cdot)\|_{L^{p'(\cdot)}(\Omega)}. \tag{3.38}$$

If we define $f(x, p) = S_t(x, |p|^2)p$ if $p \neq 0$ and $f(x, p) = 0$ if $p = 0$, then $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$ is a Carathéodory function and from (2.25a),

$$|f(x, p)| \le S_t(x, |p|^2)|p| \le s^*|p|^{p(x)-1} = s^*|p|^{p(x)/p'(x)}. \tag{3.39}$$

Therefore, it follows from Proposition 2.9 that

$$\|S_t(\cdot, |\nabla u_n(\cdot)|^2)\nabla u_n(\cdot) - S_t(\cdot, |\nabla u(\cdot)|^2)\nabla u(\cdot)\|_{L^{p'(\cdot)}(\Omega)} \to 0 \text{ as } n \to \infty, \tag{3.40}$$

so $\|d\Phi(u_n) - d\Phi(u)\|_{X^*} \to 0$, as $n \to \infty$. Hence, $d\Phi : X \to X^*$ is continuous, so Φ is Fréchet differentiable in X and the Fréchet derivative $\Phi' = d\Phi$ belongs to $C^1(X, \mathbb{R})$.

(ii) Since $[0, \infty) \ni t \mapsto S(x, t^2)$ is convex from Lemma 2.3, the functional Φ is also convex, and continuous from (i). We show that Φ is sequentially weakly lower semi-continuous on X. If it is false, then there exist a sequence $\{u_n\} \subset X$ and $u \in X$ such that $u_n \to u$ weakly in X and $\Phi(u) > \liminf_{n\to\infty} \Phi(u_n)$. Then, there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\lim_{n'\to\infty} \Phi(u_{n'}) = \liminf_{n\to\infty} \Phi(u_n)$. Hence, $\Phi(u) > \lim_{n'\to\infty} \Phi(u_{n'})$, so there exist $r \in \mathbb{R}$ and $n'_0 \in \mathbb{N}$ such that $\Phi(u) > r$ and $\Phi(u_{n'}) \le r$ for $n' \ge n'_0$. Since Φ is continuous, $M_r := \{v \in X; \Phi(v) \le r\}$ is a closed and convex subset of X. By the Mazur theorem, M_r is weakly closed. Since $u_{n'} \to u$ weakly in X, $u \in M_r$. This is a contradiction to $\Phi(u) > r$.

For $\|u\|_X > 1$,

$$\Phi(u) = \frac{1}{2}S(x, |\nabla u(x)|^2)dx \ge \frac{s_*}{p^+}\int_\Omega |\nabla u(x)|^{p(x)}dx \ge \frac{s_*}{p^+}\|u\|_X^{p^-}. \tag{3.41}$$

Since $p^- > 1$, we see that Φ is coercive.

Let $\|u\|_X \le M$. Then,

$$0 \le \Phi(u) \le \frac{s^*}{p^-}\int_\Omega |\nabla u(x)|^{p(x)}dx \le \frac{s^*}{p^-}\|u\|_X^{p^-} \vee \|u\|_X^{p^+} \le \frac{s^*}{p^-}M^{p^-} \vee M^{p^+}. \tag{3.42}$$

Therefore, Φ is bounded on every bounded subset of X.

(iii) Let $u_n \to u$ weakly in X and let $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$. Since Φ is sequentially weakly lower semi-continuous from (ii), we have $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$. Thus, $\liminf_{n\to\infty} \Phi(u_n) = \Phi(u)$. Hence, there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\lim_{n'\to\infty} \Phi(u_{n'}) = \liminf_{n\to\infty} \Phi(u_n) = \Phi(u)$. By Lemma 3.1, we see that $u_{n'} \to u$ strongly in X. □

Now, we consider the properties of the derivative Φ' of Φ.

Proposition 3.3. *The mapping $\Phi' : X \to X^*$ has the following properties.*

(i) *The mapping Φ' is strictly monotone on X, bounded on every bounded subset of X and coercive in the sense that*

$$\lim_{\|u\|_X\to\infty} \frac{\langle \Phi'(u), u\rangle_{X^*,X}}{\|u\|_X} = \infty. \tag{3.43}$$

(ii) *The mapping Φ' is of (S_+)-type, that is,*

$$u_n \to u \text{ weakly in } X \text{ and } \limsup_{n\to\infty}\langle \Phi'(u_n), u_n - u\rangle_{X^*,X} \le 0$$

imply $u_n \to u$ strongly in X.

(iii) *We can see that the mapping $\Phi' : X \to X^*$ is a homeomorphim.*

Proof. (i) For a.e. $x \in \Omega$, define $\varphi(\boldsymbol{a}) = \frac{1}{2}S(x, |\boldsymbol{a}|^2)$ for $\boldsymbol{a} \in \mathbb{R}^d$. First, it follows from (2.25a) and (2.25b) that for a.e. $x \in \Omega$, a function $G(t) = S(x, t^2)$ is a strictly monotonically increasing and strictly convex function with respect to $t \in [0, \infty)$. Indeed, $G'(t) = 2tS_s(x, t^2) \geq 2s_* t^{p(x)-1} > 0$ for $t > 0$ from (2.25a) and

$$G''(t) = 2(S_t(x, t^2) + 2t^2 S_{tt}(x, t^2)) > 0, \tag{3.44}$$

for $t > 0$ from (2.25b).

Thus, for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d$ and $0 \leq \lambda \leq 1$, we have

$$\varphi(\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}) \leq S(x, (\lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|)^2) \leq \lambda\varphi(\boldsymbol{a}) + (1-\lambda)\varphi(\boldsymbol{b}). \tag{3.45}$$

Thus, φ is convex. Moreover, let $\boldsymbol{a} \neq \boldsymbol{b}$ and $0 < \lambda < 1$. When

$$|\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}| < \lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|, \tag{3.46}$$

since the function $S(x, t^2)$ is strictly monotonically increasing with respect to t, we have

$$\begin{aligned}\varphi(\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}) &= \frac{1}{2}S(x, |\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}|^2) \\ &< \frac{1}{2}S(x, (\lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|)^2) \leq \lambda\varphi(\boldsymbol{a}) + (1-\lambda)\varphi(\boldsymbol{b}).\end{aligned}$$

When $|\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}| = \lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|$, we see that $\boldsymbol{a}$ and $\boldsymbol{b}$ are linearly dependent since $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|$. We may assume $\boldsymbol{b} \neq \boldsymbol{0}$, so we can write $\boldsymbol{a} = c\boldsymbol{b}$. Since $c|\boldsymbol{b}|^2 = |\boldsymbol{a}||\boldsymbol{b}|$, we see $c \geq 0$ and $c \neq 1$. Thus, we have $|\boldsymbol{a}| \neq |\boldsymbol{b}|$. Since $S(x, t^2)$ is strictly convex with respect to t, we have

$$S(x, |\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}|^2) = S(x, (\lambda|\boldsymbol{a}| + (1-\lambda)|\boldsymbol{b}|)^2) < \lambda S(x, |\boldsymbol{a}|^2) + (1-\lambda)S(x, t, |\boldsymbol{b}|^2). \tag{3.47}$$

Therefore, we have $\varphi(\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}) < \lambda\varphi(\boldsymbol{a}) + (1-\lambda)\varphi(\boldsymbol{b})$.

Thereby, it follows from [30, Proposition 25.10] that φ' is strictly monotone on X, that is,

$$\langle \varphi'(\boldsymbol{a}) - \varphi'(\boldsymbol{b}), \boldsymbol{a} - \boldsymbol{b} \rangle_{\mathbb{R}^d, \mathbb{R}^d} > 0 \text{ for all } \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^d \text{ with } \boldsymbol{a} \neq \boldsymbol{b}.$$

Since for $u, v \in X$, $u \neq v$ in X means that $\boldsymbol{\nabla} u \neq \boldsymbol{\nabla} v$ in $\boldsymbol{L}^{p(\cdot)}(\Omega)$, we have

$$\begin{aligned}&\langle \Phi'(u) - \Phi'(v), u - v \rangle_{X^*, X} \\ &= \int_\Omega \big(S_t(x, |\boldsymbol{\nabla} u(x)|^2)\boldsymbol{\nabla} u(x) - S_t(x, |\boldsymbol{\nabla} v(x)|^2)\boldsymbol{\nabla} v(x)\big) \cdot (\boldsymbol{\nabla} u(x) - \boldsymbol{\nabla} v(x) dx > 0.\end{aligned} \tag{3.48}$$

Thus, Φ is strictly monotone.

We show that Φ' is bounded on every bounded subset of X. Let $\|u\|_X \leq M$. Then, $\rho_{p(\cdot)}(|\boldsymbol{\nabla} u|) \leq M_1$ for some constant M_1. By the Hölder inequality (Proposition 2.2 (ii)),

$$\begin{aligned}|\langle \Phi'(u), v \rangle_{X^*, X}| &= \left| \int_\Omega S_t(x, |\boldsymbol{\nabla} u(x)|^2)\boldsymbol{\nabla} u(x) \cdot \boldsymbol{\nabla} v(x) dx \right| \\ &\leq 2\|S_t(\cdot, |\boldsymbol{\nabla} u(\cdot)|^2)\boldsymbol{\nabla} u(\cdot)\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)} \|v\|_X \text{ for all } v \in X.\end{aligned}$$

Hence, $\|\Phi'(u)\|_{X^*} \le 2\|S_t(x, |\nabla u|^2)\nabla u\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}$. Since

$$\begin{aligned}\rho_{p'(\cdot)}(S_t(x, |\nabla u|^2)\nabla u) &= \int_\Omega (S_t(x, |\nabla u(x)|^2)|\nabla u(x)|)^{p'(x)} dx \\ &\le \int_\Omega (s^* |\nabla u(x)|^{p(x)-1})^{p'(x)} dx \\ &\le \max\{(s^*)^{(p')^-}, (s^*)^{(p')^+}\} \int_\Omega |\nabla u(x)|^{p(x)} dx \\ &\le \max\{(s^*)^{(p')^-}, (s^*)^{(p')^+}\} M_1,\end{aligned}$$

we have $\|\Phi'(u)\|_{X^*} \le M_2$ for some constant M_2.

We show that Φ' is coercive. Since

$$\begin{aligned}\langle \Phi'(u), u\rangle_{X^*,X} = \int_\Omega S_t(x, |\nabla u(x)|^2)|\nabla u(x)|^2 dx \ge s_* \int_\Omega |\nabla u(x)|^{p(x)} dx \\ \ge s_* \|u\|_X^{p^-} \text{ for } \|u\|_X > 1 \end{aligned} \tag{3.49}$$

and $p^- > 1$, we can see that Φ' is coercive.

(ii) follows from [6, Proposition 9]. The proof consists of the reverse Hölder inequality in which we use (2.25a)-(2.25c). Here, we omit the proof.

(iii) We note that Φ' is coercive from (i) and clearly hemi-continuous, that is, for any $u, v, w \in X$, the mapping $[0, 1] \ni \tau \mapsto \langle \Phi'(u + \tau v), w\rangle_{X^*,X}$ is continuous. Since Φ' is strictly monotone from (i), Φ' is injective. By the Browder-Minty theorem (cf. [30, Theorem 26 A]), Φ' is surjective. Thus, $(\Phi')^{-1}$ exists. Since Φ' is continuous, it suffices to show that $(\Phi')^{-1} : X^* \to X$ is continuous. Let $f_n \to f$ in X^* as $n \to \infty$. Define $u_n = (\Phi')^{-1} f_n$ and $u = (\Phi')^{-1} f$. Then, $\Phi'(u_n) = f_n$ and $\Phi'(u) = f$. We derive that $\{u_n\}$ is bounded in X. Indeed, if $\{u_n\}$ is unbounded, there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\|u_{n'}\|_X \to \infty$ as $n' \to \infty$. Hence, there exists a constant $C > 0$ such that

$$\langle \Phi'(u_{n'}), u_{n'}\rangle_{X^*,X} = \langle f_{n'}, u_{n'}\rangle_{X^*,X} \le \|f_{n'}\|_{X^*}\|u_{n'}\|_X \le C\|u_{n'}\|_X. \tag{3.50}$$

This contradicts the coerciveness of Φ'.

Since $\{u_n\}$ is bounded in a reflexive Banach space X, there exist a subsequence $\{u_{n''}\}$ of $\{u_n\}$ and $u_0 \in X$ such that $u_{n''} \to u_0$ weakly in X, so

$$\begin{aligned}\lim_{n''\to\infty} \langle \Phi'(u_{n''}), u_{n''} - u_0\rangle_{X^*,X} &= \lim_{n''\to\infty} \langle \Phi'(u_{n''}) - \Phi'(u), u_{n''} - u_0\rangle_{X^*,X} \\ &= \lim_{n''\to\infty} \langle f_{n''} - f, u_{n''} - u_0\rangle_{X^*,X} = 0.\end{aligned} \tag{3.51}$$

Since Φ' is of (S_+)-type, we see $u_{n''} \to u_0$ strongly in X. Since Φ' is continuous, $\Phi'(u_{n''}) = f_{n''} \to \Phi'(u_0) = f = \Phi'(u)$. Hence, $\Phi'(u_0) = \Phi'(u)$. Since Φ' is injective, we see $u = u_0$. By the convergent principle (cf. [30, Thereom 10.13 (i)]), the full sequence $u_n \to u$ strongly in X. □

3.2. The properties of the functionals J and K

In this subsection, we consider the functionals J and K defined by (3.6) and (3.7), respectively.

Proposition 3.4. *Assume that* (f_i) *and* (g_i) $(i = 0, 1)$ *hold. Then, the following* (i) *and* (ii) *are verified.*

(i) *We see that* $J, K \in C^1(X, \mathbb{R})$.

(ii) *The mappings* $J', K' : X \to X^*$ *are sequentially weakly-strongly continuous, namely, if* $u_n \to u$ *weakly in* X*, then* $J'(u_n) \to J'(u)$ *and* $K'(u_n) \to K(u)$ *strongly in* X^**, so* J' *and* K' *are compact operators. Moreover, the functionals* $J, K : X \to \mathbb{R}$ *are sequentially weakly continuous.*

Proof. For brevity of notations, we write $f = f_i, g = g_i, \alpha = \alpha_i, \beta = \beta_i, F = F_i$ and $G = G_i$ for $i = 0, 1$. If we put the functionals

$$\widehat{F}(u) = \int_\Omega F(x, u(x))dx \text{ and } \widehat{G}(u) = \int_{\Gamma_2} G(x, u(x))d\sigma \text{ for } u \in X, \tag{3.52}$$

it suffices to derive that $\widehat{F}$ and $\widehat{G}$ satisfy (i) and (ii).

(i) Clearly, $\widehat{F}$ and $\widehat{G}$ are Gâteaux differentiable at every $u \in X$ and the Gâteaux derivatives $d\widehat{F}, d\widehat{G} : X \to X^*$ are given by

$$d\widehat{F}(u)(v) = \int_\Omega f(x, u(x))v(x)dx \text{ and } d\widehat{G}(u)(v) = \int_{\Gamma_2} g(x, u(x))v(x)d\sigma \tag{3.53}$$

for any $v \in X$.

We show that $d\widehat{F}, d\widehat{G} : X \to X^*$ are continuous. Let $u_n \to u$ in X. Then, by Hölder inequality (Proposition 2.2 (ii)), we have

$$\begin{aligned}|d\widehat{F}(u_n)(v) - d\widehat{F}(u)(v)| &= \left|\int_\Omega (f(x, u_n(x)) - f(x, u(x)))v(x)dx\right| \\ &\le 2\|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\|_{L^{\alpha'(\cdot)}(\Omega)}\|v\|_{L^{\alpha(\cdot)}(\Omega)}\end{aligned}$$

and

$$\begin{aligned}|d\widehat{G}(u_n)(v) - d\widehat{G}(u)(v)| &= \left|\int_{\Gamma_2} (g(x, u_n(x)) - g(x, u(x)))v(x)d\sigma\right| \\ &\le 2\|g(\cdot, u_n(\cdot)) - g(\cdot, u(\cdot))\|_{L^{\beta'(\cdot)}(\Gamma_2)}\|v\|_{L^{\beta(\cdot)}(\Gamma_2)}.\end{aligned}$$

Since $\alpha(x) < p^*(x)$ for all $x \in \Omega$ and $\beta(x) < p^\partial(x)$ for all $x \in \Gamma_2$, the embedding mappings $X \hookrightarrow L^{\alpha(\cdot)}(\Omega)$ and $X \hookrightarrow L^{\beta(\cdot)}(\Gamma_2)$ are continuous, so there exist positive constants C and D such that

$$\|v\|_{L^{\alpha(\cdot)}(\Omega)} \le C\|v\|_X \text{ and } \|v\|_{L^{\beta(\cdot)}(\Gamma_2)} \le D\|v\|_X \text{ for } v \in X. \tag{3.54}$$

Thus, we have

$$\|d\widehat{F}(u_n)(v) - d\widehat{F}(u)(v)\|_{X^*} \le 2C\|f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))\|_{L^{\alpha'(\cdot)}(\Omega)} \tag{3.55}$$

and

$$\|d\widehat{G}(u_n)(v) - d\widehat{G}(u)(v)\|_{X^*} \le 2D\|g(\cdot, u_n(\cdot)) - g(\cdot, u(\cdot))\|_{L^{\beta'(\cdot)}(\Gamma_2)}. \tag{3.56}$$

By Proposition 2.1 (iv), it suffices to show

$$\rho_{\alpha'(\cdot)}(f(\cdot, u_n(\cdot)) - f(\cdot, u(\cdot))) = \int_\Omega |f(x, u_n(x)) - f(x, u(x))|^{\alpha'(x)}dx \to 0 \tag{3.57}$$

and

$$\rho_{\beta'(\cdot),\Gamma_2}(g(\cdot,u_n(\cdot))-g(\cdot,u(\cdot)))=\int_{\Gamma_2}|g(x,u_n(x))-g(x,u(x))|^{\beta'(x)}d\sigma\to 0, \quad (3.58)$$

as $n\to\infty$. Since $\alpha(x)<p^*(x)$ for all $x\in\Omega$ and the embedding map $X\hookrightarrow W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{\alpha(\cdot)}(\Omega)$ is continuous, we have $u_n\to u$ in $L^{\alpha(\cdot)}(\Omega)$. By the convergent principle (cf. [5, Appendix]), there exist a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $0\le\widehat{u}\in L^{\alpha(\cdot)}(\Omega)$ such that $u_{n'}(x)\to u(x)$ a.e. $x\in\Omega$ as $n'\to\infty$ and $|u_{n'}(x)|\le|\widehat{u}(x)|$ for a.e. $x\in\Omega$ and all n'. Since f is a Carathéodory function, $|f(x,u_{n'}(x))-f(x,u(x))|^{\alpha'(x)}\to 0$ a.e. $x\in\Omega$ as $n'\to\infty$ and from (f_i) with $i=0,1$,

$$\begin{aligned}|f(x,u_{n'}(x))-f(x,u(x))|^{\alpha'(x)}&\le 2^{(\alpha')^+-1}(|f(x,u_{n'}(x)|^{\alpha'(x)}+|f(x,u(x))|^{\alpha'(x)})\\&\le C_1(1+|u_{n'}(x)|^{\alpha(x)}+|u(x)|^{\alpha(x)})\\&\le C_1(1+|\widehat{u}(x)|^{\alpha(x)}+|u(x)|^{\alpha(x)}).\end{aligned}$$

The last term belongs to $L^1(\Omega)$ and is independent of n'. By the Lebesgue dominated convergence theorem,

$$\rho_{\alpha'(\cdot)}(f(\cdot,u_{n'}(\cdot))-f(\cdot,u(\cdot)))\to 0 \text{ as } n'\to\infty. \quad (3.59)$$

By the convergent principle (cf. [30, Theorem 10.13 (i)]), for the full sequence $\{u_n\}$,

$$\rho_{\alpha'(\cdot)}(f(\cdot,u_n(\cdot))-f(\cdot,u(\cdot)))\to 0 \text{ as } n\to\infty. \quad (3.60)$$

Similarly, since $\beta(x)<p^\partial(x)$ for all $x\in\Gamma_2$, the embedding map $X\hookrightarrow W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{\beta(\cdot)}(\Gamma_2)$ is continuous, we can derive

$$\rho_{\beta'(\cdot),\Gamma_2}(g(\cdot,u_n(\cdot))-g(\cdot,u(\cdot)))\to 0 \text{ as } n\to\infty. \quad (3.61)$$

Thus, $\widehat{F}$ and $\widehat{G}$ are Fréchet differentiable and the Fréchet derivatives $\widehat{F}'$ and $\widehat{G}'$ satisfy $\widehat{F}'=d\widehat{F}$ and $\widehat{G}'=d\widehat{G}$, so $\widehat{F},\widehat{G}\in C^1(X,\mathbb{R})$.

(ii) Since $\alpha(x)<p^*(x)$ for all $x\in\Omega$ and $\beta(x)<p^\partial(x)$ for all $x\in\Gamma_2$, it follows from Proposition 2.3 (iii) and Proposition 2.7 that $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{\alpha(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)\hookrightarrow L^{\beta(\cdot)}(\Gamma_2)$ are compact embedding mappings. Let $u_n\to u$ weakly in X, so weakly in $W^{1,p(\cdot)}(\Omega)$. Then, $u_n\to u$ strongly in $L^{\alpha(\cdot)}(\Omega)$ and in $L^{\beta(\cdot)}(\Gamma_2)$. Repeating the arguments of the proof of (i), we see $\widehat{F}'(u_n)\to\widehat{F}'(u)$ and $\widehat{G}'(u_n)\to\widehat{G}'(u)$ strongly in X^*, so $\widehat{F}'$ and $\widehat{G}'$ are sequentially weakly-strongly continuous, so are compact operators.

We show that $\widehat{F}:X\to\mathbb{R}$ is sequentially weakly continuous. Let $u_n\to u$ weakly in X. By the mean value theorem,

$$\widehat{F}(u_n)-\widehat{F}(u)=\langle(\widehat{F}'(u+\theta_n(u_n-u)),u_n-u\rangle_{X^*,X} \quad (0<\theta_n<1). \quad (3.62)$$

Then, $u+\theta_n(u_n-u)\to u$ weakly in X. Since $\widehat{F}'$ is weakly-strongly continuous, $\widehat{F}'(u+\theta_n(u_n-u))\to\widehat{F}'(u)$ strongly in X^*. Therefore,

$$\langle\widehat{F}'(u+\theta_n(u_n-u)),u_n-u\rangle_{X^*,X}\to 0 \text{ as } n\to\infty. \quad (3.63)$$

Hence, $\widehat{F}(u_n)\to\widehat{F}(u)$ as $n\to\infty$.

Similarly, we can show $\widehat{G}(u_n)\to\widehat{G}(u)$ as $n\to\infty$. □

4. Proof of Theorem 3.1

In this section, we give a proof of Theorem 3.1. Let Ω be a bounded domain of $\mathbb{R}^d$ with a $C^{0,1}$-boundary Γ satisfying (1.4), and assume that $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies (3.10).

We apply the following result of [25, Theorem 2].

Theorem 4.1. *Let B be a separable, reflexive and real Banach space. Assume that a functional $\Phi : B \to \mathbb{R}$ is coercive, sequentially weakly lower semi-continuous, Φ is a C^1-functional belonging to $\mathcal{W}_B$, bounded on every bounded subset of B and the derivative $\Phi' : B \to B^*$ admits a continuous inverse $(\Phi')^{-1} : B^* \to B$. Moreover, assume that $J : B \to \mathbb{R}$ is a C^1-functional with a compact derivative, and assume that Φ has a strictly local minimum $u_0 \in B$ with $\Phi(u_0) = J(u_0) = 0$. Finally, put*

$$\alpha = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\}, \tag{4.1}$$

$$\beta = \sup_{u\in\Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)}, \tag{4.2}$$

and assume $\alpha < \beta$. Then, for each compact interval $[a, b] \subset \left(\frac{1}{\beta}, \frac{1}{\alpha}\right)$ (with the conventions $\frac{1}{0} = \infty$, $\frac{1}{\infty} = 0$), there exists $r > 0$ with the following property: for every $\lambda \in [a, b]$ and every C^1-functional $K : B \to \mathbb{R}$ with a compact derivative, there exists $\delta > 0$ such that for each $\mu \in [0, \delta]$, the equation $\Phi'(u) = \lambda J'(u) + \mu K'(u)$ has at least three solutions whose norms are less than r.

Proof. We note that if $u \in X$ is a critical point of the functional I, that is, $I'(u) = \Phi'(u) - \lambda J'(u) - \mu K'(u) = 0$, then u is a weak solution of (1.1). Under the hypotheses of Theorem 3.1, we derive the hypotheses of Theorem 4.1 with $B = X$ defined by (2.22) and the functionals Φ, J and K defined by (3.5), (3.6) and (3.7). Since $\Phi(u) \geq 0$ for all $u \in X$, and $\Phi(u) = 0$, if and only if $u = 0$, Φ has a strictly local minimum $u = 0$, and by the definitions of F_0 and G_0, clearly $J(0) = 0$, so $\Phi(0) = J(0) = 0$. Moreover, the hypotheses on Φ and J follows from the results of Section 3.

Fix $\varepsilon > 0$. From (3.11) and (3.12), there exist ρ_1 and ρ_2 with $0 < \rho_1 < 1 < \rho_2$ such that

$$F_0(x, t) \leq \varepsilon|t|^{p^+} \quad \text{for all } (x, t) \in \Omega \times [-\rho_1, \rho_1], \tag{4.3}$$

$$F_0(x, t) \leq \varepsilon|t|^{p^-} \quad \text{for all } (x, t) \in \Omega \times \big(\mathbb{R} \setminus [-\rho_2, \rho_2]\big) \tag{4.4}$$

and

$$G_0(x, t) \leq \varepsilon|t|^{p^+} \quad \text{for all } (x, t) \in \Gamma_2 \times [-\rho_1, \rho_1], \tag{4.5}$$

$$G_0(x, t) \leq \varepsilon|t|^{p^-} \quad \text{for all } (x, t) \in \Gamma_2 \times \big(\mathbb{R} \setminus [-\rho_2, \rho_2]\big). \tag{4.6}$$

Thus, we have

$$F_0(x, t) \leq \varepsilon|t|^{p^+} \text{ for all } (x, t) \in \Omega \times \big(\mathbb{R} \setminus ([-\rho_2, -\rho_1] \cup [\rho_1, \rho_2])\big) \tag{4.7}$$

and

$$G_0(x, t) \leq \varepsilon|t|^{p^+} \text{ for all } (x, t) \in \Gamma_2 \times \big(\mathbb{R} \setminus ([-\rho_2, -\rho_1] \cup [\rho_1, \rho_2])\big). \tag{4.8}$$

On the other hand, since f_0 and g_0 satisfy (f_0) and (g_0) respectively, we have

$$|F_0(x,t)| \le C_{1,0}|t| + \frac{C_{2,0}}{\alpha_0(x)}|t|^{\alpha_0(x)} \le C_{1,0}|t| + \frac{C_{2,0}}{\alpha_0^-}|t|^{\alpha_0(x)} \text{ for } (x,t) \in \Omega \times \mathbb{R} \quad (4.9)$$

and

$$|G_0(x,t)| \le D_{1,0}|t| + \frac{D_{2,0}}{\beta_0(x)}|t|^{\beta_0(x)} \le D_{1,0}|t| + \frac{D_{2,0}}{\beta_0^-}|t|^{\beta(x)} \text{ for } (x,t) \in \Gamma_2 \times \mathbb{R}. \quad (4.10)$$

Hence, F_0 is bounded on each bounded subset of $\Omega \times \mathbb{R}$ and G_0 is bounded on each bounded subset of $\Gamma_2 \times \mathbb{R}$. □

From hypothesis (3.10),

$$p^+ < \frac{dp^-}{d-p^-} \le \frac{dp(x)}{d-p(x)} = p^*(x) \text{ if } p(x) < d \quad (4.11)$$

and

$$p^+ < \frac{(d-1)p^-}{d-p^-} \le \frac{(d-1)p(x)}{d-p(x)} = p^\partial(x) \text{ if } p(x) < d. \quad (4.12)$$

If we choose $q \in \mathbb{R}$ such that $p^+ < q < p^\partial(x)$ for all $x \in \Gamma_2$ and $p^+ < q < p^*(x)$ for all $x \in \Omega$, then we have

$$F_0(x,t) \le \varepsilon|t|^{p^+} + c|t|^q \text{ for all } (x,t) \in \Omega \times \mathbb{R} \quad (4.13)$$

and

$$G_0(x,t) \le \varepsilon|t|^{p^+} + c|t|^q \text{ for all } (x,t) \in \Gamma_2 \times \mathbb{R} \quad (4.14)$$

for some constant $c > 0$. Since the embedding mappings $X \hookrightarrow L^{p^+}(\Omega)$, $L^{p^+}(\Gamma_2)$, $L^q(\Omega)$, $L^q(\Gamma_2)$ are continuous, there exist positive constants C_{p^+} and C_q such that

$$\begin{aligned} &\|u\|_{L^{p^+}(\Omega)} \le C_{p^+}\|u\|_X, \|u\|_{L^{p^+}(\Gamma_2)} \le C_{p^+}\|u\|_X, \\ &\|u\|_{L^q(\Omega)} \le C_q\|u\|_X \text{ and } \|u\|_{L^q(\Gamma_2)} \le C_q\|u\|_X \end{aligned} \quad (4.15)$$

for all $u \in X$. Thus, there exists a constant $c_1 > 0$ such that

$$\begin{aligned} J(u) &= \int_\Omega F_0(x,u(x))dx + \int_{\Gamma_2} G_0(x,u(x))d\sigma \\ &\le \varepsilon \int_\Omega |u(x)|^{p^+}dx + c_1\int_\Omega |u(x)|^q dx + \varepsilon\int_{\Gamma_2}|u(x)|^{p^+}d\sigma + c_1\int_{\Gamma_2}|u(x)|^q d\sigma \\ &\le 2(C_{p^+})^{p^+}\varepsilon\|u\|_X^{p^+} + 2c_1(C_q)^q\|u\|_X^q. \end{aligned}$$

When $\|u\|_X < 1$, it follows from Proposition 2.1 that

$$\frac{J(u)}{\Phi(u)} \le \frac{2(C_{p^+})^{p^+}\varepsilon\|u\|_X^{p^+} + 2c(C_q)^q\|u\|_X^q}{\frac{s_*}{p^+}\|u\|_X^{p^+}}. \quad (4.16)$$

Since $q > p^+$, we have

$$\limsup_{u\to 0}\frac{J(u)}{\Phi(u)} \le \frac{2p^+(C_{p^+})^{p^+}}{s_*}\varepsilon. \quad (4.17)$$

On the other hand, since the embedding mappings $X \hookrightarrow L^{p^-}(\Omega), L^{p^-}(\Gamma_2)$ are continuous, there exists a constant $C_{p^-} > 0$ such that

$$\|u\|_{L^{p^-}(\Omega)} \leq C_{p^-}\|u\|_X \text{ and } \|u\|_{L^{p^-}(\Gamma_2)} \leq C_{p^-}\|u\|_X \text{ for all } u \in X. \tag{4.18}$$

Since F and G are bounded on each bounded subset of $\Omega\times\mathbb{R}$ and $\Gamma_2\times\mathbb{R}$ respectively, when $\|u\|_X > 1$, it follows from (4.4) and (4.6) that there exists a constant $C_1 > 0$ such that

$$\begin{aligned} J(u) &= \int_{\{x\in\Omega;|u(x)|\leq\rho_2\}} F_0(x,u(x))dx + \int_{\{x\in\Omega;|u(x)|>\rho_2\}} F_0(x,u(x))dx \\ &\quad + \int_{\{x\in\Gamma_2;|u(x)|\leq\rho_2\}} G_0(x,u(x))d\sigma + \int_{\{x\in\Gamma_2;|u(x)|>\rho_2\}} G_0(x,u(x))d\sigma \\ &\leq 2C_1 + 2\varepsilon(C_{p^-})^{p^-}\|u\|_X^{p^-}. \end{aligned}$$

Hence,

$$\limsup_{\|u\|_X\to\infty} \frac{J(u)}{\Phi(u)} \leq \frac{2p^+(C_{p^-})^{p^-}}{s_*}\varepsilon. \tag{4.19}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (4.17) and (4.19) that

$$\max\left\{\limsup_{u\to 0}\frac{J(u)}{\Phi(u)}, \limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)}\right\} \leq 0. \tag{4.20}$$

Therefore, we have $\alpha = 0$ in Theorem 4.1. By hypothesis (3.13), we have $\beta > 0$ in (4.2). Thus, all the hypotheses of Theorem 4.1 hold. If we put $\theta = 1/\beta$, then the conclusion of Theorem 3.1 is verified. This completes the proof of Theorem 3.1.

Now, we state a corollary of Theorem 3.2. Assume that

$(f_0)'$ A Carathéodory function $f_0 : \Omega\times\mathbb{R}\to\mathbb{R}$ satisfies

$$|f_0(x,t)| \leq C_{1,0} + C_{2,0}|t|^{\alpha_0(x)-1} \text{ for a.e. } x\in\Omega \text{ and all } t\in\mathbb{R}, \tag{4.21}$$

where $C_{1,0}$ and $C_{2,0}$ are non-negative constants, and $\alpha_0 \in \mathcal{P}_+^{\log}(\Omega)$ satisfies

$$\alpha_0^+ < p^- \text{ and } \lim_{t\to 0}\frac{|f_0(x,t)|}{|t|^{p^+-1}} = 0 \text{ uniformly for a.e. } x\in\Omega. \tag{4.22}$$

$(g_0)'$ A Carathéodory function $g_0 : \Gamma_2\times\mathbb{R}\to\mathbb{R}$ satisfies

$$|g_0(x,t)| \leq D_{1,0} + D_{2,0}|t|^{\beta_0(x)-1} \text{ for a.e. } x\in\Gamma_2 \text{ and all } t\in\mathbb{R}, \tag{4.23}$$

where $D_{1,0}$ and $D_{2,0}$ are non-negative constants, and $\beta_0 \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfies

$$\beta_0^+ < p^- \text{ and } \lim_{t\to 0}\frac{|g_0(x,t)|}{|t|^{p^+-1}} = 0 \text{ uniformly for a.e. } x\in\Gamma_2. \tag{4.24}$$

(h) There exists $\delta_0 > 0$ such that

$$f_0(x,t) > 0 \text{ for } (x,t)\in\Omega\times(0,\delta_0] \text{ and } g_0(x,t)\geq 0 \text{ for } (x,t)\in\Gamma_2\times(0,\delta_0] \tag{4.25}$$

or

$$f_0(x,t) \geq 0 \text{ for } (x,t)\in\Omega\times(0,\delta_0] \text{ and } g_0(x,t) > 0 \text{ for } (x,t)\in\Gamma_2\times(0,\delta_0]. \tag{4.26}$$

Then, we obtain the following corollary of Theorem 3.1.

Corollary 4.1. *Let Ω be a bounded domain with a $C^{0,1}$-boundary Γ satisfying* (1.4) *and let $p \in \mathcal{P}_+^{\log}(\overline{\Omega})$ satisfy* (3.10). *Assume that $(f_0)'$, $(g_0)'$ and (h) hold. Then, the conclusion of Theorem 3.1 holds, that is, problem* (1.1) *has at least three weak solutions.*

Proof. From (4.22), for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|t| < \delta$, then $|f_0(x,t)| \le \varepsilon |t|^{p^+-1}$. Hence, for $|t| < \delta$, $|F_0(x,t)| \le \frac{\varepsilon}{p^+}|t|^{p^+}$, we have

$$\limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F_0(x,t)}{|t|^{p^+}} \le \frac{\varepsilon}{p^+}. \tag{4.27}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F_0(x,t)}{|t|^{p^+}} \le 0. \tag{4.28}$$

On the other hand, since f_0 is bounded on each bounded subset of $\Omega \times \mathbb{R}$ from $(f_0)'$, there exists a constant $C > 0$ such that $|f_0(x,t)| \le C$ for $(x,t) \in \Omega \times [-1,1]$. When $|t| > 1$,

$$|f_0(x,t)| \le C_{0,1} + C_{0,2}|t|^{\alpha_0(x)-1} \le C_{0,1} + C_{0,2}|t|^{\alpha_0^+-1}, \tag{4.29}$$

so we have $|f_0(x,t)| \le C'_{0,1} + C_{0,2}|t|^{\alpha_0^+-1}$. Thus, $|F_0(x,t)| \le C'_{0,1}|t| + C'_{0,2}|t|^{\alpha_0^+}$ for some constants $C'_{0,1}$ and $C'_{0,2}$. Therefore, since $\alpha_0^+ < p^-$,

$$\limsup_{|t|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} F_0(x,t)}{|t|^{p^-}} \le 0, \tag{4.30}$$

so (3.11) holds.

Similarly, using $(g_0)'$, we can derive

$$\limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Gamma_2} G_0(x,t)}{|t|^{p^+}} \le 0 \text{ and } \limsup_{|t|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Gamma_2} G_0(x,t)}{|t|^{p^-}} \le 0, \tag{4.31}$$

Therefore, (3.12) holds.

Under (h), since we can easily choose $0 \not\equiv \varphi \in X$ with $0 \le \varphi(x) \le \delta_0$ such that

$$\int_\Omega F_0(x,\varphi(x))dx + \int_{\Gamma_2} G_0(x,\varphi(x))d\sigma > 0, \tag{4.32}$$

(3.13) holds. Thus, all the hypotheses of Theorem 3.1 hold, so the conclusion of Corollary 4.1 follows from Theorem 3.1. □

Acknowledgements

The author would like to thank the anonymous reviewers for useful comments and suggestions that have helped improve this paper.

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