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Spatial Dynamics of a Lattice Lotka-Volterra Competition Model with a Shifting Habitat*

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Abstract In this paper, we concern with the spatial dynamics of the lattice Lotka-Volterra competition system in a shifting habitat. We study the impact of the environmental deterioration rate on the population density under the strong competition condition. Our results show that if the environment deteriorates rapidly, both species will become extinct. However, when the environmental degradation rate is not so fast, the species with slow diffusion will go extinct, while those with fast diffusion will survive. The extinction of species with slow diffusion can be divided into two situations: one is the extinction caused by environmental deterioration faster than its own diffusion speed, the other is the extinction caused by slow diffusion speed under the influence of strong competition.

Keywords Lattice Lotka-Volterra competition model, shifting habitat, spreading speed

MSC(2010) 34K31, 35B40, 92D40, 92D25

1. Introduction

It is well known that the evolution of a species depends on spatial location and population dispersal [2–8]. Therefore, spatial factors should be considered in relevant biological models. The famous Fisher-KPP equation, which was considered in [1], has the following form:

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial^2 x} + f(u(t,x)), \tag{1.1}$$

where $u(t,x)$ represents the population density of representative species u at position x and time t, and d represents diffusion rate of species. $f(u(t,x))$ is the function that describes population growth. Note that Fisher's equation only considers the interaction within species, but it can not account for the interaction between species. In fact, since resources and habitats are limited, competition will inevitably occur in the real world. There are many models that can well describe these phenomena, such as the Lotka-Volterra competition systems [34–36]. The Lotka-Volterra competitive

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diffusion system of two species has the following form:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1(t,x) + u_1(t,x)(r_1 - u_1(t,x) - a_1 u_2(t,x)), \quad t>0,\ x \in \Omega, \\ \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2(t,x) + u_2(t,x)(r_2 - a_2 u_1(t,x) - u_2(t,x)), \quad t>0,\ x \in \Omega, \end{cases} \tag{1.2}$$

where $x \in \Omega \subset \mathbb{R}^m$, $t > 0$ and all the parameters are non-negative, and $u_1(t,x)$ and $u_2(t,x)$ represent the densities of two competing species with diffusion rates d_1 and d_2 respectively. The dynamical properties of (1.2) have been extensively studied (see [37–41]. In fact, the spatial heterogeneity will not only affect the diffusion of species, but also the intrinsic growth rate of species. Therefore, based on this premise, Hastings [9] and Dockery [10] naturally considered the following system

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1(t,x) + u_1(r(x) - u_1 - u_2), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^m, \\ \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2(t,x) + u_1(r(x) - u_1 - u_2), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^m. \end{cases} \tag{1.3}$$

Here, the two species are the same except for their different rates of spread($d_1 \neq d_2$). This is because the authors wanted to know whether slower or faster diffusion will have a selection advantage. This will happen when one species (or strain) mutates from the other with different diffusion rate. Consequently, the two species have the same competition strength against each other and the same growth rate $r(x)$ which reflect the growth rate of the population and environmental quality.

It has been proved that if Ω is bounded and the flux at the boundary is zero, then the species with slow diffusion will win this competition, i.e., if $d_1 < d_2$, then all positive solutions of (1.3) will converge to $(u_1^*(x), 0)$, where $u_1^*(x)$ is the unique positive solution to the boundary value problem

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1(t,x) + u_1(r(x) - u_1), \quad x \in \Omega \subset \mathbb{R}^m, \\ \dfrac{\partial u_1}{\partial n} = 0, \quad x \in \partial\Omega, \end{cases}$$

When we consider the case when two species have different interspecific competition strengths, the Lotka-Volterra system (1.3) is modified to

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1 \Delta u_1(t,x) + u_1(r(x) - u_1 - a_1 u_2), \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^m, \\ \\ \dfrac{\partial u_2}{\partial t} = d_2 \Delta u_2(t,x) + u_2(r(x) - a_2 u_1 - u_2). \quad t > 0, \quad x \in \Omega \subset \mathbb{R}^m, \end{cases} \tag{1.4}$$

where the constant $a_i > 0 (i = 1,2)$ represents the competition strength of species j against species $i (i \neq j)$. There have been many studies on this more general model and many interesting results have been obtained, including the existence of the coexistence steady state under some conditions. For details, see, e.g., [9–18] and the references therein.

In recent years, the environmental degradation caused by industrialization is becoming more and more serious, and the impact on the habitat of biological population has become more and more obvious. For example, habitats are shrinking due to global warming and vegetation destruction. More and more scientists, including mathematical modelers and analysts, are paying attention and doing research in this area. A simple model of environmental change assumes that the environmental quality changes at a constant rate, and therefore the growth rate of the species will change accordingly. Based on the above considerations, Li et al. [19] considered the following reaction-diffusion equation for a single species living in the one-dimensional whole space $\mathbb{R}$

$$\frac{\partial u}{\partial t} = d\frac{\partial^2 u}{\partial^2 x} + u(r(x-ct)-u), \quad t>0, \quad x\in\mathbb{R}. \tag{1.5}$$

Here, the growth function r depends on the time t and location x, and it moves at a constant rate c. $r(t,x) = r(x-ct)$ satisfies the following hypothesis

($\mathbf{H}_1$) $r(\xi)$ is continuous, nondecreasing, bounded and piecewise continuously differentiable for $\xi \in \mathbb{R}$ with $-\infty < r(-\infty) < 0$ and $0 < r(+\infty) < +\infty$, where $r(\pm\infty) = \lim_{\xi\to\pm\infty} r(\xi)$.

From the above hypothesis, we know that the environment gradually deteriorates over time and is ultimately not suitable for biological growth($r(-\infty) < 0$). It is shown in [19] that if the environmental conditions deteriorate, and the speed $c > c^*(+\infty) := 2\sqrt{dr(+\infty)}$, then the species will go extinct in the habitat; when $c < c^*(+\infty)$, the species will persist and spread along the gradient of the shifting habitat at an asymptotic spreading speed c^* under some condition on its initial distribution. In fact, we can also consider another situation. If $r(-\infty) > 0$ is used instead of $-\infty < r(-\infty) < 0$ in ($\mathbf{H}_1$), then the environment has changed from "severely worsening" to "mildly worsening". Authors in [20] focused on this situation. They have investigated traveling waves for a general KPP type reaction diffusion equation. The much earlier work [21] also explored the traveling waves of a general reaction diffusion equation which contains (1.5) as a special case but with the growth function $r(\xi)$ having compact support.

Hence, if environmental degradation is added to the competitive system, system (1.4) will become

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\Delta u_1(t,x) + u_1(r(x-ct)-u_1-a_1u_2), t>0, x\in\Omega\subset\mathbb{R}^m, \\ \dfrac{\partial u_2}{\partial t} = d_2\Delta u_2(t,x) + u_2(r(x-ct)-a_2u_1-u_2), t>0, x\in\Omega\subset\mathbb{R}^m. \end{cases} \tag{1.6}$$

Many researchers have contributed to this model. Berestycki et al. [22] studied the nontrivial forced wave and the gap formation caused by the climate change when $a_1 = a_2 = 1$. Under the hypothesis ($\mathbf{H}_1$), Yuan et al. [23] investigated the dynamic behavior of system (1.6), they answered the question of how a species can survive when faced with the environmental degradation and competitive pressure from other species. Meanwhile, they found that a weak but faster competitor can coexist with a strong but slower competitor if the environmental degradation speed is not too fast. Further, Yuan et al. [24] discussed the case when the faster diffuser is a strong competitor with the climate change satisfy hypothesis ($\mathbf{H}_1$), and they found that two species cannot coexist under any conditions. Zhang et al. [25] established the analytical conditions for the coexistence or competitive exclusion of two competitors under the climate change, which demonstrated the ways to maintain the survival of species.

On the other hand, the distribution of species in ecology can generally be divided into three types as follows: random, uniform and aggregated dispersion. For the aggregated dispersion, it is more appropriate to use the lattice dynamical system than the continuous PDE model to describe the natural phenomena in some cases [26]. In fact, lattice dynamical systems can apply to several areas in real life, such as material science, image processing, pattern recognition, chemical reaction, biological system and so on [26–30]. When considering lattice dynamical systems of single species in shfting habitat, there will be the following model:

$$\frac{\partial u(t,x)}{\partial t} = d\mathcal{N}[u](t,x) + u(t,x)[r(x-ct) - u(t,x)], \tag{1.7}$$

where $t > 0, x \in \mathbb{R}$ or $\mathbb{Z}$ and $\mathcal{N}[u](t,x) = u(t,x+1) - 2u(t,x) + u(t,x-1)$ indicates that the diffusion mode of the species is discrete diffusion. Hu and Li [31] used the classical Bessel functions to the solutions, and ultimately got the long term behavior of solutions which relies on the environmental degradation speed c and a constant $c^*(+\infty)$ that is determined by the largest growth rate $r(+\infty)$and the diffusion coefficient d.

In this paper, we consider the following lattice Lotka-Volterra competition model with the shifting habitat:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = d_1\mathcal{N}[u_1](t,x) + u_1(r(x-ct) - u_1 - a_1u_2), & t>0,\ x \in \mathbb{R}, \\[2ex] \dfrac{\partial u_2}{\partial t} = d_2\mathcal{N}[u_2](t,x) + u_2(r(x-ct) - a_2u_1 - u_2), & t>0,\ x \in \mathbb{R}, \\[2ex] u_1(0,x) = \varphi_1(x), u_2(0,x) = \varphi_2(x), & x \in \mathbb{R}, \end{cases} \tag{1.8}$$

where $\mathcal{N}[u_i](t,x) = u_i(t,x+1) - 2u_i(t,x) + u_i(t,x-1), i = 1,2.$

Meng et.al [32] has studied the above model under the weak competition condition, *i.e*, $0 < a_i < 1, i = 1,2$. They gave the existence and uniqueness of the solution to model (1.8) and the long time dynamic behavior. Precisely, they got the following results.

Theorem 1.1. *If $\varphi_1$and φ_2 are bounded and uniformly continuous on $\mathbb{R}$, and $0 \leq \varphi_1(x), \varphi_2(x) \leq r(+\infty)$, $x \in \mathbb{R}$, then* (1.8) *admits a unique solution* $(u_1(t,x), u_2(t,x))$.

Let
$$c_i(+\infty) = \inf_{\lambda>0} \frac{d_i(e^{\lambda} - 2 + e^{-\lambda}) + r(+\infty)}{\lambda}, i = 1,2. \tag{1.9}$$

If d_2 is greater than d_1, then $c_2(+\infty)$ is greater than $c_1(+\infty)$ due to $e^{\lambda} - 2 + e^{-\lambda} > 0$ when $\lambda > 0$. Further, authors gave the long-time behavior of (1.9) under the weak competition condition.

Theorem 1.2. *Assume that* ($\mathbf{H}_1$)*holds and* $0 < a_i < 1, i = 1,2$, *let* $(u_1(t,x), u_2(t,x))$ *be the unique solution of* (1.8). *Then the following propositions hold:*

(i) *Assume that* $c > c_2(+\infty)$. *If* $\varphi_1(x)$ *and* $\varphi_2(x)$ *has a compact support, then* $\forall\varepsilon > 0, \exists t_* > 0$ *such that when* $t > t_*$,
$$(u_1(t,x), u_2(t,x)) \le (\varepsilon, \varepsilon), \quad \forall x \in \mathbb{R}.$$

(ii) *Assume that* $c \in (c_1(+\infty), c_2(+\infty))$. *If* $\varphi_1(x)$ *and* $\varphi_2(x)$ *have a compact support,* $\sup_{x\in\mathbb{R}} \varphi_1(x) < r_1(+\infty)$, *and on a closed interval,* $\varphi_2(x) > 0$, *then* $\forall l \in (0, \frac{c_2(+\infty)-c}{2})$, $\exists T_2 > 0$ *such that* $u_1(t,x) \le l$, $\forall(t,x) \in [T_2, +\infty) \times \mathbb{R}$ *and* $\lim_{t\to+\infty, x\in\mathcal{P}_t} u_2(t,x) = r_2(+\infty)$, *where*
$$\mathcal{P}_t = \{x \in \mathbb{R} : (c+l)t \le x \le (c_2(+\infty) - l)t\}.$$

(iii) *There exists a constant* $\hat{c}(+\infty) < c_1(+\infty)$ *such that* $c \in (0, \hat{c}(+\infty))$. *If* $\varphi_i(x) > 0 (i = 1, 2)$ *on a closed interval, then for* $\forall\delta \in (0, \frac{\hat{c}(+\infty)-c}{2})$, *it holds*
$$\lim_{t\to+\infty} \sup_{x\in\mathcal{Q}_t} |u_1(t,x) - \frac{1 - a_1a_2}{r_1(+\infty) - a_1r_2(+\infty)}| = 0$$
and
$$\lim_{t\to+\infty} \sup_{x\in\mathcal{Q}_t} |u_2(t,x) - \frac{1 - a_1a_2}{r_2(+\infty) - a_2r_1(+\infty)}| = 0,$$
where $\mathcal{Q}_t = \{x \in \mathbb{R} : (c+\delta)t \le x \le (\hat{c}(+\infty) - \delta)t\}$.

Besides, they gave the definition of upper and lower solutions as follow:

Definition 1.1. We refer to the functions $(\overline{u}_1, \overline{u}_2)$ and $(\underline{u}_1, \underline{u}_2)$ as a pair of the upper and lower solutions of (1.8), provided that
$$\begin{cases} \partial_t\overline{u}_1 \ge d_1\mathcal{N}[\overline{u_1}](t,x) + \overline{u}_1[r(x-ct) - \overline{u}_1 - a_1\underline{u}_2], t > 0, x \in \mathbb{R}, \\ \partial_t\underline{u}_1 \le d_1\mathcal{N}[\underline{u_1}](t,x) + \underline{u}_1[r(x-ct) - \underline{u}_1 - a_1\overline{u}_2], t > 0, x \in \mathbb{R}, \\ \overline{u}_1(0,x) \ge \underline{u}_1(0,x), x \in \mathbb{R} \end{cases}$$
and
$$\begin{cases} \partial_t\overline{u}_2 \ge d_2\mathcal{N}[\overline{u_2}](t,x) + \overline{u}_2[r(x-ct) - a_2\underline{u}_1 - \overline{u}_2], t > 0, x \in \mathbb{R}, \\ \partial_t\underline{u}_2 \le d_2\mathcal{N}[\underline{u_2}](t,x) + \underline{u}_2[r(x-ct) - a_2\overline{u}_1 - \underline{u}_2], t > 0, x \in \mathbb{R}, \\ \overline{u}_2(0,x) \ge \underline{u}_2(0,x), x \in \mathbb{R}. \end{cases}$$

After giving the definition of upper and lower solutions, they also established the comparison principle of system (1.8) as follows:

Lemma 1.1. *Let* $(\overline{u}_1, \overline{u}_2)$ *and* $(\underline{u}_1, \underline{u}_2)$ *be a pair of upper and lower solutions of* (1.8). *If* $\overline{u}_i(0,x) \ge \underline{u}_i(0,x), i = 1, 2, x \in \mathbb{R}$, *then* $\overline{u}_i(t,x) \ge \underline{u}_i(t,x), i = 1, 2, and \forall(t,x) \in [0,+\infty) \times \mathbb{R}$.

In this paper, we mainly focus on the spatial dynamics of (1.8) in strong competition condition i.e. $a_i \ge 1, i = 1, 2$. We always assume that $d_1 < d_2$. Our result can be summarized as follows:

(i) If $c > c_2(+\infty)$, then $\lim\limits_{t\to+\infty}(u_1(t,x), u_2(t,x)) = (0,0)$ for system (1.8). In other words, if the environment deteriorates too fast, both species will eventually tend to die out.

(ii) If $c_1(+\infty) < c < c_2(+\infty)$ and the initial value $\varphi(x)$ satisfies certain conditions, then $\lim_{t\to+\infty} u_1(t,x;\varphi) = 0$ and $\lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\le x\le(c_2^*(+\infty)-\varepsilon)t} |r(+\infty) - u_2(t,x;\varphi)|] = 0$ for system (1.8). This indicates that when the rate of environmental deterioration is in the middle value, species u_1 will become extinct and species u_2 will persist.

(iii) If $0 < c < c_1(+\infty)$, then species u_1 will become extinct and species u_2 will persist. This situation is different from the traditional strong competition situation. In this case, the impact of environmental degradation is small, and strong competition between species plays a major role in the long-term behavior of species. So we have the same conclusion as (ii).

The rest of this paper is organized as follows. In Section 2, we give specific conclusions about the long-term behavior of species and prove it. In Section 3, we give a discussion of our results.

2. Main results and proof

In this section, we will show our main results on the extinction and persistence of the two species. We mainly explore the spatial dynamics of system (1.8) in the case of strong competition. First, we introduce some functions and notations that we will use. For $\lambda > 0$, we define

$$\Phi_i(x,\lambda) = \frac{d_i[e^{\lambda} - 2 + e^{-\lambda}] + r(x)}{\lambda}, \qquad i = 1, 2.$$

By($\mathbf{H}_1$), we can easily check that

$$\lim_{\lambda\to 0^+} \Phi_i(x,\lambda) = +\infty \quad \textit{and} \quad \lim_{\lambda\to+\infty} \Phi_i(x,\lambda) = +\infty \quad \text{for } x>0 \text{ large enough.}$$

Then, we first introduce a lemma.

Lemma 2.1. *For $x > 0$ large enough, $\Phi_i(x,\lambda)$ has exactly one minimum point, i.e., there exist $\lambda_i(x)$ and $c_i(x)$, such that*

$$c_i(x) = \inf_{\lambda>0} \Phi_i(x,\lambda) = \Phi_i(x,\lambda_i(x)). \tag{2.1}$$

Proof. We just need to prove that $\partial_\lambda\Phi_i(x,\lambda) = 0$ has exactly one positive root for $x > 0$ large enough. In fact, we have

$$\partial_\lambda\Phi_i(x,\lambda) = \frac{1}{\lambda}[d_i(e^{\lambda} - e^{-\lambda}) - \Phi_i(x,\lambda)],$$

and

$$\begin{aligned}
\partial_\lambda(\lambda^2\partial_\lambda\Phi_i(x,\lambda)) &= \partial_\lambda(\lambda[d_i(e^{\lambda} - e^{-\lambda}) - \Phi_i(x,\lambda)]) \\
&= d_i(e^{\lambda} - e^{-\lambda}) - \Phi_i(x,\lambda) + \lambda[d_i(e^{\lambda} + e^{-\lambda}) - \partial_\lambda\Phi_i(x,\lambda)] \\
&= \lambda d_i(e^{\lambda} + e^{-\lambda}) \ge 0.
\end{aligned}$$

Hence, $\lambda^2\partial_\lambda\Phi_i(x,\lambda)$ is nondecreasing for $x > 0$ large enough. Due to $r(x) > 0$ for $x > 0$ large enough, then we have

$$\lim_{\lambda\to 0^+} \partial_\lambda\Phi_i(x,\lambda) = \lim_{\lambda\to 0^+} \frac{\lambda d_i(e^{\lambda} - e^{-\lambda}) - d_i(e^{\lambda} - 2 + e^{-\lambda}) - r(x)}{\lambda^2} = -\infty.$$

By L'Hosiptal's rule, we have

$$\begin{aligned}\lim_{\lambda\to+\infty}\partial_\lambda\Phi_i(x,\lambda) &= \lim_{\lambda\to+\infty}\frac{\lambda d_i(e^\lambda-e^{-\lambda})-d_i(e^\lambda-2+e^{-\lambda})-r(x)}{\lambda^2}\\ &= \lim_{\lambda\to+\infty}\frac{d_i(e^\lambda-e^{-\lambda})+\lambda d_i(e^\lambda+e^{-\lambda})-d_i(e^\lambda-e^{-\lambda})}{2\lambda}\\ &= \lim_{\lambda\to+\infty}\frac{d_i(e^\lambda+e^{-\lambda})}{2}=+\infty.\end{aligned}$$

Hence, we obtain that $\partial_\lambda\Phi_i(x,\lambda)=0$ has at least one positive root for x large enough. If $\partial_\lambda\Phi_i(x,\lambda)=0$ has two or more positive roots, it may be assumed that there exist λ_1 and λ_2, such that $\partial_\lambda\Phi_i(x,\lambda_1)=\partial_\lambda\Phi_i(x,\lambda_2)=0$, where λ_1 is the smallest root and λ_2 is the second smallest root of $\partial_\lambda\Phi_i(x,\lambda)=0$. Obviously, we have $\partial_\lambda\Phi_i(x,\lambda)\le 0$ in $\lambda\in(0,\lambda_1)$. If $\partial_\lambda\Phi_i(x,\lambda)>0$ in $\lambda\in(\lambda_1,\lambda_2)$, then $\lambda^2\partial_\lambda\Phi_i(x,\lambda)>0$ in $\lambda\in(\lambda_1,\lambda_2)$, but $\lambda_2^2\partial_\lambda\Phi_i(x,\lambda_2)=0$. This contradicts with the fact that $\lambda^2\partial_\lambda\Phi_i(x,\lambda)$ is nondecreasing. If $\partial_\lambda\Phi_i(x,\lambda)<0$ in $\lambda\in(\lambda_1,\lambda_2)$, then in view of $\lambda_1^2\partial_\lambda\Phi_i(x,\lambda_1)=0$, it contradicts with the fact that $\lambda^2\partial_\lambda\Phi_i(x,\lambda)$ is nondecreasing. Hence, $\partial_\lambda\Phi_i(x,\lambda)=0$ has exactly one positive root for $x>0$ large enough. This completes the proof. □

By [31], we can know that $c_i(+\infty)$ is the asymptotic spreading speed of species u_i when species u_j is absent($j\ne i$). Reviewing our previous assumptions $d_1<d_2$, we have $c_1(+\infty)<c_2(+\infty)$.

Next, we introduce a theorem to show that when the speed of environmental deterioration is fast, both species will become extinct.

Theorem 2.1 (Extinction)**.** *Assume that* $(\mathbf{H}_1)$ *holds and* $c>c_2(+\infty)$*, where* $c_2(+\infty)$ *is defined as (1.10). Let* $(u_1(t,x;\varphi_1,\varphi_2),u_2(t,x;\varphi_1,\varphi_2))$ *be the unique solution of* (1.9)*. If* $\varphi_1(x)$ *and* $\varphi_2(x)$ *has a compact support, then for any* $\varepsilon>0$*, there exists* $T>0$*, such that when* $t>T$*, we have*

$$(0,0)\le(u_1(t,x;\varphi_1,\varphi_2),u_2(t,x;\varphi_1,\varphi_2))\le(\varepsilon,\varepsilon),\quad \forall x\in\mathbb{R}.$$

Proof. We prove this theorem by constructing the upper and lower solutions. It can be easy to verify that $(r(+\infty),r(+\infty))$ and $(0,0)$ are a pair of upper and lower solutions of (1.8). Then by Lemma 1.1 we have $0\le u_i(t,x)\le r(+\infty), i=1,2$, for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. From the proof process of [32, Theorem 3.1], we can know that the following system

$$\partial_t u_i(t,x)=d_i[u_i(t,x+1)-2u_i(t,x)+u_i(t,x-1)]+u_i(t,x)[r(x-ct)-u_i(t,x)] \quad (2.2)$$

can admit a forced wave $\Psi_i(x-ct)$ with the profile function $\Psi_i(\cdot)$ nondecreasing and satisfying $\Psi_i(-\infty)=0$ and $\Psi_i(+\infty)=r_i(+\infty)$. Notice that $\varphi_i(x)<r(+\infty)$ for any $x\in\mathbb{R}$, then there exists $x_1>0$ large enough, such that $\Psi_i(x+x_1)>\varphi_i(x)$. Now we assume that $\overline{u}_i(t,x)=\Psi_i(x-ct+x_1)$ for all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. Next, we check that $(\overline{u}_1(t,x),\overline{u}_2(t,x))$ and $(\underline{u}_1,\underline{u}_2)=(0,0)$ are a pair of upper and lower solutions of system (1.8). Let $y=x-ct+x_1$. In fact, owing to $r(x)$ is nondecreasing, we have

$$\begin{aligned}\partial_t\overline{u}_i(t,x) = -c\Psi_i^{'}(y) &= d_i[\Psi_i(y+1) - 2\Psi_i(y) + \Psi_i(y-1)] + \Psi_i(y)[r(y) - \Psi_i(y)]\\ &= d_i[\overline{u}_i(t,x+1) - 2\overline{u}_i(t,x) + \overline{u}_i(t,x-1)]\\ &\quad +\overline{u}_i(t,x)[r(y) - \overline{u}_i(t,x)]\\ &\geq d_i[\overline{u}_i(t,x+1) - 2\overline{u}_i(t,x) + \overline{u}_i(t,x-1)]\\ &\quad +\overline{u}_i(t,x)[r(x-ct) - \overline{u}_i(t,x)].\end{aligned}$$

Then

$$\begin{aligned}&\partial_t\overline{u}_1(t,x) - d_1[\overline{u}_1(t,x+1) - 2\overline{u}_1(t,x)\\ &+\overline{u}_1(t,x-1)] + \overline{u}_1(t,x)[r(x-ct) - \overline{u}_1(t,x) - a_1\underline{u}_2(t,x)]\\ =\ &\partial_t\overline{u}_1(t,x) - d_1[\overline{u}_1(t,x+1) - 2\overline{u}_1(t,x) + \overline{u}_1(t,x-1)]\\ &+\overline{u}_1(t,x)[r(x-ct) - \overline{u}_1(t,x)] \geq 0\\ =\ &\partial_t\underline{u}_1(t,x) - d_1[\underline{u}_1(t,x+1) - 2\underline{u}_1(t,x) + \underline{u}_1(t,x-1)]\\ &+\underline{u}_1(t,x)[r(x-ct) - \underline{u}_1(t,x) - a_1\overline{u}_2(t,x)]\end{aligned}$$

and

$$\begin{aligned}&\partial_t\overline{u}_2(t,x) - d_2[\overline{u}_2(t,x+1) - 2\overline{u}_2(t,x) + \overline{u}_2(t,x-1)]\\ &-\overline{u}_2(t,x)[r(x-ct) - \overline{u}_2(t,x) - a_1\underline{u}_1(t,x)]\\ =\ &\partial_t\overline{u}_2(t,x) - d_2[\overline{u}_2(t,x+1) - 2\overline{u}_2(t,x) + \overline{u}_2(t,x-1)]\\ &-\overline{u}_2(t,x)[r(x-ct) - \overline{u}_2(t,x)] \geq 0\\ =\ &\partial_t\underline{u}_2(t,x) - d_2[\underline{u}_2(t,x+1) - 2\underline{u}_2(t,x) + \underline{u}_2(t,x-1)]\\ &-\underline{u}_2(t,x)[r(x-ct) - \underline{u}_2(t,x) - a_1\overline{u}_1(t,x)].\end{aligned}$$

Note that $\overline{u}_i(0,x) = \Psi_i(x+x_1) > \varphi_i(x) \geq \underline{u}_i(0,x)$, $\forall x \in \mathbb{R}$. So we are able to know that $(\overline{u}_1, \overline{u}_2)$ and $(\underline{u}_1, \underline{u}_2) = (0,0)$ is a pair of upper and lower solutions of (1.8). Then by the comparison principle, we can know that

$$u_i(t,x;\varphi_1,\varphi_2) \leq \overline{u}_i(t,x) = \Psi_i(x-ct+x_1), \quad \forall x \in \mathbb{R},\ t \geq 0.$$

Then together with $\Psi_i(-\infty) = 0$, we can choose a constant $K > 0$ large enough, such that for all $\varepsilon > 0$, $\Psi_i(-K+x_0) < \varepsilon$. Using the monotonicity of Ψ_i, we can further get

$$u_i(t,x;\varphi_1,\varphi_2) \leq \Psi_i(x-ct+x_0) \leq \Psi_i(-K+x_0) < \varepsilon, \quad \forall t > 0,\ x-ct \leq -K. \tag{2.3}$$

Next, we let $\vartheta_i(t,x)$ with $\vartheta_i(0,x) = \varphi_i(x)$, $x \in \mathbb{R}$ be the unique solution of the following equation

$$\partial_t\vartheta_i(t,x) = d_i[\vartheta_i(t,x+1) - 2\vartheta_i(t,x) - \vartheta_i(t,x-1)] + \vartheta_i(t,x)[r(+\infty) - \vartheta_i(t,x)]. \tag{2.4}$$

By [31], we can know that $c_i(+\infty)$ is the spreading speed for (2.4). And from our hypothesis, we can know that $c_2(+\infty) > c_1(+\infty)$. Then for all $c_0 \in (c_2(+\infty), c)$, we have

$$\lim_{t\to+\infty}\sup_{x\geq c_0 t}\vartheta_i(t,x) = 0.$$

Using the same method, we can easily prove that $(\vartheta_1, \vartheta_2)$ and (0,0) are a pair of upper and lower solutions of (1.8). Then by the Lemma 1.1, we have

$$\lim_{t\to+\infty} \sup_{x\geq c_0 t} u_i(t,x;\varphi_1,\varphi_2) = 0.$$

Hence, we can choose a constant $t_0 > 0$ such that

$$u_i(t,x;\varphi_1,\varphi_2) < \varepsilon, \quad \forall t \geq t_0,\ x \geq c_i t. \tag{2.5}$$

Let $T = \max\{t_0, K/(c-c_0)\}$. So for all $t > T$, we have $-K + ct \geq c_0 t \geq c_i t$. This combined with (2.3) and (2.5) yields

$$u_i(t,x;\varphi_1,\varphi_2) < \varepsilon, \quad \forall t \geq T,\ x \in \mathbb{R}.$$

This completes the proof. □

Next, we consider the situation that the worsening speed of habitat is not so fast. In fact, we can divide this situation into two cases: (1) c is between the speed of the two species, i.e., $c_1(+\infty) < c < c_2(+\infty)$; (2) c is smaller than the spreading speed of species u_1, i.e., $0 < c < c_1(+\infty)$. We will show that species u_1 will go extinct and species u_2 is able to persist in either case, and we will see that the reasons for the extinction of species u_1 in two situations are different. First, we look at the first case.

Theorem 2.2. *Assume that* (H_1) *holds and* $c_1(+\infty) < c < c_2(+\infty)$. *Let* $u(t,x;\varphi_1,\varphi_2) = (u_1(t,x;\varphi_1,\varphi_2), u_2(t,x;\varphi_1,\varphi_2))$ *be the unique solution of* (1.8). *Then the following conclusions hold.*

(i) If $\varphi_1(x) \equiv 0$ *for* x *large enough, then for* $\forall\varepsilon > 0$, $\exists T_0 > 0$,*such that*

$$u_1(t,x;\varphi_1,\varphi_2) \leq \varepsilon, \quad \forall (t,x) \in [T_0,+\infty)\times\mathbb{R}.$$

(ii) For every $\varepsilon > 0$, *we have*

$$\lim_{t\to+\infty}[\sup_{x\leq(c-\varepsilon)t} u_2(t,x;\varphi_1,\varphi_2)] = 0.$$

(iii) If $\varphi_2(x) \equiv 0$ *for* x *large enough, then for* $\forall\varepsilon > 0$, *we have*

$$\lim_{t\to+\infty}[\sup_{x\geq(c_2(+\infty)+\varepsilon)t} u_2(t,x;\varphi_1,\varphi_2)] = 0.$$

(iv) If $\varphi_1(x) \equiv 0$ *for* x *large enough and* $\varphi_2(x) > 0$ *in a close interval, then for* $\forall\varepsilon > 0$ *with* $0 < \varepsilon < (c_2(+\infty) - c)/2$, *we have*

$$\lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\leq x\leq(c_2(+\infty)-\varepsilon)t} |r(+\infty) - u_2(t,x;\varphi_1,\varphi_2)|] = 0.$$

Proof. Assume that $\overline{u}(t,x) \equiv (\overline{u}_1(t,x), \overline{u}_2(t,x))$ is the unique solution of the following decoupled system

$$\begin{cases} \partial_t u_1(t,x) = d_1\mathcal{Z}[u_1](t,x) + u_1(t,x)[r(x-ct) - u_1(t,x)], & x\in\mathbb{R}, t>0 \\ \partial_t u_2(t,x) = d_2\mathcal{Z}[u_2](t,x) + u_2(t,x)[r(x-ct) - u_2(t,x)], & \\ u(0,x) = (u_1(0,x,u_2(0,x)) = (\varphi_1(x),\varphi_2(x)) = \varphi(x), & x\in\mathbb{R}, \end{cases} \tag{2.6}$$

where $\mathcal{Z}[u_i](t,x) = u_i(t,x+1) - 2u_i(t,x) + u_i(t,x-1), i = 1,2$. Since $u_i(t,x)[r(x-ct) - u_i(t,x) - a_i u_j(t,x)] \le u_i(t,x)[r(x-ct) - u_i(t,x)]$ $(i \ne j)$, then for all $t > 0, x \in \mathbb{R}$, we have $u_i(t,x) \le \overline{u}_i(t,x), i = 1,2$, then by [31, Theorem 4.1,Theorem 5.1], we can easily get conclusion (i)-(iii). Next, we prove conclusion(iv). By (i), we can know that for any $\delta \in (0, r(+\infty)/2)$, there exists a constant $T > 0$, such that $u_1(t,x;\varphi_1,\varphi_2) \le \delta,\ t \ge T$. Consequently, for any $t > T$, we have $u_2[r(x-ct) - u_2 - a_2\delta] \le u_2[r(x-ct) - u_2 - a_2 u_1] \le u_2[r(x-ct) - u_2]$. Then, using the comparison argument, we have $\underline{u}_2(t,x) \le u_2(t,x;\varphi_1,\varphi_2) \le \overline{u}_2(t,x)$ for all $(t,x) \in [T,+\infty) \times \mathbb{R}$, where $\underline{u}_2(t,x)$ and $\overline{u}_2(t,x)$ are the solutions to

$$\begin{cases} \partial_t u_2(t,x) = d_2\mathcal{N}[u_2](t,x) + u_2(t,x)[r_\delta(x-ct) - u_2(t,x)], & t > T,\ x \in \mathbb{R}, \\ u_2(T,x) = u_2(T,x;\varphi_2), \quad x \in \mathbb{R} \end{cases}$$

and

$$\begin{cases} \partial_t u_2(t,x) = d_2\mathcal{N}[u_2](t,x) + u_2(t,x)[r(x-ct) - u_2(t,x)], & t > T,\ x \in \mathbb{R}, \\ u_2(T,x) = u_2(T,x;\varphi_2), \quad x \in \mathbb{R}, \end{cases}$$

respectively, where $r_\delta(x-ct) = r(x-ct) - a_2\delta$. If $\varphi_2(x) > 0$ on a closed interval, then $u_2(t,x;\varphi_1,\varphi_2) > 0$ on this closed interval. Therefore, by the arbitrariness of δ and [31, Theorem 5.1], we can get conclusion(iv). □

In the following content, we will focus on the situation of case (2) with strong competition. First, we consider the following system

$$\begin{cases} \partial_t u_1(t,x) = d_1\mathcal{N}[u_1](t,x) + u_1[r(x-ct) - u_1 - a_1u_2], x \in \mathbb{R}, t > 0 \\ \partial_t u_2(t,x) = d_2\mathcal{N}[u_2](t,x) + u_2[r(x-ct) - u_2], x \in \mathbb{R}, t > 0 \\ u_1(0,x) = \varphi_1(x), u_2(0,x) = \varphi_2(x), x \in \mathbb{R}, \end{cases} \tag{2.7}$$

where $\mathcal{N}[u_i](t,x) = u_i(t,x+1) - 2u_i(t,x) + u_i(t,x-1), i = 1,2,\ 0 \le \varphi_1(x), \varphi_2(x) \le r(+\infty)$, and for x large enough, $\varphi_i(x) \equiv 0, i = 1,2$, $\varphi_2(x) > 0$ on a closed interval. Denote by $(u_1^{(0)}(t,x;\varphi_1,\varphi_2), u_2^{(0)}(t,x;\varphi_1,\varphi_2))$ the solution to system (2.7). By [31, Theorem5.1], we can know that $u_2^{(0)}(t,x;\varphi_1,\varphi_2)$ satisfies

$$\lim_{t\to+\infty}[\sup_{x\le(c-\varepsilon)t} u_2^{(0)}(t,x;\varphi_1,\varphi_2)] = 0, \tag{2.8}$$

$$\lim_{t\to+\infty}[\sup_{x\ge(c_2(+\infty)+\varepsilon)t} u_2^{(0)}(t,x;\varphi_1,\varphi_2)] = 0, \tag{2.9}$$

$$\lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\le x\le(c_2(+\infty)-\varepsilon)t} |r(+\infty) - u_2^{(0)}(t,x;\varphi_1,\varphi_2)|] = 0. \tag{2.10}$$

Lemma 2.2. *For $\forall\delta > 0$, there exists a constant $T > 0$ such that for $\forall x \in \mathbb{R}, t \ge T$, and $u_1^{(0)}(t,x;\varphi_1,\varphi_2) \le \delta$.*

Proof. Without loss of generality, we can assume $\delta \in (0,1)$. Let $\rho = \dfrac{\delta}{30}$. The first equation of system (2.7) is equivalent to

$$\partial_t u_1(t,x) + \rho u_1 = d_1\mathcal{N}[u_1](t,x) + u_1(t,x)[\rho + r(x-ct) - u_1(t,x) - a_1u_2(t,x)].$$

Consequently, by [31] we can know that $u_1^{(0)}(t,x;\varphi_1,\varphi_2)$ satisfies

$$\begin{aligned} u_1^{(0)}(t,x;\varphi_1,\varphi_2) = e^{-\rho t}\sum_{m=-\infty}^{+\infty} e^{-2d_1t}I_m(2d_1t)\varphi_1(x-m) \\ +\int_0^t\sum_{m=-\infty}^{+\infty} e^{-(\rho+2d_1)(t-s)}I_m(2d_1(t-s))f_1^{(0)}(\rho,s,x-m)ds, \end{aligned} \tag{2.11}$$

where

$$I_m(2dt)=\begin{cases}\sum\limits_{j=0}^{+\infty}\dfrac{(dt)^{m+2j}}{j!(m+j)!}, & m\geq 0,\\ I_{-m}(2dt) & , m<0,\end{cases}$$

and $f_1^{(0)}(\rho,s,y)=u_1^{(0)}(s,y,\varphi)[\rho+r(y-cs)-u_1^{(0)}(s,y,\varphi_1,\varphi_2)-a_1u_2^{(0)}(s,y,\varphi_1,\varphi_2)]$, so $u_1^{(0)}(t,x;\varphi_1,\varphi_2)$ also satisfies

$$\begin{aligned} u_1^{(0)}(t,x;\varphi) = e^{-\rho t}\sum_{m=-\infty}^{+\infty} e^{-2d_1t}I_m(2d_1t)\varphi_1(x-m) \\ +\int_0^t\sum_{m=-\infty}^{+\infty} e^{-(\rho+2d_1)s}I_m(2d_1s)f_1^{(0)}(\rho,t-s,x-m)ds. \end{aligned} \tag{2.12}$$

Obviously, through a comparison discussion, we have $0\leq u_1^{(0)}(t,x;\varphi_1,\varphi_2)\leq r(+\infty)$, then $f_1^{(0)}(\rho,s,y)\leq r(+\infty)(\rho+2r(+\infty)+a_1r(+\infty))$. Note that $\int_0^{+\infty}e^{-\rho t}dt$ is convergent and by [31] we have $\sum_{m=-\infty}^{+\infty}e^{-2d_1s}I_m(2d_1s)=1$, then for the above δ, there exist $\eta>0$ and $A>\eta$ such that

$$\int_0^{\eta}\sum_{m=-\infty}^{+\infty} e^{-(\rho+2d_1)s}I_m(2d_1s)f_1^{(0)}(\rho,t-s,x-m)ds<\frac{\delta}{10} \tag{2.13}$$

and

$$\int_A^{+\infty}\sum_{m=-\infty}^{+\infty} e^{-(\rho+2d_1)s}I_m(2d_1s)f_1^{(0)}(\rho,t-s,x-m)ds<\frac{\delta}{10}. \tag{2.14}$$

For the above $\delta>0$ and any ε satisfying $0<\varepsilon<\dfrac{c_2(+\infty)-c}{4}$, there exists $T_0>0$, such that

$$e^{-\rho t}\sum_{m=-\infty}^{+\infty} e^{-2d_1t}I_m(2d_1t)\varphi(x-m)<\frac{\delta}{5},\quad \forall (t,x)\in[T_0,+\infty)\times\mathbb{R}. \tag{2.15}$$

Futhermore, by (2.10) and $a_1\geq 1$, we have

$$\begin{aligned} & r(x-m-cs)-a_1u_2^{(0)}(s,x-m)\leq r(+\infty)-u_2^{(0)}(s,x-m) \\ & <\frac{\delta}{30},\quad \forall (s,x-m)\in\{(s,x-m),s\geq T_0,(c+\varepsilon)s\leq x-m\leq (c_2(+\infty)-\varepsilon)s\}. \end{aligned} \tag{2.16}$$

Next, we prove

$$f_1^{(0)}(\rho,t,x)<\frac{\delta^2}{150},\quad \forall (t,x)\in\{(s,y)|s\geq T_0,(c+\varepsilon)s\leq y\leq (c_2(+\infty)-\varepsilon)s\}. \tag{2.17}$$

In fact, if $u_1^{(0)}(t,x;\varphi_1,\varphi_2) < \dfrac{\delta}{10}$, then when

$$(t,x) \in \{(s,y)|s \geq T_0, (c+\varepsilon)s \leq y \leq (c_2(+\infty)-\varepsilon)s\},$$

we have

$$f_1^{(0)}(\rho,t,x) < \frac{\delta}{10}(\rho+\frac{\delta}{30}) = \frac{\delta}{10}(\frac{\delta}{30}+\frac{\delta}{30}) = \frac{\delta^2}{150};$$

while if $u_1^{(0)}(t,x;\varphi_1,\varphi_2) \geq \dfrac{\delta}{10}$, then when

$$(t,x) \in \{(s,y)|s \geq T_0, (c+\varepsilon)s \leq y \leq (c_2(+\infty)-\varepsilon)s\},$$

we have

$$\begin{aligned} f_1^{(0)}(\rho,t,x) &\leq u_1^{(0)}(t,x;\varphi)(\rho+\frac{\delta}{30}-\frac{\delta}{10}) \\ &= u_1^{(0)}(t,x;\varphi)(\frac{\delta}{30}+\frac{\delta}{30}-\frac{\delta}{10}) \leq 0 < \frac{\delta^2}{150}. \end{aligned}$$

In summary, there is always (2.17) that holds true. Now, for the above $\varepsilon > 0$, we denote

$$\int_\eta^A \sum_{m=-\infty}^{+\infty} e^{-(\rho+2d_1)s} I_m(2d_1 s) f_1^{(0)}(\rho, t-s, x-m)ds = \sum_{i=1}^{3} T_i(\varepsilon,t,x),$$

where

$$T_1(\varepsilon,t,x) := \int_\eta^A \sum_{m=x-(c+\varepsilon)(t-s)}^{+\infty} e^{-(\rho+2d_1)s} I_m(2d_1 s) f_1^{(0)}(\rho, t-s, x-m)ds,$$

$$T_2(\varepsilon,t,x) := \int_\eta^A \sum_{m=x-(c_2(+\infty)-\varepsilon)(t-s)}^{x-(c+\varepsilon)(t-s)} e^{-(\rho+2d_1)s} I_m(2d_1 s) f_1^{(0)}(\rho, t-s, x-m)ds,$$

$$T_3(\varepsilon,t,x) = \int_\eta^A \sum_{m=-\infty}^{x-(c_2(+\infty)-\varepsilon)(t-s)} e^{-(\rho+2d_1)s} I_m(2d_1 s) f_1^{(0)}(\rho, t-s, x-m)ds.$$

First, we condsider $T_3(\varepsilon,t,x)$ i.e. $m \leq x-(c_2(+\infty)-\varepsilon)(t-s)$. If $x \leq (c_2(+\infty)-2\varepsilon)t$, then

$$m \leq -\varepsilon t + (c_2(+\infty)-\varepsilon)s \leq -\varepsilon t + (c_2(+\infty)-\varepsilon)A.$$

By [31, corollary 2.2] we can know that for every $\varepsilon_1 > 0$, there exists a constant $M > 1$, such that

$$\sum_{|m|>max\left\{M,\sqrt{2d_1 M t}\right\}} e^{-2d_1 t} I_m(2d_1 t) < \varepsilon_1. \tag{2.18}$$

Since $\lim\limits_{t\to+\infty} -\varepsilon t + (c_2(+\infty)-\varepsilon)A = -\infty$, there is a constant $t_1 > 0$, so that when $t > t_1$,

$$\sum_{m=-\infty}^{-\varepsilon t+(c_2(+\infty)-\varepsilon)A} e^{-2d_1 s} I_m(2d_1 s) < \varepsilon_1$$

holds. Let $t_2 = \{T_0, t_1\}$, then for the above $\delta > 0$, when $t > t_2$, there holds

$$\begin{aligned} T_3(\varepsilon,t,x) &\leq r(+\infty)[\rho + 2r(+\infty) + a_1 r(+\infty)] \\ &\quad \times \int_\eta^A e^{-\rho s} \sum_{m=-\infty}^{-\varepsilon t+(c_2(+\infty)-\varepsilon)A} e^{-2d_1 s} I_m(2d_1 s) ds \\ &\leq r(+\infty)[\rho + 2r(+\infty) + a_1 r(+\infty)]\varepsilon_1 \int_\eta^A e^{-\rho s} ds \\ &\leq r(+\infty)[\rho + 2r(+\infty) + a_1 r(+\infty)]\varepsilon_1 \frac{1}{\rho}. \end{aligned}$$

So if ε_1 is small enough, we get

$$T_3(\varepsilon,t,x) < \frac{\delta}{5}, \qquad \forall\ x \leq (c_2(+\infty) - 2\varepsilon)t, \quad t > t_2. \tag{2.19}$$

Next, we condsider $T_2(\varepsilon,t,x)$ when $x - m \in ((c+\varepsilon)(t-s), (c_2(+\infty)-\varepsilon)(t-s))$, i.e.,

$$x - (c_2(+\infty) - \varepsilon)(t-s) \leq m \leq x - (c+\varepsilon)(t-s).$$

So we can know that

$$f_1^{(0)}(\rho, t-s, x-m) < \frac{\delta^2}{150}, \ \ \forall x-(c_2(+\infty)-\varepsilon)(t-s) \leq m \leq x-(c+\varepsilon)(t-s), t \geq T_0.$$

Hence, we can know further that

$$\begin{aligned} T_2(\varepsilon,t,x) &< \tfrac{\delta^2}{150} \int_\eta^A e^{-\rho s} \sum_{x-(c_2(+\infty)-\varepsilon)(t-s)}^{x-(c+\varepsilon)(t-s)} e^{-2d_1 s} I_m(2d_1 s) ds \\ &\leq \tfrac{\delta^2}{150} \tfrac{1}{\rho} = \tfrac{\delta}{5}, \quad \forall \ \ (t,x) \in [T_0, +\infty) \times \mathbb{R}. \end{aligned} \tag{2.20}$$

Last, we consider $T_1(\varepsilon,t,x)$ i.e. $m > x - (c+\varepsilon)(t-s)$. Then if $x \geq (c+2\varepsilon)t$, we have

$$m \geq (c+2\varepsilon)t - (c+\varepsilon)(t-s) = \varepsilon t + (c+\varepsilon)s \geq \varepsilon t + (c+\varepsilon)\eta.$$

Since $\lim\limits_{t\to+\infty} \varepsilon t + (c+\varepsilon)\eta = +\infty$, then combining with (2.18) we can know that there exists a constant $t_3 > 0$, such that when $t > t_3$, we have

$$\sum_{m=\varepsilon t+(c+\varepsilon)\eta}^{+\infty} e^{-2d_1 s} I_m(2d_1 s) < \varepsilon_1.$$

Let $t_4 = \{T_0, t_3\}$, then for the above $\delta > 0$, when $t > t_4$, there holds

$$\begin{aligned} T_1(\varepsilon,t,x) &\leq r(+\infty)[\rho + 2r(+\infty) + a_1 r(+\infty)] \int_\eta^A e^{-\rho s} \sum_{m=\varepsilon t+(c+\varepsilon)\eta}^{+\infty} e^{-2d_1 s} I_m(2d_1 s) ds \\ &\leq r(+\infty)[\rho + 2r(+\infty) + a_1 r(+\infty)]\varepsilon_1 \int_\eta^A e^{-\rho s} ds \\ &\leq r(+\infty)[\rho + 2r(+\infty) + a_1 r(+\infty)]\varepsilon_1 \tfrac{1}{\rho}. \end{aligned}$$

So if ε_1 is small enough, we get

$$T_1(\varepsilon,t,x) < \frac{\delta}{5}, \qquad \forall\, x \geq (c+2\varepsilon)t, \quad t > t_4, \tag{2.21}$$

Let $t_5 = \max\{t_3, t_4\}$, then from (2.13)-(2.15) and (2.19)-(2.21), we can obtain immediately that

$$u_1^{(0)}(t,x;\varphi_1,\varphi_2) < \delta, \forall (t,x) \in \{(t,x) | t \geq t_5, (c+2\varepsilon)t \leq x \leq (c_2(+\infty)-2\varepsilon)t\}.$$

Hence, we have

$$\lim_{t\to+\infty}[\sup_{(c+2\varepsilon)t\leq x\leq(c_2(+\infty)-2\varepsilon)t} u_1^{(0)}(t,x;\varphi_1,\varphi_2)] = 0.$$

From [31] and a comparison discussion, for the above $\varepsilon > 0$, it holds that

$$\lim_{t\to+\infty}[\sup_{x\leq(c-\varepsilon)t} u_1^{(0)}(t,x;\varphi_1,\varphi_2)] = 0,$$

and

$$\lim_{t\to+\infty}[\sup_{x\geq(c_1(+\infty)+\varepsilon)t} u_1^{(0)}(t,x;\varphi_1,\varphi_2)] = 0$$

for $0 < \varepsilon < \dfrac{c_2(+\infty)-c}{4}$. By the arbitrariness of ε, we have that for $0 < \varepsilon < \dfrac{c_2(+\infty)-c}{4}$,

$$\lim_{t\to+\infty}[\sup_{x\in\mathbb{R}} u_1^{(0)}(t,x;\varphi_1,\varphi_2)] = 0.$$

This completes the proof. □

Remark 2.1. In the proof of Lemma 2.2, we only used the condition $a_1 \geq 1$, and the condition $a_2 \geq 1$ is actually not required. This actually means that the environmental deterioration and strong competition among species have led to death of species u_1.

Similarly, we consider the following system

$$\begin{cases} \partial_t u_1(t,x) = d_1\mathcal{N}_1[u](t,x) + u_1(t,x)[r(x-ct) - u_1(t,x) - a_1u_2(t,x)], \\ \partial_t u_2(t,x) = d_2\mathcal{N}_2[u](t,x) + u_2(t,x)[r(x-ct) - a_2u_1^{(0)}(t,x) - u_2(t,x)], \\ u_1(0,x) = \varphi_1(x), u_2(0,x) = \varphi_2(x), x \in \mathbb{R}. \end{cases} \tag{2.22}$$

Assume that $(u_1^{(1)}(t,x;\varphi_1,\varphi_2), u_2^{(1)}(t,x;\varphi_1,\varphi_2))$ is the unique solution of system (2.22). Apparently, by a comparison discussion we can get $u_1^{(1)}(t,x;\varphi_1,\varphi_2) \geq u_1^{(0)}(t,x;\varphi_1,\varphi_2)$ and $u_2^{(1)}(t,x;\varphi_1,\varphi_2) \leq u_2^{(0)}(t,x;\varphi_1,\varphi_2)$. Then we can know that

$$\lim_{t\to+\infty}[\sup_{x\leq(c-\varepsilon)t} u_2^{(1)}(t,x;\varphi_1,\varphi_2)] = 0, \tag{2.23}$$

$$\lim_{t\to+\infty}[\sup_{x\geq(c_2(+\infty)+\varepsilon)t} u_2^{(1)}(t,x;\varphi_1,\varphi_2)] = 0. \tag{2.24}$$

Lemma 2.3. *For any* $\varepsilon \in (0, \frac{c_2(+\infty)}{2})$*, we have*

$$\lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\le x\le(c_2(+\infty)-\varepsilon)t} |r(+\infty) - u_2^{(1)}(t,x;\varphi_1,\varphi_2)|] = 0. \tag{2.25}$$

Proof. From Lemma 2.2, we have for any δ satisfying $0 < \delta < \frac{r(+\infty)}{a_2}$, there exists $T > 0$, such that $u_1^{(0)}(t,x;\varphi_1,\varphi_2) < \delta$, $\forall (t,x) \in [T,+\infty)\times\mathbb{R}$. Denote $r_\delta(x-ct) = r(x-ct) - a_2\delta$, then $r_\delta(x)$ is continuous, nondecreasing, bounded and piecewise continuously differentiable, with $-\infty < r_\delta(+\infty) < 0$ and $0 < r_\delta(+\infty) < +\infty$. Let $\widetilde{u}_2^{(1)}(t,x;\varphi_1,\varphi_2)$ be the unique solution of the following system

$$\begin{cases} \partial_t u(t,x) = d_2\mathcal{N}[u](t,x) + u(t,x)[r_\delta(x-ct) - u(t,x)], t > T, x \in \mathbb{R}, \\ u(T,x) = u_2^{(1)}(T,x;\varphi_1,\varphi_2), x \in \mathbb{R}. \end{cases}$$

By [31] and a comparison discussion, we can know that

$$\lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\le x\le(c_\delta-\varepsilon)t} u_2^{(1)}(t,x;\varphi_1,\varphi_2)] \ge \lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\le x\le(c_\delta-\varepsilon)t} \widetilde{u}_2^{(1)}(t,x;\varphi_1,\varphi_2)]$$
$$\ge r_\delta(+\infty),$$

where $c_\delta = \dfrac{d_2(e^\lambda - 2 + e^{-\lambda}) + r_\delta(+\infty)}{\lambda}$. By the arbitrariness of δ, for $0 < \varepsilon < \dfrac{c_2(+\infty) - c}{2}$, we have

$$\lim_{t\to+\infty}[\inf_{(c+\varepsilon)t\le x\le(c_\delta-\varepsilon)t} u_2^{(1)}(t,x;\varphi_1,\varphi_2)] \ge r(+\infty).$$

This together with the fact that $u_2^{(1)}(t,x;\varphi) \le r(+\infty)$ implies

$$\lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\le x\le(c_2(+\infty)-\varepsilon)t} |r(+\infty) - u_2^{(1)}(t,x;\varphi_1,\varphi_2)|] = 0.$$

This completes the proof. □

Lemma 2.4. *For any* $\varepsilon > 0$*, there exists a constant* $T > 0$*, such that*

$$u_1^{(1)}(t,x;\varphi_1,\varphi_2) \le \varepsilon, \quad \forall x \in \mathbb{R}, t \ge T.$$

Proof. By employing the same argument as in the proof of Lemma 2.2, we can draw conclusions immediately. □

The above discussion leads us to consider the following iteration system

$$\begin{cases} \partial_t u_1^{(k)}(t,x) = d_1\mathcal{N}[u_1^{(k)}](t,x) + u_1^{(k)}(t,x)[r(x-ct) - u_1^{(k)}(t,x) - a_1u_2^{(k)}(t,x)], \\ \partial_t u_2^{(k)}(t,x) = d_2\mathcal{N}[u_2^{(k)}](t,x) + u_2^{(k)}(t,x)[r(x-ct) - a_2u_1^{(k-1)}(t,x) - u_2^{(k)}(t,x)], \\ u_1^{(k)}(0,x) = \varphi_1(x), u_2^{(k)}(0,x) = \varphi_2(x), x \in \mathbb{R}, \\ k = 1,2,\cdots, \end{cases} \tag{2.26}$$

where $\mathcal{N}[u_i^{(k)}](t,x) = u_i^{(k)}(t,x+1) - 2u_i^{(k)}(t,x) + u_i^{(k)}(t,x-1)$ and $(u_1^{(0)}(t,x;\varphi_1,\varphi_2), u_2^{(0)}(t,x;\varphi_1,\varphi_2))$ is the unique solution of system (2.7). Hence, we can get a sequence $\left\{(u_1^{(k)}(t,x;\varphi_1,\varphi_2), u_2^{(k)}(t,x;\varphi_1,\varphi_2))\right\}_{k=0}^{+\infty}$ from the iteration system. From the above discussion, we can know that $\left\{u_1^{(k)}(t,x;\varphi_1,\varphi_2)\right\}$ is a nondecreasing sequence and $\left\{u_2^{(k)}(t,x;\varphi_1,\varphi_2)\right\}$ is a decreasing sequence, which satisfies

$$0 \le u_1^{(0)} \le u_1^{(1)} \le \cdots \le u_1^{(k)} \le u_1^{(k+1)} \le \cdots \le r(+\infty),$$

and

$$r(+\infty) \ge u_2^{(0)} \ge u_2^{(1)} \ge \cdots \ge u_2^{(k)} \ge u_2^{(k+1)} \ge \cdots \ge 0.$$

By (2.8)-(2.10), (2.23)-(2.25), lemma 2.2, lemma 2.3 and Lemma 2.4, we can know that the above sequences satisfy the following properties

$$\lim_{t\to+\infty}[\sup_{x\le(c-\varepsilon)t} u_2^{(k)}(t,x;\varphi_1,\varphi_2)] = 0, \tag{2.27}$$

$$\lim_{t\to+\infty}[\sup_{x\ge(c_2(+\infty)+\varepsilon)t} u_2^{(k)}(t,x;\varphi_1,\varphi_2)] = 0, \tag{2.28}$$

$$\lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\le x\le(c_2(+\infty)-\varepsilon)t} |r(+\infty) - u_2^{(k)}(t,x;\varphi_1,\varphi_2)|] = 0 \tag{2.29}$$

and

$$\lim_{t\to+\infty}[\sup_{x\in\mathbb{R}} u_1^{(k)}(t,x;\varphi_1,\varphi_2)] = 0. \tag{2.30}$$

where $\varepsilon \in (0, \frac{c_2(+\infty)-c}{2})$. Because $\left\{u_1^{(k)}(t,x;\varphi_1,\varphi_2)\right\}_{k=0}^{+\infty}$ and $\left\{u_2^{(k)}(t,x;\varphi_1,\varphi_2)\right\}_{k=0}^{+\infty}$ are monotonous and bounded, we have $\left\{u_1^{(k)}(t,x;\varphi_1,\varphi_2)\right\}_{k=0}^{+\infty}$ and $\left\{u_2^{(k)}(t,x;\varphi_1,\varphi_2)\right\}_{k=0}^{+\infty}$ both converge pointwise, as $k\to+\infty$. Hence, there exists $u_1^*(t,x;\varphi_1,\varphi_2)$ and $u_2^*(t,x;\varphi_1,\varphi_2)$, such that

$$\begin{aligned} &\lim_{k\to+\infty} u_1^{(k)}(t,x;\varphi_1,\varphi_2) = u_1^*(t,x;\varphi_1,\varphi_2), \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}, \\ &\lim_{k\to+\infty} u_2^{(k)}(t,x;\varphi_1,\varphi_2) = u_2^*(t,x;\varphi_1,\varphi_2), \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}. \end{aligned} \tag{2.31}$$

Now, our aim is to prove the main theorem when the environment deterioration speed is slow($c < c_1(+\infty)$) with strong competition.

Theorem 2.3. *Assume that* ($\mathbf{H}_1$)*holds,* $a_i \ge 1$ *for* $i = 1,2$ *and* $0 < c < c_1(+\infty)$*. Denote* $u^*(t,x;\varphi_1,\varphi_2) = (u_1^*(t,x;\varphi_1,\varphi_2), u_2^*(t,x;\varphi_1,\varphi_2))$*. Then* $u^*(t,x;\varphi_1,\varphi_2)$ *is the solution to system* (1.8) *with* $0 \le \varphi_i(x) \le r(\infty), i = 1,2$ *and the following statements hold.*

(i) If $\varphi_1(x) \equiv 0$ *for* x *large enough, then for* $\forall\varepsilon > 0$, $\exists t_0 > 0$,*such that*

$$u_1^*(t,x;\varphi_1,\varphi_2) \le \varepsilon, \qquad \forall (t,x) \in [t_0,+\infty)\times\mathbb{R}.$$

(ii) For every $\varepsilon > 0$*, we have*

$$\lim_{t\to+\infty}[\sup_{x\le(c-\varepsilon)t} u_2^*(t,x;\varphi_1,\varphi_2)] = 0.$$

(iii) *If* $\varphi_2(x) \equiv 0$ *for* x *large enough, then for* $\forall \varepsilon > 0$, *we have*

$$\lim_{t\to+\infty}[\sup_{x\geq(c_2(+\infty)+\varepsilon)t} u_2^*(t,x;\varphi_1,\varphi_2)] = 0.$$

(iv) *If* $\varphi_1(x) \equiv 0$ *for* x *large enough and* $\varphi_2(x) > 0$ *in a close interval, then* $\forall \varepsilon \in (0, (c_2(+\infty) - c)/2)$, *we have*

$$\lim_{t\to+\infty}[\sup_{(c+\varepsilon)t\leq x\leq(c_2(+\infty)-\varepsilon)t} |r(+\infty) - u_2^*(t,x;\varphi_1,\varphi_2)|] = 0.$$

Proof. Denote by z the vector (t,x). For any given $T > 0$ and $M > 0$, let $\mho = [0,T]\times[-M,M]$. First, we prove that the convergence in (2.31) is uniform for $z \in \mho$. In fact, we have

$$\begin{aligned}
&|u_1^{(k)}(z,\varphi_1,\varphi_2) - u_1^{(k+p)}(z,\varphi_1,\varphi_2)| \\
=& \int_0^t \sum_{m=-\infty}^{+\infty} e^{-2d_1(t-s)} I_m(2d_1(t-s)) u_1^{(k)}(s,x-m) \\
&\times[r(x-m-cs) - u_1^{(k)}(s,x-m) - a_1 u_2^{(k)}(s,x-m)]ds \\
&-\int_0^t \sum_{m=-\infty}^{+\infty} e^{-2d_1(t-s)} I_m(2d_1(t-s)) u_1^{(k+p)}(s,x-m) \\
&\times[r(x-m-cs) - u_1^{(k+p)}(s,x-m) - a_1 u_2^{(k+p)}(s,x-m)]ds \\
=& \int_0^t \sum_{m=-\infty}^{+\infty} e^{-2d_1(t-s)} I_m(2d_1(t-s)) g_1(s,x-m,k,p)ds,
\end{aligned}$$

where

$$\begin{aligned}
&g_1(s,x-m,k,p) \\
=& u_1^{(k)}(s,x-m)[r(x-m-cs) - u_1^{(k)}(s,x-m) - a_1 u_2^{(k)}(s,x-m)] \\
&-u_1^{(k+p)}(s,x-m)[r(x-m-cs) - u_1^{(k+p)}(s,x-m) - a_1 u_2^{(k+p)}(s,x-m)] \\
=& [u_1^{(k+p)}(s,x-m) + u_1^{(k)}(s,x-m) + a_1 u_2^{(k+p)}(s,x-m) - r(x-m-cs)] \\
&\times[u_1^{(k+p)}(s,x-m) - u_1^{(k)}(s,x-m)] \\
&+a_1 u_1^{(k)}(s,x-m)[u_2^{(k+p)}(s,x-m) - u_2^{(k)}(s,x-m)].
\end{aligned} \tag{2.32}$$

Let $\widetilde{g}_1 = 2(3+2a_1)r^2(\infty)$. Then it is clear that $|g_1(s,x-m,k,p)| \leq \widetilde{g}_1$ for all $s \in \mathbb{R}_+$, $x-m \in \mathbb{R}$, $k = 0,1,2,\cdots$ and $p = 1,2,\cdots$. Because $\sum_{m=-\infty}^{+\infty} e^{-2d_1 t} I_m(2d_1 t) = 1$, then for any $\varepsilon > 0$, there exists $L > 0$, such that

$$\sum_{|m|\leq L} e^{-2d_1 t} I_m(2d_1 t) \geq 1 - \frac{\varepsilon}{5T\widetilde{g}_1}.$$

Therefore, we have

$$\begin{aligned}
&|u_1^{(k)}(z,\varphi_1,\varphi_2)-u_1^{(k+p)}(z,\varphi_1,\varphi_2)|\\
&=|\int_0^t \sum_{m=-\infty}^{+\infty} e^{-2d_1(t-s)}I_m(2d_1(t-s))g_1(s,x-m,k,p)ds|\\
&=|\int_0^t \sum_{m=-\infty}^{+\infty} e^{-2d_1 s}I_m(2d_1 s)g_1(t-s,x-m,k,p)ds|\\
&\le |\int_0^t \sum_{|m|\le L} e^{-2d_1 s}I_m(2d_1 s)g_1(t-s,x-m,k,p)ds|\\
&+|\int_0^t \sum_{|m|>L} e^{-2d_1 s}I_m(2d_1 s)g_1(t-s,x-m,k,p)ds|\\
&\le |\int_0^t \sum_{|m|\le L} e^{-2d_1 s}I_m(2d_1 s)g_1(t-s,x-m,k,p)ds|+\widetilde{g}_1\cdot\frac{\varepsilon}{5T\widetilde{g}_1}\cdot T\\
&=\frac{\varepsilon}{5}+|\int_0^t \sum_{|m|\le L} e^{-2d_1 s}I_m(2d_1 s)g_1(t-s,x-m,k,p)ds|.
\end{aligned}$$

Let $\beta=\dfrac{\varepsilon}{5\widetilde{g}_1}$. If $t\le\beta$, then

$$\begin{aligned}
&|u_1^{(k)}(z,\varphi_1,\varphi_2)-u_1^{(k+p)}(z,\varphi_1,\varphi_2)|\\
&\le \frac{\varepsilon}{5}+|\int_0^t \sum_{|m|\le L} e^{-2d_1 s}I_m(2d_1 s)g_1(t-s,x-m,k,p)ds|\\
&\le \frac{\varepsilon}{5}+\widetilde{g}_1|\int_0^t \sum_{|m|\le L} e^{-2d_1 s}I_m(2d_1 s)ds|\\
&\le \frac{\varepsilon}{5}+\widetilde{g}_1|\int_0^t 1ds|\le \frac{\varepsilon}{5}+\widetilde{g}_1\cdot\beta=\frac{2\varepsilon}{5},
\end{aligned}$$

while $t>\beta$, we have

$$\begin{aligned}
&|\int_{t-\beta}^t \sum_{|m|\le L} e^{-2d_1(t-s)}I_m(2d_1(t-s))g_1(s,x-m,k,p)ds|\\
&=|\int_0^\beta \sum_{|m|\le L} e^{-2d_1 s}I_m(2d_1 s)g_1(t-s,x-m,k,p)ds|\\
&\le \int_0^\beta \sum_{m=-\infty}^{+\infty} e^{-2d_1 s}I_m(2d_1 s)|g_1(t-s,x-m,k,p)|ds\\
&\le \widetilde{g}_1\cdot\beta=\frac{\varepsilon}{5}.
\end{aligned}$$

So there holds

$$\begin{aligned}
&|u_1^{(k)}(z,\varphi_1,\varphi_2)-u_1^{(k+p)}(z,\varphi_1,\varphi_2)|\\
&\le \frac{\varepsilon}{5}+|\int_0^{t-\beta} \sum_{|m|\le L} e^{-2d_1(t-s)}I_m(2d_1(t-s))g_1(s,x-m,k,p)ds|\\
&+|\int_{t-\beta}^t \sum_{|m|\le L} e^{-2d_1(t-s)}I_m(2d_1(t-s))g_1(s,x-m,k,p)ds|\\
&\le \frac{2\varepsilon}{5}+|\int_0^{t-\beta} \sum_{|m|\le L} e^{-2d_1(t-s)}I_m(2d_1(t-s))g_1(s,x-m,k,p)ds|.
\end{aligned}\tag{2.33}$$

For given $z\in\mho$, we define

$$\mho_z=\{(s,m)\ \ |\ \ 0\le s\le t-\beta, x-L\le m\le x+L\}$$

and

$$\mho_1 = [0,T] \times [-M-L, M+L].$$

Obviously, $\mho_1$ is bounded and we have $\mho_z \subset \mho_1$. Then by Egorovs Theorem, for the above $\varepsilon > 0$, there exists a measurable subset $\mho_\varepsilon$ of $\mho_1$, such that $m(\mho_1 - \mho_\varepsilon) < \frac{\varepsilon}{5T\widetilde{g}_1}$, and

$$\lim_{k\to+\infty} u_1^{(k)}(z,\varphi_1,\varphi_2) = u_1^*(z,\varphi_1,\varphi_2),$$
$$\lim_{k\to+\infty} u_2^{(k)}(z,\varphi_1,\varphi_2) = u_2^*(z,\varphi_1,\varphi_2), \quad \text{uniformly for } z \in \mho_\varepsilon.$$

Thus, for the above $\varepsilon > 0$, there exist $K_\varepsilon > 0$ and $P_\varepsilon > 0$, such that when $k > K_\varepsilon, p > P_\varepsilon, z \in \mho_\varepsilon$, we have

$$|u_1^{(k)}(z,\varphi_1,\varphi_2) - u_1^{(k+p)}(z,\varphi)| < \frac{2\varepsilon}{5T(3+2a_1)r(+\infty)},$$
$$|u_2^{(k)}(z,\varphi) - u_2^{(k+p)}(z,\varphi_1,\varphi_2)| < \frac{2\varepsilon}{5T(3+2a_1)r(+\infty)}.$$

Meanwhile, by (2.32), we have

$$\begin{aligned}
&g_1(s,x-m,k,p)\\
=\ &[u_1^{(k+p)}(s,x-m) + u_1^{(k)}(s,x-m) + a_1 u_2^{(k+p)}(s,x-m) - r(x-m-cs)]\times\\
&[u_1^{(k+p)}(s,x-m) - u_1^{(k)}(s,x-m)]\\
&+a_1 u_1^{(k)}(s,x-m)[u_2^{(k+p)}(s,x-m) - u_2^{(k)}(s,x-m)]\\
&\le (3r(+\infty) + a_1 r(+\infty) + a_1 r(+\infty)) \cdot \frac{2\varepsilon}{5T(3+2a_1)r(+\infty)}\\
=\ &\frac{2\varepsilon}{5T}.
\end{aligned}$$

So it holds that

$$\begin{aligned}
&|u_1^{(k)}(z,\varphi) - u_1^{(k+p)}(z,\varphi)|\\
\le\ &\frac{2\varepsilon}{5} + \int \textstyle\sum_{\mho_\varepsilon\cap\mho_z} e^{-2d_1(t-s)} I_{x-m}(2d_1(t-s))|g_1(s,x,k,p)|ds\\
&+ \int \textstyle\sum_{(\mho_1-\mho_\varepsilon)\cap\mho_z} e^{-2d_1(t-s)} I_{x-m}(2d_1(t-s))|g_1(s,x,k,p)|ds\\
\le\ &\frac{2\varepsilon}{5} + \frac{2\varepsilon}{5T}\int_0^t \sum_{m=-\infty}^{+\infty} e^{-2d_1(t-s)} I_{x-m}(2d_1(t-s))ds\\
&+\widetilde{g}_1 \int \textstyle\sum_{(\mho_1-\mho_\varepsilon)\cap\mho_z} e^{-2d_1(t-s)} I_{x-m}(2d_1(t-s))ds\\
\le\ &\tfrac{4\varepsilon}{5} + \widetilde{g}_1 \int \textstyle\sum_{(\mho_1-\mho_\varepsilon)\cap\mho_z} e^{-2d_1(t-s)} I_{x-m}(2d_1(t-s))ds\\
\le\ &\tfrac{4\varepsilon}{5} + \widetilde{g}_1 \cdot T \cdot m(\mho_1 - \mho_\varepsilon)\\
\le\ &\tfrac{4\varepsilon}{5} + \tfrac{\varepsilon}{5} = \varepsilon \qquad for \quad k > K_\varepsilon, p > P_\varepsilon, z \in \mho.
\end{aligned}$$

Thus the limit $\lim\limits_{k\to+\infty} u_1^{(k)}(z,\varphi_1,\varphi_2) = u_1^*(z,\varphi_1,\varphi_2)$ uniformly holds for all $z \in \mho$. In the same way, we can prove that the limit $\lim\limits_{k\to+\infty} u_2^{(k)}(z,\varphi_1,\varphi_2) = u_2^*(z,\varphi_1,\varphi_2)$ uniformly holds for all $z \in \mho$. By the arbitrariness of T and M, we can know that $\left\{(u_1^{(k)}(z,\varphi_1,\varphi_2), u_2^{(k)}(z,\varphi_1,\varphi_2))\right\}_{k=0}^{+\infty}$ is uniformly convergent to $(u_1^*(z,\varphi_1,\varphi_2),$

$u_2^*(z,\varphi_1,\varphi_2))$ on every bounded subset of $\mathbb{R}_+ \times \mathbb{R}$. This together with the iteration system (2.26) implies that $(u_1^*(z,\varphi_1,\varphi_2), u_2^*(z,\varphi_1,\varphi_2))$ is the solution to (1.8).

By $\lim\limits_{t\to+\infty}[\sup\limits_{x\in\mathbb{R}} u_1^{(k)}(t,x;\varphi_1,\varphi_2)] = 0$, we can know that for all $\delta > 0$, there exists a constant $T_\delta > 0$, such that

$$|u_1^{(k)}(t,x;\varphi_1,\varphi_2)| < \frac{\delta}{2}, \qquad z \in \mho_0 = [T_\delta, T_\delta + \tau] \times [-L, L], L > 0, \tau > 0.$$

For the above $\delta > 0$, there exists $K_{\delta,\mho_0}$ such that

$$|u_1^{(k)}(t,x;\varphi_1,\varphi_2) - u_1^*(t,x;\varphi_1,\varphi_2)| < \frac{\delta}{2} \quad \forall z \in \mho_0, k > K_{\delta,\mho_0}.$$

Therefore, for the above $\delta > 0$, there holds

$$|u_1^*(t,x;\varphi_1,\varphi_2)| \le |u_1^{(k)}(t,x;\varphi_1,\varphi_2)| + |u_1^{(k)}(t,x;\varphi_1,\varphi_2) - u_1^*(t,x;\varphi_1,\varphi_2)| < \delta, \forall z \in \mho_0.$$

Because τ and L are arbitrary, we have

$$|u_1^*(t,x;\varphi_1,\varphi_2)| < \delta, \quad \forall z \in [T_\delta, +\infty) \times \mathbb{R}.$$

Due to the arbitrariness of δ, we have

$$\lim_{t\to+\infty}[\sup_{x\in\mathbb{R}} u_1^*(t,x;\varphi_1,\varphi_2)] = 0.$$

By the same way, we can prove *(ii)*,*(iii)* and *(iv)* of the theorem. This completes the proof. □

Discussion

In this article, we mainly discuss the spatial dynamics of the lattice Lotka-Volterra competition system with the strong competition condition in a shifting habitat. We prove that when the environment deteriorates rapidly, both species will become extinct. When the deterioration rate is not so fast, the species with slow diffusion rate will become extinct, and the species with fast diffusion rate will survive. This is different from the conclusion obtained under the condition of weak competition. This is mainly because under the condition of weak competition, the slow-moving organisms face less environmental pressure and competition pressure, so they can survive. However, under the condition of strong competition, the slow-moving species will receive strong competitive pressure. Therefore, even when the environmental pressure is not so great, they cannot escape from the harsh environment and survive.

In fact, nowadays more and more countries begin to pay attention to environmental governance, so we can also consider whether it will produce different results when the environment becomes better. In addition, due to the seasonal climate change, we can also consider the case with a time period. We will further discuss the above two situations in subsequent articles.

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