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Dynamics of a Predator-Prey Model with Allee Effect and Herd Behavior

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Abstract This paper deals with dynamics of a predator-prey model with Allee effect and herd behavior. We first study the stability of non-negative constant solutions for such system. We also establish the existence of Hopf bifurcation solutions for such predator-prey model. The stability and bifurcation direction of Hopf bifurcation solution in the case of spatial homogeneity are further discussed. At the same time, several examples are given by MATLAB. Finally, the numerical simulations of the system are carried out through MATLAB, which intuitively verifies and supplements the theoretical analysis results.

Keywords Allee effect, herd behavior, stability analysis, Hopf bifurcation, numerical simulations

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1. Introduction

Biological mathematics is an interdisciplinary subject integrating mathematics and biology. Mathematical ecology is one of the most widely studied branches of mathematical biology. In particular, the ecological mathematical models in mathematical ecology are to reasonably establish the population dynamics models according to the relationship between population and population or between population and environment. So far, experts and scholars have made many important achievements in the research of ecological mathematical models.

In the ecological mathematical model, there is a model to describe the survival mode of two populations: population A depends on natural resources and population B feeds on population A, where population A is called prey and population B is called predator in ecology. Together, they form a predator-prey model. Predator-prey model is a kind of vital population model in ecological mathematical model, which has been widely studied by scholars [1–13]. The dynamic behavior of the predator-prey system is one of the focuses of mathematical ecology. There are intraspecific cooperation and competition in predator-prey system. Allee effect refers to the positive correlation between individual adaptability and population size or density [14]. In 2002, Petrovskii et al. [8] showed that Allee effect makes patchy invasion in predator-prey system possible. In 2012, Sen et al. [10] studied the bifurcation analysis of a ratio-dependent predator-prey model with Allee effect. In

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2016, Pal et al. [1] studied algae-herbivore interactions with Allee effect and chemical defense. In 2019, Xu et al. [15] cosidered the dynamics of species population for a regime switching model with environmental noises and additive Allee effect. In 2023, Kumbhakar et al. [16] investigated the dynamics of a predator-prey model with strong Allee effect in prey and a new kind of functional response by considering spatially grouped predators. At the same time, Xie and Zhang [17] studied the dynamic behaviors of a fractional order predator-prey system with Allee effect, fear effect and shelter effect.

In addition to intraspecific relationships, there are also interspecific relationships in predator-prey systems, most of which are predator-prey relationships or competition relationships between predators. Relatively few experimental or theoretical studies have explored the impact of the defensive behavior of prey against predators on the dynamics of the interaction between predators and prey. With the deepening of research, scholars began to pay attention to the defensive behavior of prey against predators, among which herd behavior is one of the most widely studied defensive behaviors of prey against predators. Herd behavior refers to that when a group shows a specific collective behavior, the individuals in the group will have the similar social behavior. Ajraldi et al. [18] expressed this idea, and proposed a predator-prey model with a single square root functional response function. Braza [3] discussed a predator-prey system with a modified Lotka-Volterra interaction term, in which the Lotka-Volterra interaction term is proportional to the square root of the prey population. In addition, they also compared the above conclusion with the dynamics of the predator-prey system with the classic Lotka-Volterra interaction term. Gimmelli et al. [6] studied a predator-prey system with herd behavior and predators carrying infectious diseases. In 2022, Brahim et al. [19] used a fractional-order model to show the effect of harvesting on a three-species predator-prey interaction in the case of prey herd behavior. Meanwhile, Shivam et al. [20] studied the temporal and spatiotemporal analysis of a prey-predator model with cooperative hunting among predators and herd behavior in prey. In 2023, Fordjour et al. [21] investigated a deterministic predator-prey model with prey herd behavior, mutual interference and the effect of fear.

At present, there are relatively few studies considering Allee effect and herd behavior in the predator-prey system at the same time [2, 7, 9, 12]. Particularly, in [12], Ye et al. studied the dynamics of the following predator-prey model with Allee effect and herd behavior in the spatially homogeneous situation

$$\begin{cases} \frac{dX}{dT} = rX(1-\frac{X}{K})(X-m) - \frac{\alpha\sqrt{X}Y}{1+T_h\alpha\sqrt{X}}, & T>0, \\ \frac{dY}{dT} = -\delta Y + \frac{c\alpha\sqrt{X}Y}{1+T_h\alpha\sqrt{X}}, & T>0, \\ X(0)=X_0 \geq 0,\ Y(0)=Y_0\geq 0, \end{cases}$$

where X and Y represent the densities of prey and predator, respectively, α is the predator's search efficiency for prey, T_h is the average processing time of each prey, c is the conversion efficiency from prey to predator, δ is the natural mortality rate of predator, r is the intrinsic growth rate, K is the environmental capacity, m is the Allee effect threshold satisfying $-K<m<K$, $rX(1-\frac{X}{K})(X-m)$ is the Allee effect term and $\frac{\alpha\sqrt{X}Y}{1+T_h\alpha\sqrt{X}}$ is the herd behavior term. In order to better study the

dynamics of the above model, the above model is dimensionless as follows

$$u = \frac{X}{K},\ v = \frac{\alpha Y}{rK\sqrt{K}},\ t = rKT,\ a = \alpha T_h\sqrt{K},\ b = \frac{c\alpha\sqrt{K}}{rK},\ s = \frac{\delta}{rK},\ \beta = \frac{m}{K}.$$

It follows that

$$\begin{cases} \frac{\partial u}{\partial t} = u(1-u)(u-\beta) - \frac{\sqrt{u}v}{1+a\sqrt{u}}, & t > 0, \\ \frac{\partial v}{\partial t} = -sv + \frac{b\sqrt{u}v}{1+a\sqrt{u}}, & t > 0, \\ u(0) = u_0 \geq 0,\ v(0) = v_0 \geq 0, \end{cases}$$

where β is the Allee effect threshold ($-1 < \beta < 0$ weak Allee effect, $0 < \beta < 1$ strong Allee effect), and other parameters are greater than zero. In order to be more consistent with the ecosystem, this paper considers the following predator-prey system with Allee effect and herd behavior in the case of spatial heterogeneity on the basis of [12],

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + u(1-u)(u-\beta) - \frac{\sqrt{u}v}{1+a\sqrt{u}}, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2\Delta v - sv + \frac{b\sqrt{u}v}{1+a\sqrt{u}}, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1.1}$$

where $d_1,\ d_2$ are diffusion coefficients of u and v, respectively. The ordinary differential system and steady-state system corresponding to system (1.1) are respectively as follows

$$\begin{cases} \frac{du}{dt} = u(1-u)(u-\beta) - \frac{\sqrt{u}v}{1+a\sqrt{u}}, & t > 0, \\ \frac{dv}{dt} = -sv + \frac{b\sqrt{u}v}{1+a\sqrt{u}}, & t > 0, \\ u(0) = u_0 \geq 0,\ v(0) = v_0 \geq 0, \end{cases} \tag{1.2}$$

and

$$\begin{cases} d_1\Delta u + u(1-u)(u-\beta) - \frac{\sqrt{u}v}{1+a\sqrt{u}} = 0, & x \in \Omega, \\ d_2\Delta v - sv + \frac{b\sqrt{u}v}{1+a\sqrt{u}} = 0, & x \in \Omega, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

The purpose of this paper is to obtain the existence, stability and bifurcation direction of periodic solutions for the system (1.1) through the stability of the non-negative constant solutions for the system (1.1) and the Hopf bifurcation theory, and to supplement and perfect the research results of [12]. Firstly, the stabilities of non-negative constant solutions in ODE system (1.2) and PDE system (1.1) are analyzed. Secondly, we study the existence of Hopf bifurcation points, and then further discuss the stability and bifurcation direction of Hopf bifurcation solutions in the case of spatial homogeneity. At the same time, numerical examples are given by MATLAB. Finally, the numerical simulations of the system are carried out through MATLAB. The phase portraits of the system (1.2) and the numerical solutions for system (1.1) are presented, which verifies and supplements the theoretical analysis results.

2. Stability of nonnegative constant solutions

This section is devoted to analyzing the stability of the non-negative constant solutions in the ODE system (1.2) and the PDE system (1.1).

Let $f(u,v) = u(1-u)(u-\beta) - \frac{\sqrt{u}v}{1+a\sqrt{u}}$, $g(u,v) = -sv + \frac{b\sqrt{u}v}{1+a\sqrt{u}}$. Then the system (1.1) can be rewritten as follows

$$\begin{cases} \frac{\partial u}{\partial t} = d_1\Delta u + f(u,v), & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} = d_2\Delta v + g(u,v), & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) \geq 0,\ v(x,0) = v_0(x) \geq 0,\ x \in \Omega. \end{cases} \tag{2.1}$$

Let (u^*, v^*) be the constant solution of the system (1.1), that is,

$$\begin{cases} u^*(1-u^*)(u^*-\beta) - \frac{\sqrt{u^*}v^*}{1+a\sqrt{u^*}} = 0, \\ -sv^* + \frac{b\sqrt{u^*}v^*}{1+a\sqrt{u^*}} = 0. \end{cases} \tag{2.2}$$

By calculation, (2.2) has the following four non-negative constant solutions:

$$(u_1^*, v_1^*) = \left(\tfrac{s^2}{(b-as)^2}, \tfrac{bs}{(b-as)^2}\left(1 - \tfrac{s^2}{(b-as)^2}\right)\left(\tfrac{s^2}{(b-as)^2} - \beta\right)\right) \triangleq \left(\alpha, \tfrac{bs}{(b-as)^2}(1-\alpha)(\alpha-\beta)\right),$$
$$(u_2^*, v_2^*) = (1,0),\ (u_3^*, v_3^*) = (\beta, 0),\ (u_4^*, v_4^*) = (0,0),$$

where

$$\alpha = \frac{s^2}{(b-as)^2}, \tag{2.3}$$

and (u_1^*, v_1^*) is positive for $\beta < \alpha < 1$.

Let $\mathbf{U} = (u,v)$ and

$$Q(\mathbf{U}) = \begin{pmatrix} f(u,v) \\ g(u,v) \end{pmatrix} = \begin{pmatrix} u(1-u)(u-\beta) - \frac{\sqrt{u}v}{1+a\sqrt{u}} \\ -sv + \frac{b\sqrt{u}v}{1+a\sqrt{u}} \end{pmatrix}.$$

Let $Q_{\mathbf{U}}(\mathbf{U}^*)$ be the Jacobian matrix of $Q(\mathbf{U})$ at $\mathbf{U}^* \triangleq (u^*, v^*)$.

$$\begin{aligned} Q_{\mathbf{U}}(\mathbf{U}^*) &= \begin{pmatrix} f_u(u^*,v^*) & f_v(u^*,v^*) \\ g_u(u^*,v^*) & g_v(u^*,v^*) \end{pmatrix} \\ &= \begin{pmatrix} (1-u^*)(u^*-\beta) - u^*(u^*-\beta) + u^*(1-u^*) - \frac{v^*}{2\sqrt{u^*}(1+a\sqrt{u^*})^2} & \frac{-\sqrt{u^*}}{1+a\sqrt{u^*}} \\ \frac{bv^*}{2\sqrt{u^*}(1+a\sqrt{u^*})^2} & \frac{b\sqrt{u^*}}{1+a\sqrt{u^*}} - s \end{pmatrix} \\ &\triangleq \begin{pmatrix} f_{u^*} & f_{v^*} \\ g_{u^*} & g_{v^*} \end{pmatrix}. \end{aligned}$$

Hence, the Jacobian matrices of $Q(\mathbf{U})$ at (u_1^*, v_1^*), (u_2^*, v_2^*) and (u_3^*, v_3^*) can be derived respectively as follows:

$$Q_{\mathbf{U}}(u_1^*, v_1^*) = \begin{pmatrix} \frac{(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})(\alpha-\beta)+2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{2(1+a\sqrt{\alpha})} & -\frac{s}{b} \\ \frac{b(1-\alpha)(\alpha-\beta)}{2(1+a\sqrt{\alpha})} & 0 \end{pmatrix} \triangleq \begin{pmatrix} f_{u_1^*} & f_{v_1^*} \\ g_{u_1^*} & g_{v_1^*} \end{pmatrix} \triangleq J,$$

$$Q_{\mathbf{U}}(u_2^*, v_2^*) = \begin{pmatrix} \beta - 1 & -\frac{1}{1+a} \\ 0 & \frac{b}{1+a} - s \end{pmatrix} \triangleq \begin{pmatrix} f_{u_2^*} & f_{v_2^*} \\ g_{u_2^*} & g_{v_2^*} \end{pmatrix},$$

$$Q_{\mathbf{U}}(u_3^*, v_3^*) = \begin{pmatrix} \beta(1-\beta) & -\frac{\sqrt{\beta}}{1+a\sqrt{\beta}} \\ 0 & \frac{b\sqrt{\beta}}{1+a\sqrt{\beta}} - s \end{pmatrix} \triangleq \begin{pmatrix} f_{u_3^*} & f_{v_3^*} \\ g_{u_3^*} & g_{v_3^*} \end{pmatrix}.$$

The stability theory in [22] will be used to analyze the stability of (u_1^*, v_1^*), (u_2^*, v_2^*) and (u_3^*, v_3^*) for the systems. Firstly, according to the standard operator theory proposed by Casten [23], the following stability conclusion of the non-negative constant solutions for ODE system (1.2) is obtained.

Theorem 2.1. (i) *There exist b_1^*, b_2^* and $\beta^* = \frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} + \alpha$. For $s + as < b < \min\{b_1^*, b_2^*\}$, if $-1 < \beta < \beta^*$, then the positive constant solution (u_1^*, v_1^*) is locally asymptotically stable in the ODE system (1.2); if $\beta^* < \beta < \alpha$, then the positive constant solution (u_1^*, v_1^*) is unstable in the ODE system (1.2).*

(ii) *If $0 < b < s(1+a)$, then the non-negative constant solution $(1, 0)$ is locally asymptotically stable in the ODE system (1.2); if $b > s(1+a)$, then the non-negative constant solution $(1, 0)$ is unstable in the ODE system (1.2).*

(iii) *The non-negative constant solution $(\beta, 0)$ is unstable in the ODE system (1.2).*

Proof. Here we only prove (i) of Theorem 2.1. However, (ii) and (iii) can be similarly proved. It can be seen from the above that

$$\begin{aligned} trace(J) &= f_{u_1^*} + g_{v_1^*} = f_{u_1^*} = \tfrac{(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})(\alpha-\beta)+2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{2(1+a\sqrt{\alpha})}, \\ \det(J) &= f_{u_1^*} g_{v_1^*} - f_{v_1^*} g_{u_1^*} = \tfrac{s(1-\alpha)(\alpha-\beta)}{2(1+a\sqrt{\alpha})}. \end{aligned}$$

It follows that $\det(J) > 0$ for $\beta < \alpha < 1$. Therefore, the cases of $trace(J)$ are discussed below. Firstly, it is analyzed that there is $\beta^* = \frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} + \alpha \in (-1, \alpha)$ such that $trace(J) = f_{u_1^*} = 0$. Moreover, $\beta^* \in (0, \alpha)$ can guarantee $\beta^* \in (-1, \alpha)$. So the sufficient conditions of $\beta^* \in (0, \alpha)$ are analyzed below.

1) The sufficient condition of $\beta^* < \alpha$, i.e., $\frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} < 0$.

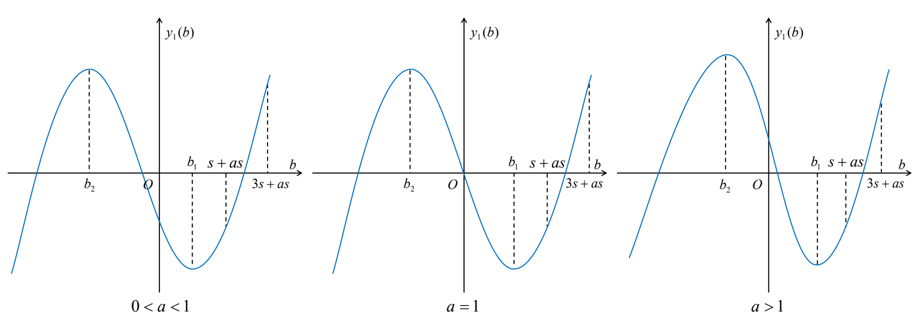

Figure 1. The graph of $y_1(b)$.

Since $2\alpha(1-\alpha)(1+a\sqrt{\alpha}) > 0$ for $\beta < \alpha < 1$, $\frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} < 0$ for $1 + 2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha} < 0$. By substituting $\alpha = \frac{s^2}{(b-as)^2}$ into $1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha} < 0$, we derive $\frac{b^3-asb^2-a^2s^2b+a^3s^3-3s^2b-as^3}{(b-as)^3} < 0$. Hence, $b^3 - asb^2 - (a^2s^2 + 3s^2)b + a^3s^3 - as^3 \triangleq y_1(b) < 0$ and $b > as$ can guarantee $\frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} < 0$. The graph of $y_1(b)$ is shown in Figure 1, and its extreme points are

$$b_1 = \frac{as + s\sqrt{4a^2+9}}{3},\ b_2 = \frac{as - s\sqrt{4a^2+9}}{3}.$$

By calculation, we derive $b_2 < 0 < b_1 < s + as$, $y_1(s + as) < 0$ and $y_1(3s + as) > 0$. As shown in Figure 1, $y_1(b)$ monotonically increases for $b > b_1$. Therefore, by Zero-Point Theorem, there is $b_1^* \in (s+as, 3s+as)$ such that $y_1(b) < 0$ for $s+as < b < b_1^*$. It follows that $\frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} < 0$ for $s + as < b < b_1^*$, i.e. $\beta^* < \alpha$ for $s + as < b < b_1^*$.

2) The sufficient condition of $\beta^* > 0$ is equivalent to the sufficient condition of $\frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} > -\alpha$.

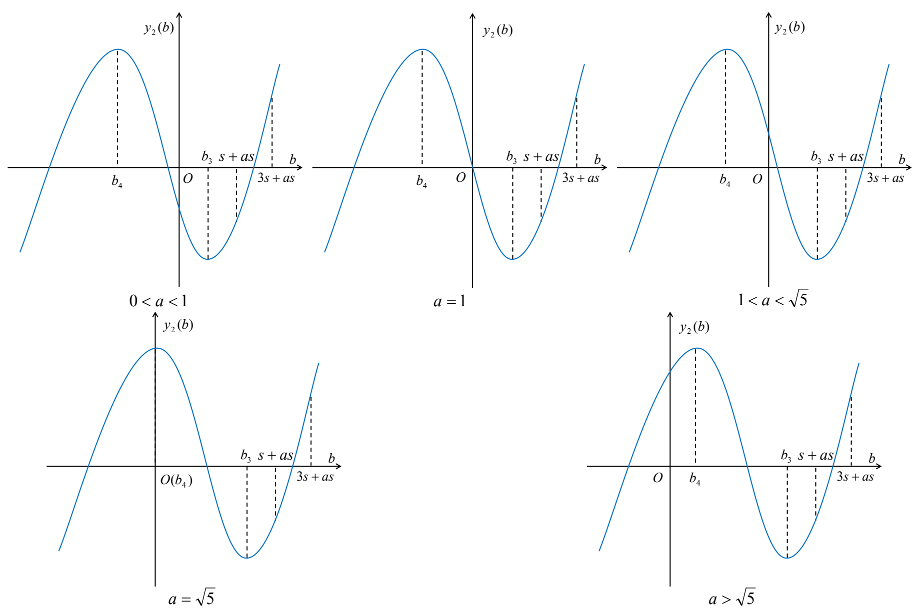

Figure 2. The graph of $y_2(b)$.

According to the previous analysis, we have that $1 + 2a\sqrt{\alpha} - 3\alpha - 4a\alpha\sqrt{\alpha} < 0$ for $s + as < b < b_1^*$. By $\alpha > 0$, we derive that $\frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} > -\alpha$ for $2(1-\alpha)(1+a\sqrt{\alpha}) < -(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})$, i.e., $3+4a\sqrt{\alpha}-5\alpha-6a\alpha\sqrt{\alpha} < 0$. By substituting $\alpha = \frac{s^2}{(b-as)^2}$ into $3 + 4a\sqrt{\alpha} - 5\alpha - 6a\alpha\sqrt{\alpha} < 0$, we derive $\frac{3b^3-5asb^2+(a^2s^2-5s^2)b+a^3s^3-as^3}{(b-as)^3} < 0$. Hence, $3b^3 - 5asb^2 + (a^2s^2 - 5s^2)b + a^3s^3 - as^3 \triangleq y_2(b) < 0$ and $b > as$ can guarantee $\frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} > -\alpha$. The graph of $y_2(b)$ is shown in Figure 2, and its extreme points are

$$b_3 = \frac{5as + s\sqrt{16a^2+45}}{9} > 0,\ b_4 = \frac{5as - s\sqrt{16a^2+45}}{18},$$

where $b_4 > 0$ for $a > \sqrt{5}$, $b_4 = 0$ for $a = \sqrt{5}$ and $b_4 < 0$ for $0 < a < \sqrt{5}$. By calculation, we derive $y_2(s+as) < 0$, $y_2(3s+as) > 0$ and $b_4 < b_3 < s+as$. As shown in Figure 2, $y_2(b)$ monotonically increases for $b > b_3$. Therefore, by Zero-Point Theorem, there is $b_2^* \in (s+as, 3s+as)$ such that $y_2(b) < 0$ for $s+as < b < b_2^*$. It follows that $\frac{2(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} > -1$ for $s+as < b < b_2^*$, i.e. $\beta^* > 0$ for $s+as < b < b_2^*$.

Hence, when $s+as < b < \min\{b_1^*, b_2^*\}$, there exists $\beta^* = \alpha + \frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} \in (0,\alpha)$ such that $f_{u_1^*} = 0$.

Moreover, from the fact that $f_{u_1^*}$ is monotonically increasing with respect to β, if $-1 < \beta < \beta^*$, then $trace(J) < 0$, which indicates that the positive constant solution (u_1^*, v_1^*) is locally asymptotically stable in the ODE system (1.2); if $\beta^* < \beta < \alpha$, then $trace(J) > 0$, which indicates that the positive constant solution (u_1^*, v_1^*) is unstable in the ODE system (1.2). $\square$

Remark 2.1. The stability of $(0,0)$ in the ODE system (1.2) can be studied by the method in Section 2.3 of [3].

After analyzing the stability of the non-negative constant solutions for the ODE system (1.2), the stability of the non-negative constant solutions for the PDE system (1.1) will be analyzed in the following.

Let $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ be the eigenvalue sequence of elliptic operator $-\Delta$ under Neumann boundary condition in Ω, where each λ_i has an algebraic multiplicity $m_i \ge 1$. Suppose that ϕ_{ij} $(1 \le j \le m_i)$ are the normalized characteristic functions corresponding to λ_i. Then $\{\phi_{ij}\}$ $(i \ge 0,\ 1 \le j \le m_i)$ forms an orthonormal basis in $L^2(\Omega)$.

The steady-state system (1.3) is linearized at $\mathbf{U}^* = (u^*, v^*)$ with respect to $\mathbf{U}$ as follows:

$$\begin{pmatrix} d_1\Delta + f_{u^*} & f_{v^*} \\ g_{u^*} & d_2\Delta + g_{v^*} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0,\ x \in \Omega,\ \frac{\partial \mathbf{U}}{\partial \nu} = 0,\ x \in \partial\Omega,$$

where f_{u^*}, f_{v^*}, g_{u^*} and g_{v^*} are as described above.

Let

$$L \triangleq \begin{pmatrix} d_1\Delta + f_{u^*} & f_{v^*} \\ g_{u^*} & d_2\Delta + g_{v^*} \end{pmatrix}.$$

Suppose that $(\phi(x), \psi(x))$ is the eigenfunction corresponding to an eigenvalue μ of L. It follows that

$$\begin{pmatrix} d_1\Delta + f_{u^*} - \mu & f_{v^*} \\ g_{u^*} & d_2\Delta + g_{v^*} - \mu \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Let $\phi = \sum\limits_{0\le i\le\infty,\ 1\le j\le m_i} a_{ij}\phi_{ij}$ and $\psi = \sum\limits_{0\le i\le\infty,\ 1\le j\le m_i} b_{ij}\phi_{ij}$. It follows that

$$\begin{aligned}&\sum_{0\le i\le\infty,\ 1\le j\le m_i}\begin{pmatrix} f_{u^*}-d_1\lambda_i-\mu & f_{v^*}\\ g_{u^*} & g_{v^*}-d_2\lambda_i-\mu\end{pmatrix}\begin{pmatrix}a_{ij}\\ b_{ij}\end{pmatrix}\phi_{ij}\\ \triangleq &\sum_{0\le i\le\infty,\ 1\le j\le m_i}(J_i-\mu I)\begin{pmatrix}a_{ij}\\ b_{ij}\end{pmatrix}\phi_{ij}=0.\end{aligned}$$

Hence, μ is an eigenvalue of L if and only if there is some $i\ge 0$ such that the determinant of $J_i-\mu I$ is equal to zero, i.e.,

$$\mu^2 - trace(J_i)\mu + \det(J_i) = 0,$$

where $trace(J_i) = -(d_1+d_2)\lambda_i + f_{u^*} + g_{v^*}$, $\det(J_i) = (f_{u^*}-d_1\lambda_i)(g_{v^*}-d_2\lambda_i) - f_{v^*}g_{u^*}$. Then, based on [24], we derive the following conclusions.

i.) $(u_1^*, v_1^*) = (\alpha, \frac{bs}{(b-as)^2}(1-\alpha)(\alpha-\beta))$

By the proof of Theorem 2.1, we have that $0<\beta^*<\alpha$ for $s+as<b<\min\{b_1^*, b_2^*\}$ and $f_{u_1^*}<0$ for $-1<\beta<\beta^*$. Moreover, since $f_{v_1^*}<0$, $g_{u_1^*}>0$, $g_{v_1^*=0}$ and $\lambda_i\ge 0$ $(i=0,\ 1,\cdots)$, $trace(J_i)<0$ and $\det(J_i)>0$, which implies that $Re(\mu)<0$ for all eigenvalues. Hence, (u_1^*, v_1^*) is locally asymptotically stable in the PDE system (1.1) for $-1<\beta<\beta^*$. In addition, by (i) of Theorem 2.1, we have that (u_1^*, v_1^*) is unstable in the ODE system (1.2) for $\beta^*<\beta<\alpha$, which implies that (u_1^*, v_1^*) is unstable in the PDE system (1.1) for $\beta^*<\beta<\alpha$.

In conclusion, (u_1^*, v_1^*) is locally asymptotically stable in the PDE system (1.1) for $-1<\beta<\beta^*$, and (u_1^*, v_1^*) is unstable in the PDE system (1.1) for $\beta^*<\beta<\alpha$.

ii.) $(u_2^*, v_2^*) = (1, 0)$

Since $g_{v_2^*}<0$ for $0<b<s(1+a)$, $trace(J_i) = -(d_1+d_2)\lambda_i + f_{u_2^*} + g_{v_2^*} < 0$ and $\det(J_i) = (f_{u_2^*}-d_1\lambda_i)(g_{v_2^*}-d_2\lambda_i) - f_{v_2^*}g_{u_2^*} > 0$ for $f_{u_2^*}<0$, $f_{v_2^*}<0$, $g_{u_2^*}=0$ and $\lambda_i\ge 0$ $(i=0,\ 1,\cdots)$, which implies that $Re(\mu)<0$ for all eigenvalues. Hence, (u_2^*, v_2^*) is locally asymptotically stable in the PDE system (1.1) for $0<b<s(1+a)$. In addition, by (ii) of Theorem 2.1, we have that (u_2^*, v_2^*) is unstable in the ODE system (1.2) for $b>s(1+a)$, which implies that (u_2^*, v_2^*) is unstable in the PDE system (1.1) for $b>s(1+a)$.

In summary, non-negative constant solution (u_2^*, v_2^*) is locally asymptotically stable in the PDE system (1.1) for $0<b<s(1+a)$, and (u_2^*, v_2^*) is unstable in the PDE system (1.1) for $b>s(1+a)$.

iii.) $(u_3^*, v_3^*) = (\beta, 0)$ $(0<\beta<1)$

By (iii) of Theorem 2.1, we have that (u_3^*, v_3^*) is unstable in the ODE system (1.2), which implies that (u_3^*, v_3^*) is unstable in the PDE system (1.1).

To sum up, the stability of (u_1^*, v_1^*), (u_2^*, v_2^*) and (u_3^*, v_3^*) for PDE system (1.1) is summarized as follows.

Theorem 2.2. (i) *There exist b_1^*, b_2^* and $\beta^* = \frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}}+\alpha$. For $s+as<b<\min\{b_1^*, b_2^*\}$, if $-1<\beta<\beta^*$, then the positive constant solution (u_1^*, v_1^*) is locally asymptotically stable in the PDE system (1.1); if $\beta^*<\beta<\alpha$, then the positive constant solution (u_1^*, v_1^*) is unstable in the PDE system (1.1).*

(ii) *If* $0 < b < s(1+a)$, *then the non-negative constant solution* $(1,0)$ *is locally asymptotically stable in the PDE system (1.1); if* $b > s(1+a)$, *then the non-negative constant solution* $(1,0)$ *is unstable in the PDE system (1.1).*

(iii) *The non-negative constant solution* $(\beta,0)$ *is unstable in the PDE system (1.1).*

Remark 2.2. Theorem 2.2 shows that there is no Turing instability for $(u_i^*, v_i^*), i = 1,2,3$.

3. Hopf bifurcation analysis

Through numerical simulation, in Figure 3, it is presented that the maxima and minima of $\|u(\cdot,t)\|_\infty$ or $\|v(\cdot,t)\|_\infty$ in $t \in [4000, 5000]$ are not equal within a certain range of β, which implies that the system (1.1) may undergo Hopf bifurcation at some β. Therefore, in this section, taking β as the Hopf bifurcation parameter and fixing the parameters $a,\ b,\ s$, we study the Hopf bifurcation solutions derived from the positive constant solution (u_1^*, v_1^*) of the system (1.1) by the Hopf bifurcation theory [13] and Theorem 2.2(i). Then the bifurcation direction and stability of the Hopf bifurcation solution for the case of spatial homogeneity are further analyzed. Moreover, numerical examples are given by MATLAB.

In this section, we consider the case of one-dimensional space $\Omega = (0, l\pi)$ $(l \in R^*)$. For the convenience, let

$$v_\beta = \frac{bs}{(b-as)^2}(1-\alpha)(\alpha-\beta). \tag{3.1}$$

Then the positive constant solution $(u_1^*, v_1^*) = (\alpha, v_\beta)$ for $\beta < \alpha < 1$, where α is given by (2.3).

Moreover, based on the stability condition of (u_1^*, v_1^*) in (i) of Theorem 2.2, it is always assumed that there are b_1^* and b_2^* such that $s + as < b < \min\{b_1^*, b_2^*\}$ in this section, which implies $0 < \beta^* < \alpha$. Next, we introduce the eigenvalue problem

$$-\phi'' = \lambda\phi,\ x \in (0, l\pi),\quad \phi' = 0,\ x = 0,\ l\pi,$$

which possesses simple eigenvalues

$$\lambda_j = (\frac{j}{l})^2, j = 0,1,2,3,... \tag{3.2}$$

and normalized eigenfunctions

$$\phi_j(x) = \begin{cases} \frac{1}{\sqrt{l\pi}}, & j = 0, \\ \sqrt{\frac{2}{l\pi}}\cos(\frac{jx}{l}), & j > 0 \end{cases} \tag{3.3}$$

corresponding to λ_j. And $\{\phi_j(x)\}$ is an orthonormal basis in $L^2(0, l\pi)$.

Let $\widehat{u} = u - \alpha$ and $\widehat{v} = v - v_\beta$, and bring them into (1.1). Moreover, we still use (u,v) to represent $(\widehat{u}, \widehat{v})$. Thus, the PDE system (1.1) is shifted to

$$\begin{cases} u_t - d_1 u_{xx} = f(\beta, u, v), & x \in (0, l\pi), t > 0, \\ v_t - d_2 v_{xx} = g(\beta, u, v), & x \in (0, l\pi), t > 0, \\ u_x(0,t) = v_x(0,t) = 0,\ u_x(l\pi,t) = v_x(l\pi,t) = 0,\ t > 0, & \end{cases} \tag{3.4}$$

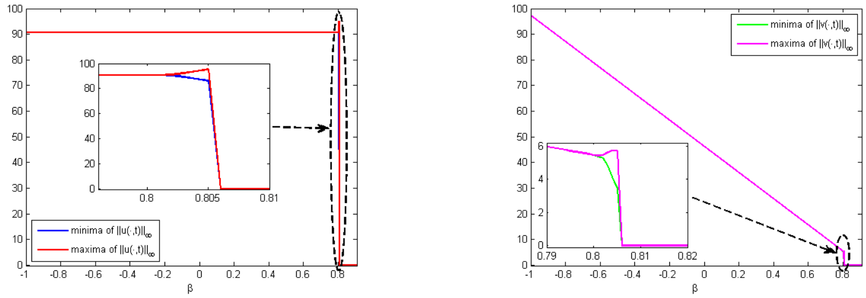

Figure 3. The bifurcation diagrams showing the maxima and minima of $\|u(\cdot,t)\|_\infty$ and $\|v(\cdot,t)\|_\infty$ in $t \in [4000, 5000]$ with the bifurcation parameter β ranging from -1 to $\alpha \approx 0.9070$. Parameter values and initial data: $d_1 = 12,\ d_2 = 1,\ a = 5,\ b = 12.1,\ s = 2$ and $(u_0(x),\ v_0(x)) = (u_1^* + 0.0763\cos(5x),\ v_1^* + 0.0763\cos(5x))$.

where

$$f(\beta, u, v) = (u+\alpha)(1-(u+\alpha))(u+\alpha-\beta) - \frac{\sqrt{u+\alpha}(v+v_\beta)}{1+a\sqrt{u+\alpha}}, \tag{3.5}$$

$$g(\beta, u, v) = -s(v+v_\beta) + \frac{b\sqrt{u+\alpha}(v+v_\beta)}{1+a\sqrt{u+\alpha}}. \tag{3.6}$$

Firstly, we consider the following linearization operator of the steady-state system corresponding to (3.4) at $(\beta, 0, 0)$.

$$L(\beta) \triangleq \begin{pmatrix} d_1 \frac{\partial^2}{\partial x^2} + A(\beta) & B \\ C(\beta) & d_2 \frac{\partial^2}{\partial x^2} + D \end{pmatrix},$$

where

$$\begin{cases} A(\beta) = f_u(\beta,0,0) = \frac{(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})(\alpha-\beta)+2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{2(1+a\sqrt{\alpha})}, \\ B = f_v(\beta,0,0) = -\frac{s}{b}, \\ C(\beta) = g_u(\beta,0,0) = \frac{b(1-\alpha)(\alpha-\beta)}{2(1+a\sqrt{\alpha})}, \\ D = g_v(\beta,0,0) = 0 \end{cases} \tag{3.7}$$

are the partial derivatives of equations (3.5) and (3.6) at $(0,0)$, respectively.

Suppose that $\begin{pmatrix} \phi \\ \psi \end{pmatrix} = \sum\limits_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{nx}{l}$ is a pair of eigenfunctions of $L(\beta)$ and its corresponding eigenvalue is η. It follows that

$$\sum_{n=0}^{\infty} L_n(\beta) \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{nx}{l} = \sum_{n=0}^{\infty} \eta \begin{pmatrix} a_n \\ b_n \end{pmatrix} \cos \frac{nx}{l},$$

where

$$L_n(\beta) = \begin{pmatrix} A(\beta) - \frac{d_1 n^2}{l^2} & B \\ C(\beta) & -\frac{d_2 n^2}{l^2} \end{pmatrix}.$$

Thus, the eigenvalues of $L(\beta)$ can be given by the eigenvalues of $L_n(\beta)$. The characteristic equations of $L_n(\beta)$ are

$$\eta^2 - T_n(\beta)\eta + D_n(\beta) = 0,\ n = 0, 1, 2, \cdots, \tag{3.8}$$

where

$$T_n(\beta) = A(\beta) - \frac{(d_1 + d_2)n^2}{l^2},\ D_n(\beta) = \frac{d_1 d_2 n^4}{l^4} - d_2 \frac{n^2}{l^2} A(\beta) - BC(\beta).$$

Then, by substituting (3.7) into $T_n(\beta)$ and $D_n(\beta)$, we obtain

$$T_n(\beta) = \frac{(1 + 2a\sqrt{\alpha} - 3\alpha - 4a\alpha\sqrt{\alpha})(\alpha - \beta) + 2\alpha(1 - \alpha)(1 + a\sqrt{\alpha})}{2(1 + a\sqrt{\alpha})} - \frac{(d_1 + d_2)n^2}{l^2}, \tag{3.9}$$

$$\begin{aligned} D_n(\beta) =& \frac{d_1 d_2 n^4}{l^4} - d_2 \frac{n^2}{l^2} \frac{(1 + 2a\sqrt{\alpha} - 3\alpha - 4a\alpha\sqrt{\alpha})(\alpha - \beta) + 2\alpha(1 - \alpha)(1 + a\sqrt{\alpha})}{2(1 + a\sqrt{\alpha})} \\ &+ \frac{s(1 - \alpha)(\alpha - \beta)}{2(1 + a\sqrt{\alpha})}. \end{aligned} \tag{3.10}$$

Next, according to the following condition (A) for the existence of Hopf bifurcation in [13], we find the Hopf bifurcation point β_0.

(A) There exist $n \in N$ and β_0 such that

$$T_n(\beta_0) = 0,\ D_n(\beta_0) > 0,\ \text{and } T_j(\beta_0) \neq 0,\ D_j(\beta_0) \neq 0,\ j \neq n \tag{3.11}$$

and

$$\theta'(\beta_0) \neq 0 \tag{3.12}$$

for the only pair of complex eigenvalues $\theta(\beta) \pm i\omega(\beta)$ near the imaginary axis.

It can be seen from (3.9)-(3.10) that $T_n(\beta) < 0$ and $D_n(\beta) > 0$ for $-1 < \beta < \beta^*$, which indicates that (α, v_β) is locally asymptotically stable for $-1 < \beta < \beta^*$. Hence all potential bifurcation points β_0 must be in $[\beta^*, \alpha)$.

If $\theta(\beta) \pm i\omega(\beta)$ are the eigenvalues of $L_n(\beta)$, then

$$\theta(\beta) = \frac{T_n(\beta)}{2} = \frac{A(\beta)}{2} - \frac{(d_1 + d_2)n^2}{2l^2}, \tag{3.13}$$

$$\omega(\beta) = \frac{\sqrt{4D_n(\beta) - T_n^2(\beta)}}{2} = \sqrt{D_n(\beta) - \theta^2(\beta)}. \tag{3.14}$$

Moreover, since $1 + 2a\sqrt{\alpha} - 3\alpha - 4a\alpha\sqrt{\alpha} < 0$,

$$\theta'(\beta) = \frac{A'(\beta)}{2} = -\frac{1 + 2a\sqrt{\alpha} - 3\alpha - 4a\alpha\sqrt{\alpha}}{4(1 + a\sqrt{\alpha})} > 0. \tag{3.15}$$

Therefore, the condition (3.12) is always true.

Let $\beta_0^H = \beta^* = \frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} + \alpha$. Then, we derive for any $j \geq 1$, $T_0(\beta_0^H) = 0$ and $T_j(\beta_0^H) < 0$. Moreover, $D_s(\beta_0^H) > 0$ for any $s \in N$. Hence, β_0^H is the Hopf bifurcation point of the system (1.1).

In the following, we analyze the bifurcation direction and stability of Hopf bifurcation solution at β_0^H according to the discriminant method in [13]. By [13], it is easy to know that the bifurcation direction and stability of Hopf bifurcation solution are determined by the following $\beta''(0)$.

$$\beta''(0) = -\frac{1}{\theta'(\beta_0^H)} Re(C_1(\beta_0^H)), \tag{3.16}$$

where $\theta'(\beta_0^H) > 0$ and

$$Re(C_1(\beta_0^H)) = Re\{\frac{i}{2\omega_0} < q^*, Q_{qq} >< q^*, Q_{q\bar{q}} > + < q^*, Q_{w_{11}q} > + \frac{1}{2} < q^*, Q_{w_{20}\bar{q}} > + \frac{1}{2} < q^*, C_{qq\bar{q}} >\}.$$

Take

$$q \triangleq \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{b\omega_0 i}{s} \end{pmatrix}, \ q^* \triangleq \begin{pmatrix} a_0^* \\ b_0^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2l\pi} \\ -\frac{si}{2b\omega_0 l\pi} \end{pmatrix}.$$

And $q, \ q^*$ satisfy $< q^*, q >= 1, \ < q^*, \bar{q} >= 0, \ L(\beta_0^H)q = i\omega_0 q$ and $L^*(\beta_0^H)q^* = -i\omega_0 q^*$, where $\omega_0 \triangleq \sqrt{-\frac{s\alpha(1-\alpha)^2}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}}}$ and $< U_1, U_2 >=< (u_1, v_1), (u_2, v_2) >= \int_0^{l\pi} (\bar{u}_1 u_2 + \bar{v}_1 v_2)dx$ represents the complex-valued inner product in $L^2(0, l\pi)\times(0, l\pi)$.

Next, the partial derivatives of $f(\beta, u, v)$ and $g(\beta, u, v)$ at $(\beta_0^H, 0, 0)$ in (3.5)-(3.6) are given below.

$$\begin{aligned}
f_{11} &\triangleq f_{uu}(\beta_0^H, 0, 0) = -4\alpha + 2 + \tfrac{8\alpha(1-\alpha)(1+a\sqrt{\alpha})^2 - (1-\alpha)^2(1+3a\sqrt{\alpha})}{2(1+a\sqrt{\alpha})(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})}, \\
f_{111} &\triangleq f_{uuu}(\beta_0^H, 0, 0) = -6 + \tfrac{(1-\alpha)^2(24a\sqrt{\alpha}+6+30a^2\alpha)}{8\alpha(1+a\sqrt{\alpha})^2(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})}, \\
f_{112} &\triangleq f_{uuv}(\beta_0^H, 0, 0) = \tfrac{1+3a\sqrt{\alpha}}{4\alpha\sqrt{\alpha}(1+a\sqrt{\alpha})^3}, \\
f_{122} &\triangleq f_{uvv}(\beta_0^H, 0, 0) = 0, \ f_{22} \triangleq f_{vv}(\beta_0^H, 0, 0) = 0, \ f_{222} \triangleq f_{vvv}(\beta_0^H, 0, 0) = 0, \\
g_{11} &\triangleq g_{uu}(\beta_0^H, 0, 0) = \tfrac{b(1-\alpha)^2(1+3a\sqrt{\alpha})}{2(1+a\sqrt{\alpha})(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})}, \\
g_{112} &\triangleq g_{uuv}(\beta_0^H, 0, 0) = -\tfrac{b(1+3a\sqrt{\alpha})}{4\alpha\sqrt{\alpha}(1+a\sqrt{\alpha})^3}, \\
g_{111} &\triangleq g_{uuu}(\beta_0^H, 0, 0) = -\tfrac{(1-\alpha)^2(24ab\sqrt{\alpha}+30a^2b\alpha+6b)}{8\alpha(1+a\sqrt{\alpha})^2(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})}, \\
g_{12} &\triangleq g_{uv}(\beta_0^H, 0, 0) = \tfrac{b}{2\sqrt{\alpha}(1+a\sqrt{\alpha})^2}, \ g_{122} \triangleq g_{uvv}(\beta_0^H, 0, 0) = 0, \\
g_{22} &\triangleq g_{vv}(\beta_0^H, 0, 0) = 0, \ g_{222} \triangleq g_{vvv}(\beta_0^H, 0, 0) = 0.
\end{aligned}$$

In addition, define $Q_{qq} = \cos^2 \frac{n}{l}x \begin{pmatrix} c_n \\ d_n \end{pmatrix}$, $Q_{q\bar{q}} = \cos^2 \frac{n}{l}x \begin{pmatrix} e_n \\ p_n \end{pmatrix}$ and $C_{qq\bar{q}} =$

$\cos^3 \frac{n}{l}x \begin{pmatrix} q_n \\ h_n \end{pmatrix}$, where

$$\begin{aligned}
c_n &= f_{11}a_n^2 + 2f_{12}a_n b_n + f_{22}b_n^2,\ d_n = g_{11}a_n^2 + 2g_{12}a_n b_n + g_{22}b_n^2,\\
e_n &= f_{11}|a_n|^2 + f_{12}(a_n\bar{b}_n + \bar{a}_n b_n) + f_{22}|b_n|^2,\\
p_n &= g_{11}|a_n|^2 + g_{12}(a_n\bar{b}_n + \bar{a}_n b_n) + g_{22}|b_n|^2,\\
q_n &= f_{111}|a_n|^2 a_n + f_{112}(2|a_n|^2 b_n + a_n^2\bar{b}_n) + f_{122}(2|b_n|^2 a_n + b_n^2\bar{a}_n) + f_{222}|b_n|^2 b_n,\\
h_n &= g_{111}|a_n|^2 a_n + g_{112}(2|a_n|^2 b_n + a_n^2\bar{b}_n) + g_{122}(2|b_n|^2 a_n + b_n^2\bar{a}_n) + g_{222}|b_n|^2 b_n.
\end{aligned}$$

When $n = 0$, we derive

$$\begin{aligned}
c_0 &= f_{11} + f_{12}\tfrac{-2bi\omega_0}{s},\ e_0 = f_{11},\ p_0 = f_{111} + f_{112}\tfrac{-bi\omega_0}{s},\\
d_0 &= g_{11} + g_{12}\tfrac{-2bi\omega_0}{s},\ q_0 = g_{11},\ h_0 = g_{111} + g_{112}\tfrac{-bi\omega_0}{s}.
\end{aligned}$$

It follows that

$$\begin{aligned}
< q^*, Q_{qq} > &= \tfrac{f_{11}}{2} + g_{12} + (\tfrac{s}{2b\omega_0}g_{11} - \tfrac{b\omega_0}{s}f_{12})i,\\
< q^*, Q_{q\bar{q}} > &= \tfrac{f_{11}}{2} + \tfrac{s}{2b\omega_0}g_{11}i,\\
< \bar{q^*}, Q_{qq} > &= \tfrac{f_{11}}{2} - g_{12} + (\tfrac{-s}{2b\omega_0}g_{11} - \tfrac{b\omega_0}{s}f_{12})i,\\
< \bar{q^*}, Q_{q\bar{q}} > &= \tfrac{f_{11}}{2} - \tfrac{s}{2bw_0}g_{11}i,\\
< q^*, C_{qq\bar{q}} > &= \tfrac{f_{111}}{2} + \tfrac{g_{112}}{2} + (\tfrac{s}{2b\omega_0}g_{111} - \tfrac{b\omega_0}{2s}f_{112})i.
\end{aligned}$$

Define $w_{20} = [2i\omega_0 I - L(\beta_0^H)]^{-1}H_{20}$ and $w_{11} = -[L(\beta_0^H)]^{-1}H_{11}$, where $H_{20} = Q_{qq} - < q^*, Q_{qq} > q - < \bar{q^*}, Q_{qq} > \bar{q}$ and $H_{11} = Q_{q\bar{q}} - < q^*, Q_{q\bar{q}} > q - < \bar{q^*}, Q_{q\bar{q}} > \bar{q}$. By calculation, we derive $H_{20} = \mathbf{0}$ and $H_{11} = \mathbf{0}$, which implies that $w_{20} = w_{11} = \mathbf{0}$. Hence, $< q^*, Q_{w_{11}q} > = < q^*, Q_{w_{20}\bar{q}} > = 0$. Thus

$$\begin{aligned}
Re(C_1(\beta_0^H)) &= Re\{\tfrac{i}{2\omega_0} < q^*, Q_{qq} >< q^*, Q_{q\bar{q}} > + \tfrac{1}{2} < q^*, C_{qq\bar{q}} >\}\\
&= \tfrac{s(1+a\sqrt{\alpha})A_1 + b\sqrt{\alpha}A_2}{16s\alpha(1+a\sqrt{\alpha})^3(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})} \triangleq \delta,
\end{aligned} \tag{3.17}$$

where $A_1 = -40\alpha - 180a\alpha\sqrt{\alpha} + 90\alpha^2 - 334a\alpha^2\sqrt{\alpha} - 264a^2\alpha^2 + 414a^2\alpha^3 - 120a^3\alpha^2\sqrt{\alpha} + 168a^3\alpha^3\sqrt{\alpha} + 30a\sqrt{\alpha} + 50a^2\alpha + 24a^3\alpha\sqrt{\alpha} + 6$ and $A_2 = 10\alpha + 30a\alpha\sqrt{\alpha} - 15\alpha^2 - 37a\alpha^2\sqrt{\alpha} + 24a^2\alpha^2 - 24a^2\alpha^3 - 9a\sqrt{\alpha} - 8a^2\alpha - 3$.

In conclusion, the system (1.1) undergoes a spatially homogeneous Hopf bifurcation at β_0^H. Moreover, the bifurcation direction and stability of Hopf bifurcation solution at β_0^H can be obtained as follows.

Theorem 3.1. *Based on $\theta'(\beta) > 0$, if δ in (3.17) is less than zero, then the direction of Hopf bifurcation at $\beta = \beta_0^H$ is supercritical and the bifurcation periodic solution is stable; if δ in (3.17) is greater than zero, then the direction of Hopf bifurcation at $\beta = \beta_0^H$ is subcritical and the bifurcation periodic solution is unstable.*

Next, we discuss spatially inhomogeneous Hopf bifurcation at β_k^H $(k \geq 1)$. Let $l_n = n\sqrt{\frac{d_1+d_2}{\alpha(1-\alpha)}}$. For $l_n < l \leq l_{n+1}$, there exists $n \in N^+$ such that the equation $A(\beta)=\frac{(d_1+d_2)j^2}{l^2}$ $(1\le j\le n)$ has a unique positive root

$$\beta_j^H=\alpha-\frac{(1+a\sqrt{\alpha})(2j^2(d_1+d_2)-2\alpha+2\alpha^2)}{l^2(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})}>\beta_0^H,$$

where $-1<\beta_0^H=\beta^*<\beta_1^H<\beta_2^H<\cdots<\beta_n^H<\alpha$. Hence, $T_j(\beta_j^H)=0$ and $T_i(\beta_j^H)\ne 0\ (1\le j\le n,\ i\ne j)$.

By (3.11) and (3.12), if $D_i(\beta_j^H)>0$ for any $i\in N$ and $1\le j\le n$, then β_j^H are the Hopf bifurcation points of the system (1.1). Next, we analyse the sufficient condition of $D_i(\beta)>0$ for $\beta\in(\beta^*,\alpha)$, which leads to the sufficient condition of $D_i(\beta_j^H)>0$ $(i\in N,\ 1\le j\le n)$. It is easy to see

$$\begin{aligned}D_i(\beta)&=\tfrac{s(1-\alpha)(\alpha-\beta)}{2(1+a\sqrt{\alpha})}-\tfrac{i^2d_2}{l^2}\tfrac{(1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha})(\alpha-\beta)+2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{2(1+a\sqrt{\alpha})}+\tfrac{d_1d_2i^4}{l^4}\\&>\tfrac{s(1-\alpha)(\alpha-\beta)}{2(1+a\sqrt{\alpha})}-\tfrac{i^2}{l^2}d_2\alpha(1-\alpha)+\tfrac{d_1d_2i^4}{l^4}.\end{aligned}$$

Let $p_i=\frac{i^2}{l^2}$ and $v(p_i)=d_1d_2p_i^2-d_2\alpha(1-\alpha)p_i+\frac{s(1-\alpha)(\alpha-\beta)}{2(1+a\sqrt{\alpha})}$. It is obvious that $D_i(\beta_j^H)>0$ $(1\le j\le n)$ if $v(p_i)\ge 0$ for $i\in N$. Since $v(p_i)\ge 0$ is equivalent to $\beta\le\alpha-p_i(d_2\alpha(1-\alpha)-d_1d_2p_i)\frac{2(1+a\sqrt{\alpha})}{s(1-\alpha)}\triangleq Q(p_i)$, we study the latter as follows.

Case (1) When $d_1\ge\frac{\alpha(1-\alpha)}{p_1}=l^2\alpha(1-\alpha)$, we have $Q(p_i)\ge\alpha>\beta$, which implies that $D_i(\beta)>0$ always holds for $\beta\in(\beta^*,\alpha)$ and $i\in N$. According to the previous analysis for $T_j(\beta_j^H)=0$ and $T_i(\beta_j^H)\ne 0$ $(1\le j\le n,\ i\ne j)$, it requires $l_n<l\le l_{n+1}$ which implies $d_1\le n^2d_1<l^2\alpha(1-\alpha)$ $(n\in N^+)$. This contradicts $d_1\ge\frac{\alpha(1-\alpha)}{p_1}=l^2\alpha(1-\alpha)$. Hence, the case of $d_1\ge\frac{\alpha(1-\alpha)}{p_1}$ will not occur.

Case (2) When $d_1<\frac{\alpha(1-\alpha)}{p_1}=l^2\alpha(1-\alpha)$, there exists $i_0\in N^+$ such that $\frac{\alpha(1-\alpha)}{p_{i_0+1}}\le d_1<\frac{\alpha(1-\alpha)}{p_{i_0}}$. For $1\le i\le i_0$, we have $Q(p_i)<\alpha$. For $i>i_0$, it is easy to derive that $D_i(\beta)>0$ always holds for $\beta\in(\beta^*,\alpha)$. Hence, the following discussion will be conducted in the case of $1\le i\le i_0$.

(i) By calculation, we have $Q'(p_i)=\frac{-2(1+a\sqrt{\alpha})}{s(1-\alpha)}(d_2\alpha(1-\alpha)-2d_1d_2p_i)\ge 0$ $(1\le i\le i_0)$ for $d_1\ge\frac{\alpha(1-\alpha)}{2p_1}$. It follows that $D_i(\beta)>0$ $(1\le i\le i_0)$ for $\beta\le Q(p_1)$. For any $m\in N^+$, we derive

$$\begin{aligned}(\beta^*<)\beta_m^H\le Q(p_1)\Leftrightarrow(\frac{m}{l})^2(d_1+d_2)&=A(\beta_m^H)\le A(Q(p_1))\\&\Leftrightarrow l\ge m\sqrt{\frac{d_1+d_2}{A(Q(p_1))}}\triangleq l_m'.\end{aligned}$$

Hence, when $l\ge l_m'$ and $\frac{\alpha(1-\alpha)}{2p_1}\le d_1<\frac{\alpha(1-\alpha)}{p_1}$, there exist at least m values $\beta_1^H<\beta_2^H<\cdots<\beta_m^H$ such that $D_i(\beta_k^H)>0$ $(i\in N,\ 1\le k\le m)$.

(ii) Moreover, by calculation, $Q'(p_i)=\frac{-2(1+a\sqrt{\alpha})}{s(1-\alpha)}(d_2\alpha(1-\alpha)-2d_1d_2p_i)\le 0$ $(1\le i\le i_0)$ for $d_1\le\frac{\alpha(1-\alpha)}{2p_{i_0}}$. It follows that $D_i(\beta)>0$ $(1\le i\le i_0)$ for $\beta\le Q(p_{i_0})$. For any $m\in N^+$, we derive

$$(\beta^*<)\beta_m^H\le Q(p_{i_0})\Leftrightarrow(\frac{m}{l})^2(d_1+d_2)=A(\beta_m^H)\le A(Q(p_{i_0}))$$

$$\Leftrightarrow l \geq m\sqrt{\frac{d_1 + d_2}{A(Q(p_{i_0}))}} \triangleq l_m^*.$$

Hence, when $l \geq l_m^*$ and $d_1 \leq \frac{\alpha(1-\alpha)}{2p_{i_0}}$, there exist at least m values $\beta_1^H < \beta_2^H < \cdots < \beta_m^H$ such that $D_i(\beta_k^H) > 0$ $(i \in N,\ 1 \leq k \leq m)$.

To sum up, the main conclusions of spatially inhomogeneous Hopf bifurcation at β_k^H $(k \geq 1)$ are as follows.

Theorem 3.2. *Suppose that* $d_1,\ d_2,\ a,\ b,\ s > 0$ *and* $l_n = n\sqrt{\frac{d_1+d_2}{\alpha(1-\alpha)}}$. *There exist* $b_1^*,\ b_2^*$ *such that* $s + as < b < \min\{b_1^*, b_2^*\}$ *holds. If there exist* $i_0,\ m,\ n \in N^+$ *such that* $\frac{\alpha(1-\alpha)}{2p_1} \leq d_1 < \frac{\alpha(1-\alpha)}{p_1}$ *and* $\max\{l_n, l_m'\} < l \leq l_{n+1}$ *or* $d_1 \leq \frac{\alpha(1-\alpha)}{2p_{i_0}}$ *and* $\max\{l_n, l_m^*\} < l \leq l_{n+1}$, *then there exist* $\min\{m, n\}$ *spatially inhomogeneous Hopf bifurcation points* β_j^H $(j = 1,\ 2, \cdots, \min\{m, n\})$ *for the system (1.1). Moreover,* $-1 < \beta_0^H = \beta^* = \frac{2\alpha(1-\alpha)(1+a\sqrt{\alpha})}{1+2a\sqrt{\alpha}-3\alpha-4a\alpha\sqrt{\alpha}} + \alpha < \beta_1^H < \beta_2^H < \cdots < \beta_{\min\{m,n\}}^H < \alpha$ *and the bifurcation periodic solutions generated at the bifurcation points can be written in the parametric form (2.32) in [13].*

Remark 3.1. (i) The bifurcation periodic solution of system (1.1) at $\beta = \beta_0^H$ is spatially homogeneous, which is consistent with the periodic solution of its corresponding ODE system (1.2).

(ii) The bifurcation periodic solutions of the system (1.1) at $\beta = \beta_j^H$ $(j = 1,\ 2, \cdots, \min\{m, n\})$ are spatially nonhomogeneous.

Example 3.1. (i) When $a = 1,\ b = 2.1$ and $s = 1$, we derive $\alpha \approx 0.8264$, $\beta = \beta_0^H \approx 0.6211$ and $\delta \approx 1.3530 > 0$. By Theorem 3.1, it follows that the direction of Hopf bifurcation at $\beta = \beta_0^H$ is subcritical and the bifurcation periodic solution is unstable.

(ii) When $a = 5,\ b = 12.1$ and $s = 2$, we derive $\alpha \approx 0.9070$, $\beta = \beta_0^H \approx 0.8045$ and $\delta \approx -1.5534 < 0$. By Theorem 3.1, it follows that the direction of Hopf bifurcation at $\beta = \beta_0^H$ is supercritical and the bifurcation periodic solution is stable.

Example 3.2. (i) When $a = 5,\ b = 12.1,\ s = 2,\ l = 1.2,\ d_1 = 0.07$ and $d_2 = 0.01$, we derive $\alpha \approx 0.9070$, $\beta^* \approx 0.8045$, $\frac{\alpha(1-\alpha)}{p_1} \approx 0.1214$, $\frac{\alpha(1-\alpha)}{2p_1} \approx 0.0607$, $\frac{\alpha(1-\alpha)}{p_2} \approx 0.0304$, $l_1 \approx 0.9740$, $l_2 \approx 1.9480$, $l_1' \approx 1.0564$ and $l_2' \approx 2.1127$, which satisfy that $n = i_0 = m = 1$, $s + as < b < \min\{b_1^*, b_2^*\}$, $\frac{\alpha(1-\alpha)}{2p_1} \leq d_1 < \frac{\alpha(1-\alpha)}{p_1}$ and $\max\{l_n, l_m'\} < l \leq l_{n+1}$. It follows that there exist two Hopf bifurcation points $\beta_0^H \approx 0.8045$ and $\beta_1^H \approx 0.9034$.

(ii) When $a = 5,\ b = 12.1,\ s = 2,\ l = 1,\ d_1 = 0.04$ and $d_2 = 0.01$, we derive $\alpha \approx 0.9070$, $\beta^* \approx 0.8045$, $\frac{\alpha(1-\alpha)}{p_1} \approx 0.0843$, $\frac{\alpha(1-\alpha)}{2p_1} \approx 0.0422$, $\frac{\alpha(1-\alpha)}{p_2} \approx 0.0211$, $l_1 \approx 0.7700$, $l_2 \approx 1.5400$, $l_1^* \approx 0.8999$ and $l_2^* \approx 1.7998$, which satisfy that $n = i_0 = m = 1$, $s + as < b < \min\{b_1^*, b_2^*\}$, $d_1 \leq \frac{\alpha(1-\alpha)}{2p_{i_0}}$ and $\max\{l_n, l_m^*\} < l \leq l_{n+1}$. It follows that there exist two Hopf bifurcation points $\beta_0^H \approx 0.8045$ and $\beta_1^H \approx 0.8653$.

4. Numerical simulation

In this section, for verifying and supplementing the previous theoretical analysis results, the systems (1.2) and (1.1) will be numerically simulated by MATLAB. Thus the phase portraits of the system (1.2) and the numerical solutions for system (1.1) are presented.

Theorem 2.1 shows the stability conclusion of the non-negative constant solutions in ODE system (1.2) corresponding to system (1.1). In Figure 4, we further present the phase portraits of system (1.2) to verify the stability conclusion by selecting parameters a, b, s and β that meet the conditions of Theorem 2.1. It can be observed from Figure 4 that the equilibrium solutions $(0,0)$ and $(1,0)$ always exist. Moreover, when $(\beta,0)$ is present, it is always unstable, which is consistent with Theorem 2.1(iii). Detailed description under specific parameters values is as follows(Table 1).

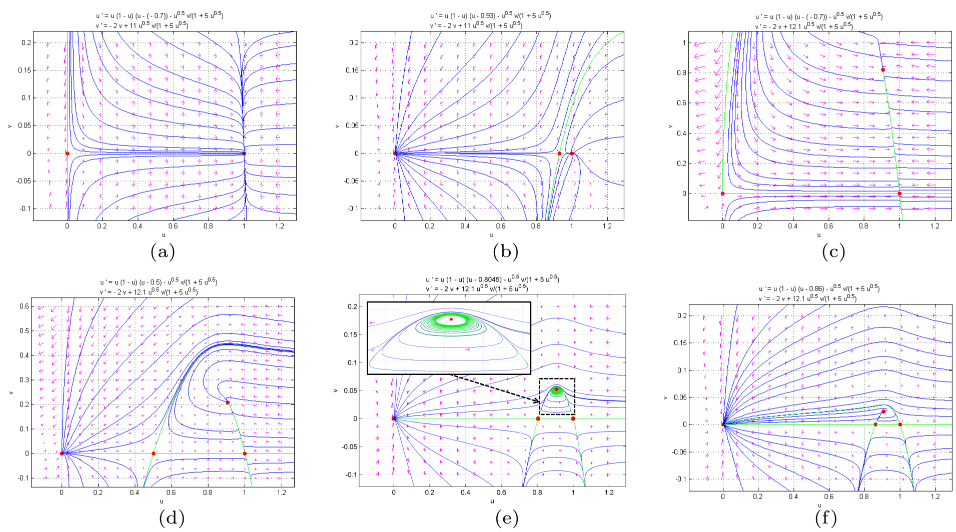

Figure 4. The phase portraits of system (1.2). The red points and green curves represent equilibrium solutions, stable and unstable orbits of system (1.2), respectively.

(i) Take $a=5,\ b=11$ and $s=2$ so that $0<b<as+s$(Figure 4(a-b)).

  (1) Take $\beta=-0.7\in(-1,0)$. In Figure 4(a), we observe that there exist two equilibrium solutions $(0,0)$ and $(1,0)$. Moreover, $(0,0)$ is unstable and $(1,0)$ is locally asymptotically stable, which is consistent with Theorem 2.1(ii).

  (2) Take $\beta=0.93\in(0,1)$. Figure 4(b) shows that there are three equilibrium solutions $(0,0)$, $(1,0)$ and $(\beta,0)=(0.93,0)$. Moreover, $(0,0)$ and $(1,0)$ are locally asymptotically stable, and $(\beta,0)$ is unstable. The above results are consistent with Theorem 2.1(ii) and (iii).

(ii) Take $a=5,\ b=12.1$ and $s=2$ so that $b>as+s$, $\beta^*\approx 0.8045$ and $\alpha\approx 0.9070$(Figure 4(c-f)).

  (1) Take $\beta=-0.7\in(-1,0)$. In Figure 4(c), we observe that there exist three equilibrium solutions $(0,0)$, $(1,0)$ and $(u_1^*,v_1^*)\approx(0.9070,0.8199)$.

Moreover, $(0,0)$ and $(1,0)$ are unstable, and (u_1^*, v_1^*) is locally asymptotically stable. The above results are consistent with Theorem 2.1(i) and (ii).

(2) Take $\beta = 0.5 \in (0, \beta^*)$. Figure 4(d) shows that there are four equilibrium solutions $(0,0)$, $(1,0)$, $(\beta,0) = (0.5,0)$ and $(u_1^*, v_1^*) \approx (0.9070, 0.2077)$. Moreover, we observe that $(0,0)$ and (u_1^*, v_1^*) are locally asymptotically stable, while $(\beta,0)$ and $(1,0)$ are unstable. The above results are consistent with Theorem 2.1(i-iii).

(3) Take $\beta = \beta^*$. In Figure 4(e), we observe that there are four equilibrium solutions$(0,0)$, $(1,0)$, $(\beta,0) = (\beta^*,0)$ and $(u_1^*, v_1^*) \approx (0.9070, 0.0523)$.$(\beta,0)$ and $(1,0)$ are unstable, and $(0,0)$ is locally asymptotically stable. Moreover, a limit cycle is generated at (u_1^*, v_1^*). The above results are consistent with Theorem 2.1(i-iii) and the existence of Hopf bifurcation solutions in the case of spatial homogeneity in Chapter 3.

(4) Take $\beta = 0.86 \in (\beta^*, \alpha)$. Figure 4(f) shows that there are four equilibrium solutions $(0,0)$, $(1,0)$, $(\beta,0) = (0.86,0)$ and $(u_1^*, v_1^*) \approx (0.9070, 0.0240)$. Moreover, we observe that $(0,0)$ is locally asymptotically stable, while $(\beta,0)$, $(1,0)$ and (u_1^*, v_1^*) are unstable. The above results are consistent with Theorem 2.1(i-iii).

Table 1. Stability of the non-negative constant solutions in ODE system (1.2), where U,S stand for Unstable and Stable, respectively

b	β	$(0,0)$	$(1,0)$	$(\beta,0)$	(u_1^*, v_1^*)
11	-0.7	U	S	Non-existent	Non-existent
11	0.93	S	S	U	Non-existent
12.1	-0.7	U	U	Non-existent	S
12.1	0.5	S	U	U	S
12.1	0.8045	S	U	U	Hopf Bifurcation
12.1	0.86	S	U	U	U

In Table 2, we choose $d_1 = 12,\ d_2 = 1,\ a = 5,\ b = 12.1$ and $s = 2$. By numerical calculations, we derive $\alpha \approx 0.9070$ and $\beta^* \approx 0.8045$. We show the value of v_1^* for different β. Moreover, the numerical solutions for system (1.1) are shown in Figure 5. In Figure 5(a-c), we observe that the solution of system (1.1) converges to the equilibrium solution (u_1^*, v_1^*) for $-1 < \beta < \beta^*$. And Figure 5(d-f) show that the system (1.1) exhibits the nonconstant positive solution for $\beta > \beta^*$, which implies that (u_1^*, v_1^*) is unstable. The above results meet Theorem 2.2(i).

Furthermore, when $\beta = 0$, the system (1.1) is a predator-prey system with only herd behavior term and no Allee effect term. When $a = 5,\ b = 12.1,\ s = 2$ and $\beta = 0$, there are $u_1^* \approx 0.9070$ and $v_1^* \approx 0.4627$. As shown in Figure 6, it can be observed that when t is large enough, the solutions of the system (1.1) tend to be the positive constant solution (u_1^*, v_1^*). This numerical simulation verifies the case of $\beta = 0$ in Theorem 2.2. Moreover, it indicates that the Allee effect term may be a factor driving the periodic solution of the predator-prey system with only herding behavior term but no Allee effect term.

Table 2. (u_1^*, v_1^*) for different β

β	-0.7	0	0.5	0.8045	0.806
u_1^*	0.9070	0.9070	0.9070	0.9070	0.9070
v_1^*	0.8199	0.4627	0.2077	0.0523	0.0515

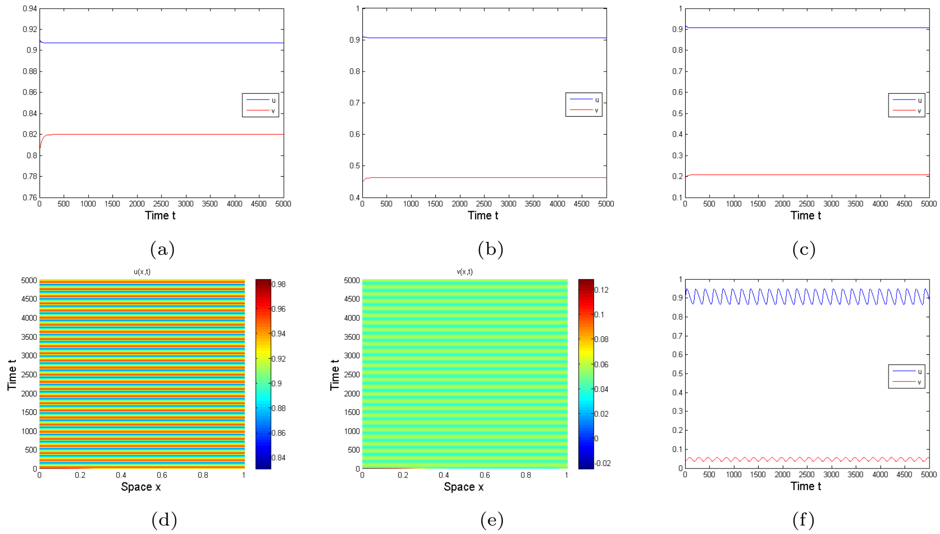

Figure 5. The numerical solutions for system (1.1) vary with β. (a) $\beta = -0.7$, (b) $\beta = 0$, (c) $\beta = 0.5$, and (d-f) $\beta = 0.806$.

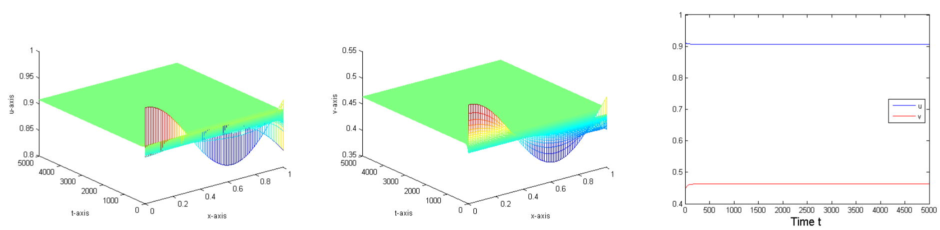

Figure 6. The numerical solutions to system (1.1) for $\beta = 0$. Initial data and parameter values: $(u_0(x), v_0(x)) = (u_1^* + 0.0763\cos(5x),\ v_1^* + 0.0763\cos(5x))$ and $d_1 = 12, d_2 = 1, a = 5, b = 12.1, s = 2, \beta = 0$.

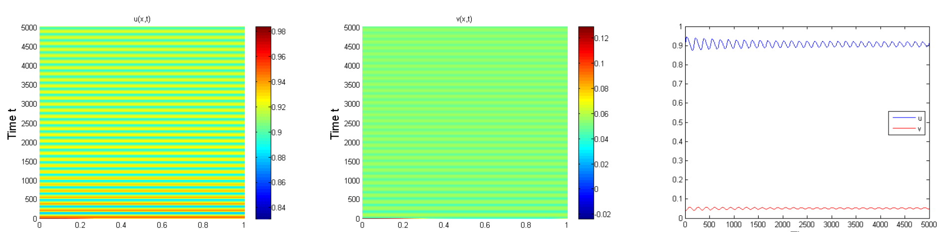

Figure 7. The periodic solution to system (1.1). Initial data and parameter values: $(u_0(x), v_0(x)) = (u_1^* + 0.0763\cos(5x),\ v_1^* + 0.0763\cos(5x))$ and $d_1 = 12, d_2 = 1, a = 5, b = 12.1, s = 2, \beta = \beta_0^H$.

In Figure 7, for $d_1 = 12,\ d_2 = 1,\ a = 5,\ b = 12.1,\ s = 2$ and $\beta = \beta_0^H \approx 0.8045$, we obtain $\alpha \approx 0.9070,\ \beta_0^H \approx 0.8045$ and $\delta \approx -1.5534 < 0$. Moreover, in Figure 7, it can be observed that the Hopf bifurcation solution of the system (1.1) at $\beta = \beta_0^H \approx 0.8045$ is stable, which is consistent with the result of Example 3.1(ii) and verifies Theorem 3.1.

In Figure 5(d-f) and Figure 7, it can be clearly observed that the densities of predator and prey fluctuate, which is consistent with the actual relationship between predator and prey in the ecosystem.

5. Discussion

In this paper, we mainly study the predator-prey model with Allee effect and herd behavior. First, we calculated the four non-negative constant solutions of the model, and proved the stability of $(u_1^*, v_1^*),\ (u_2^*, v_2^*)$ and (u_3^*, v_3^*) in the ODE system (1.2) and the PDE system (1.1) by using the stability theory. Secondly, based on the stability of the positive constant solution, the Hopf bifurcation theory was used to study the existence of the Hopf bifurcation solution emanating from the positive constant solution for the system (1.1). Then, the bifurcation direction and stability of Hopf bifurcation solution for the spatial homogeneous system were further studied. Finally, the system (1.2) and (1.1) were numerically simulated by MATLAB. Thus the phase portraits of the system (1.2) and the numerical solutions for system (1.1) were presented. Figure 4 showed the stability of the non-negative constant solutions for different b and β in ODE system (1.2). Figure 5 presented that the solution of system (1.1) converges to the equilibrium solution (u_1^*, v_1^*) for $-1 < \beta < \beta^*$, and it showes that the system (1.1) exhibits the nonconstant positive solution for $\beta > \beta^*$. The results meet Theorem 2.2(i). In Figure 6, it can be observed that when t is large enough, the solutions of the system (1.1) tend to be the positive constant solution (u_1^*, v_1^*), which verifies the case of $\beta = 0$ in Theorem 2.2. This indicates that the Allee effect term may be a factor driving the periodic solution of the predator-prey system with only herd behavior term but no Allee effect term. In Figure 7, it can be observed that the Hopf bifurcation solution of the system at $\beta = \beta_0^H$ is stable, which is consistent with the result of Example 3.1(ii) and verifies Theorem 3.1. In Figure 5(d-f) and Figure 7, it can be clearly observed that the densities of predator and prey fluctuate, which is consistent with the actual relationship between predator and prey in the ecosystem.

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