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Existence and Approximate Controllability of Solutions for an Impulsive Evolution Equation with Nonlocal Conditions in Banach Space*

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Abstract In this article, we study the existence of mild solutions and approximate controllability for non-autonomous impulsive evolution equations with nonlocal conditions in Banach space. The existence of mild solutions and some conditions for approximate controllability of these non-autonomous impulsive evolution equations are given by using the Krasnoselskii's fixed point theorem, the theory of evolution family and the resolvent operator. In particular, the impulsive functions are supposed to be continuous and the nonlocal item is divided into Lipschitz continuous and completely bounded. An example is given as an application of the results.

Keywords Impulsive evolution equation, approximate controllability, nonlocal conditions, resolvent operator, evolution family

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1. Introduction

Recently, the evolution equation is used to describe the state or process that changes with time in physics, mechanics or other natural sciences. It is well known that the nonlocal problems are more widely used in applications than the classical ones. Byszewski [1] first investigated the nonlocal problems. They obtained the existence and uniqueness of mild solutions for nonlocal differential equations without impulsive conditions. Deng [3] pointed out that the nonlocal initial condition can be applied in physics with a better effect than the classical initial condition $u(0) = u_0$ and used the nonlocal conditions $u(0) = \sum_{k=1}^{m} c_k u(t_k)$ to describe the diffusion

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phenomenon on a small amount of gas in a transparent tube. The above findings have encouraged more authors to focus on differential equations with non-local conditions. The differential equations with non-local conditions are often applied to models of processes subject to abrupt changes in a specific time. They have a wide range of applications in areas such as control, mechanical, electrical engineering fields and so on. Fan [4] discussed the existence results for semilinear differential equations with nonlocal and impulsive conditions. Tai [5] studied the exact controllability of a fractional impulsive neutral functional integro-differential systems with nonlocal conditions by using the fractional power of operators and the Banach contraction mapping theorem. When describing some physical phenomena, the nonlocal condition can be more useful than the standard initial condition $u(0) = u_0$. The importance of nonlocal conditions has also been discussed in [1,2,5,6,14–17,19,22–28,30,31]. When discussing the problem of evolution equations, it is often necessary to explore their application and combine them with the controllability.

The controllability and approximate controllability of evolution equations are considered by many authors owing to their wide applications in the field of physics, biology and medicine, see [5,6,9–14,17,21,29]. The concept of controllability, after being first introduced by Kalman [7] in 1963, has become an active area of research due to its enormous applications in physics. There are various studies on the approximate controllability of systems represented by differential equations, integral differential equations, differential inclusion, neutral-type generalized differential equations and integer-order impulsive differential equations in Banach spaces. Mahmudov [8] in 2008 studied the approximate controllability for the abstract evolution equations with nonlocal conditions in Hilbert spaces and obtained sufficient conditions for the approximate controllability of the semi-linear evolution equation. In 2018, Chen [12] discussed the approximate controllability of non-autonomous evolution system with nonlocal conditions and introduced a new Green's function to prove the existence of mild solutions. The approximate controllability of the development equation with impulse makes the application of the development equation more practical and representative.

Impulsive differential equations are commonly used for modelling processes that change abruptly at some point in time. They have a wide range of applications in control, mechanical, electrical and other fields. These changes of state are caused by transient forces (perturbations). Differential systems that use transient forces as impulsive conditions appear in many applications, such as biological phenomena involving thresholds, sudden rhythm models in medicine and biology, optimal control models in economics and frequency modulation systems. For these reasons, Hernández and O'Regan [9] discussed on a new class of abstract impulsive differential equations, introduced a new model named as differential equations without instantaneous impulses. It shows that the action of drugs in the blood and their absorption into the body is a gradual and continuous process. Wang [10] studied a general impulsive evolution equation and discussed periodic solutions and Ulam's type stability to a new generalized evolution equation without instantaneous impulses in the infinite-dimensional spaces.

In 2018, Chen [12] studied the approximate controllability of non-autonomous evolution system with nonlocal conditions in Banach space X as follows.

$$\begin{cases} u'(t) - A(t)u(t) = Bv(t) + f(t, u(t)), \ t \in [0, a] := J, \\ u(0) = \sum_{k=1}^{m} c_k u(t_k), \end{cases}$$

where $a > 0$ is a constant, $A(t)$ is a family of (possibly unbounded) linear operators depending on time and having the domains $D(A(t))$ for every $t \in J$, the control function $v(t)$ is given in Banach space $L^2(J; U)$ of admissible control functions, U is also a Banach space, $h(t) \in C(J, J)$, $f : J \times X \times X \to X$ is a continuous nonlinear mapping, and B is a bounded linear operator from U to X, $0 < t_1 < t_2 < \cdots < t_m < a$, $m \in \mathbb{N}$, c_k are real numbers, $c_k \neq 0, k = 1, 2, \cdots, m$. By using a new Green's function and constructing a control function involving a Gramian controllability operator, the author studied the existence of mild solutions and approximate controllability.

In 2015, Liang [13] obtained the controllability of fractional integro-differential evolution equations with nonlocal conditions

$$\begin{cases} D^q u(t) + Au(t) = f(t, u(t), Gu(t)) + Bv(t), \ t \in J, \\ u(0) = \sum_{k=1}^{m} c_k u(t_k), \end{cases}$$

where D^q denotes the Caputo fractional derivative of order $q \in (0, 1), -A : D(A) \subset X \to X$ is the infinitesimal generator of a C_0-semigroup $T(t)(t \geqslant 0)$ of the uniformly bounded linear operator, the control function v is given in $L^2(J, U), U$ is a Banach space, B is a linear bounded operator from U to X, f is a given function which will be specified later and

$$Gu(t) = \int_0^t K(t, s)u(s)ds$$

is a Volterra integral operator with integral kernel $K \in C(\Delta, \mathbb{R}^+), \Delta = \{(t, s) : 0 \leqslant s \leqslant t \leqslant b\}$. He used the fixed point theorem of Mönch's type and studied the controllability for a class of fractional integro-differential evolution equations with nonlocal initial conditions. There is less research [12, 13] on the approximate controllability of evolution equations with both nonlocal conditions and impulses.

Motivated by all of the above-mentioned aspects, in this paper we consider the existence of mild solutions as well as the approximate controllability for the following non-autonomous evolution equation with nonlocal conditions

$$\begin{cases} u'(t) = A(t)u(t) + Bv(t) + f(t, u(t), u(h(t))), \ t \in J := [0, a], t \neq t_k, \\ \Delta u(t_k) = u(t_k^+) - u(t_k^-) = I_k(u(t_k)), \ k = 1, 2, \cdots, m, \\ u(0) = g(u) + u_0, \end{cases} \tag{1.1}$$

in Banach space X, where $a > 0$ is a constant. $A(t) : D(A) \subseteq X \to X$ is a family of densely defined and closed linear operator which generates an evolution system $\{H(t, s) : 0 \leqslant s \leqslant t \leqslant a\}$ on X, $D(A)$ is independent of t, the control function $v(t)$ is given in Banach space $L^2(J; U)$ of admissible control functions, U is also a Banach space, $h(t) \in C(J, J)$, $f : J \times X \times X \to X$ is a continuous nonlinear mapping, and B is a bounded linear operator from U to X, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = a$, $I_k, k = 1, 2, \cdots, m$ are impulsive functions, $m \in \mathbb{N}$. The existence of mild solutions to non-autonomous impulsive evolution equations (1.1) is proved by using the Krasnoselskii's fixed point theorem as well as the theory of evolution family. In addition, the conditions of approximate controllability are given by using the resolvent operator. In particular, the nonlocal item g is divided into Lipschitz continuous and completely bounded. The approximate controllability of the evolution equation with impulse is studied, which makes the application of the evolution equation more universal and representative. An example is given as an application of the results. Compared with Yang [18] and Luo [24], the equation in this paper has more universality and generality. Yang [18] and Luo [24] can be considered as a special case of this article.

The rest of this paper is organized as follows. In Section 2, we introduce some basic definitions, lemmas and properties. The existence of mild solutions as well as approximate controllability for evolution equation (1.1) is discussed by using the nonlinear function and the control operator in Section 3. We give an example to illustrate the feasibility of our results in the last section.

2. Preliminaries

Let X and U be two real Banach spaces with norms $\|\cdot\|$ and $\|\cdot\|_U$. We denote by $C(J,X)$ the Banach space of all continuous functions from interval J into X equipped with the supremum norm

$$\|u\|_C = \sup_{t\in J}\|u(t)\|,\ u\in C(J,X),$$

and by $L^p(X)$ the Banach space of all X-valued p-order Bochner integrable functions on J equipped with the norm

$$\|f\|_{L^p} = \left(\int_0^b \|f(t)\|^p dt\right)^{\frac{1}{p}} \text{ for } p \geqslant 1.$$

We put $J_0 = [0,t_1]$, and $J_k = (t_k, t_{k+1}]\,, k = 1,\cdots,m$. Let $PC(J,X) := \{u : J \to X : u$ be continuous on J_k, and the right limit $u\left(t_k^+\right)$ exists, $k = 1,2,\cdots,m\}$. It is easy to check that $PC(J,X)$ be a Banach space endowed with the norm $\|u\|_{PC} = \sup\{\|u(t)\|, t\in J\}$ and $C(J,X)\subseteq PC(J,X)\subseteq L^1(J,X)$.

Suppose that a family of linear operators $\{A(t) : 0\le t\le a\}$ satisfies the following assumptions:

(A_1) The family $\{A(t) : 0\le t\le a\}$ is a closed linear operator;
(A_2) for each $t\in[0,a]$, the resolvent $R(\lambda, A(t)) = (\lambda - A(t))^{-1}$ of linear operator $A(t)$ exists for all λ such that $\mathrm{Re}\,\lambda\leqslant 0$, and there also exists $K>0$ such that $\|R(\lambda, A(t))\|\leqslant K/(|\lambda|+1)$;
(A_3) there exist $0<\delta\leqslant 1$ and $K>0$ such that $\left\|(A(t)-A(s))A^{-1}(\tau)\right\|\leqslant K|t-s|^{\delta}$ for all t,s and $\tau\in[0,a]$;
(A_4) for each $t\in[0,a]$ and some $\lambda\in\rho(A(t))$, the resolvent set $R(\lambda, A(t))$ of linear operator $A(t)$ is compact.

Since these conditions, we know that the family $\{A(t) : 0\leqslant t\leqslant T\}$ generates a unique linear evolution system, or called a linear evolution family $\{H(t,s) : 0\leqslant s\leqslant t\leqslant T\}$, and there exists a family of bounded linear operators $\{\Psi(t,\tau)\mid 0\leqslant\tau\leqslant t\leqslant T\}$ with norm $\|\Psi(t,\tau)\|\leqslant C|t-\tau|^{\delta-1}$ such that $H(t,s)$ can be represented as

$$H(t,s) = e^{-(t-s)A(t)} + \int_s^t e^{-(t-\tau)A(\tau)}\Psi(\tau,s)d\tau, \tag{2.1}$$

where $e^{-\tau A(t)}$ denotes the analytic semigroup with infinitesimal generator $(-A(t))$.

Lemma 2.1. [14] *The family of linear operators $\{H(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$ satisfies the following conditions:*

(i) the mapping $(t,s) \to H(t,s)x$ is continuous, for each $x \in X$, $H(t,s) \in L(X)$ and $0 \le s \le t \le T$;
(ii) $H(t,s)H(s,\tau) = H(t,\tau)$ for $0 \le \tau \le s \le t \le T$, and $H(t,t) = I$;
(iii) $H(t,s)$ is a compact operator whenever $t - s > 0$;
(iv) There holds

$$\|H(t+h,\tau) - H(t,\tau)\| \le \frac{Kh^\gamma}{|t-\tau|^\gamma}, \text{ for } 0 < h < 1, 0 < \gamma < 1 \text{ and } t - \tau > h.$$

Condition (A_4) *ensures the generated evolution operator satisfies (iii) (see [12], Proposition 2.1). Hence, there exists a constant $M \ge 1$, such that*

$$\|H(t,s)\| \le M \quad \text{for all } 0 \le s \le t \le T. \tag{2.2}$$

Definition 2.1. [14] The evolution family $\{H(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$ is continuous and maps bounded subsets of X into pre-compact subsets of X.

Lemma 2.2. [18] *For each $t \in [0,a]$ and some $\lambda \in \rho(A(t))$, if the resolvent $R(\lambda, A(t))$ is a compact operator, then $H(t,s)$ is a compact operator whenever $0 \le s < t \le a$.*

Lemma 2.3. [18] *Let $\{H(t,s), 0 \leqslant s \leqslant t \leqslant a\}$ be a compact evolution system in X. Then for each $s \in [0,a]$, $t \mapsto H(t,s)$ is continuous by operator norm for $t \in (s,a]$.*

Let Y be another separable reflexive Banach space, whose norm is also denoted by $\|\cdot\|$, in which the control function $v(t)$ takes its values, U is a bounded subset of Y. Denoted by $P_c(Y)$ a class of nonempty closed and convex subsets of Y. We suppose that the multi-valued map $\psi : J \to P_c(Y)$ is graph-measurable, and $\psi(\cdot) \subset U$. The admissible control set V_{ad} is defined by

$$V_{ad} = \{v \in L^p(J,U) : v(t) \in \psi(t), \text{ a.e.t } \in J\}, \ p > 1.$$

Obviously, $V_{ad} \ne \emptyset$ (see [20]) and $V_{ad} \subset L^p(J,Y)(p > 1)$ is bounded, closed and convex. For any $r > 0$, let $\Omega_r := \{u \in C(J,X) : \|u(t)\| \le r, t \in J\}$.

Definition 2.2. A function $u \in C(J,X)$ is said to be a mild solution of nonlocal evolution equation *(1.1)*, if for any $v \in L^2(J,U)$, $u(t)$ satisfies the integral equation

$$\begin{aligned} u(t) = {} & H(t,0)\left(u_0 + g(u)\right) + \int_0^t H(t,s)[f(s,u(s),u(h(s))) + Bv(s)]ds \\ & + \sum_{0<t_k<t} H\left(t,t_k\right) I_k\left(u\left(t_k\right)\right), \ t \in J. \end{aligned}$$

Lemma 2.4. [18] *(Krasnoselskii's fixed point theorem). Let W be a closed, convex and nonempty subset of Banach space X. Let operators $Q_1, Q_2 : W \to X$ satisfy*
(i) if $x, y \in W$, then $Q_1x + Q_2y \in W$;
(ii) Q_1 is a contraction;
(iii) Q_2 is compact and continuous.
Then the operator $Q := Q_1 + Q_2$ has at least one fixed point in W.

Lemma 2.5. [18] *If Ω is a compact subset of a Banach space X, its convex closure is compact.*

Definition 2.3. Let u be a mild solution of evolution equation *(1.1)* corresponding to the control $v \in L^2(J,U)$. Nonlocal evolution equation *(1.1)* is said to be approximately controllable on the interval J if $\overline{\mathcal{K}_a(f)} = X$, where the set

$$\mathcal{K}_a(f) = \left\{u(a) \in X : v \in L^2(J,U)\right\}$$

is called the reachable set of nonlocal evolution equation *(1.1)*.

3. Main results

In this section, we will present and prove the approximate controllability of nonlocal evolution euation (1.1) by using the Krasnoselskii's fixed point theorem and the theory of evolution system. For this purpose, we first introduce the following two operators defined on Banach space X by

$$\Gamma_0^a = \int_0^a H(a,s)BB^*H^*(a,s)ds, \tag{3.1}$$

$$R\left(\lambda, \Gamma_0^a\right) = \left(\lambda I + \Gamma_0^a\right)^{-1}, \ \lambda > 0, \tag{3.2}$$

where B^* and $H^*(t,s)$ denote the adjoint operators of B and $H(t,s)$. Let u be the mild solution of evolution equation (1.1) corresponding to the control $v \in L^2(J,U)$. Then, the evolution equation (1.1) is said to be approximately controllable on interval $[0,a]$ if for every desired final state $u_a \in X$ and $\epsilon > 0$, there exists a control $v \in L^2(J,U)$ such that u satisfies $\|u(a) - u_a\| < \epsilon$.

Next, we will show that for every $\lambda > 0$ and $u_a \in X$ there exists a continuous function $u \in C(J,X)$ such that

$$\begin{aligned} u(t) = {} & H(t,0)\left(u_0 + g(u)\right) + \int_0^t H(t,s)[f(s,u(s),u(h(s))) + Bv_\lambda(s)]ds \\ & + \sum_{0<t_k<t} H\left(t,t_k\right) I_k\left(u\left(t_k\right)\right), \ t \in J, \end{aligned} \tag{3.3}$$

where the function v_λ is the control function defined by

$$v_\lambda = B^*H^*(a,t)R(\lambda,\Gamma_0^a)z(u(\cdot)), \tag{3.4}$$

and

$$z(u(\cdot)) = u_a - \int_0^a H(a,s)f(s,u(s),u(h(s)))ds - H(a,0)(u_0 + g(u)). \tag{3.5}$$

First, we consider the existence of mild solutions of the non-autonomous problem (1.1). So we further assume the following conditions:

(H$_1$) There exists a function $\psi \in L\left(J,\mathbb{R}^+\right)$such that $\|Bv(t)\| \le \psi(t)$ for all $v \in L^2(J,U)$ and $t \in J$;

(H_2) the function $f : J \times X \times X \to X$ satisfies :
(i) for every $t \in J$, the function $f(t, \cdot, \cdot) : X \times X \to X$ is continuous and for each $(u, v) \in X \times X$, the function $f(\cdot, u, v) : J \to X$ is strongly measurable;
(ii) for any $r > 0$, there exists a function $\varphi \in L^1(J, \mathbb{R}^+)$ such that $\sup\{\|f(t, u, u)\| : \|u\| \leqslant r\} \leqslant \varphi(t)$, for all $u \in \Omega_r$, $t \in J$ and $\lim_{r\to+\infty} \frac{\|\varphi\|_{L^1}+\|\psi\|_{L^p}}{r} = \sigma < \infty$,

(H_3) the function $g : PC(J, X) \to X$ is supposed to be $g(0) = 0$ and Lipschitz continuous with a Lipschitz constant $x > 0$ such that $\|g(u) - g(v)\| \leqslant x\|u - v\|_{PC}$, $\forall u, v \in \Omega_r$,

(H_4) for $k \in \{1, 2, \cdots, m\}$, the function $I_k : X \to X$ and there exists a constant $y_k > 0$ such that

$$\|I_k(u) - I_k(v)\| \leqslant y_k\|u - v\|, \ \forall u, v \in X.$$

Theorem 3.1. *Let the evolution family* $\{H(t, s) : 0 \leqslant s \leqslant t \leqslant T\}$ *generated by* $\{A(t) : 0 \le t \le a\}$ *be compact. Suppose that the assumption* (H_1) − (H_4) *are also satisfied. Then the evolution equation* (1.1) *has at least one mild solution on* J *provided that*

$$Mx + M\sigma + M \sum_{1<k<M} y_k < 1. \tag{3.6}$$

Proof. Defined the operator $Q = Q_1 + Q_2$, where

$$(Q_1u)(t) = H(t, 0)(u_0 + g(u)) + \sum_{0<t_k<t} H(t, t_k) I_k(u(t_k)), \ t \in J, \tag{3.7}$$

$$(Q_2u)(t) = \int_0^t H(t, s)[Bv(s) + f(s, u(s), u(h(s)))]ds, \ t \in J. \tag{3.8}$$

By Definition 2.2, we can know that the mild solution of evolution equation (1.1) is equivalent to the fixed point of operator Q. In the following, we will prove that the operator Q admits a fixed point by applying the Krasnoselskii's fixed point theorem. The proof is divided into five steps.

Step 1. $Q(\Omega_r) \subseteq \Omega_r$ for some $r > 0$.

If this is not true, for each $r > 0$, there exists $u_r \in \Omega_r$, $\|(Qu_r)t\| > r$ for all $t \in J$. From the definition of Q and hypotheses (H_1) − (H_4), we have

$$\begin{aligned} r < \|(Qu_r)t\| \le \| \ H(t,0)(u_0 + g(u))\| &+ \Big\| \sum_{0<t_k<t} H(t, t_k) I_k(u_r(t_k)) \Big\| \\ &+ \Big\| \int_0^t H(t, s)[Bv(s) + f(s, u_r(s), u_r(h(s)))]ds \Big\| \\ &\le M\|u_0\| + M(xr + \|g(0)\|) + M \sum_{k=1}^{m} \Big(y_k r + \|I_k(0)\| \Big) \\ &+ M\|\varphi\|_{L^1} + M\|\psi\|_{L^p}. \end{aligned}$$

Dividing on both sides by r and taking the lower limit as $r \longrightarrow +\infty$, we obtain

$$Mx + M\sigma + M \sum_{1<k<M} y_k < 1,$$

which is a contradiction to the condition in Theorem 3.1. Thus, $Q(\Omega_r) \subseteq \Omega_r$ for some $r > 0$.

Step 2. $Q_1 : \Omega_r \to \Omega_r$ is a contraction operator.

For any $t \in J$, $u, v \in \Omega_r$, (3.7), (2.2), (H_3) and (H_4) imply

$$\begin{aligned}
&\|(Q_1 u)(t) - (Q_1 v)(t)\| \\
&\leqslant \|H(t,0)(g(u) - g(v))\| + \left\| \sum_{0<t_k<t} H(t,t_k)\left(I_k(u(t_k)) - I_k(v(t_k))\right) \right\| \\
&\leqslant Mx\|u - v\|_{PC} + M\sum_{k=1}^{m} y_k \|u - v\|_{PC},
\end{aligned}$$

which yields that

$$\|Q_1 u - Q_1 v\|_{PC} \leqslant \left(Mx + M\sum_{k=1}^{m} y_k \right) \|u - v\|_{PC}.$$

Hence Q_1 is a contraction operator in Ω_r.

Step 3. $Q_2 : \Omega_r \to \Omega_r$ is continuous. Let $\{u_n\}_{n=1}^{\infty} \subset C(J,X)$ with $\lim_{n\to+\infty} u_n = u$ in $C(J,X)$. Then by the continuity of f, we have

$$\lim_{n\to+\infty} f(s, u_n(s), u_n(h(s))) = f(s, u(s), u(h(s))), \ \forall s \in J. \tag{3.9}$$

In addition, since

$$\|f(s, u_n(s), u_n(h(s))) - f(s, u(s), u(h(s)))\| \leq 2\varphi(s), \tag{3.10}$$

and the Lebesgue's dominated convergence theorem follows that

$$\begin{aligned}
\|(Q_2 u_n)(t) - (Q_2 u)(t)\| \ &\leq \int_0^t \|H(t,s)\| \|f(s, u_n(s), u_n(h(s))) \\
&\quad - f(s, u(s), u(h(s)))\| ds \\
&\leq M \int_0^t \|f(s, u_n(s), u_n(h(s))) - f(s, u(s), u(h(s)))\| ds \\
&\to 0 \ \text{ as } \ n \to \infty,
\end{aligned}$$

which means that

$$\|(Qu_n) - (Qu)\|_C = \sup_{t\in J} \|(Qu_n)(t) - (Qu)(t)\| \to 0 \ \text{ as } \ n \to \infty. \tag{3.11}$$

Therefore, by (3.11), we know that $Q_2 : \Omega_r \to \Omega_r$ is continuous.

Step 4. Q_2 is equi-continuous in Ω_r. For any $u \in \Omega_r$ and $0 \leq t_1 \leq t_2 \leq a$, by (3.8) and ($H_1$) – ($H_4$), we have

$$\begin{aligned}
\|(Q_2 u)(t_2) - (Q_2 u)(t_1)\| \ &= \ \| \int_0^{t_2} H(t_2, s)\,[f(s, u(s), u(h(s))) + Bv(s)] ds \\
&\quad - \int_0^{t_1} H(t_1, s)\,[f(s, u(s), u(h(s))) + Bv(s)] ds \|
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^{t_1} \|H(t_2,s)-H(t_1,s)\| \, \|f(s,u(s),u(h(s)))\|ds \\
&\quad + \int_0^{t_1} \|H(t_2,s)-H(t_1,s)\| \, \|Bv(s)\|ds \\
&\quad + \int_{t_1}^{t_2} \|H(t_2,s)\| \|f(s,u(s),u(h(s)))\|ds \\
&\quad + \int_{t_1}^{t_2} \|H(t_2,s)\| \|Bv(s)\|ds \\
&\le \int_0^{t_1} \|H(t_2,s)-H(t_1,s)\| \\
&\quad \|Bv(s)+f(s,u(s),u(h(s)))\|ds \\
&\quad + \int_{t_1}^{t_2} \|H(t_2,s)\| \|Bv(s)+f(s,u(s),u(h(s)))\|ds \\
&:= I_1 + I_2,
\end{aligned}$$

where

$I_1 = \int_0^{t_1} \|H(t_2,s)-H(t_1,s)\| \, \|Bv(s)+f(s,u(s),u(h(s)))\|ds;$

$I_2 = \int_{t_1}^{t_2} \|H(t_2,s)\| \|Bv(s)+f(s,u(s),u(h(s)))\|ds.$

If $t_1 \equiv 0$, and $0 < t_2 \le a$, the conclusion is obvious. If $0 < t_1 < a$, we choose $\varepsilon \in (0,t_1)$ small enough, by the conditions (H_0), (H_1) and (H_2), we have

$$\begin{aligned}
I_1 \le & \int_0^{t_1-\varepsilon} \|H(t_2,s)-H(t_1,s)\| \, \|f(s,u(s),u(h(s)))\|ds \\
& + \int_{t_1-\varepsilon}^{t_1} \|H(t_2,s)-H(t_1,s)\| \, \|f(s,u(s),u(h(s)))\|ds \\
\le & \sup_{s\in[0,t_1-\varepsilon]} \|H(t_2,s)-H(t_1,s)\|_{L(X)} \int_0^{t_1-\varepsilon} [\varphi(s)+\psi(s)]ds \\
& + M\int_{t_1-\varepsilon}^{t_1} \varphi(s)ds + M\int_{t_1-\varepsilon}^{t_1} \psi(s)ds \to 0 \ \text{ as } \ t_2-t_1 \to 0 \ \text{ and } \ \varepsilon \to 0, \\
I_2 \le & M\int_{t_1}^{t_2} \varphi(s)+\psi(s)ds \to 0, as\ t_2-t_1 \to 0.
\end{aligned}$$

Therefore, $\|(Q_2u)(t_2)-(Q_2u)(t_1)\| \to 0$, as $t_2-t_1 \to 0$, which means that the operator $Q_2 : \Omega_r \to \Omega_r$ is equi-continuous.

Step 5. The set $W(t) := \{(Q_2u)(t) : u \in \Omega_r\}$ is relatively compact in X for each $t \in J$. Obviously, the set $W(0) = \{(Q_2u)(0) : u \in \Omega_r\}$ is relatively compact in X. Let $t \in (0,a]$, for any $x \in \mathrm{H_{ad}}, \mathrm{u} \in \Omega_\mathrm{r}$ and $\epsilon \in (0,t-s)$, we define an operator Q_2^ϵ by

$$(Q_2^\epsilon u)(t) := \int_0^{t-\epsilon} H(t,s)[Bv(s)+f(s,u(s),u(h(s)))]ds.$$

It follows from the boundedness of $\mathrm{H_{ad}}$ and (H_1) that the set $Z_\epsilon = \{H(t,s)[Bv(s)+f(s,u(s),u(h(s)))] : 0 \leqslant s < t-\epsilon\}$ is relatively compact and depend on the compactness of $H(t,s)(t-s>0)$. Then, $\overline{co}(W_\epsilon)$ is a compact set depending on Lemma 2.5. By the mean value theorem of Bochner integrals, we can get $(Q_2^\epsilon u)(t) \in (t-\epsilon)\overline{co}(W_\epsilon)$ for all $t \in J$. Thus, the set $W_\epsilon(t) = \{(Q_2^\epsilon u)(t) : u \in \Omega_r\}$

is relatively compact in X for every $t \in J$. Moreover, by (3.8), (2.2) and (H$_1$), we have

$$\begin{aligned}
&\|(Q_2u)(t) - (Q_2^{\epsilon}u)(t)\| \\
=&\| \int_0^t H(t,s)[Bv(s) + f(s,u(s),u(h(s)))]ds \\
&- \int_0^{t-\epsilon} H(t,s)[Bv(s) + f(s,u(s),u(h(s)))]ds\| \\
\leq & \int_{t-\epsilon}^t \|H(t,s)[Bv(s) + f(s,u(s),u(h(s)))]\|ds \\
\leq & M\int_{t-\epsilon}^t \varphi(s) + \psi(s)ds,
\end{aligned}$$

which means that $\lim_{\epsilon\to 0} \|(Q_2u)(t) - (Q_2^{\epsilon}u)(t)\| = 0$. So we have proved that there is a family of relatively compact sets $W_{\epsilon}(t)$ arbitrarily close to the set $W(t)$. Thus, the set $W(t)$ is relatively compact in X for every $t \in [0,a]$.

By Steps 3-5, thanks to the Ascoli-Arzela theorem, we deduce that the operator $Q_2 : \Omega_r \to \Omega_r$ is compact and continuous. By the Krasnoselskii's fixed point theorem, we can get the operator Q has at least one fixed point in Ω_r, which is the mild solution of the evolution equation (1.1) on J. This completes the proof of Theorem 3.1.

□

Next, we present an existence result for control problem (1.1) when the nonlocal function g is completely continuous in $PC(J,X)$.

(H$_3$)$'$ The function $g : PC(J,X) \to X$ is completely continuous.

Remark 3.1. From (H$_3$)$'$, the set $\{g(u) : u \in \Omega_r\}$ is completely bounded. Hence $\sup_{u\in\Omega_r} \|g(u)\|$ exists and $\lim_{r\to\infty} \frac{\sup_{u\in\Omega_r}\|g(u)\|}{r} = 0$.

Theorem 3.2. *Let the conditions* (H$_1$), (H$_2$), (H$_3$)$'$ *and* (H$_4$) *hold. If*

$$M\sigma + M \sum_{1<k<M} y_k < 1, \tag{3.12}$$

then the control problem (1.1) *has at least one mild solution in* Ω_r*.*

Proof. For every $v \in V_{ad}$, we define an operator $P = P_1 + P_2$: $\Omega_r \to PC(J,X)$, where

$$(P_1u)(t) = \sum_{0<t_k<t} H(t,t_k) I_k(u(t_k)),\ t \in J,$$

$$(P_2u)(t) = H(t,0)(u_0 + g(u)) + \int_0^t H(t,s)[f(s,x(s),x(h(s))) + Bv(s)]ds,\ t \in J.$$

By Definition 2.2, we can know the mild solution of (1.1) is equivalent to the fixed point of P on J. The same is true for Theorem 3.1, we can prove that $P(\Omega_r) \subseteq \Omega_r$, $P_1 : \Omega_r \to \Omega_r$ is a contraction operator and $P_2 : \Omega_r \to \Omega_r$ is compact and continuous. By Lemma 2.4, the operator P admits a fixed point in Ω_r. Here, we omit the detail. This completes the proof of Theorem 3.2.

□

Theorem 3.3. *Let the evolution family* $\{H(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$ *generated by* $\{A(t) : 0 \leq t \leq a\}$ *be compact. If the assumptions* (H_1), (H_3), (H_4) *and the following assumptions are satisfied:*

$(H_2)'$ *The function* $f : J \times X \times X \to X$ *satisfies :*

(i) for every $t \in J$ *, the function* $f(t,\cdot,\cdot) : X \times X \to X$ *is continuous and for each* $(u,v) \in X \times X$ *, the function* $f(\cdot,u,v) : J \to X$ *is strongly measurable;*

(ii) for any $r > 0$*, there exists a function* $\varphi \in L^2(J,\mathbb{R}^+)$ *such that* $\sup\{\|f(t,u,u)\| : \|u\| \leqslant r\} \leqslant \varphi(t)$*, for all* $u \in \Omega_r$*,* $t \in J$ *and* $\lim_{r\to+\infty} \frac{\|\varphi\|_{L^2}+\|\psi\|_{L^p}}{r} = \sigma < \infty$*,*

(H_5) $\lambda R(\lambda,\Gamma_0^a) \to 0$ *as* $\lambda \to 0^+$ *in the strong operator topology.*

Then the evolution equation (1.1) *is approximately controllable on* J*.*

Proof. It is easily to know that the assumption $(\mathrm{H}_2)' \Rightarrow (\mathrm{H}_2)$. Therefore, by Theorem 3.1, we know that the evolution equation (1.1) has at least one mild solution $u_\lambda \in \Omega_r$, which means that

$$\begin{aligned} u_\lambda(t) =& H(t,0)\left(u_0 + g(u_\lambda)\right) + \int_0^t H(t,s)[f(s,u_\lambda(s),u_\lambda(h(s))) + Bv_\lambda(s)]ds \\ &+ \sum_{0<t_k<t} H\left(t,t_k\right) I_k\left(u_\lambda\left(t_k\right)\right), \ t \in J, \end{aligned} \tag{3.13}$$

with

$$v_\lambda(t) = B^* H^*(a,t) R\left(\lambda,\Gamma_0^a\right) z\left(u_\lambda(\cdot)\right), \tag{3.14}$$

and

$$\begin{aligned} z\left(u_\lambda(\cdot)\right) =& u_a - \int_0^a H(a,s) f\left(s,u_\lambda(s),u_\lambda(h(s))\right) ds \\ &- \sum_{0<t_k<a} H\left(a,t_k\right) I_k\left(u_\lambda\left(t_k\right)\right) - H(a,0)(u_0 + g(u_\lambda)). \end{aligned} \tag{3.15}$$

Therefore, by (3.13), (3.14) and (3.15), we can combine with an easy computation to get

$$\begin{aligned} u_\lambda(a) =& H(a,0)(u_0 + g(u_\lambda)) + \int_0^a H(a,s)\left[f\left(s,u_\lambda(s),u_\lambda(h(s))\right) + Bv_\lambda(s)\right] ds \\ &+ \sum_{0<t_k<a} H\left(a,t_k\right) I_k\left(u_\lambda\left(t_k\right)\right) \\ =& u_a - z\left(u_\lambda(\cdot)\right) + \int_0^a H(a,s) BB^* H^*(a,s) R\left(\lambda,\Gamma_0^a\right) z\left(u_\lambda(\cdot)\right) ds \\ =& u_a - z\left(u_\lambda(\cdot)\right) + \Gamma_0^a R\left(\lambda,\Gamma_0^a\right) z\left(u_\lambda(\cdot)\right) \\ =& u_a - \left(\lambda I + \Gamma_0^a\right) R\left(\lambda,\Gamma_0^a\right) z\left(u_\lambda(\cdot)\right) + \Gamma_0^a R\left(\lambda,\Gamma_0^a\right) z\left(u_\lambda(\cdot)\right) \\ =& u_a - \lambda R\left(\lambda,\Gamma_0^a\right) z\left(u_\lambda(\cdot)\right). \end{aligned} \tag{3.16}$$

According to the condition $(H_2)'$, one gets that

$$\left(\int_0^a \left\|f\left(s,u_\lambda(s),u_\lambda(h(s))\right)\right\|^2 ds\right)^{\frac{1}{2}} \leq \left(\int_0^a \varphi^2(s) ds\right)^{\frac{1}{2}} < \infty.$$

Then the boundedness of the sequence $\{f(\cdot, u_\lambda(\cdot), u_\lambda(\cdot)) \mid \lambda > 0\}$ in $L^2(J, X)$ shows that there exists a subsequence of $\{f(\cdot, u_\lambda(\cdot), u_\lambda(\cdot)) \mid \lambda > 0\}$ which converges weakly to some $K(\cdot) \in L^2(J, X)$. Define

$$\omega := u_a - \int_0^a H(a,s)K(s)ds. \tag{3.17}$$

It follows that

$$\begin{aligned} \|z(u_\lambda) - \omega\| \leq & \|H(a,0)(u_0 + g(u_\lambda))\| \\ & + \left\| \int_0^a H(a,s)\left[f(s, u_\lambda(s), u_\lambda(h(s))) - K(s)\right] ds \right\|. \end{aligned} \tag{3.18}$$

According to the fact, the evolution family $H(t,s)$ is compact operators for $0 \leq s < t \leq a$. This means that the mapping

$$k(t) \to \int_0^t H(t,s)k(s)ds$$

is compact for $t \in J$, which implies that

$$\int_0^a H(a,s)\left[f(s, u_\lambda(s), u_\lambda(h(s))) - K(s)\right] ds \to 0 \quad \text{as} \quad \lambda \to 0^+. \tag{3.19}$$

Hence, from (3.16) and (3.19), we know that

$$\|z(u_\lambda) - \omega\| \to 0 \quad \text{as } \lambda \to 0^+. \tag{3.20}$$

In the following, (3.16), (3.20) and assumption (H5) imply that

$$\begin{aligned} \|u_\lambda(a) - u_a\| &\leq \|\lambda R(\lambda, \Gamma_0^a) z(u_\lambda)\| \\ &\leq \|\lambda R(\lambda, \Gamma_0^a)\omega\| + \|\lambda R(\lambda, \Gamma_0^a)\| \cdot \|z(u_\lambda) - \omega\| \\ &\to 0 \text{ as } \lambda \to 0^+. \end{aligned} \tag{3.21}$$

In conclution, the evolution equation (1.1) is approximately controllable. The proof is complete.

□

4. Example

In this section, we provide a correct example to illustrate our abstract results.

Example 4.1. Consider the following non-autonomous partial differential equation with nonlocal problem:

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial^2 x}u(x,t) + a(t)u(x,t) + \frac{t^2\sin(2\pi t)}{1+|u(x,t)|} \cdot \frac{1}{1+|u(x,\sin t)|} \\ \qquad\qquad + 2v(x,t), x \in [0,\pi], t \in J\backslash\{\frac{1}{2}\}, \\ u(0,t) = u(1,t) = u(\pi, \sin t) = 0, \ t \in J\backslash\{\frac{1}{2}\}, \\ u(x,0) = u_0(x) + \sum_{i=1}^2 c_i[\sin(u(x,t_i)) \\ \qquad\qquad - \sin(u(0,t_i))], 0 < t_1 < t_2 < a, x \in [0,\pi], \\ \Delta u\left(x, \frac{1}{2}\right) = \frac{\left|u\left(x,\frac{1}{2}\right)\right|}{2+\left|u\left(x,\frac{1}{2}\right)\right|}, \end{cases} \tag{4.1}$$

where $J := [0, a]$, $a > \frac{1}{2}$, $a(t) : J \to \mathbb{R}$ is a continuously differentiable function and satisfies

$$a_{\min} := \min_{t\in[0,1]} a(t) < 1, \tag{4.2}$$

and $a > 0$ is a constant, $v \in L^2\left(J, L^2(0,\pi;\mathbb{R})\right)$. Let $X = L^2(0,\pi;\mathbb{R})$ with the norm $\|\cdot\|_2$ and inner product $\langle\cdot,\cdot\rangle$. Consider the operator A on X defined by

$$Au := \frac{\partial^2}{\partial x^2}u,\ u \in D(A),$$

where

$$D(A) := \left\{u \in L^2(0,\pi;\mathbb{R}), u'' \in L^2(0,\pi;\mathbb{R}), u(0) = u(\pi) = 0\right\}.$$

The A generates a compact and analytic C_0-semigroup in C, A has a discrete spectrum, and its eigenvalues are $-n^2, n \in \mathbb{N}^+$ with the corresponding normalized eigenvectors $v_n(x) = \sqrt{\frac{2}{\pi}}\sin(nx)$. Define the operator $A(t)$ on A by

$$A(t)u = Au - a(t)u,$$

with domain

$$D(A(t)) = D(A),\ t \in [0,1].$$

The family $\{A(t) : 0 \le t \le a\}$ generates an strongly continuous evolution family $\{H(t,s) : 0 \le s \le t \le a\}$ defined by

$$H(t,s)u = \sum_{n=1}^{\infty} e^{-\left(\int_s^t a(\tau)d\tau + n^2(t-s)\right)} \langle u, v_n\rangle\, v_n,\ 0 \le s \le t \le 1, u \in X. \tag{4.3}$$

A direct calculation gives

$$\|H(t,s)\|_{L(X)} \le e^{-(1+a_{\min})(t-s)},\ 0 \le s \le t \le 1.$$

(4.2) and (4.3) mean that

$$M := \sup_{0\le s\le t\le a} \|H(t,s)\|_{L(X)} = 1.$$

(see [12])

For any $t \in [0,a]$, we define

$$\begin{aligned}
u(t)(x) &= u(x,t);\\
f(t,u(t),u(h(t))) &= \frac{\sqrt{u(x,t)}}{1+t^2}\cdot\frac{1}{1+|u(x,\sin t)|};\\
g(u(t))(x) &= \sum_{i=1}^{2} c_i\left[\sin\left(u\left(x,t_i\right)\right) - \sin\left(u\left(0,t_i\right)\right)\right];\\
Bv(t)(x) &= 2v(x,t);\\
I_k u\left(x,\frac{1}{2}\right) &= \frac{\left|u\left(x,\frac{1}{2}\right)\right|}{2+\left|u\left(x,\frac{1}{2}\right)\right|}.
\end{aligned}$$

For any $r > 0$, let $\Omega_r := \{u \in PC(J, X) : \|u(t)\|_{PC} \leq r, t \in J\}$. For any $u, v \in \Omega_r$ and $t \in J$, we have

$$\|f(t, u(t), u(h(t)))\| = \|\frac{\sqrt{u(x,t)}}{1+t^2} \cdot \frac{1}{1+|u(x, \sin t)|}\| \leq \sqrt{r};$$

$$\left\|I_k\left(u\left(x, \frac{1}{2}\right)\right) - I_k\left(v\left(x, \frac{1}{2}\right)\right)\right\| = \left\|\frac{\left|u\left(x, \frac{1}{2}\right)\right|}{2+\left|u\left(u, \frac{1}{2}\right)\right|} - \frac{\left|v\left(v, \frac{1}{2}\right)\right|}{2+\left|v\left(x, \frac{1}{2}\right)\right|}\right\|$$
$$\leqslant \frac{1}{2}\left\|u\left(x, \frac{1}{2}\right) - v\left(x, \frac{1}{2}\right)\right\|.$$

We know $g : PC(0, a; X) \to X$ is a continuous function defined by

$$g(u)(x) = \sum_{i=1}^{q} c_i \left[\sin\left(u\left(t_i\right)\right)(x) - \sin\left(u\left(t_i\right)\right)(0)\right], 0 < t_1 < t_2 < a, 0 \leq u \leq \pi,$$

where $\sin(u(t))(x) = \sin(u(x,t))$, $0 \leq t \leq a$, and $0 \leq x \leq \pi$. It follows from $|\sin(a) - \sin(b)| \leq |a - b|$ which holds for all $a, b \in \mathbb{R}$ that the function g satisfies the Lipschitz constant $l = \max\{|c_i|, i = 1, 2\}$.

Based on the definition of nonlinear term f and the bounded linear operator A combined with the above discussion, we can easily verify that the assumptions $(\mathrm{H}_1) - (\mathrm{H}_4)$ hold true when $\psi(t) = 2v(t)$.

Therefore, the non-autonomous partial differential equation (4.1) is equivalent to the evolution equation (1.1). According to theorem 3.1, we know that (4.1) has at least one mild solution $u \in [C(0, \pi) \times (0, a)]$. By theorem 3.3, we know that (4.1) is approximately controllable on J .

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