Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 3, September 2024, 589–601 DOI:10.12150/jnma.2024.589

The Blow-up Dynamics for the L^2-Critical Hartree Equation with Harmonic Potential*

Mao Zhang[1], Jingjing Pan[2] and Jian Zhang[1,†]

Abstract In this paper, we study the L^2-critical Hartree equation with harmonic potential which arises in quantum theory of large system of nonrelativistic bosonic atoms and molecules. Firstly, by using the variational characteristic of the nonlinear elliptic equation and the Hamilton conservations, we get the sharp threshold for global existence and blow-up of the Cauchy problem. Then, in terms of a change of variables, we first find the relation between the Hartree equation with and without harmonic potential. Furthermore, we prove the upper bound of blow-up rate in $\mathbb{R}^3$ as well as the mass concentration of blow-up solution for the Hartree equation with harmonic potential in $\mathbb{R}^N$.

Keywords Hartree equation, harmonic potential, blow-up rate, upper bound, mass concentration

MSC(2010) 35Q55, 35B44, 35B40.

1. Introduction

In a non-relativistic setting where the number of boson is very large, the Hartree equation with harmonic potential is a model describing a quantum mechanical boson system, and arises from many-body quantum mechanics in a mean-field limit [7] and in Bose-Einstein condensate (BEC) with long range interaction [8,16]. A Bose condensate can be represented by a wave function that obeys the following Hartree equation with harmonic potential

$$iu_t + \Delta u - \omega^2 |x|^2 u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^p \right) |u|^{p-2} u = 0, \tag{1.1}$$

where $u = u(t,x) : [0,T) \times \mathbb{R}^N \to \mathbb{C}$ is a complex-valued wave function, $2 \le p < 2^*$ ($2^* = 1 + \frac{\gamma+2}{N-2}$ if $N \ge 3$ or $2^* = \infty$ if $N = 1,2$), N is the space dimension, $0 < \gamma < N$, $0 < T \le \infty$, $i = \sqrt{-1}$, Δ is the Laplace operator, $\omega > 0$ and $*$ denotes the convolution operator in $\mathbb{R}^N$.

When $\omega = 0$, Eq.(1.1) reduces to the focusing Schrödinger-Hartree equation

$$iu_t + \Delta u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^p \right) |u|^{p-2} u = 0. \tag{1.2}$$

† the corresponding author.
Email address: zhangmao1012@163.com(M. Zhang), jingjingpan2018@163.com(J. Pan), zhangjian@uestc.edu.cn(J. Zhang)
[1] School of Mathematical Sciences, University of Electronic Science and Technology of China, Chengdu 611731, China
[2] School of Mathematics, Zhengzhou University of Aeronautics, Zhengzhou 450046, China
* The authors were supported by National Natural Science Foundation of China (11871138).

For $p = 2$, the above equation (1.2) becomes the well-known standard Hartree equation, which can be considered as a model describing a quantum mechanical boson system in non-relativistic setting where the number of boson is very large. And it arises in the study of long range interaction between the molecules, which goes back to the work of [3, 6, 10, 11]. The equation (1.2) can be written as the Schrödinger-possion system of the form

$$\begin{cases} iv_t + \Delta v + W\,|v|^{p-2}\,v = 0, \\ -\Delta W = (N-2)|S^{N-1}|\,|v|^p\,, \end{cases}$$

where $|S^{N-1}|$ denotes the surface of the unit sphere in $\mathbb{R}^N$. This can be viewed as an electrostatic version of the Maxwell-Schrödinger system, it describing the interaction between the electromagnetic field and the wave function related to a quantum non-relativistic charged particle (see [15, 17]).

In [5], Cazenave established the local well-posedness for (1.2) in $H^1(\mathbb{R}^N)$ (also see Ginibre and Velo [10] for $p = 2$). In [1], Arora and Roudenko studied the global and finite time blow-up solution under the mass-energy assumption, and obtained the sharp threshold for global existence and finite time blow-up in mass super-critical and energy sub-critical regime. In [9], Genev and Venkov proved the concentration properties and the existence of standing wave solutions, and got conditions for formation of singularities in dependence of the values of $p \geq 2$ and $\gamma = 2$. In [13], Krieger, Lenzmann and Raphaël showed the existence of critical mass finite time blowup solution $u(t,x)$ that demonstrates the pseudo-conformal blowup rate $||\nabla v(t)||_{L^2} \sim \frac{1}{|t|}$ as $t \to 0$. In [24], Yang, Roudenko and Zhao obtained that a generic blow-up has a self-similar structure and exhibited log-log blow-up rate for $\gamma = 2$ and $p = 1 + \frac{4}{N}$ in $N = 3 \sim 7$ by numerical simulations.In [2], Arora, Roudenko and Yang showed the spectral property for $\gamma = 2$ and $p = 1 + \frac{4}{N}$ in dimension $N = 3$. By using the spectral results above, Arora and Roudenko showed the upper bounds on the blow-up rate for the blow-up solutions: there is a constant $C > 0$ such that

$$||\nabla v(t,x)||_{L^2} \leq C \left(\frac{|\ln(T-t)|}{T-t} \right)^{\frac{1}{2}}, \quad as \quad t \to T.$$

Let $u(t,x) = e^{it}\varphi(x)$ be a standing wave solution of (1.2). Then $\varphi(x)$ satisfies the following nonlinear elliptic equation

$$\Delta\varphi + \left(\frac{1}{|x|^{N-\gamma}} * |\varphi|^p \right) |\varphi|^{p-2}\,\varphi = \varphi. \tag{1.3}$$

The equation (1.3) is also called the nonlinear Choquard or Choquard-pekar equation. In [14], Lieb first proved the existence and uniqueness of the minimizing solution to (1.3) for $p = 2$ and $\gamma = 2$ in $\mathbb{R}^3$. In [19], Moroz and Van Schaftingen proved the general existence of positive solutions along with regularity and radial symmetry of solutions to (1.3) (also see [20]). In [13] , Krieger, Lenzmann and Raphaël proved the uniqueness of the ground state solution in dimension $N = 4$ for $p = 2$ and $\gamma = 2$. In [1], Arora and Roudenko proved the uniqueness of the ground state solution in dimension $2 < N < 6$ for $p = 2$ and $\gamma = 2$. In [23], Xiang proved the uniqueness of ground state solution for $p = 2 + \varepsilon$ and $\gamma = 2$. In the general case, the uniqueness is still an open problem.

In this paper, the blow-up dynamics of (1.1) is described by the variational characteristics of the elliptic equation (1.3) without harmonic potential. We first prove the existence of blow-up solution in a finite time of (1.1) and obtain the sharp threshold of global existence and blow-up by using Hamilton conservation and a Gagliardo-Nirenberg inequality with best constant. In the light of [4, 25, 26], we next find that there exists a transform between the solutions of (1.1) and (1.2). Moreover, by using the transform, we prove the upper bound on the blow-up rate for the blow-up solution of (1.1) in $\mathbb{R}^3$. The result is as follows.

Theorem 1.1. *Let $N = 3$ and $\gamma = 2$. Assume that $u_0 \in H$ and there exists an universal constant $C^* > 0$ and for some $\alpha > 0$, such that*

$$||Q||_{L^2(\mathbb{R}^3)} < ||u_0||_{L^2(\mathbb{R}^3)} < ||Q||_{L^2(\mathbb{R}^3)} + \alpha, \quad E^*(u_0) < 0. \tag{1.4}$$

Let $u(t,x)$ be the corresponding solution of the Cauchy problems (1.1)-(2.1) *and blow up in finite time $0 < T < \frac{1}{2\omega}$. Then there holds:*

$$||\nabla u||_{L^2(\mathbb{R}^3)} \leq C^* \left(\frac{|\ln 2\omega(T-t)|}{2\omega(T-t)} \right)^{\frac{1}{2}} \quad as \quad t \to T. \tag{1.5}$$

Furthermore, we prove the concentration property for the blow-up solution of (1.1) in $\mathbb{R}^N$.

Theorem 1.2. *Assume $u_0 \in H$. Let $u(t,x)$ be the blow-up solution of the Cauchy problems* (1.1)-(2.1)*, and let $\lambda(t)$ be a real-valued positive function in $[0,T)$ satisfying $\lambda(t)\cdot \parallel \nabla u \parallel_{L^2} \to +\infty$ as $t \to T$. Then, there exists $x(t)$ in $\mathbb{R}^N$ such that*

$$\liminf_{t \to T} \int_{|x-x(t)| \leq \lambda(t)} |u(t,x)|^2 dx \geq \parallel Q \parallel^2_{L^2(\mathbb{R}^N)}, \tag{1.6}$$

where Q is the ground state solution of (1.3)*.*

This paper is organized as follows. In section 2, we give some preliminaries. In section 3, we prove the existence of blow-up solution in a finite time and obtain the sharp threshold of global existence and blow-up for equation (3.1). In section 4, we obtain a vital transform between the solutions of (1.1) and (1.2) in L^2-critical power, and prove the upper bound of blow-up rate of the blow-up solution for (3.1) in $\mathbb{R}^3$. In section 5, we prove the concentration property in $\mathbb{R}^N$.

For simplicity, we denote for the remainder of this paper that $\int_{\mathbb{R}^N} \cdot dx$ by $\int \cdot dx$, $|| \cdot ||_{L^p(\mathbb{R}^N)}$ by $|| \cdot ||_{L^p}$, and various positive constants are simply denoted by C.

2. Preliminaries

For the equation (1.1), we impose the initial data

$$u(0,x) = u_0(x), \tag{2.1}$$

and define the natural energy space

$$H := \{u \in H^1(\mathbb{R}^N), \ \int |x|^2 \, |u|^2 \, dx < \infty\}. \tag{2.2}$$

Thus H becomes a Hilbert space, continuously embedded in $H^1(\mathbb{R}^N)$, when endowed with the inner product

$$< u, v >_H = \int \left(\nabla u \nabla \bar{v} + u\bar{v} + |x|^2 u\bar{v}\right) dx, \tag{2.3}$$

whose associated norm is denoted by $||\cdot||_H$.

In [5], Cazenave established the local well-posedness for the Cauchy problems (1.1)-(2.1) and (1.2)-(2.1) in the corresponding energy space.

Proposition 2.1. *Assume $2 \le p < 2^*$ and $u_0 \in H$. Then there exists a unique solution $u(t,x)$ for the Cauchy problems (1.1)-(2.1) in $C([0,T);H)$ for some $T \in (0,\infty]$ (maximal existence time). At the same time, we have the alternatives $T=\infty$ (global existence) or else $T<\infty$ and $\lim_{t\to T}\|u\|_H = \infty$ (blow-up). Moreover, for all $t \in [0,T)$, the solution u satisfies the following two conservation laws of mass and energy:*

$$\int |u|^2\, dx = \int |u_0|^2\, dx, \tag{2.4}$$

$$E(u) := \int \left(|\nabla u|^2 + \omega^2 |x|^2 \, |u|^2 - \frac{1}{p}\left(\frac{1}{|x|^{N-\gamma}} * |u|^p\right)|u|^p \right) dx = E(u_0). \tag{2.5}$$

Let $v(t,x)$ be the corresponding solution of the following Cauchy problems (1.2)-(2.1) in $C[(0,\tau);H)$, for some $\tau \in (0,+\infty]$(maximal existence time), then for all $t \in [0,\tau)$, $v(t,x)$ satisfies the following two conservation:

$$\int |v|^2\, dx = \int |u_0|^2\, dx, \tag{2.6}$$

$$E^*(v) := \int \left(|\nabla v|^2 - \frac{1}{p}\left(\frac{1}{|x|^{N-\gamma}} * |v|^p\right)|v|^p \right) dx = E^*(u_0). \tag{2.7}$$

The equation (1.2) admits several invariances: if $v(t,x)$ solves (1.2), then so does $\widetilde{v}(t,x)$.

- $Time\ translatioin\ invariance:\ \widetilde{v}(t,x) = v(t+t_0,x), \quad for\ \forall t_0 \in \mathbb{R}.$
- $Space\ translatioin\ invariance:\ \widetilde{v}(t,x) = v(t,x+x_0), \quad for\ \forall x_0 \in \mathbb{R}^N.$
- $Time\ reversal\ invariance:\ \widetilde{v}(t,x) = \bar{v}(-t,x).$
- $Phase\ invariance:\ \widetilde{v}(t,x) = e^{i\theta}v(t,x), \quad for\ \forall \theta \in \mathbb{R}.$
- $Galilean\ invariance:\ \widetilde{v}(t,x) = e^{i(x\cdot\xi_0 - t|\xi_0|^2)}v(t,x-\xi_0 t), \quad for\ \forall \xi \in \mathbb{R}^N.$
- $Scaling\ invariance:\ \widetilde{v}(t,x) = \lambda^{\frac{\gamma+2}{2(p-1)}}v(\lambda^2 t, \lambda x), \quad for\ \forall \lambda > 0.$
- $conformal\ invariance:\ If\ p = 1+\frac{\gamma+2}{N},\ then\ \widetilde{v}(t,x) = \frac{1}{|t|^{\frac{N}{2}}}v(-\frac{1}{t},\frac{x}{t})e^{\frac{i|x|^2}{4t}}.$

Equation (1.1) only admits time translation, time reversal and phase invariance. Next, we state the Gagliardo-Nirenberg inequality of convolution from [1]. Define the Weinstein-type functional for $\psi \in H\backslash\{0\}$

$$I(\psi) = \frac{\|\nabla\psi\|_{L^2}^{Np-(N+\gamma)}\|\psi\|_{L^2}^{N+\gamma-(N-2)p}}{\int\left(\frac{1}{|x|^{N-\gamma}} * |\psi(t,x)|^p\right)|\psi|^p\, dx}. \tag{2.8}$$

Lemma 2.1. *Assume that* $2 \le p < 1 + \frac{\gamma+2}{N-2}$ *and* $0 < \gamma < N$. *Then*

$$\inf_{\psi \in H} I(\psi) = C_{GN} \|Q\|_{L^2}^{2(p-1)}, \tag{2.9}$$

where $C_{GN} = \sigma^{-p\sigma}(1-\sigma)^{1-p\sigma}$, $\sigma = \frac{N(p-1)+\gamma}{2p}$, *and* Q *is the positive radial of nonlinear elliptic equation* (1.3)

$$\Delta\varphi + \left(\frac{1}{|x|^{N-\gamma}} * |\varphi|^p\right)|\varphi|^{p-2}\varphi = \varphi.$$

Remark 2.1. For Lemma 2.1, we can get the following Gagliardo-Nirenberg inequality with a best constant,

$$\int \left(\frac{1}{|x|^{N-\gamma}} * |\psi|^p\right)|\psi|^p\,dx \le C_{GN}\|Q\|_{L^2}^{-2(p-1)} \|\nabla\psi\|_{L^2}^{Np-(N+\gamma)} \|\psi\|_{L^2}^{N+\gamma-(N-2)p}. \tag{2.10}$$

Moreover, we can get the Pohozaev's identities related to (1.3) by a direct calculation

$$\|\nabla Q\|_{L^2}^2 = \frac{1}{p}\int \left(\frac{1}{|x|^{N-\gamma}} * |Q|^p\right)|Q|^p\,dx, \tag{2.11}$$

$$\|Q\|_{L^2}^2 = \frac{p-1}{p}\int \left(\frac{1}{|x|^{N-\gamma}} * |Q|^p\right)|Q|^p\,dx. \tag{2.12}$$

Remark 2.2. The existence of the positive solution along with the regularity and radial symmetry of solution to Eq (1.3) has been proved by Moroz and Van Schafingen [19]. But the uniqueness problem of the positive radial solution is still an open problem. Genev and Venkov [9] showed that all the positive radial solutions have the same mass. We denote the same mass by $||Q||_{L^2}$, and denote the positive radial solutions by Q (ground state solution).

In addition, by a direct calculation ([5]), we have the following result.

Proposition 2.2. *Assume* $u_0 \in H$. *Let* $u(t,x)$ *be a solution of the Cauchy problems* (1.1)-(2.1) *in* $C([0,T),H)$, *and put* $J(t) := \int |x|^2|u|^2 dx$. *Then* $J'(t) = 4\Im \int x\bar{u}\nabla u dx$ *and*

$$J''(t) = 8\int |\nabla u|^2 - \omega^2|x|^2|u|^2 - \frac{N(p-1)-\gamma}{2p}\left(\frac{1}{|x|^{N-\gamma}} * |u|^p\right)|u|^p\,dx. \tag{2.13}$$

From mass conservation (2.4), energy conservation (2.5) and Gagliardo-Nirenberg inequality (2.10), $H(\mathbb{R}^N)$ solution of the Cauchy problems (1.1)-(2.1) with $2 \le p < 1 + \frac{2+\gamma}{N}$ (L^2-subcritical case) is global and bounded in $H(\mathbb{R}^N)$ for any initial value. The solution of the Cauchy problems (1.1)-(2.1) with $1 + \frac{2+\gamma}{N} < p < 2^*$ (L^2-supercritical case) may blow-up in a finite time for any initial value. Then we are interested in investigating blow-up dynamics in L^2-critical case $p = 1 + \frac{2+\gamma}{N} \ge 2$.
Spectral Property. We define two real schrödinger operators L_1 and L_2 as

$$L_1\varepsilon_1 = \frac{1}{2}\left[L_+(\Lambda\varepsilon_1) - \Lambda(L_+\varepsilon_1)\right]; \quad L_2\varepsilon_2 = \frac{1}{2}\left[L_-(\Lambda\varepsilon_2) - \Lambda(L_-\varepsilon_2)\right],$$

where

$$L_+ := -\Delta + 1 - \frac{4}{N}\left(\frac{1}{|y|^{(N-2)}} * Q^{1+\frac{4}{N}}\right) Q^{\frac{4}{N}-1} - \left(1+\frac{4}{N}\right)\left(\frac{1}{|y|^{(N-2)}} * \left(Q^{\frac{4}{N}}(\cdot)\right)\right) Q^{\frac{4}{N}},$$

$$L_- := -\Delta + 1 - \left(\frac{1}{|y|^{(N-2)}} * Q^{1+\frac{4}{N}}\right) Q^{\frac{4}{N}-1},$$

and $\Lambda f := \frac{N}{2} f + x \cdot \nabla f$. Define the real valued quadratic form as

$$\mathcal{B}(\varepsilon, \varepsilon) = (L_1 \varepsilon_1, \varepsilon_1) + (L_2 \varepsilon_2, \varepsilon_2), \quad for \;\; \varepsilon = \varepsilon_1 + i\varepsilon_2 \in H^1.$$

The Spectral property has been proved in [2] only for N=3, as follows.

Lemma 2.2. *Let $N = 3$. If $(\varepsilon_1, Q) = (\varepsilon_1, \Lambda Q + \Lambda^2 Q) = (\varepsilon_2, \Lambda Q) = (\varepsilon_2, \Lambda^2 Q) = 0$ under the radial symmetric assumption, then there exists a universal constant $\delta_0 > 0$ such that*

$$\mathcal{B}(\varepsilon, \varepsilon) \geq \delta_0 \int \left(|\nabla \varepsilon|^2 + |\varepsilon|^2 e^{-2^- |y|}\right) dy, \quad \forall \varepsilon \in H^1(\mathbb{R}^3).$$

By using the spectral property, Arora, Roudenko and Yang [2] obtained the upper bound of the blow-up solutions for the Cauchy problems (1.2)-(2.1) in the spirit of [18].

Lemma 2.3. *Let $N = 3$ and the spectral property holds true. Then there exists $\alpha > 0$ and a positive constant $C > 0$ such that the following is true. Let $u_0 \in H$ such that*

$$||Q||_{L^2} < ||u_0||_{L^2} < ||Q||_{L^2} + \alpha, \quad E^*(u_0) < 0.$$

Let $v(t, x)$ be the corresponding solution of (1.2)-(2.1) in $\mathbb{R}^3$. Then $v(t, x)$ blows up in a finite time $0 < T < +\infty$ and there holds:

$$||\nabla v(t, x)||_{L^2} \leq C \left(\frac{|\ln(T-t)|}{T-t}\right)^{\frac{1}{2}} \quad as \quad t \to T.$$

3. The sharp threshold of global existence

In this section, we study the global existence and blow-up for the Cauchy problems (1.1)-(2.1) in L^2-critical case $p = 1 + \frac{2+\gamma}{N} \geq 2$,

$$iu_t + \Delta u - \omega^2 |x|^2 u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^{1+\frac{2+\gamma}{N}}\right) |u|^{\frac{2+\gamma}{N}-1} u = 0. \tag{3.1}$$

Firstly, we prove the threshold of global existence for the Cauchy problems (3.1)-(2.1) by using the Hamilton conservations and the sharp Gagliardo-Nirenberg inequality (2.10). We obtain a threshold by using the positive solution of the elliptic equation (1.3) without harmonic potential in spirit of Zhang [25].

Theorem 3.1. *Let $N - 2 \leq \gamma < N$. Assume that $u_0 \in H$ and*

$$||u_0||_{L^2} < ||Q||_{L^2}. \tag{3.2}$$

Then the solution $u(t, x)$ of the Cauchy problems (3.1)-(2.1) exists globally in time.

Proof. Let $u(t,x)$ be a solution for the Cauchy problems (3.1)-(2.1), and $u(t,x) \in C([0,T);H)$. By the mass conservation (2.4), energy conservation (2.5) and the critical Gagliardo-Nirenberg inequality (2.10), we deduce that

$$\left(1-\left(\frac{||u_0||_{L^2}}{||Q||_{L^2}}\right)^{\frac{4+2\gamma}{N}}\right)||\nabla u||_{L^2}^2+\omega^2\|xu\|_{L^2}^2\leq E(u_0).$$

Furthermore, by the condition (3.2), $\omega^2\|xu\|_{L^2}<+\infty$ and

$$||\nabla u||_{L^2}<+\infty,\quad for\ \ \forall t>0.$$

Thus we can get $||\nabla u||_{L^2}$ and $\|xu\|_{L^2}$ are bounded for any $T<\infty$ and $t\in[0,T)$. By Proposition 2.1, the solution $u(t,x)$ exists globally in time. □

Next, by a direct calculation, we prove the sufficient condition of the existence of blow-up solution in a finite time.

Proposition 3.1. *Let $N-2\leq\gamma<N$, and let $u(t,x)$ be the solutions of the Cauchy problems (3.1)-(2.1). If the initial value $u_0\in H$ satisfies*

$$(i)E(u_0)<0;$$

$$(ii)E(u_0)=0,\ and\ Im\int x\bar{u_0}\nabla u_0dx<0;$$

$$(iii)E(u_0)>0,\ and\ Im\int x\bar{u_0}\nabla u_0dx\leq-\sqrt{E(u_0)J(0)}.$$

Then $u(t,x)$ blows up in a finite time $0<T<\infty$.

Proof. Let u be a solution for the Cauchy problems (3.1)-(2.1) in $C([0,T);H)$. By Proposition 2.2, we see that $J^{''}(t)\leq 8E(u_0)$. Then we have

$$\begin{aligned}J(t)&=J(0)+J'(0)t+\int_0^t(t-s)J''(s)ds\\&\leq J(0)+J'(0)t+4E(u_0)t^2,\ 0\leq t<\infty\end{aligned}$$

under hypothesis $(i),(ii),(iii)$, which implies that there is a $0<T<\infty$ such that

$$\lim_{t\to T}\int|x|^2|u|^2dx=0.$$

And we know that $-N\int|u|^2dx=2Re\int x\cdot\nabla u\bar{u}dx$. By calculation and using the Cauchy-Schwarz inequality, the following estimate holds:

$$\int|u|^2dx\leqslant\frac{2}{N}\left(\int|\nabla u|^2dx\right)^{\frac{1}{2}}\left(\int|x|^2|u|^2dx\right)^{\frac{1}{2}}.$$

Thus $\lim_{t\to T}\|\nabla u(t)\|_{L^2}=+\infty$, and the solution $u(t,x)$ blows up in finite positive time. □

Last, we show a blow-up result in $||u_0||_{L^2}>||Q||_{L^2}$ for the Cauchy problems (3.1)-(2.1). The following result will show that the threshold of global existence in Theorem 3.1 is sharp.

Theorem 3.2. *Let $N-2 \le \gamma < N$. Assuming $u_0 \in H$, there exists the initial value u_0, such that*

$$||u_0||_{L^2} = ||Q||_{L^2} + \varepsilon, \ \ for \ \forall \varepsilon > 0. \tag{3.3}$$

Then the solution $u(t,x)$ for the Cauchy problems (3.1)-(2.1) *blows up in finite time. Here Q is the positive radial solution of Eq.*(1.3) *with $p = 1 + \frac{2+\gamma}{N}$.*

Proof. For arbitrary $\lambda > 0$ and $\mu > 0$, we set $Q_{\lambda,\mu}(x) = \mu \lambda^{\frac{N}{2}} Q(\lambda x)$ by scaling and $F(Q) = \int \left(\frac{1}{|x|^{N-\gamma}} * |Q|^{1+\frac{2+\gamma}{N}} \right) |Q|^{1+\frac{2+\gamma}{N}} dx$. Then we have

$$\int \left[Q_{\lambda,\mu}(x)\right]^2 dx = \mu^2 \int Q^2 dx, \tag{3.4}$$

$$\int |x|^2 [Q_{\lambda,\mu}(x)]^2 dx = \mu^2 \lambda^{-2} \int |x|^2 Q^2 dx, \tag{3.5}$$

$$F(Q_{\lambda,\mu}(x)) = \mu^{2(1+\frac{2+\gamma}{N})} \lambda^2 F(Q), \tag{3.6}$$

$$\int |\nabla Q_{\lambda,\mu}(x)|^2 dx = \mu^2 \lambda^2 \int |\nabla Q|^2 dx. \tag{3.7}$$

By Pohozaev's identities (2.11) and (2.12), we have

$$\int |\nabla Q|^2 dx = \frac{N}{N+2+\gamma} \int \left(\frac{1}{|x|^{N-\gamma}} * |Q|^{1+\frac{2+\gamma}{N}} \right) |Q|^{1+\frac{2+\gamma}{N}} dx = \frac{N}{2+\gamma} \int |Q|^2 dx. \tag{3.8}$$

Now, we take that $u_0(x) = \mu \lambda^{\frac{N}{2}} Q(\lambda x)$, and

$$\mu = \left[\frac{(\int Q^2 dx + \varepsilon)}{\int Q^2 dx} \right]^{\frac{1}{2}}, \ \lambda > \left[\frac{\int \omega^2 |x|^2 Q^2 dx}{(\mu^{\frac{4+2\gamma}{N}} - 1) \int |\nabla Q|^2 dx} \right]^{\frac{1}{4}}.$$

Then it is obvious that $u_0 \in H$ and $\int |x|^2 |u_0|^2 dx < \infty$ (see [21]). From equation (3.4), it follows that

$$\|u_0\|_2 = \|Q\|_2 + \varepsilon.$$

From energy conservation (2.5) and (3.4)−(3.7), we have

$$E(u_0) = \mu^2 \lambda^2 \int \left[(1 - \mu^{\frac{4+2\gamma}{N}}) |\nabla Q|^2 + \lambda^{-4} \omega^2 |x|^2 Q^2 \right] dx < 0. \tag{3.9}$$

By Proposition 3.1, one obtains that there exists $0 < T < \infty$ such that $\lim_{t \to T} ||\nabla u(t)||_2 = +\infty$, then the solution $u(t,x)$ for the Cauchy problems (3.1)-(2.1) blows up in a finite time. This proof is completed. □

Remark 3.1. From Theorem 3.1 and Theorem 3.2, we can know that "$\|u_0\|_2 = \|Q\|_2$" is a sharp threshold of global existence and blow-up in finite time for the Cauchy problems (3.1)-(2.1).

4. The upper bound for blow-up rate

In this section, we prove the upper bound of the blow-up solutions for the Cauchy problems (3.1)-(2.1). Firstly, in the spirit of Carles [4] we discuss the following transform between the Cauchy problems (3.1)-(2.1) and (1.2)-(2.1) , which is necessary to prove the rate of blow-up and concentration result.

Theorem 4.1. *Let $p = 1 + \frac{2+\gamma}{N}$ and $N-2 \le \gamma < N$. Assume that v is a solution of the Cauchy problems* (1.2)-(2.1) *in $C([0,T);H)$. For any $\omega > 0$, define*

$$u(t,x) = (\cos(2\omega t))^{-\frac{N}{2}} e^{-i\frac{\omega}{2}|x|^2\tan(2\omega t)} v\left(\frac{\tan(2\omega t)}{2\omega}, \frac{x}{\cos(2\omega t)}\right). \tag{4.1}$$

Then $u(t,x) \in C([0, \arctan(2\omega T)/2\omega); H)$ is a solution of the Cauchy problems (3.1)-(2.1)*. Reciprocally, if u is a solution of the Cauchy problems* (3.1)-(2.1)*, then v defined by*

$$v(t,x) = \left(1+(2\omega t)^2\right)^{-\frac{N}{4}} e^{i\frac{\omega^2 t}{1+(2\omega t)^2}|x|^2} u\left(\frac{\arctan(2\omega t)}{2\omega}, \frac{x}{\sqrt{1+(2\omega t)^2}}\right) \tag{4.2}$$

solves (1.2)-(2.1).

Proof. For simplicity, let $s := \frac{\tan(2\omega t)}{2\omega}$ and $y := \frac{x}{\cos(2\omega t)}$. Then by a direct calculation, we have

$$\begin{aligned} & iu_t + \Delta u - \omega^2|x|^2 u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^{\frac{2+\gamma}{N}+1}\right)|u|^{\frac{2+\gamma}{N}-1} u \\ =& (\cos(2\omega t))^{-\frac{N}{2}-2} e^{-i\frac{\omega}{2}|x|^2\tan(2\omega t)} \left[iv_s + \Delta v + \left(\frac{1}{|y|^{N-\gamma}} * |v|^{\frac{2+\gamma}{N}+1}\right)|v|^{\frac{2+\gamma}{N}-1} v\right]. \end{aligned}$$

If $v(s,y) \in C([0,T), H)$ is a solution of the Cauchy problems (1.2)-(2.1), then

$$iu_t + \Delta_x u - \omega^2|x|^2 u + \left(\frac{1}{|x|^{N-\gamma}} * |u|^{1+\frac{2+\gamma}{N}}\right)|u|^{\frac{2+\gamma}{N}-1} u = 0,$$

and the initial

$$u(0,x) = v(0,x) = u_0.$$

In the progress of the above calculation, we know

$$s = \frac{\tan(2\omega t)}{2\omega} \in [0,T),$$

then one has

$$t \in \left[0, \frac{\arctan(2\omega T)}{2\omega}\right).$$

We conclude the proof of the transform (4.1). The proof of (4.2) follows by a similar argument. □

Combining with the transform (4.1) and Lemma 2.3, we give the **proof of Theorem 1.1**.

Proof. Letting $u(t,x)$ be a solution to the Cauchy problems (1.1)-(2.1) and blow up in finite time $0 < T < \frac{1}{2\omega}$, and using the transform (4.1), then $v(t,x) \in C([0, \frac{tan(2\omega)}{2\omega}); H)$ is a blow-up solution of the Cauchy problems (1.2)-(2.1), and

$$u(t,x) = (\cos(2\omega t))^{-\frac{N}{2}} e^{-i\frac{\omega}{2}|x|^2\tan(2\omega t)} v(\frac{\tan(2\omega t)}{2\omega}, \frac{x}{\cos(2\omega t)}),\ t \in [0,T).$$

For simplicity, let $s := \frac{\tan(2\omega t)}{2\omega}$ and $y := \frac{x}{\cos(2\omega t)}$. Then for $t \in [0,T)$ and $T \in (0, \frac{1}{2\omega})$, we have

$$||\nabla_x u(t,x)||_{L^2} \le \omega \sin(2\omega t)||yv(s,y)||_{L^2} + \frac{1}{\cos(2\omega t)}||\nabla_y v(s,y)||_{L^2}. \tag{4.3}$$

Let $J(s)=\int|yv|^2dy$. Then $J'(s)=4\Im\int y\bar{v}\nabla vdy$ and

$$J''(s)=8E^*(u_0).$$

From $E^*(u_0)<0$, there exists a constant $C>0$, such that

$$|J(s)|=J(0)+J'(0)s+\int_0^s J''(t)(s-t)dt<C \quad \forall s\in[0,\frac{\tan(\omega T)}{\omega}). \tag{4.4}$$

By using Lemma 2.3, there exist two constants $C_1,\quad C_2>0$, such that

$$||\nabla_x u(t,x)||_{L^2}\le C_1+C_2\left(\frac{\left|\ln(\frac{\tan(2\omega T)-\tan(2\omega t)}{\omega})\right|}{\frac{\tan(2\omega T)-\tan(2\omega t)}{2\omega}}\right)^{\frac{1}{2}}.$$

Since $\tan(2\omega T)-\tan(2\omega t)=\frac{\sin(2\omega(T-t))}{\cos(2\omega T)\cos(2\omega t)}\ge\sin(2\omega(T-t))$, there is a constant $C^*>0$, such that

$$||\nabla_x u(t,x)||_{L^2}\le C^*\left(\frac{|\ln\sin(2\omega(T-t))|}{\sin(2\omega(T-t))}\right)^{\frac{1}{2}} \quad as \quad t\to T.$$

As $t\to T$, we have $\sin(2\omega(T-t))\sim 2\omega(T-t)$, then

$$\left(\frac{|\ln\sin(2\omega(T-t))|}{\sin(2\omega(T-t))}\right)^{\frac{1}{2}}\sim\left(\frac{|\ln(2\omega(T-t))|}{2\omega(T-t)}\right)^{\frac{1}{2}} \quad as \quad t\to T,$$

which concludes the proof. □

Remark 4.1. For the Cauchy problems (1.2)-(2.1) with $N=3$, there exists a stable blow-up solution v with the log-log blow-up rate by numerical simulations (see Yang, Roudenko and Zhao [24])

$$||\nabla v||_{L^2}\sim\left(\frac{\ln|\ln(T-t)|}{T-t}\right)^{\frac{1}{2}} \quad as \quad t\to T.$$

By using the transform (4.1), we can obtain the existence of blow-up solutions with log-log blow-up rate

$$||\nabla u||_{L^2}\sim\left(\frac{\ln|\ln 2\omega(T-t)|}{T-t}\right)^{\frac{1}{2}} \quad as \quad t\to T,$$

which can be proved by the similar proof of Theorem 1.1.

5. The concentration of blow-up solution

In this section, we prove the concentration property of blow-up solution for the Cauchy problems (1.1)-(2.1). In [9], Genev and Venkov showed a compactness result adapted to the analysis of the mass concentration property for the Hartree equation (1.2). In the spirit of Hmidi and Keraani [12], we also can prove the mass concentration property for the blow-up solution to the Cauchy problems (1.1)-(2.1). However, we prove the mass concentration property by applying the vital transform (4.1), which exhibits the relationship between the Hartree equation (1.2) and the Hartree equation with harmonic potential (1.1). Firstly, we give the known concentration result to Eq.(1.2) as follows ([9]).

Lemma 5.1. *Let v be a solution of the Cauchy problems* (1.2)-(2.1) *which blows up in a finite time T_1, and let $\lambda_1(t) > 0$ be a real-valued positive function in* $[0, T_1)$ *such that $\lambda_1(t)||\nabla v||_{L^2} \to +\infty$ as $t \to T_1$. Then there exists $x_1(t) \in \mathbb{R}^N$ such that*

$$\liminf_{t\to T} \int_{|x-x_1(t)|\le\lambda_1(t)} |v(t,x)|^2 dx \ge ||Q||_{L^2}^2. \tag{5.1}$$

Next, by using Lemma 5.1 and the transform (4.1), we give the **proof of Theorem 1.2**.

Proof. Let $u(t, x)$ be a blow-up solution for the Cauchy problems (1.1)-(2.1) in $[0, T)$ (we assume that $0 < T < \frac{\pi}{2}$ by time translation invariance), and let $\lambda(t)$ be a real-valued positive function satisfying

$$\lambda(t)||\nabla u||_{L^2} \to +\infty \quad as \quad t \to T.$$

For simplicity, we set $s := \frac{\tan(\omega t)}{\omega}$ and $y := \frac{x}{\cos(\omega t)}$. By (4.3) and (4.4), we get

$$||\nabla_x u(t,x)||_{L^2} \le C + \frac{1}{\cos(\omega t)}||\nabla_y v(s,y)||_{L^2}.$$

Then the solution v is a blow-up solution of the Cauchy problems (1.2)-(2.1) in $[0, \frac{\tan(\omega T)}{\omega})$ and

$$\frac{\lambda(t)}{\cos(\omega t)}||\nabla_y v(s,y)||_{L^2} \to +\infty \quad as \quad t \to T.$$

Let $\lambda_1(s) := \frac{\lambda(t)}{\cos(\omega t)}$, where $t = \frac{\arctan(\omega s)}{\omega}$, such that

$$\lambda_1(s)||\nabla_y v(s,y)||_{L^2} \to +\infty \quad as \quad s \to \frac{\tan(\omega T)}{\omega}.$$

From the argument above, the solution $v(s, y)$ satisfies the condition of Lemma 5.1, then there exists $x_1(s) \in \mathbb{R}^N$, such that

$$\liminf_{s\to\frac{\tan(\omega T)}{\omega}} \int_{|y-x_1(s)|\le\lambda_1(s)} |v(s,y)|^2 dy \ge ||Q||_{L^2}^2. \tag{5.2}$$

Moreover, we have

$$\liminf_{s\to\frac{\tan(\omega T)}{\omega}} \int_{|y-x_1(t)|\le\lambda_1(s)} |v(s,y)|^2 dy = \liminf_{t\to T} \int_{|x-x_1(s)cos(\omega t)|\le\lambda(t)} |u(t,x)|^2 dx. \tag{5.3}$$

Let $x(t) = x_1(\frac{\tan(\omega t)}{\omega})cos(\omega t)$. From (5.2) and (5.3), there exists $x(t) \in \mathbb{R}^N$, such that

$$\liminf_{t\to T} \int_{|x-x(t)|\le\lambda(t)} |u(t,x)|^2 dx \ge ||Q||_{L^2}^2.$$

We complete the proof. □

Remark 5.1. From the result of Theorem 1.2, we can obtain that for every $A > 0$, there exists a function $x(t) \in \mathbb{R}^N$, such that the blow-up solution in a finite time $T > 0$ satisfies

$$\liminf_{t\to T} \int_{|x-x(t)|\le A} |u(t,x)|^2 dx \ge \| Q \|_{L^2}^2 . \tag{5.4}$$

Indeed, if we take $0 < \sigma < 1$ and constant $C > 0$, it is apparent that $\frac{C}{||\nabla u||_{L^2}^{1-\sigma}}$ satisfies the hypothesis of Theorem 1.2 and $\lim_{t\to T} \frac{C}{||\nabla u||_{L^2}^{1-\sigma}} \to 0$. For every $A > 0$, $A > \frac{C}{||\nabla u||_{L^2}^{1-\sigma}}$ as $t \to T$. Then the concentration result (5.4) holds.

References

[1] A. K. Arora and S. Roudenko, *Global behavior of solutions to the focusing generalized Hartree equation*, Michigan Mathematical Journal, 2021, DOI: 10.1307/mmj/20205855.

[2] A. K. Arora, S. Roudenko, and K. Yang, *On the focusing generalized Hartree equation*, Mathematics in Applied Sciences and Engineering, 2020, 1(4), 383–402.

[3] C. Bardos, L. Erdös, F. Golse, N. Mauser, and H.-T. Yau, *Derivation of the Schrödinger–Poisson equation from the quantum N-body problem*, Comptes Rendus Mathematique, 2002, 334 (6), 515–520.

[4] R. Carles, *Critical nonlinear Schrödinger equations with and without harmonic potential*, Mathematical Models and Methods in Applied Sciences, 2002, 12 (10), 1513–1523.

[5] T. Cazenave, *Semilinear Schrödinger Equations*, vol. 10, American Mathematical Soc., 2003.

[6] J. Fröhlich, S. Graffi, and S. Schwarz, *Mean-field-and classical limit of many-body Schrödinger dynamics for bosons*, Communications in Mathematical Physics, 2007, 271 (3), 681–697.

[7] J. Fröhlich and E. Lenzmann, *Mean-field limit of quantum Bose gases and nonlinear Hartree equation*, Séminaire Équations aux dérivées partielles (Polytechnique) dit aussi "Séminaire Goulaouic-Schwartz", 2003, 1–26.

[8] J. J. Garcıa-Ripoll, V. V. Konotop, B. Malomed, and V. M. Pérez-Garcıa, *A quasi-local Gross–Pitaevskii equation for attractive Bose–Einstein condensates*, Mathematics and Computers in Simulation, 2003, 62 (1), 21–30.

[9] H. Genev and G. Venkov, *Soliton and blow-up solutions to the time-dependent Schrödinger-Hartree equation*, Discrete & Continuous Dynamical Systems-S, 2012, 5 (5), 903.

[10] J. Ginibre and G. Velo, *On a class of nonlinear Schrödinger equations with nonlocal interaction*, Mathematische Zeitschrift, 1980, 170 (2), 109–136.

[11] K. Hepp, *The classical limit for quantum mechanical correlation functions*, Communications in Mathematical Physics, 1974, 35 (4), 265–277.

[12] T. Hmidi and S. Keraani, *Blowup theory for the critical nonlinear Schrödinger equations revisited*, International Mathematics Research Notices, 2005, 46, 2815–2828.

[13] J. Krieger, E. Lenzmann, and P. Raphaël, *On stability of pseudo-conformal blowup for L^2-critical Hartree NLS*, Annales Henri Poincaré, 2009, 10 (6), 1159–1205.

[14] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation*, Studies in Applied Mathematics, 1977, 57 (2), 93–105.

[15] E. H. Lieb, *The stability of matter and quantum electrodynamics*, Milan Journal of Mathematics, 2003, 71 (1), 199–217.

[16] P. M. Lushnikov, *Collapse and stable self-trapping for Bose-Einstein condensates with $1/r^b$-type attractive interatomic interaction potential*, Physical Review A, 2010, 82 (2), DOI: 10.1103/PhysRevA.82.023615.

[17] G. Maria, C. And, and V. Georgiev, *Solitary waves for Maxwell-Schrödinger equations*, Electronic Journal of Differential Equations, 2004, 31(94).

[18] F. Merle and P. Raphael, *The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation*, Annals of Mathematics, 2005,161(1), 157–222.

[19] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, Journal of Functional Analysis, 265 (2013), 153–184.

[20] V. Moroz and J. Van Schaftingen, *A guide to the Choquard equation*, Journal of Fixed Point Theory and Applications, 2017, 19 (1), 773–813.

[21] T. Ogawa and Y. Tsutsumi, *Blow-up of H^1 solution for the nonlinear Schrödinger equation*, Journal of Differential Equations, 1991, 92 (2), 317–330.

[22] M. I. Weinstein, *Nonlinear Schrödinger equations and sharp interpolation estimates*, Communications in Mathematical Physics, 1983, 87 (4), 567–576.

[23] C.-L. Xiang, *Uniqueness and nondegeneracy of ground states for Choquard equations in three dimensions*, Calculus of Variations and Partial Differential Equations, 2016, 55 (6), 1–25.

[24] K. Yang, S. Roudenko, and Y. Zhao, *Stable blow-up dynamics in the-critical and-supercritical generalized Hartree equation*, Studies in Applied Mathematics, 2020, 145 (4), 647–695.

[25] J. Zhang, *Stability of attractive Bose–Einstein condensates*, Journal of Statistical Physics, 2000, 101 (3), 731–746.

[26] S. Zhu and X. Li, *Sharp upper and lower bounds on the blow-up rate for nonlinear Schrödinger equation with potential*, Applied Mathematics and Computation, 2007, 190 (2), 1267–1272.