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Optimal Control of a Fractional-Order New Psychoactive Substance Model*

Xinxin Wang[1], Xiaoyun Wang [1,†]

Abstract In this paper, we develop a fractional-order model of new psychoactive substance (NPS) transmission. We first prove the non-negativity and boundedness of the model. Then, prevention function and treatment function are introduced into the model to establish the fractional-order optimal control model. The existence of the optimal control pair is proved, and the solution of the model is obtained. Finally, the numerical simulation of the optimal control model is carried out. The results show that the fractional model is helpful for us to analyze the NPS dynamics model more deeply. Through the analysis of three control strategies, the optimal control strategy is the combination of prevention and treatment measures. In addition, it is worth noting that this paper obtains a new insight that prevention is more effective than treatment in the early stage of NPS transmission, and the control strategy at this time is to invest in maximum prevention measures and moderate treatment measures. However, when the spread of NPS shows an epidemic trend, the influence of preventive measures is greatly reduced, and the control strategy at this time is to focus on increasing the investigation of addicts and improving the treatment rate of psychological addicts.

Keywords New psychoactive substances, fractional order model, optimal control, numerical simulation

MSC(2010) 34A08, 34H15.

1. Introduction

Over the past decade, new psychoactive substances have become a global phenomenon [1]. According to the United Nations Office on Drugs and Crime (UNODC), new psychoactive substances (NPS) are defined as substances that are not controlled by international drug control conventions but are subject to abuse and pose a threat to public health. These substances, which are generally obtained by modifying the chemical structure of existing drugs, not only have narcotic, excitatory or hallucinogenic effects similar to those of listed drugs, but also can evade legal control. As a result, the production, trafficking and abuse of these substances are becoming increasingly serious [2]. As of December 2021, the total number of new psychoactive substances reported by national authorities and forensic laboratories

†the corresponding author.
Email address:xywang0708@126.com(Xiaoyun. Wang)
[1]College of Mathematics, Taiyuan University of Technology, Taiyuan , 030024 Shanxi, China
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was 1,127, which is more than three times the 302 psychoactive substances under international control at the end of 2021 [3] (Fig.1).

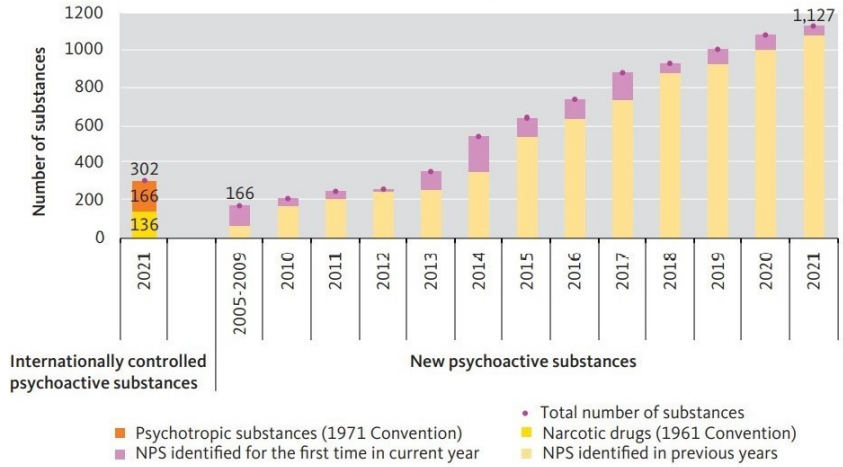


Figure 1. [3] The number of internationally controlled drugs in 2021, and the number of new psychoactive substances identified at the global level, 2005-2021 (cumulative figures)

These new psychoactive substances can not only seriously endanger the physical health of users, infecting them with various infectious diseases and even jeopardizing their lives, but they can also damage their nervous system, which in turn leads to psychotic symptoms [4, 5]. More seriously, NPS directly affects the central nervous system, exhibiting greater dependence and infectiousness [6–8]. Studies have shown that the abuse of these substances leads to seven times more violent crimes than controlled substances [9]. In conclusion, new psychoactive substances pose a major threat to global public health [10].

Existing papers have shown that mathematical models can not only better explain drug transmission patterns but also provide predictive tools for the behavior of various types of drug users. White and Comiskey proposed a model in 2007 that divided drug users into two groups: those who are not in treatment (U_1) and those who are in treatment (U_2). Through the sensitivity analysis of the model, the stability of the system and the conditions for the existence of backward branches are proved. At the end of the article, a key result is that prevention is better than cure [11]. G. P. Samanta analyzed the heroin model with distributed time delay by improving the White and Comiskey heroin model [12]. In 2015, F. Nyabadza and J. Mushanyu studied two types of rehabilitation trends: inpatient rehabilitation and outpatient rehabilitation [13]. In 2017, Mingju Ma et al. proposed a new synthetic drug model that includes both psychological and physiological addicts. By analyzing the model, it can be concluded that controlling the spread of synthetic drugs is more effective than treating addicts [14]. Pengyan Liu established a model of synthetic drug transmission by dividing susceptible people into those with a history of drug use and those without [15].

Given limited resources, policy makers must consider minimizing overall costs while controlling the spread of disease. The optimal control system has been studied in many fields and achieved remarkable results. Neilan et al. (2010) proposed a SEIR epidemic model that applied optimal control theory to disease modeling.

An optimal control problem with the objective of minimizing the number of infections and the cost of vaccination was developed by using the vaccination rate as the control variable [16]. Alfred Hugo et al. established the optimal control model of Newcastle disease in Tanzania, analyzed the model using the Pontryagin maximum principle, and proposed three control strategies. According to an incremental cost-benefit analysis, the best strategy to control ND is to combine fake vaccination and human education campaign strategies [17]. S. Sangeeta and G. P. Samanta (2019) proposed a synthetic drug transmission model with Holling Type-11 function and formulated an optimal control scheme. In this article, the importance of health education is emphasized, pointing out that schools and families should take action to raise the awareness of the young generation. Finally, the numerical simulation shows that counseling treatment can minimize the cost of treatment and minimize the number of drug users [18]. In 2004, Agarwal extended the classical control theory to the fractional dynamic system for the first time and provided the general formula and solution of the fractional optimal control system problem [19]. Yongsheng Ding established the optimal control of the fractional HIV immune system in 2011. Numerical simulation results show that the fractional optimal control scheme can improve the processing quality [20]. Meghadri Das (2021) studied a fractional synthetic drug transmission model considering memory effects. The analysis of the model concluded that controlling the spread of drugs is more effective than treating addicts. Based on this conclusion, the authors take counseling and publicity activities as control variables and established an optimal control system corresponding to the synthetic drug transmission model [21].

Fractional derivatives, as a generalization of integer derivatives, have been used to study the properties of various diseases. Because fractional systems have memory effect, it provides an excellent tool for describing the genetic properties of various systems [22]. The fractional model can analyze the transmission of NPS more deeply, so we have developed a fractional-order NPS transmission model. The structure of the paper is as follows. In section 2, the model is established. In Section 3, some definitions of fractional order equations are reviewed and the nonnegativity and boundedness of the solution are systematically discussed. In Section 4, the optimal control system corresponding to the new psychoactive substance transmission model is studied and numerical simulations of the optimal control system are performed. Finally, a summary and discussion are given.

2. Mathematical model

2.1. NPS transmission model

We divide the total population into seven compartments: high-risk susceptible individuals (S_1), low-risk susceptible individuals (S_2), psychological addicts (I_1), physiological addicts (I_2), addicts treated in the community (T_1), addicts treated in compulsory detoxification centers (T_2) and permanently detoxified individuals (R), then propose a fractional-order new psychoactive substances (NPS) transmission model with anti-drug education and media coverage. The model equations are as

follows

$$
\begin{aligned}
&{}_{t_0}^{C}D_t^{\varepsilon}S_1(t) = \Lambda^{\varepsilon} - \beta_1^{\varepsilon}S_1I_1 - \beta_2^{\varepsilon}S_1I_2 - b^{\varepsilon}S_1 - \mu^{\varepsilon}S_1,\\
&{}_{t_0}^{C}D_t^{\varepsilon}S_2(t) = b^{\varepsilon}S_1 - \beta_3^{\varepsilon}S_2I_1 - \beta_4^{\varepsilon}S_2I_2 - \mu^{\varepsilon}S_2,\\
&{}_{t_0}^{C}D_t^{\varepsilon}I_1(t) = \left(\beta_1^{\varepsilon}S_1 + \beta_3^{\varepsilon}S_2\right)I_1 + \left(\beta_2^{\varepsilon}S_1 + \beta_4^{\varepsilon}S_2\right)I_2 + \eta^{\varepsilon}T_2 - \left(\alpha^{\varepsilon} + \gamma^{\varepsilon} + \mu^{\varepsilon}\right)I_1,\\
&{}_{t_0}^{C}D_t^{\varepsilon}I_2(t) = \gamma^{\varepsilon}I_1 - \left(k_1^{\varepsilon} + k_2^{\varepsilon} + \mu^{\varepsilon}\right)I_2,\\
&{}_{t_0}^{C}D_t^{\varepsilon}T_1(t) = \alpha^{\varepsilon}I_1 + k_1^{\varepsilon}I_2 - \left(\rho^{\varepsilon} + \mu^{\varepsilon} + \delta_1^{\varepsilon}\right)T_1,\\
&{}_{t_0}^{C}D_t^{\varepsilon}T_2(t) = \rho^{\varepsilon}T_1 + k_2^{\varepsilon}I_2 - \left(\eta^{\varepsilon} + \mu^{\varepsilon} + \delta_2^{\varepsilon}\right)T_2,\\
&{}_{t_0}^{C}D_t^{\varepsilon}R(t) = \delta_1^{\varepsilon}T_1 + \delta_2^{\varepsilon}T_2 - \mu^{\varepsilon}R,
\end{aligned}
\tag{2.1}
$$

where $0 < \varepsilon < 1$, and ${}_{t_0}^{C}D_t^{\varepsilon}$ is the notation due to Caputo fractional derivative. We assumed that proportion b of high-risk susceptible individuals would consciously avoid exposure to NPS after receiving education and media reports on the dangers of NPS, hence in turn become low-risk susceptible individuals. The susceptible individuals will first become psychologically addicted when they come into contact with a new psychoactive substance addict. If psychological addicts are caught, then they will be sent to the community for treatment. Otherwise, if the psychological addicts continue to use, they will generate physical dependence and become physiological addicts, assuming that the effective contact rates of S_1, S_2, and I_1, I_2 are β_1, β_2, β_3, β_4, respectively. Two treatment models are introduced in the model, and if community-based treatment fails, the addict will be sent to a compulsory detoxification center for treatment. The duration of community treatment is typically three years, and the duration of compulsory detoxification treatment is typically two years. According to the director of the compulsory drug treatment center, after the compulsory detoxification treatment, there will be no more physical dependence. The failure of the compulsory detoxification treatment is a failure in the psychological sense, which means that they become psychological addicts again. We define a drug addict's relapse within 2 years of leaving a compulsory detoxification center as a failure of detoxification, or permanent detoxification if the detoxification period exceeds 2 years. The new psychoactive substances (NPS) transmission model diagram is shown in Fig.2.

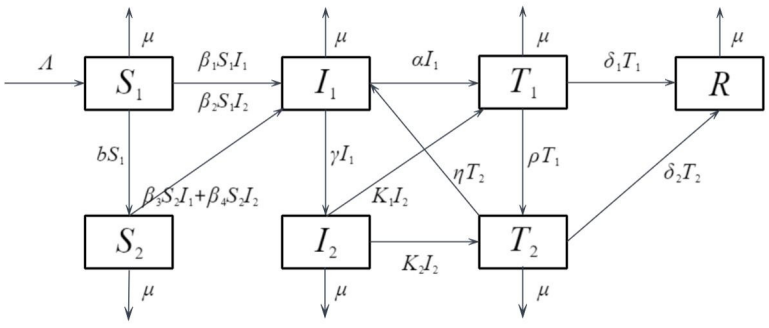


Figure 2. Compartmental diagram of new psychoactive substances (NPS) transmission model

In general, high-risk susceptible individuals are more likely to be infected than low-risk susceptible individuals, and physiological addicts have higher levels of ad-

diction and harmfulness than psychological addicts. Form this, $\beta_3 < \beta_1$, $\beta_4 < \beta_2$, $\beta_3 < \beta_4$, $\beta_1 < \beta_2$ and then we assume that $\beta_1 = \beta$, $\beta_2 = q\beta$, $\beta_3 = \xi\beta$, $\beta_4 = q\xi\beta$, where $q > 1, 0 < \xi < 1$. Considering $t_0 = 0$ and $\varepsilon = 1$, then we have the following system

$$\begin{aligned}
&{}_0^C D_t^\varepsilon S_1(t) = \Lambda - \beta S_1 I_1 - q\beta S_1 I_2 - bS_1 - \mu S_1,\\
&{}_0^C D_t^\varepsilon S_2(t) = bS_1 - \xi\beta S_2 I_1 - \xi q\beta S_2 I_2 - \mu S_2,\\
&{}_0^C D_t^\varepsilon I_1(t) = (\beta S_1 + \xi\beta S_2) I_1 + (q\beta S_1 + \xi q\beta S_2) I_2 + \eta T_2 - (\alpha + \gamma + \mu) I_1,\\
&{}_0^C D_t^\varepsilon I_2(t) = \gamma I_1 - (k_1 + k_2 + \mu) I_2,\\
&{}_0^C D_t^\varepsilon T_1(t) = \alpha I_1 + k_1 I_2 - (\rho + \mu + \delta_1) T_1,\\
&{}_0^C D_t^\varepsilon T_2(t) = \rho T_1 + k_2 I_2 - (\eta + \mu + \delta_2) T_2,\\
&{}_0^C D_t^\varepsilon R(t) = \delta_1 T_1 + \delta_2 T_2 - \mu R,
\end{aligned} \tag{2.2}$$

where $S_1(0) > 0$, $S_2(0) > 0$, $I_1(0) > 0$, $I_2(0) > 0$, $T_1(0) > 0$, $T_2(0) > 0$, $R(0) > 0$.

The definitions of variables and parameters are shown in Table.1.

Table 1. Definition of variables and parameters

Variable Parameter	Description
Λ	Inflow rate into high-risk susceptible individuals
b	Conversion rate of high-risk susceptible individuals to low-risk susceptible individuals
β_1	Effective contact rate between psychological addicts and high-risk susceptible individuals
β_2	Effective contact rate between physiological addicts and high-risk susceptible individuals
β_3	Effective contact rate between psychological addicts and low-risk susceptible individuals
β_4	Effective contact rate between physiological addicts and low-risk susceptible individuals
γ	Rate of psychological addicts who continue to take drugs and become physiological addicts
α	Rate of psychological addicts entering community treatment
k_1	Rate of physiological addicts entering community treatment
k_2	Rate of physiological addicts entering compulsory detoxification treatment
ρ	Conversion rate from community treatment to compulsory detoxification treatment
η	Relapse rate of compulsory detoxification treatment
δ_1	Success rate of community treatment
δ_2	Success rate of compulsory detoxification treatment
μ	Natural death rate

2.2. Preliminaries

Definition 2.1. ([23,24]) The fractional order derivative in the Caputo case with order α for a function $f \in C^n$ is defined as

$${}_{t_0}^C D_t^\varepsilon f(t) = \begin{cases} \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}} dx, n-1 < \alpha < n,\\ \frac{d^n}{dt^n} f(t), \alpha = n, \end{cases}$$

where $\Gamma(\cdot)$ is the Gamma function, $n \in \mathbb{N}$. In particular, for $\alpha \in (0,1)$

$$ {}_{t_0}^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} \frac{f'(x)}{(t-x)^{\alpha}} dx. $$

Lemma 2.1 (Lemma 1, [25]). *Let $0 < \xi < 1$, $f(t) \in C[a,b]$ and ${}_0^{C}D_t^{\varepsilon} f(t) \in C[a,b]$, then we have*

$$ f(t) = f(a) + \frac{1}{\Gamma(\eta)} {}_0^{C}D_t^{\varepsilon} f(\xi)(t-a)^{\epsilon}, $$

where $0 \le \xi \le t, \quad \forall t \in [a,b]$.

Remark 2.1. If $f(t) \in C[a,b]$ and ${}_0^{C}D_t^{\varepsilon} f(t) \in C[a,b]$ for $0 < \epsilon \le 1$. It is clear that if ${}_0^{C}D_t^{\varepsilon} f(t) \ge 0$ (${}_0^{C}D_t^{\varepsilon} f(t) \le 0$), $t \in (a,b)$, then $f(t)$ is non-decreasing (non-increasing) for all $t \in [a,b]$.

3. Basic properties

3.1. Non-negativity

Theorem 3.1. *The solutions $X(t) = (S_1, S_2, I_1, I_2, T_1, T_2, R)$ of system (2) that start in $\mathbb{R}_+^7$ are non-negative for all $t > 0$.*

Proof. From system (2), we have

$$ \begin{aligned} &{}_0^{C}D_t^{\varepsilon} S_1(t) \mid_{S_1(t)=0} = \Lambda \ge 0, \\ &{}_0^{C}D_t^{\varepsilon} S_2(t) \mid_{S_2(t)=0} = bS_1 \ge 0, \\ &{}_0^{C}D_t^{\varepsilon} I_1(t) \mid_{I_1(t)=0} = q\beta S_1 I_2 + \xi q\beta S_2 I_2 + \eta T_2 \ge 0, \\ &{}_0^{C}D_t^{\varepsilon} I_2(t) \mid_{I_2(t)=0} = \gamma I_1 \ge 0, \\ &{}_0^{C}D_t^{\varepsilon} T_1(t) \mid_{T_1(t)=0} = \alpha I_1 + k_1 I_2 \ge 0, \\ &{}_0^{C}D_t^{\varepsilon} T_2(t) \mid_{T_2(t)=0} = \rho T_1 + k_2 I_2 \ge 0, \\ &{}_0^{C}D_t^{\varepsilon} R(t) \mid_{R(t)=0} = \delta_1 T_1 + \delta_2 T_2 \ge 0. \end{aligned} \tag{3.1} $$

According to (3) and Remark 1, the solutions $S_1(t)$, $S_2(t)$, $I_1(t)$, $I_2(t)$, $T_1(t)$, $T_2(t)$, $R(t)$ are non-decreasing and can not cross the hyperplanes of $S_1 = 0$, $S_2 = 0$, $I_1 = 0$, $I_2 = 0$, $T_1 = 0$, $T_2 = 0$ and $R = 0$. Therefore, all solutions of system (2) are non-negative for all $t > 0$. □

3.2. Boundedness

Theorem 3.2. *All the solutions $X(t) = (S_1, S_2, I_1, I_2, T_1, T_2, R)$ of system (2) are bounded.*

Proof. Let $N(t) = S_1(t) + S_2(t) + I_1(t) + I_2(t) + T_1(t) + T_2(t) + R(t)$. Then

$$ {}_0^{C}D_t^{\varepsilon} N(t) = \Lambda - \mu N(t). $$

Applying Laplace transformation, we have

$$ N(s) \le \frac{\frac{\Lambda}{s} + N(0)s^{\varepsilon-1}}{s^{\varepsilon} + \mu}. $$

Taking inverse Laplace transformation,

$$\begin{aligned} N(t) &\leq \frac{\Lambda}{\mu}[1 - E_\varepsilon(-\mu t^\varepsilon)] + N(0)E_\varepsilon(-\mu t^\varepsilon) \\ &\leq M[E_{\varepsilon,1}(-\mu t^\varepsilon) + \mu t^\varepsilon E_{\varepsilon,\varepsilon+1}(-\mu t^\varepsilon)] = \frac{M}{\Gamma(1)} = M, \end{aligned}$$

where $M = \max\{\frac{\Lambda}{\mu}, N(0)\}$. Thus the solutions of the model (2) are bounded.

□

4. Optimal control problem

Most of the previous studies used parametric control strategies to control drug transmission but did not consider the control cost and time dependence, resulting in the lack of thoroughness and accuracy in the study of control measures. Our aim is to combine prevention and treatment measures investing proportionately in the different stages of NPS transmission, to maximize the control of the number of NPS users while minimizing the cost. Therefore, we construct the fractional-order optimal control model corresponding to the system (2), and introduce two control variables into the model: prevention function $u_1(t)$ and treatment function $u_2(t)$. The prevention function $u_1(t)$ represents the efforts to increase awareness of susceptible individuals through media reports, public education, etc. The treatment function $u_2(t)$ represents the efforts to increase the proportion of psychological addicts entering treatment at time t. In addition, we mainly study the number of NPS addicts under control, and considering that the last equation of system (2) does not affect the results of the remaining equations, we establish a six-dimensional fractional order optimal control model as follows

$$\begin{aligned} &{}_0^C D_t^\varepsilon S_1(t) = \Lambda - (1 - u_1(t))\beta S_1(I_1 + qI_2) - bS_1 - \mu S_1, \\ &{}_0^C D_t^\varepsilon S_2(t) = bS_1 - (1 - u_1(t))\xi\beta S_2(I_1 + qI_2) - \mu S_2, \\ &{}_0^C D_t^\varepsilon I_1(t) = (\beta S_1 + \xi\beta S_2)\, I_1 + (q\beta S_1 + \xi q\beta S_2)\, I_2 + \eta T_2 - (\alpha + \gamma + \mu)\, I_1 - u_2(t)I_1, \\ &{}_0^C D_t^\varepsilon I_2(t) = \gamma I_1 - (k_1 + k_2 + \mu)\, I_2, \\ &{}_0^C D_t^\varepsilon T_1(t) = \alpha I_1 + k_1 I_2 + u_2(t)I_1 - (\rho + \mu + \delta_1)\, T_1, \\ &{}_0^C D_t^\varepsilon T_2(t) = \rho T_1 + k_2 I_2 - (\eta + \mu + \delta_2)\, T_2, \end{aligned} \tag{4.1}$$

where $S_1(0) > 0$, $S_2(0) > 0$, $I_1(0) > 0$, $I_2(0) > 0$, $T_1(0) > 0$, $T_2(0) > 0$.

The objective function of the control model (4) is as follows

$$J(u_1(t), u_2(t)) = \int_0^{t_f} [A_1 I_1 + A_2 u_1^2(t) + A_3 u_2^2(t)]dt,$$

where A_1, A_2 and A_3 represent the balancing cost factors related to the size and importance of the three segments of the objective function. We assume that the cost caused by prevention and treatment measures is nonlinear due to the diversity of implementable measures, and takes a quadratic relationship [26, 27], where $A_1 I_1$ represents the human, financial, and other costs caused by psychological addicts; $A_2 u_1^2(t)$ represents the cost of intervention measures such as public education, family

education and media publicity; $A_3u_2^2(t)$ represents the cost of treatment measures such as psychological counseling, community treatment and drug withdrawal. In summary, to reduce the number of NPS users while keeping costs very low, the goal is to find an optimal control pair $u^* = (u_1^*, u_2^*)$ to make

$$J(u_1^*, u_2^*) = \min\{J(u_1(t), u_2(t)) \mid u_1, u_2 \in U\},$$

where $U = \{u(t) \mid u_1(t) \in [0,1],\ u_2(t) \in [0,1],\ t \in [0, t_f]\}$, $u_1(t),\ u_2(t)$ are measurable, t_f is the final time.

4.1. Existence of Optimal Control Pair

Theorem 4.1. *There exists an optimal control pair $u^* = (u_1^*, u_2^*)$ in U, and corresponding solution $S_1^*,\ S_2^*,\ I_1^*,\ I_2^*,\ T_1^*,\ T_2^*$ to the state initial value (4) such that $J(u_1^*, u_2^*) = min\{J(u_1(t), u_2(t))\}$, $u_1(t),\ u_2(t)$ are measurable and bounded.*

Proof. To prove the existence of an optimal control , the following conditions must be satisfied.

(a) The set of solutions to the system (4) are non-negative values.

(b) The control set U is convex and closed.

(c) Each right hand side of the state system is continuous, and can be written as a linear function of u with coefficients depending on time and the state.

(d) The integrand of the objective functional $J(u_1, u_2)$ is convex, and there exist c_1, c_2 such that the integrand of $J(u_1, u_2)$ is bounded below $c_1(u_1^2 + u_2^2)^2 - c_2$.

According to the biological significance, it is obvious that all the solutions of system (4) are non-negative, and the total population is:

$$N = S_1 + S_2 + I_1 + I_2 + T_1 + T_2, N(0) = S_1(0) + S_2(0) + I_1(0) + I_2(0) + T_1(0) + T_2(0),$$

it follows that: $\dfrac{dN}{dt} \le \Lambda - \mu N$, further $0 < N(t) \le N(0)e^{-\mu t} + \dfrac{\Lambda}{t}(1 - e^{-\mu t})$, with $t \to \infty, 0 \le N \le \dfrac{\Lambda}{\mu}$. Therefore, condition (a) and condition (b) are satisfied.

To prove conditions (c) and (d), the integrand of $J(u_1, u_2)$ is $g = A_1I_1 + A_2u_1^2(t) + A_3u_2^2(t) \ge \min\{A_2, A_3\}(u_1^2(t) + u_2^2(t)) - c_2$, letting $c_1 = \min\{A_2, A_3\},\ c_2 > 0$. Therefore, conditions (c) and (d) have been accomplished. □

4.2. The characterization of Optimal Control Pair

Theorem 4.2. *Given an optimal control pairs $u^* = (u_1^*, u_2^*)$ and state variables of $S_1^*,\ S_2^*,\ I_1^*,\ I_2^*,\ T_1^*,\ T_2^*$ that minimize the objective functional $J(u_1(t), u_2(t))$. Then there exist adjoint variables $\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_6$ satisfying*

$$\begin{aligned}
{}_0D_{t_f}^{\varepsilon}\lambda_1(t) &= \lambda_1(1 - u_1(t))\beta(I_1 + qI_2) + \lambda_1(b + \mu) - b\lambda_2,\\
{}_0D_{t_f}^{\varepsilon}\lambda_2(t) &= \lambda_2(1 - u_1(t))\xi\beta(I_1 + qI_2) + \mu\lambda_2,\\
{}_0D_{t_f}^{\varepsilon}\lambda_3(t) &= -A_1 + (1 - u_1(t))\beta S_1(\lambda_1 - \lambda_3) + (1 - u_1(t))\xi\beta S_2(\lambda_2 - \lambda_3)\\
&\quad + \lambda_3(\alpha + \gamma + \mu + u_2(t)) - \gamma\lambda_4 - \lambda_5(\alpha + u_2(t)),\\
{}_0D_{t_f}^{\varepsilon}\lambda_4(t) &= (1 - u_1(t))q\beta S_1(\lambda_1 - \lambda_3) + (1 - u_1(t))q\xi\beta S_2(\lambda_2 - \lambda_3)\\
&\quad + \lambda_4(k_1 + k_2 + \mu) - \lambda_5k_1 - \lambda_6k_2,\\
{}_0D_{t_f}^{\varepsilon}\lambda_5(t) &= \lambda_5(\rho + \mu + \delta_1) - \lambda_6\rho,\\
{}_0D_{t_f}^{\varepsilon}\lambda_6(t) &= \lambda_6(\eta + \mu + \delta_2) - \eta\lambda_3,
\end{aligned} \tag{4.2}$$

where $\lambda_i(t_f) = 0,\ i = 1, 2, 3, 4, 5, 6$. *Moreover,* $u^* = (u_1^*, u_2^*)$ *are given by*

$$u_1^* = min\{1, \max\{0, \frac{1}{2A_2}[\beta S_1(I_1 + qI_2)(\lambda_3 - \lambda_1) + \xi\beta S_2(I_1 + qI_2)(\lambda_3 - \lambda_2)]\}\},$$
$$u_2^* = min\{1, \max\{0, \frac{1}{2A_3}(\lambda_3 - \lambda_5)I_1\}\}.$$

Proof. The Hamiltonian function for the optimal control problem is defined as

$$\begin{aligned} H = {} & A_1 I_1 + A_2 u_1^2(t) + A_3 u_2^2(t) + \lambda_1[\Lambda - (1 - u_1(t))\beta S_1(I_1 + qI_2) - bS_1 - \mu S_1] \\ & + \lambda_2[bS_1 - (1 - u_1(t))\xi\beta S_2(I_1 + qI_2) - \mu S_2] \\ & + \lambda_3[(1 - u_1(t))[(\beta S_1 + \xi\beta S_2)\, I_1 + (q\beta S_1 + \xi q\beta S_2)\, I_2 + \eta T_2 \\ & - (\alpha + \gamma + \mu)\, I_1 - u_2(t) I_1] \\ & + \lambda_4[\gamma I_1 - (k_1 + k_2 + \mu)\, I_2] + \lambda_5[\alpha I_1 + k_1 I_2 + u_2(t) I_1 - (\rho + \mu + \delta_1)\, T_1] \\ & + \lambda_6[\rho T_1 + k_2 I_2 - (\eta + \mu + \delta_2)\, T_2], \end{aligned}$$

where $\lambda_i(t),\ i = 1, 2, 3, 4, 5, 6$ are the adjoint variable with $\lambda_i(t_f) = 0$, which satisfy

$$\begin{aligned} {}_0D_{t_f}^{\varepsilon}\lambda_1(t) &= -\frac{\partial H}{\partial S_1} = \lambda_1(1 - u_1(t))\beta(I_1 + qI_2) + \lambda_1(b + \mu) - b\lambda_2, \\ {}_0D_{t_f}^{\varepsilon}\lambda_2(t) &= -\frac{\partial H}{\partial S_2} = \lambda_2(1 - u_1(t))\xi\beta(I_1 + qI_2) + \mu\lambda_2, \\ {}_0D_{t_f}^{\varepsilon}\lambda_3(t) &= -\frac{\partial H}{\partial I_1} = -A_1 + (1 - u_1(t))\beta S_1(\lambda_1 - \lambda_3) + (1 - u_1(t))\xi\beta S_2(\lambda_2 - \lambda_3) \\ &\quad + \lambda_3(\alpha + \gamma + \mu + u_2(t)) - \gamma\lambda_4 - \lambda_5(\alpha + u_2(t)), \\ {}_0D_{t_f}^{\varepsilon}\lambda_4(t) &= -\frac{\partial H}{\partial I_2} = (1 - u_1(t))q\beta S_1(\lambda_1 - \lambda_3) + (1 - u_1(t))q\xi\beta S_2(\lambda_2 - \lambda_3) \\ &\quad + \lambda_4(k_1 + k_2 + \mu) - \lambda_5 k_1 - \lambda_6 k_2, \\ {}_0D_{t_f}^{\varepsilon}\lambda_5(t) &= -\frac{\partial H}{\partial T_1} = \lambda_5(\rho + \mu + \delta_1) - \lambda_6\rho, \\ {}_0D_{t_f}^{\varepsilon}\lambda_6(t) &= -\frac{\partial H}{\partial T_2} = \lambda_6(\eta + \mu + \delta_2) - \eta\lambda_3. \end{aligned} \tag{4.3}$$

Then, we have

$$\frac{\partial H}{\partial u_1} \mid u_1^* = 2A_2 u_1(t)(\beta S_1(I_1 + qI_2)(\lambda_1 - \lambda_3) + \xi\beta S_2(I_1 + qI_2)(\lambda_2 - \lambda_3)) = 0,$$
$$\frac{\partial H}{\partial u_2} \mid u_2^* = 2A_3 u_2(t) - \lambda_3 I_1 + \lambda_5 I_1 = 0.$$

Using the property of the control space, the solutions of u_1^* and u_2^* are

$$u_1^* = \min\{1, \max\{0, \frac{1}{2A_2}[\beta S_1(I_1 + qI_2)(\lambda_3 - \lambda_1) + \xi\beta S_2(I_1 + qI_2)(\lambda_3 - \lambda_2)]\}\},$$
$$u_2^* = \min\{1, \max\{0, \frac{1}{2A_3}(\lambda_3 - \lambda_5)I_1\}\}.$$

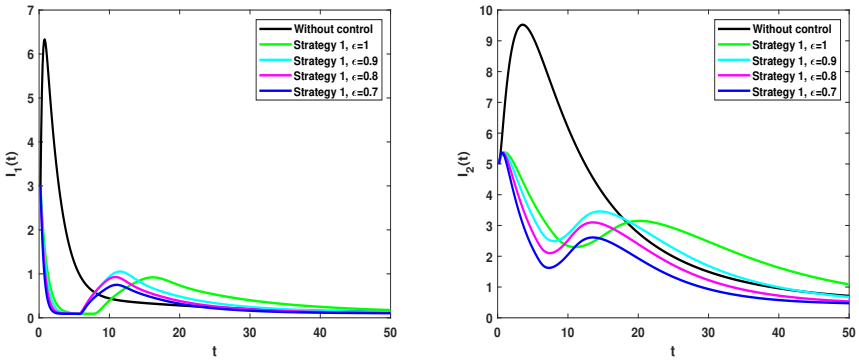
□

Table 2. Parameter values used in numerical simulations

Parameter	Value	Reference	Parameter	Value	Reference
Λ	0.1	[28]	k_2	0.1	[30]
μ	0.07	[28]	ρ	0.035	[30]
q	1.2	[29]	η	0.04	[30]
γ	0.55	[29]	δ_1	0.05	[30]
α	0.1	[29]	δ_2	0.1	[30]
k_1	0.02	[30]			

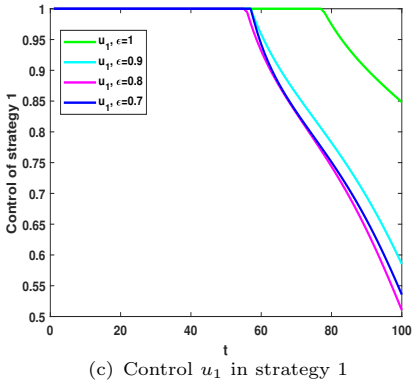
4.3. Numerical simulation of Optimal Control System

We now numerically simulate the optimal control problem and explain the effect of controls u_1 and u_2 on the spread of the new psychoactive substance (NPS). We use the Adams type predictor-corrector method [31] to obtain the optimal control pair and state variables, and use the forward iterative method to solve the state variables and the backward iterative method to solve the co-state equations. Next, we consider three strategies and perform numerical simulations to propose the best strategy for controlling NPS transmission.



(a) Number of psychological addicts

(b) Number of physiological addicts

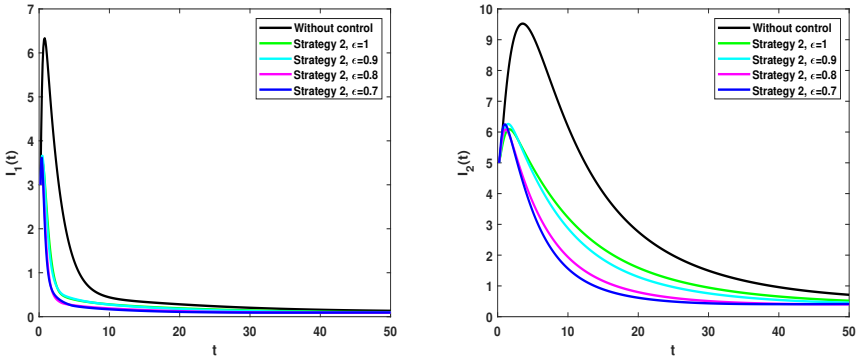


(c) Control u_1 in strategy 1

Figure 3. Strategy 1 (Effect of prevention measure and the fractional derivative order ε on the spread of NPS)

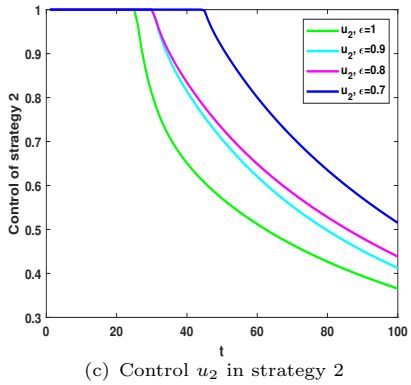
Strategy 1 (Prevention Strategy)

In this case, we consider the preventive measure only, i.e., $u_2 = 0$ and $u_1 \neq 0$. Using the parameter values in Table.2, the results are shown in Fig.3. As seen in Fig.3, the memory effect of the system increases as the fractional order derivative decreases from 1. It is seen that an oscillation occurs in the number of both psychological addicts and physiological addicts. After the implementation of preventive measures such as anti-drug education and media coverage, both I_1 and I_2 experienced a process of first decreasing, then increasing and finally decreasing to a stable level. This suggests that the maximum investment in prevention during the early stages of NPS transmission leads to a reduction in drug abuse behavior, but the effect of prevention is not unlimited. Over a long period of time, people will become accustomed to such information, and the influence of drug prevention will gradually decline, leading to a slight rebound in the number of addicts. As a whole, over time the number of both psychological addicts and physiological addicts will remain stable after converging to zero under the influence of control u_1.



(a) Number of psychological addicts

(b) Number of physiological addicts



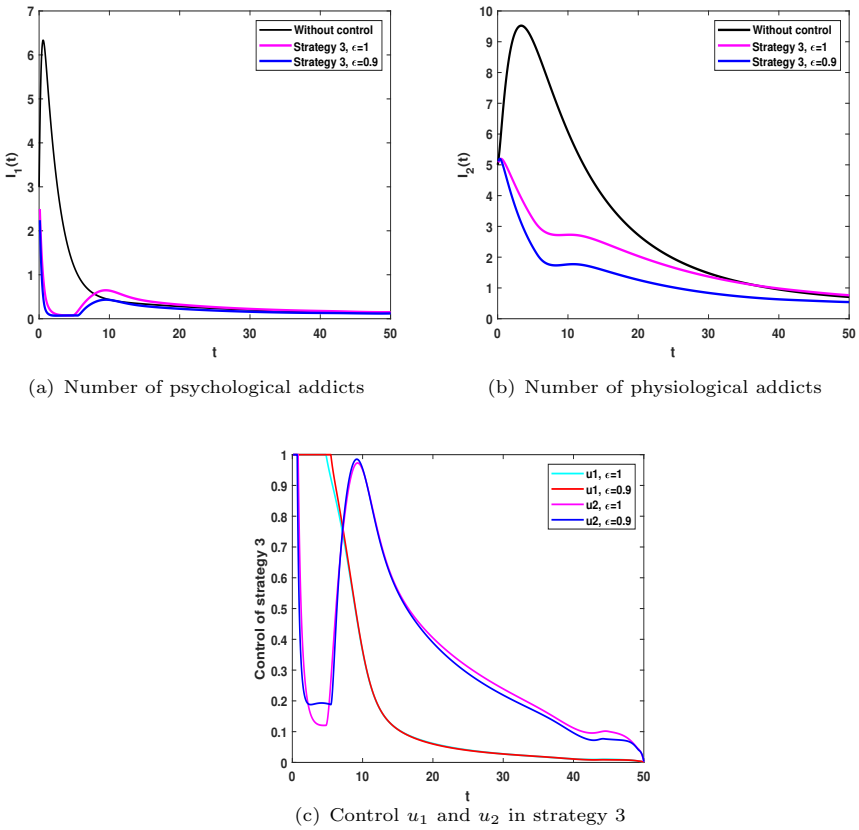
(c) Control u_2 in strategy 2

Figure 4. Strategy 2 (Effect of treatment measure and the fractional derivative order ε on the spread of NPS)

Strategy 2 (Treatment Strategy)

In this case, we consider the treatment measure only, i.e., $u_1 = 0$ and $u_2 \neq 0$.

Using the parameter values in Table.2, the results are shown in Fig.4. As can be seen from Fig.4, with the decrease of the fractional derivative, the number of addicts also decreases, which also verifies the role of memory effect. Unlike strategy 1, the number of addicts in strategy 2 initially increases, quickly reaches an extreme value, and then decreases to zero as treatment is implemented. This indicates that treatment measures have a short delay effect. It can be seen from Fig.3 and Fig.4 that the single implementation of prevention and treatment measures can control the spread of NPS, but the control effectiveness of both strategies has limitations and shortcomings.



(a) Number of psychological addicts

(b) Number of physiological addicts

(c) Control u_1 and u_2 in strategy 3

Figure 5. Strategy 3 (Effect of combining prevention and treatment measures on the spread of NPS)

Strategy 3 (Combination of prevention strategy and treatment strategy)

In this case, a combination of preventive and treatment measures was used to control the spread of NPS, i.e., $u_1 \neq 0$ and $u_2 \neq 0$. Using the parameter values in Table.2, the results are shown in Fig.5. As can be seen in Fig.5(c), the combination of the two measures has made the number of addicts stabilize after a rapid decrease. It is worth noting that, as seen in Fig.5, at the beginning, u_1 and u_2 are invested in the same proportion, and subsequently, the proportion of control u_2 decreases rapidly, which indicates that control u_1 plays a more important role in the early stage of NPS transmission. Over time, the control u_2 continues to rise and u_1

continues to fall, which suggests that treatment becomes more important as NPS spreads. All of the above results suggest that a large proportion of control u_1 and appropriate u_2 should be invested in the early stages of NPS transmission, that is, at which point it is more effective to control the effective contact rate of addicts and susceptible individuals than to treat them. With the tendency toward the spread of NPS transmission, the impact of preventive measures is substantially reduced and attention should be turned to increasing the investigation of NPS addicts and improving the treatment rate of addicts. In conclusion, combining preventive and treatment measures is the most effective strategy to control the spread of NPS. In addition, when considering the cost of control, the investment ratio of preventive and treatment measures should be adjusted according to the different stages of NPS transmission to maximize the reduction of NPS addicts while minimizing the cost.

5. Conclusions and discussion

Considering that fractional-order models provide a better understanding of epidemiological dynamics processes than integer-order models, we develop a fractional-order new psychoactive substance (NPS) transmission model, which divides the total population into seven categories: high-risk susceptible individuals (S_1), low-risk susceptible individuals (S_2), psychological addicts (I_1), physiological addicts (I_2), addicts treated in the community (T_1), addicts treated in compulsory detoxification centers (T_2) and permanently detoxified individuals (R). Firstly, the nonnegativity and boundedness of the model are proved. Next, we try to maximize the control of the number of addicts through anti-drug education, media coverage and surveys of NPS addicts. For this, we introduce prevention function and treatment function as control variables in the model, and establish a fractional-order optimal control model corresponding to the new psychoactive substance transmission model. Firstly, the necessary conditions for the optimal value of control measures are found by using Pontryagin's maximum principle. Then, the existence of the optimal control pair is proved, and the optimal control pair is obtained by the fractional-order optimal control principle. Finally, according to the optimal control model, three control strategies are proposed, and numerical simulation is carried out. Simulation results show that fractional derivatives can capture memory effects, which is helpful for further simulation and analysis of NPS transmission model. In addition, through the simulation of different strategies, the results show that the optimal control strategy is a combination of preventive measures and treatment measures, which minimizes the number of drug users while minimizing the cost. In addition, it is worth noting that we have gained new insights into the control of NPS transmission — investing in different proportions of preventive and treatment measures at different stages of NPS transmission can economically and effectively control the NPS epidemic. That is, in the early stage of NPS transmission, prevention is better than treatment. During this time we should invest in maximum preventive measures and moderate treatment measures, whereas, when NPS shows a tendency to spread, we should reduce the investment in preventive measures and focus on increasing the investment in treatment measures.

The highlight of this paper is that previous models of drug abuse have mostly considered heroin or synthetic drugs, but in today's society, the biggest threat to global public health has been shifted to new psychoactive substances. Our study first proposed a new psychoactive substance (NPS) transmission model and added

fractional order theory to the model. The fractional order model is better than the integer order model for understanding the NPS transmission epidemic, and it helps us to analyze the NPS dynamics behavior more deeply. Furthermore, we introduced a combination of prevention and treatment measures into the optimal control model to gain new insights into the control of NPS transmission. While previous models have concluded that prevention is better than treatment or a combination of prevention and treatment to control drug transmission, our model proposes corresponding optimal strategies based on the different stages of NPS transmission, which not only minimizes the number of addicts but also minimizes the cost. Our findings are expected to provide a powerful tool for controlling the spread of NPS. Due to the hidden nature of drug trade, we could not get the real data, which is also the shortcoming of our model, and we hope to get more real data and make more valuable contributions in the future.

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