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Traveling Wave of Three-Species Stochastic Lotka-Volterra Competitive System*

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Abstract This paper is devoted to a three-species stochastic competitive system with multiplicative noise. The existence of stochastic traveling wave solution can be obtained by constructing sup/sub-solution and using random dynamical system theory. Furthermore, under a more restrict assumption on the coefficients and by applying Feynman-Kac formula, the upper/lower bounds of asymptotic wave speed can be achieved.

Keywords Stochastic competitive system, white noise, traveling wave solution, asymptotic wave speed

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1. Introduction

In this paper, we are interested in the following stochastic three-species competition model driven by Itô type multiplicative noise

$$\begin{cases} u_t = u_{xx} + u(1 - u - a_1 v - b_1 w) + \epsilon u dW_t, \\ v_t = v_{xx} + v(1 - v - a_2 u) + \epsilon(v - 1)dW_t, \\ w_t = w_{xx} + w(1 - w - b_2 u) + \epsilon(w - 1)dW_t, \\ u(0) = u_0, v(0) = 1 - \chi_{(-\infty,0]}, w(0) = 1 - \chi_{(-\infty,0]}, \end{cases} \tag{1.1}$$

where $u = u(t, x)$, $v = v(t, x)$ and $w = w(t, x)$ denote the species densities of three competing species at location $x \in R$ and time $t > 0$ respectively. Moreover, $a_i > 0$ and $b_i > 0$ represent the interspecific competition coefficients, and the environment carrying capacity of each species is ruled to be "1". Further, $W(t)$ is the white noise. Let $\epsilon = 0$, $a_2 = b_2$ and dispersal terms be replaced by nonlocal dispersal functions. Then equation (1.1) is reduced to the model proposed by Dong, Li and Wang in [2], and they showed the existence, monotonicity and asymptotic behavior of traveling waves with bistable dynamics. Based on their work, Wang, Chen and Wu [24] used a three-species competition model to expand Lotka-Volterra model to empirical analysis, and concluded that cooperative action is better than competitive strategy. Furthermore, He and Zhang [6] studied the linear determinacy of critical wave speed of three-species competitive system with nonlocal dispersal by constructing more precise conditions and suitable upper solutions. Moreover,

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Liu et al., [14] studied three-species competition-diffusion model in a general case where every species competes with each other, and they pointed out that the wave speed of the slowest species is dependent on the other two faster species.

Throughout the paper, we always assume the coefficients of three-species competitive system (1.1) as follows

- (C1) $a_1 < \frac{1}{2}$, $b_1 < \frac{1}{2}$, $a_2 \geq 2$, $b_2 \geq 2$;
- (C2) $2\max\{a_1a_2, b_1b_2\} < 2 - a_1 - b_1$;
- (C3) $2\min\{a_1a_2 + b_1b_2\} + (a_1 + b_1 - 1)^2 \geq 1$;
- (C4) $\max\{a_2 - 1, b_2 - 1\} \leq \frac{1}{1-a_1-b_1}$.

Obviously, $(C1) \cap (C2) \cap (C3) \cap (C4)$ is not empty. Under condition (C1), there exist five nonnegative equilibria $P_1 = (0, 1, 1)$, $P_2 = (1, 0, 0)$, $P_3 = (0, 1, 0)$, $P_4 = (0, 0, 1)$ and $P_5 = (0, 0, 0)$, where P_2 is the only stable equilibrium, and the traveling wave solution is a trajectory connecting P_1 and P_2. More precisely, it reflects that the species u wins the competition rather than the pair (v, w).

Letting $v := 1 - \tilde{v}$, $w := 1 - \tilde{w}$ and dropping the tilde, we have

$$\begin{cases} u_t = u_{xx} + u(1 - a_1 - b_1 - u + a_1 v + b_1 w) + \epsilon u dW_t, \\ v_t = v_{xx} + (1 - v)(a_2 u - v) + \epsilon v dW_t, \\ w_t = w_{xx} + (1 - w)(b_2 u - w) + \epsilon w dW_t, \\ u(0) = \chi_{(-\infty,0]}, v(0) = \chi_{(-\infty,0]}, w(0) = \chi_{(-\infty,0]}, \end{cases} \tag{1.2}$$

and it is easy to see that (1.2) is a stochastic cooperative system, and the two equilibria P_1 and P_2 turn to be

$$\tilde{P}_1 = (0, 0, 0), \quad \tilde{P}_2 = (1, 1, 1) \tag{1.3}$$

respectively.

It is worth mentioning that most existing results for stochastic traveling wave solution deal with the scaler Fisher-KPP equation. For instance, Tribe [23] studied the KPP equation with nonlinear multiplicative noise $\sqrt{u}dW_t$, and Muëller et al., [16–18] studied the KPP equation with $\sqrt{u(1-u)}dW_t$. Both of their works take the Heaviside function as the initial data, and the main contribution of Muëller is that he explicitly described the influence brought by the noise, whether it is weak or strong, and successfully estimated the wave speed with an upper bound and a lower bound. Zhao et al., [3, 20, 21] confirmed that only if the strength of noise is moderate, and when the multiplicative noise is $k(t)dW_t$, the effects of noise would present or the solution would tend to be zero or converge to the deterministic traveling wave solution. Huang and Liu [8] studied the KPP equation driven by dual noises $k_1 u dW_1(t)$ and $k_2(K - u)dW_2(t)$, and revealed the bifurcations of solution induced by the strength of noise. For stochastic two-species cooperative system, Wen, Huang and Li [27] used random monotone dynamical systems and the Kolmogorov tigheness criterion to obtain the existence of stochastic traveling wave solution, and then by constructing the upper and lower solution and applying Feynman-Kac formula, they obtained the estimation of the upper bound and lower bound for wave speed respectively. The novelty of this paper not only in the three-species competitive system we study, for which there is no relevant work, but also in our confirmation that the lower bound of wave speed depends on the impact of vulnerable groups on powerful groups.

This paper is organized as follows. In Section 2, we present some notations and crucial lemmas. In Section 3, the existence of stochastic traveling wave solution is established. Finally, in Section 4, we estimate the wave speed by determining the upper bound and lower bound with the sup-solution and the sub-solution respectively.

2. Preliminaries and notation

Throughout this paper, we set Ω be the space of temperature distributions, $\mathcal{F}$ be the σ-algebra on Ω, and $(\Omega, \mathcal{F}, \mathbb{P})$ be the white noise probability space. Denoted by $\mathbb{E}$, the expectation is with respect to $\mathbb{P}$. Denoted by $\phi_\lambda(x) = exp(-\lambda|x|)$, here are some notations:

- $C^+ = \{f | f : R \to [0, \infty)\ and\ f\ is\ continuous\}$;
- $||f||_\lambda = \sup\limits_{x\in R}(|f(x)\phi_\lambda(x)|)$;
- $C^+_\lambda = \{f \in C^+ | f\ is\ continuous, and\ |f(x)\phi_\lambda(x)| \to 0\ as\ x \to \pm\infty\}$;
- $C^+_{tem} = \bigcap\limits_{\lambda>0} C^+_\lambda$;
- $C^+_{C[0,1]} = \{f | f : R \to [0, 1]\}$ is the space of nonnegative functions with compact support;
- $\Phi = \{f : ||f||_\lambda < \infty\ for\ some\ \lambda < 0\}$ is the space of functions with exponential decay.

Lemma 2.1 ([23]). *A set $K \subset C^+_\lambda$ is called relatively compact, if and only if*

(a) K is equicontinuous on a compact set;

(b) $\lim\limits_{R\to\infty} \sup\limits_{f\in K} \sup\limits_{|x|\geq R} |f(x)e^{-\lambda|x|}| = 0$.

Lemma 2.2 ([23]). *$K \subset C^+_{tem}$ is (relatively) compact, if and only if it is (relatively) compact in C^+_λ for all $\lambda > 0$.*

Lemma 2.3 ([23]). *(Kolmogorov tightness criterion) For $C < \infty, \delta > 0, \mu < \lambda, \gamma > 0$, define*

$$K(C, \delta, \gamma, \mu) = \{f : |f(x) - f(x')| \leq C|x - x'|^\gamma e^{\mu|x|}\ for\ all\ |x - x'| \leq \delta\}.$$

Then with the above conditions, we know that $K(C, \delta, \gamma, \mu) \cap \{f : \int_R f(x)\phi_1 dx \leq a\}$ is compact in C^+_λ, where a is a constant.

(1) If $\{X_n(\cdot)\}$ are C_λ-valued processes, with $\{\int_R X_n\phi_1 dx\}$ tight, and there are $C_0 < \infty, p > 0, \gamma > 1, \mu < \lambda$ such that for all $n \geq 1$, $|x - y| \leq 1$,

$$\mathbb{E}(|X_n(x) - X_n(y)|^p) \leq C_0|x - y|^\gamma e^{\mu p|x|},$$

then $\{X_n\}$ are tight.

(2) Similarly, if $\{X_n\}$ are $C([0, T], C^+_\lambda)$-valued processes, with $\{\int_R X_n(0)\phi_1 dx\}$ tight, and there are $C_0 < \infty, p > 0, \gamma > 2, \mu < \lambda$ such that for all $n \geq 1, |x - y| \leq 1, |t - t'| \leq 1, t, t' \in [0, T]$,

$$\mathbb{E}(|X_n(x, t) - X_n(y, t')|^p) \leq C_0(|x - y|^\gamma + |t - t'|^\gamma)e^{\mu p|x|},$$

then $\{X_n\}$ are tight.

3. Existence of traveling wave solution

For simplicity, we denote by $Y = (u, v, w)^T$, $F(Y) = (u(1 - a_1 - b_1 - u + a_1 v + b_1 w), (1 - v)(a_2 u - v), (1 - w)(b_2 u - w))^T$, $H(Y) = (u, v, w)^T$, $F_1(Y) = u(1 - a_1 - u + a_1 v + b_1 w)$, $F_2(Y) = (1 - v)(a_2 u - v)$, $F_3(Y) = (1 - w)(b_2 u - w)$, $H_1(Y) = u$, $H_2(Y) = v$, $H_3(Y) = w$, and we can rewrite the transformed system (1.2)

$$\begin{cases} Y_t = Y_{xx} + F(Y) + \epsilon H(Y) dW_t, \\ Y(0, x) = Y_0 = (u_0, v_0, w_0)^T, \end{cases} \tag{3.1}$$

where $u_0 = v_0 = w_0 = \chi_{(-\infty,0]}$.

For any matrix $M = (m_{ij})_{n\times m}$, define the norm $|\cdot|$ as $|M| = \Sigma_{i,j=1}|m_{ij}|$, and the vector norm is $||A||_\infty = \max_i(A_i)$.

Lemma 3.1. *For $u_0, v_0, w_0 \in C^+_{tem}$, and a.e. $\omega \in \Omega$, there exists a unique solution $Y(t, x)$ to (1.2) with the form*

$$\begin{aligned} Y(t, x) = &\int_R G(t, x, y) Y_0 dy \\ &+ \int_0^t \int_R G(t - s, x, y) F(Y) ds dy + \epsilon \int_0^t \int_R G(t - s, x, y) H(Y) dW_s dy, \end{aligned}$$

where $G(t, x, y)$ is the Green function.

Proof. Set $F_1^n(Y^n) = (1 - a_1 - b_1)u^n - (u^n)^2 \wedge n + a_1(u^n \wedge \sqrt{n})(v^n \wedge \sqrt{n}) + b_1(u^n \wedge \sqrt{n})(w^n \wedge \sqrt{n})$, $F_2^n(Y_n) = a_2 u^n - v^n - a_2(u^n \wedge \sqrt{n})(v^n \wedge \sqrt{n}) + (v^n)^2 \wedge n$, $F_3^n(Y^n) = b_2 u^n - w^n - b_2(u^n \wedge \sqrt{n})(w^n \wedge \sqrt{n}) + (w^n)^2 \wedge n$. Then there exists a pathwise unique solution $Y^n(t) \in C^+_{tem}$ solving

$$\begin{cases} Y^n_t = Y^n_{xx} + F^n(Y^n) + \epsilon H(Y^n) dW_t, \\ Y^n_0 = Y_0. \end{cases} \tag{3.2}$$

Thus, referring to [23,27], one can easily finish the proof by Kolmogorov tightness criterion (Lemma 2.3), and any limit point of sequence $\{Y^n(t) : n \geq 1\}$ is a solution to equation (3.1). Similarly, we know that $Y(t, x) \in C^+_{tem}$. □

Furthermore, along the idea of Tribe [23], we have the following conclusion which contributes to verifying that $Y(t, x)$ is stationary.

Lemma 3.2 ([23]). *All solutions to (3.1) started at Y_0 have the same law which we denote by Q^{Y_0,a_1,a_2,b_1,b_2}, and the map $(Y_0, a_1, a_2, b_1, b_2) \to Q^{Y_0,a_1,a_2,b_1,b_2}$ is continuous. The law Q^{Y_0,a_1,a_2,b_1,b_2} for $Y_0 \in C^+_{tem}$ forms a strong Markov family.*

First, referring to [10,19,23], we introduce the comparison methods for stochastic reaction-diffusion equations as follows.

Lemma 3.3. *There is a coupling solution $Y(t, x)$ to (3.1) started at $Y_0 \in C^+_{tem}$ with $\Theta(t, x)$ a solution to*

$$\begin{cases} \Theta_t = \Theta_{xx} + P(\Theta) + \epsilon H(\Theta) dW_t, \\ \Theta_0 = Y_0. \end{cases}$$

If $P(Y) \geq F(Y)$ and $P(Y)$ is Lipschitz continuous, then for any $Y_0, \Theta_0 \in C^+_{tem}$, the following assertions hold.

(1) Fix $\Theta_0^{(1)}, \Theta_0^{(2)} \in C_{tem}^+$ with $\Theta_0^{(1)} \leq \Theta_0^{(2)}$, then for every $t > 0, x \in R$, and for a.e. $\omega \in \Omega$

$$\Theta^{(1)}(t,x) \leq \Theta^{(2)}(t,x).$$

(2) For every $t > 0, x \in R$, and for a.e. $\omega \in \Omega$

$$Y(t,x) \leq \Theta(t,x).$$

Next, we will show how to construct sup-solution and do some estimation about $Y(t,x)$, which is of great importance for our further research.

Lemma 3.4. *Let $(\hat{u}(t,x), \hat{w}(t,x))$ be the solution to*

$$\begin{cases} \hat{u}_t = \hat{u}_{xx} + (1-a_1-b_1)\hat{u}(1 - \frac{\alpha+\beta}{2(1-a_1-b_1)}\hat{u}) + \epsilon \hat{u} dW_t, \\ \hat{u}(0,x) = \frac{2(1-a_1-b_1)}{\alpha+\beta}\chi_{(-\infty,0]}, \end{cases} \tag{3.3}$$

where $\alpha, \beta \in (\frac{1-2\min\{a_1a_2,b_1b_2\}}{1-a_1-b_1}, \frac{2-a_1-b_1-2\max\{a_1a_2,b_1b_2\}}{2-a_1-b_1})$ are both positive constants, then we have $\hat{u}(t,x) \in C_{tem}^+$. Let $(u,v,w) = (\hat{u}, \frac{1-\alpha}{2a_1}\hat{u}, \frac{1-\beta}{2b_1}\hat{u})$, then (u,v,w) is a sup-solution to equation (1.2).

Proof. First, referring to [27], the solution to equation (3.3) satisfies $\hat{u}(t,x) \in C_{tem}^+$, which completes the first part of this lemma. Moreover, for u component, we have

$$\begin{aligned} & u_{xx} - u_t + u(1 - a_1 - u + a_1 v + b_1 w) + \epsilon u dW_t \\ =& \hat{u}_{xx} - \hat{u}_t + \hat{u}(1 - a_1 - b_1 - \hat{u} + \frac{1-\alpha}{2}\hat{u} + + \frac{1-\beta}{2}\hat{u}) + \epsilon \hat{u} dW_t \\ =& \hat{u}_{xx} - \hat{u}_t + (1-a_1-b_1)\hat{u}(1 - \frac{\alpha+\beta}{2(1-a_1-b_1)}\hat{u}) + \epsilon \hat{u} dW_t \\ =& 0. \end{aligned}$$

For v component, we have

$$\begin{aligned} & v_{xx} - v_t + (1-v)(a_2 u - v) + \epsilon v dW_t \\ =& \frac{1-\alpha}{2a_1}[\hat{u}_{xx} - \hat{u}_t + \hat{u}(1 - a_1 - b_1 - \frac{\alpha+\beta}{2}\hat{u}) + \epsilon \hat{u} dW_t] \\ & + (1 - \frac{1-\alpha}{2a_1}\hat{u})(a_2 - \frac{1-\alpha}{2a_1})\hat{u} - \frac{1-\alpha}{2a_1}\hat{u}(1 - a_1 - b_1 - \frac{\alpha+\beta}{2}\hat{u}) \\ =& [a_2 - \frac{1-\alpha}{2a_1} - \frac{(1-\alpha)(1-a_1-b_1)}{2a_1}]\hat{u} + \frac{1-\alpha}{2a_1}(\frac{\alpha+\beta}{2} - a_2 + \frac{1-\alpha}{2a_1})\hat{u}^2 \\ =& \frac{1}{2a_1}[2a_1a_2 - (2-a_1-b_1)(1-\alpha)]\hat{u} + \frac{\alpha-1}{4a_1^2}[a_1(\alpha+\beta) - \alpha - 2a_1a_2 + 1]\hat{u}^2, \end{aligned}$$

and for w component, we have

$$\begin{aligned} & w_{xx} - w_t + (1-w)(b_2 u - w) + \epsilon w dW_t \\ =& \frac{1-\beta}{2a_1}[\hat{u}_{xx} - \hat{u}_t + \hat{u}(1 - a_1 - b_1 - k\hat{u}) + \epsilon \hat{u} dW_t] \\ & + (1 - \frac{1-\beta}{2b_1}\hat{u})(b_2 - \frac{1-\beta}{2b_1})\hat{u} - \frac{1-\beta}{2b_1}\hat{u}(1 - a_1 - b_1 - \frac{\alpha+\beta}{2}\hat{u}) \end{aligned}$$

$$
\begin{aligned}
=&[b_2 - \frac{1-\beta}{2b_1} - \frac{(1-\beta)(1-a_1-b_1)}{2b_1}]\hat{u} + \frac{1-\beta}{2b_1}(\frac{\alpha+\beta}{2} - a_2 + \frac{1-\beta}{2b_1})\hat{u}^2 \\
=&\frac{1}{2b_1}[2b_1b_2 - (2-a_1-b_1)(1-\beta)]\hat{u} + \frac{\beta-1}{4b_1^2}[b_1(\alpha+\beta) - \beta - 2b_1b_2 + 1]\hat{u}^2.
\end{aligned}
$$

It can be deduced from (C3) that

$$
2a_1a_2 - (2-a_1-b_1)(1-\alpha) \le 0,\ 2b_1b_2 - (2-a_1-b_1)(1-\beta) \le 0
$$

and

$$
\frac{\alpha-1}{4a_1^2}[a_1(\alpha+\beta) - \alpha - 2a_1a_2 + 1] \le 0,\ \frac{\beta-1}{4b_1^2}[b_1(\alpha+\beta) - \beta - 2b_1b_2 + 1] \le 0.
$$

Moreover, since $\alpha, \beta \in (\frac{1-2\min\{a_1a_2, b_1b_2\}}{1-a_1-b_1}, \frac{2-a_1-b_1-2\max\{a_1a_2, b_1b_2\}}{2-a_1-b_1})$, combined with (C2), we have obtained that $\frac{2(1-a_1-b_1)}{\alpha+\beta} > 1$. Therefore, we have verified our claim, and $(u, v, w) = (\hat{u}, \frac{1-\alpha}{2a_1}\hat{u}, \frac{1-\beta}{2b_1}\hat{u})$ is a sup-solution to equation (1.2). □

In the same way, we can construct a sub-solution to equation (1.2).

Lemma 3.5. *Let $\tilde{u}(t,x)$ be the solution to*

$$
\begin{cases}
\tilde{u}_t = \tilde{u}_{xx} + \tilde{u}(1-a_1-b_1-\tilde{u}) + \epsilon\hat{u}dW_t, \\
\tilde{u}(0,x) = (1-a_1-b_1)\chi_{(-\infty,0]}.
\end{cases}
\tag{3.4}
$$

Then we have $\tilde{u}(t,x) \in C^+_{tem}$. Set $a_2 - 1 \le \gamma_1 \le \frac{a_2}{2-a_1-b_1}$, $b_2 - 1 \le \gamma_2 \le \frac{b_2}{2-a_1-b_1}$, and let $(u, v, w) = (\tilde{u}, \gamma_1\tilde{u}, \gamma_2\tilde{w})$. Then (u, v, w) is a sub-solution to equation (1.2).

Proof. Similar to Lemma 3.4, we know that $\tilde{u}(t,x) \in C^+_{tem}$. For u component, we have

$$
\begin{aligned}
&u_{xx} - u_t + u(1-a_1-b_1-u+a_1v+b_1w) + \epsilon u dW_t \\
=&\tilde{u}_{xx} - \tilde{u}_t + \tilde{u}(1-a_1-b_1-\tilde{u}+a_1\gamma_1\tilde{u}+b_1\gamma_2\tilde{u}) + \epsilon\tilde{u}dW_t \\
\ge&0.
\end{aligned}
$$

For v component,

$$
\begin{aligned}
&v_{xx} - v_t + (1-v)(a_2u - v) + \epsilon v dW_t \\
=&\alpha[\tilde{u}_{xx} - \tilde{u}_t + \tilde{u}(1-a_1-b_1-\tilde{u}) + \epsilon\tilde{u}dW_t] \\
&+ (1-\gamma_1\tilde{u})(a_2-\gamma_1)\tilde{u} - \gamma_1\tilde{u}(1-a_1-b_1-\tilde{u}) \\
=&[a_2 - \gamma_1 - (1-a_1-b_1)\gamma_1]\tilde{u} + \gamma_1(1-a_2+\gamma_1)\tilde{u}^2 \\
=&[a_2 - (2-a_1-b_1)\gamma_1]\tilde{u} + \gamma_1(1-a_2+\gamma_1)\hat{u}^2,
\end{aligned}
$$

and for w component, we have

$$
\begin{aligned}
&w_{xx} - w_t + (1-w)(b_2u - w) + \epsilon w dW_t \\
=&\gamma_2[\tilde{u}_{xx} - \tilde{u}_t + \tilde{u}(1-a_1-b_1-\tilde{u}) + \epsilon\tilde{u}dW_t] \\
&+ (1-\gamma_2\tilde{u})(b_2-\gamma_2)\tilde{u} - \gamma_2\tilde{u}(1-a_1-b_1-\tilde{u}) \\
=&[b_2 - \gamma_2 - (1-a_1-b_1)\gamma_2]\tilde{u} + \gamma_2(1-a_2+\gamma_2)\tilde{u}^2 \\
=&[b_2 - (2-a_1-b_1)\gamma_2]\tilde{u} + \gamma_2(1-b_2+\gamma_2)\hat{u}^2.
\end{aligned}
$$

Since $a_2 - 1 \le \gamma_1 \le \frac{a_2}{2-a_1-b_1}$ and $b_2 - 1 \le \gamma_2 \le \frac{b_2}{2-a_1-b_1}$, then (C4) ensures that

$$a_2 - (2 - a_1 - b_1)\gamma_1 \ge 0,\ b_2 - (2 - a_1 - b_1)\gamma_2 \ge 0 \tag{3.5}$$

and

$$\gamma_1(1 - a_2 + \gamma_1) \ge 0,\ \gamma_2(1 - b_2 + \gamma_2) \ge 0. \tag{3.6}$$

Now, we can say that $(u, v, w) = (\tilde{u}, \gamma_1\tilde{u}, \gamma_2\tilde{u})$ is a sub-solution to equation (1.2). □

Theorem 3.1. *For any $u_0, v_0, w_0 \in C^+_{tem} \setminus \{0\}$, and any $t > 0$, a.e. $\omega \in \Omega$, it permits that*

$$\mathbb{E}[u(t,x) + v(t,x) + w(t,x)] \le C(\epsilon),\ \forall x \in R,$$

where $C(\epsilon)$ is a constant.

Proof. From Lemma 3.4, we know that $(u, v, w) = (\hat{u}, \frac{1-\alpha}{2a_1}\hat{u}, \frac{1-\beta}{2b_1}\hat{u})$ is a sup-solution to equation (1.2), and $u(t,x) \le \hat{u}(t,x)$ a.s., $v(t,x) \le \alpha\hat{u}(t,x)$ a.s. and $w(t,x) \le \beta\hat{u}(t,x)$ a.s. Therefore, by doing some estimations of $\hat{u}(t,x)$, we can obtain the boundedness of $Y(t,x)$. For stochastic KPP equation (3.3), referring to [3,8], we can easily find that for any $\hat{u}_0 \in C^+_{tem}$, a.e. $\omega \in \Omega$ and $t > 0$ fixed, we have

$$\mathbb{E}[\hat{u}(t,x)] \le C_1, \tag{3.7}$$

where C_1 is a positive constant depending on $\hat{u}_0$ and ϵ. Accordingly, for any $u_0, v_0, w_0 \in C^+_{tem}$, a.e. $\omega \in \Omega$ and $t > 0$ fixed, we have

$$\begin{aligned}\mathbb{E}[u(t,x) + v(t,x) + w(t,x)] \le& (1 + \frac{1-\alpha}{2a_1} + \frac{1-\beta}{2b_1})\mathbb{E}[\hat{u}(t,x)]\\ \le& (1 + \frac{1-\alpha}{2a_1} + \frac{1-\beta}{2b_1})C_1\\ :=& C_2,\end{aligned}$$

where C_2 is also a positive constant.

Moreover, for equation (3.3), letting $V(t) = \hat{u}^2(t)$, via Itô formula, we have

$$\begin{aligned}dV(t) =& 2\langle \hat{u}, \hat{u}_{xx}\rangle dt + 2\langle \hat{u}, \hat{u}(1 - a_1 - b_1 - \frac{\alpha+\beta}{2}\hat{u})\rangle dt\\ &+ \epsilon^2\hat{u}^2 dt + 2\epsilon\hat{u}^2 dW_t.\end{aligned}$$

Then, integrating and taking the expectation gives that

$$\begin{aligned}\mathbb{E}[V(t)] =& \mathbb{E}[|\hat{u}_0|^2] + 2\mathbb{E}\int_0^t \langle \hat{u}, \hat{u}_{xx}\rangle ds + \epsilon^2\mathbb{E}\int_0^t \hat{u}^2 ds\\ &+ 2\mathbb{E}\int_0^t \langle \hat{u}, \hat{u}(1 - a_1 - b_1 - \frac{\alpha+\beta}{2}\hat{u})\rangle ds\\ \le& \mathbb{E}[|\hat{u}_0|^2] - 2\mathbb{E}\int_0^t |\nabla\hat{u}|^2 ds + \epsilon^2\mathbb{E}\int_0^t \hat{u}^2 ds\end{aligned}$$

$$
\begin{aligned}
&+2(1-a_1-b_1)\mathbb{E}\int_0^t \hat{u}^2 ds-(\alpha+\beta)\mathbb{E}\int_0^t \hat{u}^3 ds\\
\leq &\mathbb{E}[|\hat{u}_0|^2]-(\alpha+\beta)\mathbb{E}\int_0^t \hat{u}^3 ds+[2(1-a_1-b_1)+\epsilon^2]\mathbb{E}\int_0^t \hat{u}^2 ds\\
&+\mathbb{E}\int_0^t \hat{u}^2 ds-\mathbb{E}\int_0^t \hat{u}^2 ds.
\end{aligned}
$$

Thus, by Young inequality, there exists a positive constant $C_k > 0$ such that

$$
[2(1-a_1-b_1)+\epsilon^2+1]\mathbb{E}\int_0^t \hat{u}^2 ds \leq (\alpha+\beta)\mathbb{E}\int_0^t \hat{u}^3 ds + C_k t.
$$

By Gronwall inequality, we have

$$
\mathbb{E}\sup_{0\leq t\leq T}[|\hat{u}(t)|^2] \leq \mathbb{E}[|\hat{u}_0|^2]e^{-t}+C_k(1-e^{-t}).
$$

Similarly, we have the conclusion that

$$
\mathbb{E}[|u(t)|^2+|v(t)|^2+|w(t)|^2] \leq (1+\frac{1}{4a_1^2}+\frac{1}{4b_1^2})[\mathbb{E}[|\hat{u}_0|^2]+C_k].
$$

Thus, we complete the proof. □

As far as we know, we have obtained the boundedness of $Y(t,x)$ which contributes to the use of comparison method, proving the boundedness of the wavefront marker and estimating the wave speed through sup-solution and sub-solution. Obviously, compared with the two-species competitive system, it causes more difficulties for us because of the relationship among the three different species, which reflects the properties of equilibria. Next, the following lemma estimates how fast the compact support of solution $Y(t)$ can spread.

Lemma 3.6. *Let $Y(t,x)$ be a solution to (1.2) starting at Y_0, and suppose for some $R>0$ that Y_0 is supported outside $(-R-2, R+2)$. Then for any $t\geq 1$,*

$$
\begin{aligned}
\mathbb{P}(\int_0^t\int_{-R}^R ||Y(s,x)||_\infty ds dx > 0) \leq & Ce^t\int \frac{\sqrt{t}}{|x|-(R+1)}\\
&\times exp(-\frac{(|x|-(R+1))^2}{2t})||Y_0||_\infty dx.
\end{aligned}
$$

Proof. From Lemma 3.4 and Theorem 3.1, we first construct a sup-solution $(u^\star, v^\star, w^\star)\in C^+_{tem}$ solving

$$
\begin{cases}
u_t^\star = u_{xx}^\star + u^\star(\kappa-u^\star)+\epsilon u^\star dW_t,\\
v_t^\star = v_{xx}^\star + (1-v^*)(a_2u^\star-v^\star)+\epsilon v^\star dW_t,\\
w_t^\star = w_{xx}^\star + (1-w^\star)(b_2u^\star-w^\star)+\epsilon w^\star dW_t,\\
u^\star(0)=u_0, v^\star(0)=v_0, w^\star(0)=w_0,
\end{cases}
$$

where $\kappa > \frac{a_2}{a_2-a_1}+\frac{b_2}{b_2-b_1}$ is a constant satisfying $F_1(Y)\leq u(\kappa-a_1u)$. Frequently, we suppose the solution $u^*\in C^+_{tem}$ solving

$$
\begin{cases}
u_t^* = u_{xx}^* + u^*(\kappa-u^*)+\epsilon u^* dW_t,\\
u^*(0)=\kappa u_0,
\end{cases}
$$

similar to Lemma 3.4, $(u^\star, v^\star, w^\star) = (u^*, k_1 u^*, k_2 w^*)$, where $\frac{a_2}{a_2-a_1} \le k_1 \le \kappa+1$, and $\frac{b_2}{b_2-b_1} \le k_2 \le \kappa+1$ is a sup-solution to equation (3.8),. Moreover, for a.e. $\omega \in \Omega$ and $x \in R$, we have

$$\mathbb{E}[u^\star(t,x) + v^\star(t,x) + w^\star(t,x)] \le C(\kappa),\ C(\kappa) > 0 \text{ is a constant.}$$

Since $u(t,x) \le u^\star(t,x)$ a.s., $v(t,x) \le v^\star(t,x)$ a.s. and $w(t,x) \le v^\star(t,x)$ a.s., refer to [1,23], we have

$$\mathbb{P}(\int_0^t \int_{-R}^R u(s,x)dsdx > 0) \le Ce^t \int \frac{\sqrt{t}}{|x|-(R+1)} \times exp(-\frac{(|x|-(R+1))^2}{2t})u_0 dx,$$

and it leads to

$$\mathbb{P}(\int_0^t \int_{-R}^R v(s,x)dsdx > 0) \le Ce^t \int \frac{\sqrt{t}}{|x|-(R+1)} \times exp(-\frac{(|x|-(R+1))^2}{2t})v_0 dx.$$

Besides, we also have

$$\mathbb{P}(\int_0^t \int_{-R}^R w(s,x)dsdx > 0) \le Ce^t \int \frac{\sqrt{t}}{|x|-(R+1)} \times exp(-\frac{(|x|-(R+1))^2}{2t})w_0 dx.$$

With these three inequalities, we complete the proof. □

In order to construct a tight measure sequence in Lemma 3.9, it essentially requires that $Y(t,x)$ satisfy Kolmogorov tightness criterion, and $Y(t,x) \in K(C,\delta,\mu,\gamma)$. For more universality, we will verify this property with a p-moment estimation.

Lemma 3.7. *For any $u_0, v_0, w_0 \in C^+_{tem} \setminus \{0\}$, $t > 0$, fixed $p \ge 2$ and a.e. $\omega \in \Omega$, if $|x-x'| \le 1$, there exists a positive constant $C(t)$, such that*

$$Q^{Y_0}(|Y(t,x) - Y(t,x')|^p) \le C(t)|x-x'|^{p/2-1}.$$

Proof. Since the solution $Y(t,x)$ can be expressed as

$$\begin{aligned} Y(t,x) = &\int_R G(t,x-y)Y_0 dy \\ &+ \int_0^t \int_R G(t-s,x,y)F(Y)dsdy + \epsilon \int_0^t \int_R G(t-s,x,y)H(Y)dW_s dy, \end{aligned}$$

through direct calculation, we get

$$\begin{aligned}
&|Y(t,x)-Y(t,x')|^p \\
&\le 3^{p-1}|\int_R (G(t,x-y)-G(t,x'-y))u_0 dy|^p \\
&\quad + 3^{p-1}|\int_R (G(t,x-y)-G(t,x'-y))v_0 dy|^p \\
&\quad + 3^{p-1}|\int_R (G(t,x-y)-G(t,x'-y))w_0 dy|^p \\
&\quad + \underbrace{3^{p-1}|\int_R\int_0^t (G(t-s,x-y)-G(t-s,x'-y))F_1(Y)dsdy|^p}_{I} \\
&\quad + \underbrace{3^{p-1}|\int_R\int_0^t (G(t-s,x-y)-G(t-s,x'-y))F_2(Y)dsdy|^p}_{II} \\
&\quad + \underbrace{3^{p-1}|\int_R\int_0^t (G(t-s,x-y)-G(t-s,x'-y))F_3(Y)dsdy|^p}_{III} \\
&\quad + \underbrace{3^{p-1}\epsilon^p|\int_R\int_0^t (G(t-s,x-y)-G(t-s,x'-y))H_1(Y)dW_s dy|^p}_{IV} \\
&\quad + \underbrace{3^{p-1}\epsilon^p|\int_R\int_0^t (G(t-s,x-y)-G(t-s,x'-y))H_2(Y)dW_s dy|^p}_{V} \\
&\quad + \underbrace{3^{p-1}\epsilon^p|\int_R\int_0^t (G(t-s,x-y)-G(t-s,x'-y))H_3(Y)dW_s dy|^p}_{VI}.
\end{aligned}$$

Referring to Shiga [22] Lemma 6.2, as

$$\int_0^t \int_R (G(t-s,x-y)-G(t-s,x'-y))^2 dsdy \le C(t)|x-x'|,$$

for IV, with Theorem 3.1, we obtain

$$\begin{aligned}
\mathbb{E}[III] \le & C(p)\epsilon^p \mathbb{E}(\int_0^t \int_R (G(t-s,x-y)-G(t-s,x'-y))^2 dsdy)^{p/2-1} \\
& \times (\int_0^t \int_R (G(t-s,x-y)-G(t-s,x'-y))^2 u^p dsdy) \\
\le & C_1(p,t)|x-x'|^{p/2-1}.
\end{aligned}$$

Similarly, for V and VI, we have

$$\mathbb{E}[V] \le C_2(p,t)|x-x'|^{p/2-1}, \quad \mathbb{E}[VI] \le C_3(p,t)|x-x'|^{p/2-1}.$$

For I, with Hölder inequality, we get

$$\begin{aligned}
\mathbb{E}[I] =& 3^{p-1}\mathbb{E}|\int_0^t\int_R (G(t-s,x-y)-G(t-s,x'-y))(u-a_1u^2+b_1uv)dsdy|^p \\
\leq& 3^{p-1}(\int_0^t\int_R (G(t-s,x-y)-G(t-s,x'-y))^2 dsdy)^{p/2-1} \\
&\times (\int_0^t\int_R |G(t-s,x-y)-G(t-s,x'-y)|^2 \mathbb{E}[(u-a_1u^2+b_1uv)^p]dsdy) \\
\leq& C_4(p,t)|x-x'|^{p/2-1}.
\end{aligned}$$

Similarly,

$$\mathbb{E}[II] \leq C_5(p,t)|x-x'|^{p/2-1}, \quad \mathbb{E}[III] \leq C_6(p,t)|x-x'|^{p/2-1}.$$

For the rest terms, we have

$$\begin{aligned}
\mathbb{E}|\int_R (G(t,x-y)-G(t,x'-y))u_0dy|^p =& \mathbb{E}|\int_R\int_{x'}^x \frac{(y-r)}{2t\sqrt{4\pi y}}\exp(-\frac{(y-r)^2}{4t})u_0drdy|^p \\
\leq& K(t)(\int_R\int_{x'}^x \frac{1}{\sqrt{t}}\exp(-\frac{(y-r)^2}{5t})u_0drdy)^p \\
\leq& K(t)|x-x'|^p\int_R \frac{1}{\sqrt{t}}\exp(-\frac{(y-x)^2}{5t})|u_0|^p dy \\
\leq& C_7(p,t)|x-x'|^{p/2-1}, \quad (since\ |x-x'| \leq 1)
\end{aligned}$$

and

$$\mathbb{E}|\int_R (G(t,x-y)-G(t,x'-y))v_0dy|^2 \leq C_8(p,t)|x-x'|^{p/2-1},$$

$$\mathbb{E}|\int_R (G(t,x-y)-G(t,x'-y))w_0dy|^2 \leq C_9(p,t)|x-x'|^{p/2-1}.$$

In summary, we complete the proof with the above inequalities. That is,

$$\mathbb{E}[|Y(t,x)-Y(t,x')|^p] \leq C(p,t)|x-x'|^{p/2-1}.$$

□

Remark 3.1. Lemma 3.7 verifies that $Y(t,x) \in K(C,\delta,\mu,\gamma)$, and $Y(t,x)$ satisfies Kolmogrov tightness criterion. Thus, we can start to construct a traveling wave solution.

Define Q^{Y_0} as the law of the unique solution to equation (1.2) with an initial data $Y(0) = Y_0$. For a probability measure ν on C^+_{tem}, we define

$$Q^\nu(A) = \int_{C^+_{tem}} Q^{Y_0}(A)\nu(dY_0).$$

In order to construct a traveling wave solution to equation (1.2), we must ensure that the translation of the solution with respect to a wavefront marker is stationary, and that the solution poses SCP property. However, $R_0(Y(t))$ does not meet this demand. Therefore, we have to choose a new suitable wavefront marker. As the solution to (1.2) with Heaviside initial condition is exponentially small almost surely as $x \to \infty$, with the stochastic Feynmac-Kac formula, we may turn to $R_1(t)$: $C^+_{tem} \to [-\infty, \infty]$ defined as

$$R_1(f) = ln \int_R e^x f dx,$$

$$R_1(u(t)) = ln \int_R e^x u(t) dx,$$

and

$$R_1(t) := R_1(Y(t)) = \max\{R_1(u(t)), R_1(v(t)), R_1(w(t))\}.$$

The marker $R_1(t)$ is an approximation to $R_0(Y(t)) = \max\{R_0(u(t)), R_0(v(t)), R_0(w(t))\}$.

Let $Z(t) = Y(t, \cdot + R_1(t)) = (Z_1(t), Z_2(t), Z_3(t))^T$, $Z_0(t) = Y(t, \cdot + R_0(Y(t)))$, and define

$$Z(t) = \begin{cases} (0,0,0)^T, & R_1(t) = -\infty, \\ (u(t, \cdot + R_1(t)), v(t, \cdot + R_1(t)), w(t, \cdot + R_1(t)))^T, & -\infty < R_1(t) < \infty, \\ (1,1,1)^T, & R_1(t) = \infty. \end{cases}$$

Hence, $Z(t)$ is the wave shifted so that the wavefront marker $R_1(t)$ lies at the origin. Note that whenever $R_0(Y_0) < \infty$, the compact support property in Lemma 3.6 implies that $R_0(t) < \infty$, $\forall t > 0$, Q^{Y_0}-a.s.

Remark 3.2. Here, we define $R_1(t)$ in the maximum form, not only since it simplifies the discussion about boundedness, but also because the asymptotic wave speed is the minimum wave speed which keeps the traveling wave solution monotonic. As mentioned before, we calculate the asymptotic wave speed via $c = \lim\limits_{t\to\infty} \frac{R_1(t)}{t}$. Therefore, the wavefront marker $R_1(t)$ defined in such a form can ensure that the traveling wave solutions of the two subsystems are monotonic.

Next, define

$$\nu_T = \textit{ the law of } \frac{1}{T}\int_0^T Z(s) ds \textit{ under } Q^{Y_0}.$$

Now, we summarise the method for constructing traveling wave solution. With the initial data $(u_0 = \chi_{(-\infty,0]}, v_0 = \chi_{(-\infty,0]}, w_0 = \chi_{(-\infty,0]}) \in C^+_{tem}$ taken as Heaviside function, we shall show that the sequence $\{\nu_T\}_{T\in\mathbb{N}}$ is tight (see Lemma 3.9) and any limit point is nontrivial (see Theorem 3.2). Hence, for any limit point ν (the limit is not unique), Q^ν is the law of a traveling wave solution. Two parts constituting the proof of tightness are Kolmogorov tightness criterion for the unshifted waves (see Lemma 3.7) and the control on the movement of the wavefront marker $R_1(t)$ to ensure that the shifting will not destroy the tightness (see Lemma 3.8).

Firstly, we complete preparations to prove the tightness of the sequence $\{\nu_T\}_{T\in\mathbb{N}}$.

Lemma 3.8. *For any $u_0, v_0, w_0 \in C^+_{tem} \setminus \{0\}$, $t \geq 0$, $d > 0$, $T \geq 1$, and a.e. $\omega \in \Omega$ there exists a positive constant $C(t) < \infty$, such that*

$$Q^{\nu_T}(|R_1(t)| > d) \leq \frac{C(t)}{d}.$$

Proof. First, we again construct a sup-solution satisfying

$$\begin{cases} \check{u}_t = \check{u}_{xx} + k_0\check{u} + \epsilon\check{u}dW_t, \\ \check{v}_t = \check{v}_{xx} + F_2(\check{u},\check{v}) + \epsilon\check{v}dW_t, \\ \check{w}_t = \check{w}_{xx} + F_3(\check{u},\check{w}) + \epsilon\check{w}dW_t, \\ \check{u}_0 = u_0, \check{v}_0 = v_0, \check{w}_0 = w_0, \end{cases} \tag{3.8}$$

where $k_0 > 0$ is a constant which can be obtained by Theorem 3.1 such that $F_1(Y) < k_0 u$. Let $\xi(t,x) \in C_{tem}^+$ be the solution to

$$\begin{cases} \xi_t = \xi_{xx} + k_0\xi + \epsilon\xi dW_t, \\ \xi_0 = u_0. \end{cases}$$

Then we can claim that $(\check{u},\check{v},\check{w}) := (\xi, p_1\xi, p_2\xi)$, is a sup-solution to equation (3.8), where $\frac{a_2}{1+k_0} \leq p_1 \leq a_2$ and $\frac{b_2}{1+k_0} \leq p_2 \leq b_2$, and $\check{v}(t,x) \leq p_1\check{u}(t,x)$ a.s., $\check{w}(t,x) \leq p_2\check{u}(t,x)$ a.s.

Therefore, we only need to study the property of $\tilde{u}(t)$. According to the comparison principle, we have $u(t) \leq \check{u}(t)$ holding on $[0,T]$ uniformly, and for a.e. $\omega \in \Omega$, the solution $\check{u}(t,x)$ can be expressed as

$$\check{u}(t,x) = \int_R e^{k_0 t} G(t, x-y) u_0(y) dy + \epsilon \int_R \int_0^t G(t-s, x-y)\check{u} dW_s dy.$$

Applying the comparison method, we obtain

$$\begin{aligned} Q^{u_0}(\int_R u(t,x)e^x dx) &\leq \mathbb{E}[\int_R \check{u}(t,x)e^x dx] \\ &= \mathbb{E}[\int_R \int_R e^{k_0 t} G(t,x-y)u_0(y)dy e^x dx] = e^{k_0 t + t}\int_R u_0(x)e^x dx. \end{aligned}$$

Similarly, we have

$$Q^{v_0}(\int_R v(t,x)e^x dx) \leq p_1 e^{k_0 t + t}\int_R u_0(x)e^x dx,$$

$$Q^{w_0}(\int_R w(t,x)e^x dx) \leq p_2 e^{k_0 t + t}\int_R u_0(x)e^x dx.$$

Without generality, we assume that $R_1(t) = R_1(u(t))$. Then it permits that

$$\int_R u(t, x + R_1(t))e^x dx = e^{-R_1(t)}\int_R u(t,x)e^x dx = 1.$$

On the other hand, we have

$$\int_R v(t, x + R_1(t))e^x dx \leq p_1, \ \int_R w(t, x+R_1(t))e^x dx \leq p_2.$$

Associating with inequalities above, we have

$$\begin{aligned}
Q^{\nu_T}(R_1(t) \geq d) =& \frac{1}{T}\int_0^T Q^{u_0}(Q^{u(s)}(R_1(t) \geq d))ds \\
=& \frac{1}{T}\int_0^T Q^{u_0}(Q^{u(s)}(e^{-d}\int_R u(t,x)e^x dx \geq 1))ds \\
\leq& e^{-d}\frac{1}{T}\int_0^T Q^{u_0}(Q^{u(s)}(\int_R u(t,x)e^x dx))ds \\
\leq& e^{-d}e^{k_0 t+t}\frac{1}{T}\int_0^T \int_R u(s,x+R_1(s))e^x dxds \\
=& e^{-d}e^{k_0 t+t}.
\end{aligned}$$

Next, Jensen's inequality gives

$$Q^{u_0}(R_1(t)) \leq ln(e^{k_0 t+t}\int_R u_0(x)e^x dx) \leq k_0 t + t + R_1(u_0).$$

In addition, we have such an estimation

$$\begin{aligned}
&\frac{1}{T}Q^{u_0}(\int_t^{T+t} R_1(s)ds - \int_0^T R_1(s)ds) \\
=& \frac{1}{T}Q^{u_0}(\int_0^T R_1(t+s) - R_1(s)ds) \\
=& \frac{1}{T}\int_0^T \int_{\{R_1(t+s)-R_1(s)>-d\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\
&+ \frac{1}{T}\int_0^T \int_{\{R_1(t+s)-R_1(s)\leq -d\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\
\leq& \frac{1}{T}\int_0^T \int_{\{R_1(t+s)-R_1(s)>0\}} (R_1(t+s) - R_1(s))Q^{u_0}(du)ds \\
&- \frac{d}{T}\int_0^T Q^{u_0}(R_1(t+s) - R_1(s) \leq -d)ds \\
\leq& \frac{1}{T}\int_0^T \int_0^\infty Q^{u_0}(R_1(t+s) - R_1(s) \geq y)dyds \\
&- \frac{d}{T}\int_0^T Q^{u_0}(R_1(t+s) - R_1(s) \leq -d)ds \\
=& \int_0^\infty Q^{\nu_T}(R_1(t) \geq y)dy - dQ^{\nu_T}(R_1(t) \leq -d).
\end{aligned}$$

Rearranging the inequalities gives

$$\begin{aligned}
&Q^{\nu_T}(R_1(t) \leq -d) \\
\leq& \frac{1}{d}\int_0^\infty Q^{\nu_T}(R_1(t) \geq y)dy + \frac{1}{dT}\int_0^T Q^{\nu_T}(R_1(s))ds - \frac{1}{dT}\int_t^{T+t} Q^{u_0}(R_1(s))ds \\
\leq& \frac{1}{d}\int_0^\infty e^{-y+k_0 t+t}dy + \frac{1}{dT}\int_0^T k_0 s + s + R_1(u_0)ds \\
\leq& \frac{C(t)}{d}.
\end{aligned}$$

The proof is complete. □

Now, we can say the marker $R_1(t)$ is bounded, which helps prove that the sequence $\{\nu_T : T \in \mathbb{N}\}$ is tight and wavefront marker $R_0(t)$ is bounded. Then, we will show the tightness of $\{\nu_T : T \in \mathbb{N}\}$ with $Y(t,x) \in K(C,\delta,\mu,\gamma)$.

Lemma 3.9. *For any $u_0, v_0, w_0 \in C^+_{tem} \setminus \{0\}$, and a.e. $\omega \in \Omega$, the sequence $\{\nu_T : T \in \mathbb{N}\}$ is tight.*

Proof.
Similar to Theorem 3.8, we discuss with $u(t,x)$. According to Lemma 3.7, $Y(t,x) \in K(C,\delta,\mu,\gamma)$ gives $u(t,x) \in K(C,\delta,\mu,\gamma)$, then we obtain

$$\begin{aligned}
\nu_T(K(C,\delta,\gamma,\mu)) =& \frac{1}{T}\int_0^T Q^{u_0}(u(t,\cdot + R_1(t)) \in K(C,\delta,\gamma,\mu))ds \\
\geq& \frac{1}{T}\int_0^T Q^{u_0}((u(t,\cdot + R_1(t-1)) \in K(Ce^{-\mu d},\delta,\gamma,\mu)) \\
& \times |R_1(t) - R_1(t-1)| \leq d)ds \\
\geq& \frac{1}{T}\int_1^T Q^{u_0}(Q^{Z_1(t-1)}(u(1) \in K(Ce^{-\mu d},\delta,\gamma,\mu)))dt \\
& - \frac{1}{T}\int_1^T Q^{u_0}(|R_1(t) - R_1(t-1)| \geq d)dt \\
:=& I - II.
\end{aligned}$$

With Lemma 3.8, $II \to 0$ as $d \to \infty$. Via Kolmogorov tightness and Lemma 3.7, for given $d, \mu > 0$, one can choose C, δ, γ to make I as close to $\frac{T-1}{T}$ as desired. In addition, we have

$$\nu_T\{u_0 : \int_R u_0(x)e^{-|x|}dx \leq \int_R u_0(x)e^x dx = 1\} = 1.$$

By the definition of tightness, for given $\mu > 0$, one can choose C, δ, γ such that $\nu_T(K(C,\delta,\mu,\gamma) \cap \{u_0 : \int_R u_0(x)e^{-|x|}dx\})$ as close to 1 as desired for T and d sufficiently large, which implies that the sequence $\{\nu_T : T \in \mathbb{N}\}$ is tight. □

Theorem 3.2. *For any $u_0, v_0, w_0 \in C^+_{tem} \setminus \{0\}$, and a.e. $\omega \in \Omega$, there is a traveling wave solution to equation (1.2), and Q^ν is the law of traveling wave solution.*

Proof. Denote by $(f,g) = \int_R fg dx$. First, taking a subsequence $\{\nu_{T_n}\}$ converging to ν, then we choose $g(x) \in C^+_{tem}$ satisfying $\int_R g(x)e^x dx = 1$. Choose $g_1(x), g_2(x) \in C^+_{tem}$ with $g = g_1 + g_2$, $(g_1, I_{(d/3,\infty)}) = 0$ and $(g_2, I_{(-\infty,2d/3)}) = 0$. Taking ϱ_1, ϱ_2 independent solutions to (3.8) with respect to $\tilde{u}$ starting at g_1, g_2, then using the comparison method shows that $\varrho \leq \varrho_1 + \varrho_2$ is a solution to (3.8) with respect to $\check{u}$ starting at g. Applying Lemma 3.6 and taking large d, we have

$$\begin{aligned}
Q^g((u(t,x), I_{(d,\infty)}) > 0) \leq& \mathbb{P}((\varrho_1(t,x), I_{(d,\infty)}) > 0) + \mathbb{P}((\varrho_2(t,x), 1) > 0) \\
\leq& C(k_0,t)e^{-d/3}.
\end{aligned}$$

Taking $h(x)$ with $\int_R h(x)e^x dx = 1$, we also have

$$Q^h((v(t,x), I_{(d,\infty)}) > 0) \leq C(k_0,t)e^{-d/3}, \quad Q^h((w(t,x), I_{(d,\infty)}) > 0) \leq C(k_0,t)e^{-d/3}.$$

Therefore,

$$
\begin{aligned}
&\nu_T(u_0 : (u_0, I_{(2d,\infty)}) = 0) = \frac{1}{T}\int_0^T Q^{u_0}((Z_1(t), I_{(2d,\infty)}) = 0)dt \\
&\geq \frac{1}{T}\int_0^T Q^{u_0}((u(t), I_{(d+R_1(t-1),\infty)}) = 0, -|R_1(t) - R_1(t-1)| \leq d)dt \\
&\geq \frac{1}{T}\int_1^T Q^{u_0}(Q^{Z_1(t-1)}((u(1), I_{(d,\infty)}) = 0)dt - Q^{\nu_T}(|R_1(1) \geq d) \\
&\geq \frac{T-1}{T} - \frac{C(1)}{d}.
\end{aligned}
$$

By Lemma 3.8, we have

$$\lim_{T\to\infty}\lim_{d\to\infty} \nu_T(u_0 : (u_0, I_{(2d,\infty)}) = 0) = 1.$$

Similarly, we have

$$\lim_{T\to\infty}\lim_{d\to\infty} \nu_T(v_0 : (v_0, I_{(2d,\infty)}) = 0) = 1$$

and

$$\lim_{T\to\infty}\lim_{d\to\infty} \nu_T(w_0 : (v_0, I_{(2d,\infty)}) = 0) = 1.$$

In order to prove the boundedness of $R_0(t)$, from $\nu_{T_n}(u_0 : (u_0, e^x) = 1) = 1$, we get

$$\nu(u_0 : (u_0, e^x) \leq 1) = 1.$$

Taking $e_1^d(x) = \exp(d - |x - d|)$, then

$$
\begin{aligned}
\nu(u_0 : (u_0, e^x) \geq 1) \geq &\nu(u_0 : (u_0, e_1^d) \geq 1) \\
\geq &\limsup_{n\to\infty} \nu_{T_n}(u_0 : (u_0, e_1^d) = 1) \\
= &\limsup_{n\to\infty} \nu_{T_n}(u_0 : (u_0, I_{(d,\infty)}) = 0) \to 1,\ as\ d \to \infty.
\end{aligned}
$$

Since $\nu(u_0 : (u_0, e^x) = 1) = 1$, we obtain $\nu(u_0 : R_0(u_0) > -\infty) = 1$.

Similarly, we have $\nu(v_0 : R_0(v_0) > -\infty) = 1$ and $\nu(w_0 : R_0(w_0) > -\infty) = 1$.

Now, we continue to prove the boundedness of wavefront marker $R_0(t)$. Taking $\psi_d \in \Phi$ with $(\psi_d > 0) = (d, \infty)$, then

$$
\begin{aligned}
\nu(u_0 : R_0(u_0) \leq d) = &\nu(u_0 : (u_0, \psi_d) = 0) \\
\geq &\limsup_{n\to\infty} \nu_{T_n}(u_0 : (u_0, \psi_d) = 0) \\
= &\limsup_{n\to\infty} \nu_{T_n}(u_0 : (u_0, I_{(d,\infty)}) = 0) \to 1\ as\ d \to \infty.
\end{aligned}
$$

Therefore, we have $\nu(Y_0 : -\infty < R_0(Y_0) < \infty) = 1$ and complete the proof of boundedness of wavefront marker $R_0(t)$.

To verify that the solution $Y(t)$ is nontrivial, taking $R_1^d(t) = \ln \int ||Y(t)||_\infty e_1^d dx$, we have

$$\begin{aligned} Q^\nu(\exists s \le t,\ |Y(s)| = 0) \le & Q^\nu(R_1^d(t) < -d) \\ \le & \limsup_{n\to\infty} Q^{\nu_{T_n}}(R_1^d(t) < -d) \\ \le & \limsup_{n\to\infty} (Q^{\nu_{T_n}}(R_1(t) < -d) + Q^{\nu_{T_n}}((u(t), I_{(d,\infty)}) > 0)) \\ \le & \frac{C(T)}{d} \to 0\ as\ d \to \infty. \end{aligned}$$

Now, we show that $Z(t)$ is a stationary process, and Q^ν is the law of a traveling wave solution to (1.2). Letting $F : C_{tem}^+ \to R$ be bounded and continuous, and taking $u(t,x)$ as an example, for any $t > 0$ fixed,

$$\begin{aligned} & |Q^{\nu_{T_n}}(F(Z_1(t))) - Q^\nu(F(Z_1(t)))| \\ & \quad \le |Q^{\nu_{T_n}}(F(u(t, \cdot + R_1^d(t)))) - Q^\nu(F(u(t, \cdot + R_1^d(t))))| \\ & \qquad + \sup_{x\in R} |F(u_0)|(Q^{\nu_{T_n}}(R_1(t) \ne R_1^d(t)) + Q^\nu(R_1(t) \ne R_1^d(t))). \end{aligned}$$

Since $\nu_{T_n}(u_0 : (u_0, e^x) = 1) = 1$, we have

$$Q^{\nu_{T_n}}(R_1(t) \ne R_1^d(t)) \le Q^{\nu_{T_n}}((u(t), I_{(d,\infty)}) > 0) \le C(k_0, t)/d.$$

With $\nu(u_0 : (u_0, e^x) = 1) = 1$, we have

$$Q^\nu(R_1(t) \ne R_1^d(t)) \le Q^\nu((u(t), I_{(d,\infty)}) > 0) \le C(k_0, t)/d.$$

By the continuity of $u_0 \to Q^{u_0}$, one has $Q^{\nu_{T_n}} \to Q^\nu$. Since F is bounded and continuous, then

$$|Q^{\nu_{T_n}}(F(u(t, \cdot + R_1^d(t)))) - Q^\nu(F(u(t, \cdot + R_1^d(t))))| \to 0,\ as\ n \to \infty.$$

Therefore, we have

$$\begin{aligned} Q^\nu(F(Z_1(t))) &= \lim_{n\to\infty} Q^{\nu_{T_n}}(F(Z_1(t))) \\ &= \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} Q^{u_0}(F(Z_1(s+t)))ds \\ &= \lim_{n\to\infty} \frac{1}{T_n} \int_0^{T_n} Q^{u_0}(F(Z_1(s)))ds \\ &= \nu(F). \end{aligned}$$

Similarly, we have $Q^\nu(F(Z_2(t))) = \nu(F)$ and $Q^\nu(F(Z_3(t))) = \nu(F)$. It is straightforward to check that $\{Z(t) : t \ge 0\}$ is Markov. Hence, $\{Z(t) : t \ge 0\}$ is stationary. Since the map $Y_0 \to Y_0(\cdot - R_0(Y_0))$ is measurable on C_{tem}^+, the process $\{Z_0(t) : t \ge 0\}$ is also stationary, which implies that Q^ν is the law of traveling wave solution to equation (1.2). $\square$

4. Asymptotic wave speed

In this section, we investigate the asymptotic property of the traveling wave solution. By constructing the sup-solution and the sub-solution, we obtain the asymptotic wave speed for the two traveling wave solutions respectively. Then, we have the estimation of the wave speed of traveling wave solutions to (1.2). Since the asymptotic wave speed c of the traveling wave solution is defined as

$$c = \lim_{t\to\infty} \frac{R_0(t)}{t} \ a.s.,$$

we denote by $R_0(u(t)) = \sup\{x \in \mathbb{R} : u(t,x) > 0\}$, $R_0(v(t)) = \sup\{x \in \mathbb{R} : v(t,x) > 0\}$ and $R_0(w(t)) = \sup\{x \in \mathbb{R} : w(t,x) > 0\}$for the sub-systems of the transformed cooperative system. Since the wavefront marker $R_0(t)$ of the cooperative system is $R_0(t) = \max\{R_0(u(t)), R_0(v(t)), R_0(w(t))\}$, and the asymptotic wave speed defined as the maximum value of $\lim_{t\to\infty} \frac{R_0(u(t))}{t}$, $\lim_{t\to\infty} \frac{R_0(v(t))}{t}$ and $\lim_{t\to\infty} \frac{R_0(w(t))}{t}$, we can define the wave speed $c^\star$ as

$$c^\star = \lim_{t\to\infty} \frac{R_0(Y(t))}{t} \ a.s.$$

Now, we construct a sup-solution by Lemma 3.4 and Theorem 3.1. Let $\bar{Y}(t,x) = (\bar{u}(t,x), \bar{v}(t,x), \bar{w}(t,x))^T$ satisfy

$$\begin{cases} \bar{u}_t = \bar{u}_{xx} + \bar{u}(k_m - \bar{u}) + \epsilon\bar{u}dW_t, \\ \bar{v}_t = \bar{v}_{xx} + (1-\bar{v})(a_2\bar{u} - \bar{v}) + \epsilon\bar{v}dW_t, \\ \bar{w}_t = \bar{w}_{xx} + (1-\bar{w})(b_2\bar{u} - \bar{w}) + \epsilon\bar{w}dW_t, \\ \bar{u}_0 = k_m u_0, \bar{v}_0 = v_0, \bar{w}_0 = w_0, \end{cases} \tag{4.1}$$

where $F_1(Y) \le u(k_m - u)$, k_m is the upper bound from Theorem 3.1 and $k_m > \frac{a_2}{a_2-a_1} + \frac{b_2}{b_2-b_1}$.

Then, we construct a sub-solution and let $\underline{Y}(t,x) = (\underline{u}(t,x), \underline{v}(t,x), \underline{w}(t,x))^T$ satisfy

$$\begin{cases} \underline{u}_t = \underline{u}_{xx} + \underline{u}(1 - a_1 - b_1 - \underline{u}) + \epsilon\underline{u}dW_t, \\ \underline{v}_t = \underline{v}_{xx} + (1-\underline{v})(a_2\underline{u} - \underline{v}) + \epsilon\underline{v}dW_t, \\ \underline{w}_t = \underline{w}_{xx} + (1-\underline{w})(b_2\underline{u} - \underline{w}) + \epsilon\underline{w}dW_t, \\ \underline{u}_0 = (1-a_1-b_1)u_0, \underline{v}_0 = v_0, \underline{w}_0 = w_0. \end{cases} \tag{4.2}$$

Obviously, $F_1(Y) \ge u(1-a_1-b_1-u)$. With equation (4.1) and equation (4.2), we can state the following results.

Theorem 4.1. *For any $u_0, v_0, w_0 \in C^+_{tem} \setminus \{0\}$, let $c^\star$ be the asymptotic wave speed of equation* (1.2). *Then*

$$\sqrt{4(1-a_1-b_1) - 2\epsilon^2} \le c^\star \le \sqrt{4k_m - 2\epsilon^2} \ a.s.$$

In order to prove Theorem 4.1, we need the following lemmas. First, we introduce the comparison method for the asymptotic wave speed.

Lemma 4.1 ([27]). *Let $\underline{Y}(t,x)$ and $\bar{Y}(t,x)$ be solutions to* (4.2) *and* (4.1) *respectively. If $\underline{c}$ is the asymptotic wave speed of $\underline{Y}(t, \cdot + R_0(\underline{Y}(t)))$, and $\bar{c}$ is the asymptotic wave speed of $\bar{Y}(t, \cdot + R_0(\bar{Y}(t)))$, then*

$$\underline{c} \le c^\star \le \bar{c} \quad a.s.$$

4.1. Asymptotic wave speed of sup-solution

Now, we show the asymptotic property of wavefront marker of the sup-solution. Let $\zeta(t,x)$ be the solution to

$$\begin{cases} \zeta_t = \zeta_{xx} + \zeta(k_m - \zeta) + \epsilon \zeta dW_t, \\ \zeta_0 = \bar{u}_0. \end{cases}$$

Obviously, $u(k_m - u) \geq F_1(u,v)$. Similar to equation (3.8), we can construct a sup-solution $(\bar{u}, \bar{v}, \bar{w}) = (\zeta, d_1\zeta, d_2\zeta)$ to equation (4.1), where $\frac{a_2}{a_2-a_1} \leq d_1 \leq k_m + 1$ and $\frac{b_2}{b_2-b_1} \leq d_2 \leq k_m + 1$, then we have $\bar{u}(t,x) \leq \zeta(t,x)$ a.s., $\bar{v}(t,x) \leq d_1\zeta(t,x)$ a.s. and $\bar{w}(t,x) \leq d_2\zeta(t,x)$ a.s. Thus, thanks to the definition of wave speed and Lemma 4.1, we can obtain how the traveling waves spread by estimating $c(\bar{u})$.

Theorem 4.2. *For any $u_0, v_0, w_0 \in C^+_{tem} \setminus \{0\}$, $\bar{Y}(t,x)$ is a solution to (4.1), then the asymptotic wave speed $c(\bar{Y})$ satisfies*

$$c(\bar{Y}) = \sqrt{4k_m - 2\epsilon^2} \ a.s.$$

Proof. For any $h > 0$, take $\kappa \in (0, \frac{h^2}{4} + \sqrt{1 - \frac{\epsilon^2}{2} + 1}h)$. Defining

$$\eta_t(\omega) = \exp(\int_0^t \epsilon dW_s - \frac{1}{2}\int_0^t \epsilon^2 ds), \ 0 \leq t \leq \infty$$

constructing new probability space $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$, $\bar{W} = (\bar{W}(t) : t \geq 0)$ is Brownian motion. Then, there exists $T_1 > 0$, such that for $t \geq T_1$ and a.e. $\omega \in \Omega$

$$\exp(-\frac{\epsilon^2}{2}t - \kappa t) \leq \eta_t(\omega) \leq \exp(-\frac{\epsilon^2}{2}t + \kappa t).$$

Thus, the stochastic Feynman-Kac formula gives

$$\begin{aligned} \bar{u}(t,x) \leq & \exp(t - \frac{1}{2}\epsilon^2 t + \kappa t)\tilde{\mathbb{P}}(\tilde{W}(t) \leq -\frac{x}{\sqrt{2}}) \\ \leq & \exp(t - \frac{1}{2}\epsilon^2 t + \kappa t - \frac{x^2}{4t}) \ a.s., \end{aligned}$$

for $t \geq T_1$. Set $x \geq (k+h)t$, where k is a constant. Multiplying e^x with both sides and integrating in $[(k+h)t, \infty)$, then we have

$$\begin{aligned} \int_{(k+h)t}^{\infty} \bar{u}(t,x)e^x dx \leq & \int_{(k+h)t}^{\infty} \exp(t - \frac{1}{2}\epsilon^2 t + \kappa t - \frac{x^2}{4t} + x)dx \\ =& 2\sqrt{t}\exp(t - \frac{1}{2}\epsilon^2 t + \kappa t + t)\int_{\frac{(k+h)t-2t}{\sqrt{4t}}}^{\infty} e^{-x^2}dx \\ \leq & \sqrt{t}\exp((1 + \kappa - \frac{k^2}{4} - \frac{kh}{2} - \frac{h^2}{4} - k - h - \frac{\epsilon^2}{2})t) \ a.s., \end{aligned}$$

for $t \geq T_1$. Let $k = \sqrt{4 - 2\epsilon^2 + 4} - 2$. Then, we obtain

$$\lim_{t \to \infty} \int_{(k+h)t}^{\infty} \bar{u}(t,x)e^x dx = 0 \ a.s.$$

Similarly, integrating $\bar{u}(t,x)e^x$ in $[(\sqrt{4k_m-2\epsilon^2}+h)t,(k-h)t)$, we have

$$\begin{aligned}
&\int_{(\sqrt{4k_m-2\epsilon^2}+h)t}^{(k-h)t}\bar{u}(t,x)e^x dx\\
\leq&\int_{(\sqrt{4k_m-2\epsilon^2}+h)t}^{(k-h)t}\exp(t-\frac{1}{2}\epsilon^2 t+\kappa t-\frac{x^2}{4t}+x)dx\\
=&2\sqrt{t}\exp(t-\frac{1}{2}\epsilon^2 t+\kappa t+t)\int_{\frac{(\sqrt{4k_m-2\epsilon^2}+h)t-2t}{2\sqrt{t}}}^{\frac{(k-h)t-2t}{2\sqrt{2}}}e^{-x^2}dx\\
\leq&\sqrt{t}\exp(t-\frac{\epsilon^2}{2}t+\kappa t-\frac{4k_m-2\epsilon^2}{4}t-\frac{(\sqrt{4k_m-2\epsilon^2})h}{2}t-\frac{h^2}{4}t+\sqrt{4k_m-2\epsilon^2}t+ht)\\
&-\sqrt{t}\exp(t-\frac{\epsilon^2}{2}t+\kappa t-\frac{k^2}{4}t+\frac{kh}{2}t-\frac{h^2}{4}t+kt-ht)\\
\leq&\sqrt{t}\exp(\kappa t+\sqrt{4k_m-2\epsilon^2}t-\frac{(\sqrt{4k_m-2\epsilon^2})h}{2}t-\frac{h^2}{4}t+ht)\\
&-\sqrt{t}\exp(\kappa t+\frac{kh}{2}t-\frac{h^2}{4}t-ht)\ a.s.,
\end{aligned}$$

for $t\geq T_1$. Analogously, we have

$$\begin{aligned}
&\int_{(\sqrt{4k_m-2\epsilon^2}-h)t}^{(\sqrt{4k_m-2\epsilon^2}+h)t}\bar{u}(t,x)e^x dx\\
&\qquad\leq\sqrt{t}\exp(\kappa t+\sqrt{4k_m-2\epsilon^2}t+\frac{\sqrt{4k_m-2\epsilon^2}h}{2}t-\frac{h^2}{4}t-ht)\\
&\qquad\quad-\sqrt{t}\exp(\kappa t+\sqrt{4k_m-2\epsilon^2}t-\frac{\sqrt{4k_m-2\epsilon^2}h}{2}t-\frac{h^2}{4}t+ht)\ a.s.
\end{aligned}$$

and

$$\begin{aligned}
\int_{(k-h)t}^{(k+h)t}\bar{u}(t,x)e^x dx\leq&\sqrt{t}\exp(\kappa t+\frac{kh}{2}t-\frac{h^2}{4}t-ht)\\
&-\sqrt{t}\exp(\kappa t-\frac{kh}{2}t-\frac{h^2}{4}t+ht)\ a.s.,
\end{aligned}$$

for $t\geq T_1$. Referring to [21], there exists $T_2>0$, such that for all $t\geq T_2$ and $x<(\sqrt{4-2\epsilon^2}-h)t$, there exist $\rho_1,\rho_2>0$ satisfying

$$\exp(-\rho_1\sqrt{2t\ln\ln t})\leq\bar{u}(t,x)\leq\exp(\rho_2\sqrt{2t\ln\ln t})\ a.s.,$$

which goes into

$$\int_{-\infty}^{(\sqrt{4k_m-2\epsilon^2}-h)t}\bar{u}(t,x)e^x dx\leq\exp(\rho_2\sqrt{2t\ln\ln t}+(\sqrt{4k_m-2\epsilon^2}-h)t)\ a.s.$$

Since $\int_{(k+h)t}^{\infty}\bar{u}(t,x)e^x dx\leq 1$, we have

$$\int_R\bar{u}(t,x)e^x dx\leq\exp(\rho_2\sqrt{2t\ln\ln t}+(\sqrt{4k_m-2\epsilon^2}-h)t)(2+H(t)+G(t))\ a.s.,$$

where

$$H(t) = \sqrt{t}\exp(\frac{1}{2}\epsilon^2 - \frac{\epsilon^2}{2}t + \kappa t + \frac{kh}{2}t - \frac{h^2}{4}t - \rho_2\sqrt{2t\ln\ln t} - \sqrt{4k_m - 2\epsilon^2}t)$$

and

$$G(t) = \sqrt{t}\exp(\frac{1}{2}\epsilon^2 - \frac{\epsilon^2}{2}t + \kappa t - \frac{\sqrt{4k_m - 2\epsilon^2}h}{2}t - \rho_2\sqrt{2t\ln\ln t} - \frac{h^2}{4}t + 2ht).$$

With the arbitrariness of h and κ we know that $H(t) \leq 1$ a.s. for large t, and simple calculation shows that

$$\begin{aligned}\frac{1}{t}\ln G(t) =& \frac{1}{2t}\ln 4t - \frac{1}{t}(\ln 2 - \frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}t) \\ &+ \kappa - \frac{4k_m - 2\epsilon^2}{4}h - \frac{h^2}{4} + 2h - \frac{1}{t}\rho_2\sqrt{2t\ln\ln t}\ a.s.\end{aligned}$$

Again, with the arbitrariness of h and κ,

$$\lim_{t\to\infty}\frac{1}{t}\ln G(t) = 0\ a.s.$$

In summary, we obtain the upper bound of the asymptotic wave speed of traveling wave solution to (4.1),

$$\frac{R_1(t)}{t} \leq \frac{1}{t}\rho_2\sqrt{2t\ln\ln t} + \sqrt{4 - 2\epsilon^2} - h + \frac{1}{t}\ln 2 + \frac{1}{t}\ln G(t)\ a.s.$$

Furthermore, we have

$$\limsup_{t\to\infty}\frac{R_1(t)}{t} \leq \sqrt{4k_m - 2\epsilon^2}\ a.s. \tag{4.3}$$

In addition, we have

$$\frac{R_1(t)}{t} \geq -\frac{1}{t}\rho_1\sqrt{2\ln\ln t} + \sqrt{4k_m - 2\epsilon^2} - h\ a.s.$$

Thus, the upper bounded can be obtained that

$$\liminf_{t\to\infty}\frac{R_1(t)}{t} \geq \sqrt{4k_m - 2\epsilon^2}\ a.s. \tag{4.4}$$

Combining (4.3) with (4.4) gives

$$\lim_{t\to\infty}\frac{R_1(t)}{t} = \sqrt{4k_m - 2\epsilon^2}\ a.s.$$

As for $c(\bar{v})$, by the definition of wave speed and $R_1(t) = \ln\int e^x\bar{v}(t,x)dx$, we have

$$c(\bar{v}) = \lim_{t\to\infty}\frac{\ln\int_R \bar{v}(t,x)e^x dx}{t} \leq \lim_{t\to\infty}\frac{\ln\int_R q\bar{u}(t,x)e^x dx}{t} = \sqrt{4k_m - 2\epsilon^2}\ a.s.$$

Analogously, for $c(\bar{w})$, we have

$$c(\bar{w}) \leq \sqrt{4k_m - 2\epsilon^2}\ a.s.$$

Then, we achieve the conclusion that $c(\bar{Y}) = \sqrt{4k_m - 2\epsilon^2}$ a.s. □

4.2. Asymptotic wave speed of sub-solution

By the method used in Theorem 4.2, we refer to Lemma 3.5 and have that $(\underline{u}(t,x), \underline{v}(t,x), \underline{w}(t,x) = (\tilde{u}(t,x), \gamma_1\tilde{u}(t,x), \gamma_2\tilde{u}(t,x))$ is a sub-solution to equation (4.2). Similarly, we construct a new probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, and $\tilde{W} = (\tilde{W}(t) : t \geq 0)$ is a Brownian motion defined on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. For any $h > 0$, choosing $0 < \tau < \frac{h^2}{4} + \sqrt{1 - a_1 - b_1 - \frac{\epsilon^2}{2} + 1}h$ and defining

$$\eta_t(\omega) = \exp(\int_0^t \epsilon dW_s - \frac{1}{2}\int_0^t \epsilon^2 ds),\ 0 \leq t \leq \infty,$$

there exists $T_1 > 0$ such that

$$\underline{u}(t,x) \leq \exp((1 - a_1 - b_1)t - \frac{1}{2}\epsilon^2 t + \tau t - \frac{x^2}{4t})\ a.s.,$$

for $t \geq T_1$. Thus, similar to Theorem 4.2, we have the conclusion.

Theorem 4.3. *For any $u_0, v_0, w_0 \in C_{tem}^+ \setminus \{0\}$, $\underline{Y}(t,x)$ is a solution to (4.2), then the asymptotic wave speed $c(\underline{Y})$ satisfies*

$$c(\underline{Y}) \geq \sqrt{4(1 - a_1 - b_1) - 2\epsilon^2}\ a.s.$$

Proof. The proof is similar to Theorem 4.2, and we omit it. However, it is noteworthy that

$$c(\underline{v}) \geq \lim_{t\to\infty} \frac{\ln \int_R \gamma_1 \underline{u}(t,x)e^x dx}{t} = \sqrt{4(1 - a_1 - b_1) - 2\epsilon^2}\ a.s. \tag{4.5}$$

and

$$c(\underline{w}) \geq \sqrt{4(1 - a_1 - b_1) - 2\epsilon^2}\ a.s. \tag{4.6}$$

Based on the definition of the wave speed which keeps the traveling wave solution monotonic, $c(\underline{Y}) = \max\{c(\underline{u}), c(\underline{v}), c(\underline{w})\} \geq \sqrt{4(1 - a_1 - b_1) - 2\epsilon^2}$ a.s. □

Proof of Theorem 4.1. Associating Theorem 4.2 and Theorem 4.3 with Lemma 4.1, we can achieve the conclusion

$$\sqrt{4(1 - a_1 - b_1) - 2\epsilon^2} \leq c^\star \leq \sqrt{4k_m - 2\epsilon^2}\ a.s.$$

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