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The Longtime Behavior of an SIR Epidemic Model with Free Boundaries*

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Abstract In [6] , the authors studied an SIR epidemic model with free boundary. They first proved the global existence and uniqueness of solution, and then gave the criteria for spreading and vanishing. They also obtained the longtime behavior for the case of vanishing. However, the longtime behavior when spreading happens remains open. In this short paper, we aim to solve this open problem.

Keywords SIR model, free boundary, longtime behavior

MSC(2010) 35K57, 35B40, 92D30.

1. Introduction

The classical SIR model has been received great attention because it can describe the development of an infectious disease. In such a model, the population is separated into three categories: susceptible, infectious and recovered individuals, denoted by S, I and R respectively. If we assume that the disease incubation period is negligible so that each susceptible individual becomes infectious and later recovers with a permanently acquired immunity, then the SIR model has the following form:

$$\begin{cases} S' = b - \beta SI - \mu_1 S, \\ I' = \beta SI - \gamma I - \mu_2 I, \\ R' = \gamma I - \mu_3 R, \end{cases} \tag{1.1}$$

where b is the constant recruitment rate, β stands for the constant effective contact rate, γ represents the recovery rate of the infectious, and $\mu_i (i = 1, 2, 3)$ are the death rates of S, I and R. It was shown in [5] that the number $\widetilde{R}_0 = \frac{\beta b}{(\mu_2+\gamma)\mu_1}$ is a threshold value for the long-time dynamical behaviour of (1.1): the epidemic will eventually die out if $\widetilde{R}_0 < 1$, and remain endemic if $\widetilde{R}_0 > 1$.

In (1.1), the spatial factor is ignored. Various types of the corresponding SIR model with spatial diffusion was considered by many researchers. For example, Wang and Wang [9] considered the traveling wave phenomena in a diffusive SIR Model; Yang et al. [8] studied traveling waves in a nonlocal dispersal SIR epidemic model; Enatsu et al. [4] investigated the traveling wave solution for a diffusive simple

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epidemic model with a free boundary; Djilali et al. [2] analysed the asymptotic behavior of SIR epidemic model with nonlocal diffusion and generalized nonlinear incidence functional, and so on. However, these works are not good enough to obtain an accurate estimate of the spreading front in the spreading of an epidemic. In order to better describe the spreading of the epidemic front, Kim et al. [6] followed the approach of [3] and investigated this problem via the free boundary version. If we further assume that $\mu_1 = \mu_2 = \mu_3 = k$, then the problem in [6] for one-dimensional case becomes the following free boundary model:

$$\begin{cases} S_t = dS_{xx} + b - \beta SI - kS, & t > 0,\ x \in \mathbb{R}, \\ I_t = dI_{xx} + \beta SI - \gamma I - kI, & t > 0,\ x \in (g(t), h(t)), \\ R_t = dR_{xx} + \gamma I - kR, & t > 0,\ x \in (g(t), h(t)), \\ I(t,x) = R(t,x) = 0, & t > 0,\ x \le g(t) \text{ or } x \ge h(t), \\ g'(t) = -\mu I_x(t, g(t)),\ h'(t) = -\mu I_x(t, h(t)), & t > 0, \\ S(0,x) = S_0(x), & x \in \mathbb{R}, \\ -g(0) = h(0) = h_0, & \\ I(0,x) = I_0(x),\ R(0,x) = R_0(x), & x \in [-h_0, h_0], \end{cases} \tag{1.2}$$

where $S_0(x), I_0(x)$ and $R_0(x)$ satisfy

$$\begin{aligned} & S_0(x) \in C^2(\mathbb{R}),\ I_0(x), R_0(x) \in C^2([-h_0, h_0]), \\ & I_0(x) = R_0(x) = 0,\ x \in \mathbb{R} \backslash (-h_0, h_0), \\ & S_0(x) > 0, x \in \mathbb{R},\ I_0(x) > 0, R_0(x) > 0, x \in (-h_0, h_0). \end{aligned} \tag{1.3}$$

Denote

$$R_0 = \frac{\beta b}{(k+\gamma)k}.$$

It follows from [6] that the solution of (1.2) exists and is unique for all $t > 0$, and the following criteria hold:

(i) if $R_0 < 1$, then the disease will vanish;

(ii) if $R_0 > 1$, then the disease will vanish for sufficiently small h_0 and μ;

(iii) if $R_0 > 1$, then the disease will spread for suitably large h_0.

Furthermore, the results in [6] show that if $h_\infty - g_\infty < \infty$, then

$$\begin{aligned} & \lim_{t\to\infty} S(t,x) = b/k \text{ locally uniformly in } \mathbb{R}; \\ & \lim_{t\to\infty} \|I(t,x)\|_{C([g(t),h(t)])} = \lim_{t\to\infty} \|R(t,x)\|_{C([g(t),h(t)])} = 0. \end{aligned}$$

Just as pointed out by Kim et al. [6], the study of the asymptotic spreading speed when spreading happens is an interesting question. To answer this question, we should firstly show the longtime behavior of (1.2) when spreading happens. Motivated by [6], Ding et al. [1] considered the effect of relapse, while we [11] considered the nonlocal version of (1.2) in 2020, but both of these two works do not give the longtime behavior when spreading happens. In this short paper, we aim to solve this problem for the cases of local diffusion. We believe this method can be used to deal with the longtime behavior in [1, 11]. It is worth mentioning that Li et al. [7] proposed a new SIR epidemic model recently, and considered the dynamical properties. They obtained the longtime behavior of this model when the diseases spread.

2. Some auxiliary lemmas

We first consider

$$u'' + A - Bu = 0,\ x \in \mathbb{R}, \tag{2.1}$$

where A and B are positive constants. We can obtain the following lemma:

Lemma 2.1. *Let u be a non-negative and bounded solution of (2.1). Then $u(x) \equiv A/B$.*

Proof. By solving (2.1), we have

$$u(x) = c_1 e^{\sqrt{B}x} + c_2 e^{-\sqrt{B}x} + \frac{A}{B}.$$

If $u(x)$ is bounded, then $c_1 = c_2 = 0$. Thus $u(x) \equiv A/B$. □

In the following, we study the problem

$$\begin{cases} u'' + A - Bu = 0, & x \in (-l, l), \\ u(-l) = u(l) = K, \end{cases} \tag{2.2}$$

where K is a nonnegative constant.

Lemma 2.2. *The problem (2.2) has a unique positive solution $u_l(x)$, and*

$$\lim_{l\to\infty} u_l(x) = A/B \text{ locally uniformly in } \mathbb{R}.$$

Proof. By solving (2.2), we have

$$u(x) = \frac{K - A/B}{e^{\sqrt{B}l} + e^{-\sqrt{B}l}}(e^{\sqrt{B}x} + e^{-\sqrt{B}x}) + A/B,$$

then

$$\lim_{l\to\infty} u_l(x) = A/B \text{ locally uniformly in } \mathbb{R}.$$

□

For some $T > 0$ and $l > 0$, we next consider

$$\begin{cases} v_t = v_{xx} + A - Bv, & t > T, x \in (-l, l), \\ v(t, -l) = K, v(t, l) = K, & t > T, \\ v(T, x) = v_0(x), & x \in [-l, l]. \end{cases} \tag{2.3}$$

We first give the comparison principle.

Lemma 2.3. *If u and v satisfy*

$$\begin{cases} u'' + A - Bu \le 0 \le v'' + A - Bv, & x \in (-l, l), \\ u(-l) \ge v(-l),\ u(l) \ge v(l), \end{cases}$$

then $u(x) \ge v(x)$ for $x \in [-l, l]$.

Lemma 2.4. *If u and v satisfy*

$$\begin{cases} u_t - u_{xx} - A + Bu \ge 0 \ge v_t - v_{xx} - A + Bv, & t > T,\ x \in (-l, l), \\ u(t, -l) \ge v(t, -l),\ u(t, l) \ge v(t, l), & t > T, \\ u(T, x) \ge v(T, x), & x \in [-l, l], \end{cases}$$

then $u(t, x) \ge v(t, x)$ for $t \ge T$ and $x \in [-l, l]$.

Lemma 2.5. *For some $l > 0$, let $v_0(x) \in C([-l,l])$ and v be the solution of* (2.3). *Then we have*

$$\lim_{t\to\infty} v(t,x) = v_l(x),$$

where $v_l(x)$ is the unique positive solution of

$$\begin{cases} v'' + A - Bv = 0, & x \in (-l,l), \\ v(-l) = K, v(l) = K. \end{cases}$$

Moreover,

$$\lim_{l\to\infty} v_l(x) = A/B \text{ locally uniformly in } \mathbb{R}.$$

Proof. Since the comparison principle of (2.3) is true, we can use Lemma 2.1 and 2.2 to prove this lemma by the same arguments in [10, Proposition 8.1]. We omit the proof here. □

Lemma 2.6. *Let $w(t,x)$, $g(t)$ and $h(t)$ be the continuous functions. Suppose that u satisfies*

$$\begin{cases} u_t = u_{xx} + w(t,x) - bu, & t > 0, x \in (g(t), h(t)), \\ u(t, g(t)) \ge 0, u(t, h(t)) \ge 0, & t > 0, \\ u(0,x) = u_0(x), \ -g(0) = h(0) = h_0, & x \in [-h_0, h_0], \end{cases}$$

and there exists some constant $C > 0$ such that $u(t,x) \le C$ for $t > 0$ and $x \in [g(t), h(t)]$. If $\lim_{t\to\infty} h(t) = -\lim_{t\to\infty} g(t) = \infty$, then the following statements hold:

(i) *if $\liminf_{t\to\infty} w(t,x) \ge m$ locally uniformly in $\mathbb{R}$ for some $m > 0$, then*

$$\liminf_{t\to\infty} u(t,x) \ge \frac{m}{b} \text{ locally uniformly in } \mathbb{R};$$

(ii) *if $\limsup_{t\to\infty} w(t,x) \le M$ locally uniformly in $\mathbb{R}$ for some $M > 0$, then*

$$\limsup_{t\to\infty} u(t,x) \le \frac{M}{b} \text{ locally uniformly in } \mathbb{R};$$

(iii) *if $w(t,x) \equiv a$ for some constant $a > 0$, then*

$$\lim_{t\to\infty} u(t,x) = \frac{a}{b} \text{ locally uniformly in } \mathbb{R}.$$

Proof. (i) For any given $\varepsilon > 0$ and $L > 0$, there exists T depending on ε and L such that

$$w(t,x) \ge m - \varepsilon, \ g(t) < -L, \ h(t) > L \text{ for } t > T \text{ and } x \in (-L, L).$$

Then u satisfies

$$\begin{cases} u_t \ge u_{xx} + m - \varepsilon - bu, & t > T, x \in (-L, L), \\ u(t,-L) \ge 0, u(t,L) \ge 0, & t > T, \\ u(T,x) \ge 0, & x \in [-L, L]. \end{cases}$$

Let $v(t,x)$ be the solution of

$$\begin{cases} v_t = v_{xx} + m - \varepsilon - bv, & t > T, x \in (-L, L), \\ v(t,-L) = 0, v(t,L) = 0, & t > T, \\ v(T,x) = v_T(x), & x \in [-L, L], \end{cases}$$

where $v_T(-L) = v_T(L) = 0$ and $v_T(x) \le u(T,x)$ for $x \in [-L, L]$. By the comparison principle, we have

$$u(t,x) \ge v(t,x) \text{ for } t > T \text{ and } x \in [-L, L].$$

Let $v_L(x)$ be the solution of

$$\begin{cases} v'' + m - \varepsilon - bv = 0, & x \in (-L, L), \\ v(-L) = 0, v(L) = 0. \end{cases}$$

It follows from Lemma 2.5 that

$$\lim_{t\to\infty} v(t,x) = v_L(x) \text{ uniformly in } [-L, L],$$

and

$$\lim_{L\to\infty} v_L(x) = \frac{m-\varepsilon}{b} \text{ locally uniformly in } \mathbb{R}.$$

By the arbitrariness of ε, we have

$$\liminf_{t\to\infty} u(t,x) \ge \frac{m}{b} \text{ locally uniformly in } \mathbb{R}.$$

(ii) For any given $\varepsilon > 0$ and $L > 0$, there exists T depending on ε and L such that

$$w(t,x) \le M + \varepsilon,\ \ g(t) < -L,\ \ h(t) > L \text{ for } t > T \text{ and } x \in (-L, L).$$

Then u satisfies

$$\begin{cases} u_t \le u_{xx} + M + \varepsilon - bu, & t > T, x \in (-L, L), \\ u(t,-L) \le C, u(t,L) \le C, & t > T, \\ u(T,x) \le C, & x \in [-L, L]. \end{cases}$$

Applying the comparison principle, we have $u(t,x) \le v(t,x)$ for $t > T$ and $x \in (-L, L)$, where $v(t,x)$ is the solution of

$$\begin{cases} v_t = v_{xx} + M + \varepsilon - bv, & t > T, x \in (-L, L), \\ v(t,-L) = C, v(t,L) = C, & t > T, \\ v(T,x) = C, & x \in [-L, L]. \end{cases}$$

By the similar arguments in (i), we have

$$\limsup_{t\to\infty} u(t,x) \le \frac{M}{b} \text{ locally uniformly in } \mathbb{R}.$$

(iii) This result can be obtained from (i) and (ii) directly. □

3. Longtime behavior

Theorem 3.1. *Assume that $R_0 > 1$ and $\gamma < k$. If $h_\infty = -g_\infty = \infty$, then*

$$\lim_{t\to\infty}(S, I, R)(t, x) = (S^*, I^*, R^*) \text{ uniformly in any bounded subset of } \mathbb{R},$$

where $S^* = \frac{k+\gamma}{\beta}$, $I^* = \frac{b}{k+\gamma}(1 - \frac{1}{R_0}) = \frac{b}{k+\gamma} - \frac{k}{\beta}$ and $R^* = \frac{\gamma I^*}{k}$.

Proof. The theorem can be proved by the similar methods in [7, Theorem 3.9] and [10, Theorem 4.3]. We give the details as follows.

Claim 1. We first show that

$$\lim_{t\to\infty}(I, R)(t, x) = (I^*, R^*) \text{ locally uniformly in } \mathbb{R}.$$

Let $N = S + I + R$, then N satisfies

$$\begin{cases} N_t = dN_{xx} + b - kN, & t > 0, x \in (g(t), h(t)), \\ N(t, g(t)) \geq 0,\ \ N(t, h(t)) \geq 0, & t > 0, \\ N(0, x) = S_0(x) + I_0(x) + R_0(x),\ -g(0) = h(0) = h_0, & x \in [-h_0, h_0]. \end{cases}$$

It follows from Lemma 2.6 (iii) that

$$\lim_{t\to\infty} N(t, x) = \frac{k}{b} =: N^* \text{ locally uniformly in } \mathbb{R}.$$

We will prove Claim 1 by using the following iteration method.

Step 1. For any small $\varepsilon > 0$ and large L_1, there exists T_1 such that

$$N(t, x) \leq N^* + \varepsilon \text{ for } t > T_1 \text{ and } x \in [-L_1, L_1]$$

and $g(t) < -L_1$ and $h(t) > L_1$ for $t > T_1$ by $h_\infty = -g_\infty = \infty$. By the second equation of (1.2), we have, for $t > T_1$ and $x \in [-L_1, L_1]$,

$$\begin{aligned} I_t &= dI_{xx} + \beta(N - I - R)I - \gamma I - kI \\ &\leq dI_{xx} + \beta(N^* + \varepsilon - I)I - \gamma I - kI \\ &= dI_{xx} + [\beta(N^* + \varepsilon) - \gamma - k - \beta I]I. \end{aligned}$$

It follows from [10, Proposition 8.1] that

$$\limsup_{t\to\infty} I(t, x) \leq \frac{\beta(N^* + \varepsilon) - \gamma - k}{\beta} \text{ for } x \in [-L_1, L_1].$$

By the arbitrariness of ε and L_1, we have

$$\limsup_{t\to\infty} I(t, x) \leq \frac{\beta N^* - \gamma - k}{\beta} = \frac{b}{k} - \frac{\gamma + k}{\beta} = \left(\frac{b}{\gamma + k} - \frac{k}{\beta}\right)\frac{\gamma + k}{k} = I^*\frac{\gamma + k}{k} = \overline{I}_1$$

locally uniformly in $\mathbb{R}$.

Step 2. By the third equation of (1.2), it follows from Lemma 2.6 (ii) that

$$\limsup_{t\to\infty} R(t, x) \leq \frac{\gamma\overline{I}_1}{k} = \frac{\gamma}{k}I^*\frac{\gamma + k}{k} = \overline{R}_1 \text{ locally uniformly in } \mathbb{R}. \tag{3.1}$$

Step 3. For any small $\varepsilon > 0$ and large L_2, there exists T_2 such that

$$N(t,x) \geq N^* - \varepsilon \text{ for } t > T_2 \text{ and } x \in [-L_2, L_2]$$

and $g(t) < -L_2$ and $h(t) > L_2$ for $t > T_2$ by $h_\infty = -g_\infty = \infty$. Thanks to (3.1), for any ε_1, there exists $T_3 > T_2$ such that

$$R(t,x) \leq \overline{R}_1 + \varepsilon_1 \text{ for } t > T_3 \text{ and } x \in [-L_2, L_2].$$

By the second equation of (1.2), we have, for $t > T_3$ and $x \in [-L_2, L_2]$,

$$I_t \geq dI_{xx} + [\beta(N^* - \varepsilon) - \beta(\overline{R}_1 + \varepsilon_1) - \gamma - k - \beta I]I.$$

It follows from [10, Proposition 8.1] that

$$\liminf_{t\to\infty} I(t,x) \geq \frac{\beta(N^* - \varepsilon) - \beta(\overline{R}_1 + \varepsilon_1) - \gamma - k}{\beta} \text{ for } x \in [-L_2, L_2].$$

By the arbitrariness of ε, ε_1 and L_2, we have

$$\liminf_{t\to\infty} I(t,x) \geq \frac{\beta N^* - \beta\overline{R}_1 - \gamma - k}{\beta} = \overline{I}_1 - \overline{R}_1 = I^*\frac{\gamma + k}{k}(1 - \frac{\gamma}{k}) = \underline{I}_1$$

locally uniformly in $\mathbb{R}$.

Step 4. By the third equation of (1.2), it follows from Lemma 2.6 (i) that

$$\liminf_{t\to\infty} R(t,x) \geq \frac{\gamma\underline{I}_1}{k} = \frac{\gamma}{k}I^*\frac{\gamma + k}{k}(1 - \frac{\gamma}{k}) = \underline{R}_1 \text{ locally uniformly in } \mathbb{R}.$$

Step 5. By the similar arguments in Step 3, we have

$$\limsup_{t\to\infty} I(t,x) \leq \frac{\beta N^* - \gamma - k - \beta\underline{R}_1}{\beta} = \overline{I}_1 - \underline{R}_1 = I^*\frac{\gamma + k}{k}[1 - \frac{\gamma}{k} + (\frac{\gamma}{k})^2] = \overline{I}_2$$

locally uniformly in $\mathbb{R}$.

Step 6. By the third equation of (1.2), it follows from Lemma 2.6 (ii) that

$$\limsup_{t\to\infty} R(t,x) \leq \frac{\gamma\overline{I}_2}{k} = \frac{\gamma}{k}I^*\frac{\gamma + k}{k}[1 - \frac{\gamma}{k} + (\frac{\gamma}{k})^2] = \overline{R}_2$$

locally uniformly in $\mathbb{R}$.

Step 7. By the similar arguments in Step 3, we have

$$\liminf_{t\to\infty} I(t,x) \geq \frac{\beta N^* - \gamma - k - \beta\overline{R}_2}{\beta} = \overline{I}_1 - \overline{R}_2 = I^*\frac{\gamma + k}{k}[1 - \frac{\gamma}{k} + (\frac{\gamma}{k})^2 - (\frac{\gamma}{k})^3] = \underline{I}_2$$

locally uniformly in $\mathbb{R}$.

Step 8. By the third equation of (1.2), it follows from Lemma 2.6 (i) that

$$\liminf_{t\to\infty} R(t,x) \geq \frac{\gamma\underline{I}_2}{k} = \frac{\gamma}{k}I^*\frac{\gamma + k}{k}[1 - \frac{\gamma}{k} + (\frac{\gamma}{k})^2 - (\frac{\gamma}{k})^3] = \underline{R}_2$$

locally uniformly in $\mathbb{R}$.

Repeating the above arguments, we have

$$\underline{I}_i = I^* \frac{\gamma + k}{k} \cdot \frac{1 - (-\gamma/k)^{2i}}{1 - (-\gamma/k)}, \quad \overline{I}_i = I^* \frac{\gamma + k}{k} \cdot \frac{1 - (-\gamma/k)^{2i-1}}{1 - (-\gamma/k)},$$
$$\underline{R}_i = \frac{\gamma}{k} I^* \frac{\gamma + k}{k} \cdot \frac{1 - (-\gamma/k)^{2i}}{1 - (-\gamma/k)}, \quad \overline{R}_i = \frac{\gamma}{k} I^* \frac{\gamma + k}{k} \cdot \frac{1 - (-\gamma/k)^{2i-1}}{1 - (-\gamma/k)},$$

and

$$\underline{I}_1 \le \underline{I}_2 \le \cdots \le I \le \cdots \le \overline{I}_2 \le \overline{I}_1,$$
$$\underline{R}_1 \le \underline{R}_2 \le \cdots \le R \le \cdots \le \overline{R}_2 \le \overline{R}_1.$$

If $\gamma < k$, then

$$\lim_{i \to \infty} \underline{I}_i = \lim_{i \to \infty} \overline{I}_i = I^*, \quad \lim_{i \to \infty} \underline{R}_i = \lim_{i \to \infty} \overline{R}_i = R^*.$$

Therefore,

$$\lim_{t \to \infty} (I, R)(t, x) = (I^*, R^*).$$

Claim 2. Then we show that

$$\lim_{t \to \infty} S(t, x) = S^* \text{ locally uniformly in } \mathbb{R}.$$

By Claim 1, we have

$$I^* - \varepsilon \le \liminf_{t \to \infty} I(t, x) \le \liminf_{t \to \infty} I(t, x) \le I^* + \varepsilon \text{ locally uniformly in } \mathbb{R}.$$

Applying the similar arguments in Lemma 2.6, we can prove Claim 2. Here we omit it. □

4. Discussion

In this paper, we give the longtime behavior of (1.2). We will obtain the asymptotic spreading speed when spreading happens in another paper. Particularly, we assume that $\mu_i = k (i = 1, 2, 3)$ in this paper. If this condition is not true, the argument in this paper will not be suitable.

References

[1] Q. Ding, Y. Liu, Y. Chen and Z. Guo, *Dynamics of a reaction-diffusion SIRI model with relapse and free boundary*, Mathematical Biosciences and Engineering, 2020, 17(2), 1659–1676.

[2] S. Djilali, Y. Chen and S. Bentout, *Asymptotic analysis of SIR epidemic model with nonlocal diffusion and generalized nonlinear incidence functional*, Mathematical Methods in the Applied Sciences, 2023, 46(5), 6279–6301.

[3] Y. Du and Z. Lin, *Spreading-vanishing dichotomy in the diffusive logistic model with a free boundary*, SIAM Journal on Mathematical Analysis, 2010, 42(1), 377–405. Erratum: SIAM Journal on Mathematical Analysis, 2013, 45(3), 1995–1996.

[4] Y. Enatsu, E. Ishiwata and T. Ushijima, *Traveling wave solution for a diffusive simple epidemic model with a free boundary*, Discrete and Continuous Dynamical Systems. Series S, 2021, 14(3), 835–850.

[5] H.W. Hethcote, *Qualitative analyses of communicable disease models*, Mathematical Biosciences, 1976, 28(3-4), 335–356.

[6] K.I. Kim, Z. Lin and Q. Zhang, *An SIR epidemic model with free boundary*, Nonlinear Analysis: Real World Applications, 2013, 14(5), 1992–2001.

[7] L. Li, W. Ni and M. Wang, *Dynamical properties of a new SIR epidemic model*, Discrete and Continuous Dynamical Systems. Series S, 2024, 17(2), 690-707.

[8] F.Y. Yang, W.T. Li and Z.C. Wang, *Traveling waves in a nonlocal dispersal SIR epidemic model*, Nonlinear Analysis: Real World Applications, 2015, 23, 129–147.

[9] H. Wang and X.S. Wang, *Traveling wave phenomena in a Kermack-McKendrick SIR model*, Journal of Dynamics and Differential Equations, 2016, 28(1), 143–166.

[10] M. Wang and J. Zhao, *A free boundary problem for the predator-prey model with double free boundaries*, Journal of Dynamics and Differential Equations, 2017, 29(3), 957–979.

[11] M. Zhao, W.T. Li and J.F. Cao, *Dynamics for an SIR epidemic model with nonlocal diffusion and free boundaries*, Acta Mathematica Scientia. Series B. English Edition, 2021, 41(4), 1081–1106.