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Unique Solution for a General Coupled System of Fractional Differential Equations*

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Abstract This paper discusses a new coupled system of Riemann-Liouville fractional differential equations, in which the nonlinear terms include the Riemann-Liouville fractional integrals and the boundary value problems involve three-points. We seek also for the existence and uniqueness of solutions for this new system. We first get some useful properties of the Green's function generated by the system, and then we apply a fixed point theorem of increasing φ-(h,e)-concave operators to this new coupled system. Finally, we gain the existence and uniqueness results of the solution for this problem. In the end, a concrete example is structured to illustrate the main result.

Keywords Existence and uniqueness, coupled system of fractional differential equations, fixed point theorem, φ-(h,e)-concave operators

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1. Introduction

This paper considers the existence and uniqueness of solutions for a new coupled system of fractional differential equations:

$$\begin{cases} D^{\alpha}u(t)+f(t,v(t),I^{\gamma}v(t))=a, t\in(0,1), \\ D^{\beta}v(t)+g(t,u(t),I^{\delta}u(t))=b, t\in(0,1), \\ u(0)=u'(0)=0, u'(1)=\mu u'(\xi), \\ v(0)=v'(0)=0, v'(1)=\mu v'(\xi), \end{cases} \tag{1.1}$$

where $2<\alpha,\beta\leq 3$, $1<\gamma,\delta\leq 2$, and $\alpha-\delta\geq 1$, $\beta-\gamma\geq 1$, $0<\xi<1$, $0<\mu\xi^{\alpha-2}<1$, $0<\mu\xi^{\beta-2}<1$, a,b are nonnegative constants. D^{α} and D^{β} are the standard Riemann-Liouville fractional derivatives, and $f,g:[0,1]\times[0,+\infty)\times[0,+\infty)\to[0,+\infty)$ are continuous functions. We employ a fixed point theorem of increasing φ-(h,e)-concave operators by Zhai and Wang to study the system (1.1).

In recent decades, fractional calculus has been widely used as a tool to study many problems in different research fields, such as engineering, biology, physics,

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and so on. Because the extensive development and application of fractional calculus equation theory, the theme of fractional differential equation system has developed into a crucial research field. Based on it, the theoretical research on the existence of solutions for fractional problems has attracted widespread attention. For some gained studies on fractional differential equations, we can refer to [3, 7–13, 19–23, 27–39] and others. Many problems of coupled systems involving fractional differential equations have been investigated extensively, see [9, 15, 22, 24] and others. From literature, we know that the mathematical definition of the differintegral operator of fractional order has been the subject of different approaches, and the most used are the Riemann-Liouville (RL), the Grünwald-Letnikov (GL), and the Caputo's (C) definitions. Although they are different in differential equations, some fixed methods can be used to study these problems and some similar results under different conditions are also obtained. Based upon this reason, here we consider (1.1) only involving Riemann-Liouville fractional derivative rather than other derivatives.

Nowadays, many researchers devoted themselves to determining the solvability of system of fractional differential equations with different boundary conditions, specifically to the study of existence of solutions to some systems of fractional differential equations, see [4–6, 9–14, 16–21, 23] for example and the references therein. The usual methods used are Schauder's fixed point theorem, Banach's fixed point theorem, Guo-Krasnoselskii's fixed point theorem on cone, monotone iterative technique, the method of lower and upper solutions, nonlinear differentiation of Leray-Schauder type and others. In [10], the authors considered a system of fractional differential equations:

$$\begin{cases} D^{\alpha}u(t) + f(t, v(t), D^{\gamma}v(t)) = 0, 0 < t < 1, \\ D^{\beta}v(t) + g(t, u(t), D^{\delta}u(t)) = 0, 0 < t < 1, \\ u(0) = u'(0) = 0, u'(1) = \mu u'(\xi), \\ v(0) = v'(0) = 0, v'(1) = \mu v'(\xi), \end{cases}$$

where, $2 < \alpha, \beta \leq 3$, $1 < \gamma, \delta \leq 2$, and $\alpha - \delta \geq 1$, $\beta - \gamma \geq 1$, $0 < \xi < 1$, $0 < \mu\xi^{\alpha-2} < 1$, $0 < \mu\xi^{\beta-2} < 1$. D^{α} and D^{β} are the Riemann-Liouville fractional derivatives, and $f, g : [0, 1] \times [0, +\infty) \times (-\infty, +\infty) \to [0, +\infty)$ are continuous functions and depend on the unknown functions as well as their lower order fractional derivatives. The authors got the existence and uniqueness of positive solutions for the system by utilizing fixed point theorems due to Schauder and Banach.

This paper considers the existence and uniqueness of solutions for system (1.1). As is well known, no papers have considered the solutions of system (1.1) from literatures. So it is worthwhile to investigate (1.1).

2. Preliminaries and previous results

Definition 2.1. [1, 2] The Riemann-Liouville fractional integral of order $\alpha > 0$, for a function $f : (0, +\infty) \to (-\infty, +\infty)$, is given by

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, t > 0,$$

provided the integral exists.

Definition 2.2. [1, 2] For a function $f : (0, +\infty) \to (-\infty, +\infty)$, the Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$D^\alpha f(t) = (\frac{d}{dt})^n (I^{n-\alpha} f)(t) = \frac{1}{\Gamma(n-\alpha)} (\frac{d}{dt})^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, t > 0,$$

where $n = [\alpha] + 1$, in which $[\alpha]$ denotes the integer part of the number α.

Lemma 2.1. *[1] If $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ belongs to $C(0,1) \cap L(0,1)$, then*

$$I^\alpha D^\alpha u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_n t^{\alpha-N},$$

for some $C_i \in (-\infty, +\infty), i = 1, 2, \ldots, N$, where $N = [\alpha] + 1$.

Lemma 2.2. *[10] Let $y_1 \in C[0,1]$ and $\alpha \in (2,3]$, then the problem*

$$\begin{cases} D^\alpha u(t) + y_1(t) = 0, 0 < t < 1, \\ u(0) = u'(0) = 0, u'(1) = \mu u'(\xi), \end{cases}$$

has a unique solution $u(t) = \int_0^t G_1(t,s) y_1(s) ds$, where

$$G_1(t,s) = \begin{cases} \frac{(1-s)^{\alpha-2} t^{\alpha-1} - \mu(\xi-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1}(1-\mu\xi^{\alpha-2})}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}, 0 \le s \le t \le 1, s \le \xi, \\ \frac{(1-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1}(1-\mu\xi^{\alpha-2})}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}, 0 < \xi \le s \le t \le 1, \\ \frac{(1-s)^{\alpha-2} t^{\alpha-1} - \mu(\xi-s)^{\alpha-2} t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}, 0 \le t \le s \le \xi < 1, \\ \frac{(1-s)^{\alpha-2} t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}, 0 \le t \le s \le 1, \xi \le s. \end{cases} \tag{2.1}$$

Lemma 2.3. *The above Green's function $G_1(t,s)$ defined by (2.1) has some properties:*

(i) $G_1(t,s) > 0$, for $t, s \in (0,1)$;

(ii) $t^{\alpha-1} G_1(1,s) \le G_1(t,s) \le \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})}, t, s \in (0,1)$.

Proof. From Lemma 4 in Reference [10], we know that the condition (i) is true. In addition, when $0 < s \le t < 1, s \le \xi$, there is

$$\begin{aligned} G_1(t,s) &= \frac{t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}[(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-\frac{s}{t})^{\alpha-1}(1-\mu\xi^{\alpha-2})] \\ &\ge \frac{t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}[(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-s)^{\alpha-1}(1-\mu\xi^{\alpha-2})] \\ &= t^{\alpha-1} G_1(1,s). \end{aligned}$$

When $0 < \xi \le s \le t < 1$, we have

$$\begin{aligned} G_1(t,s) &= \frac{t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}[(1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-\frac{s}{t})^{\alpha-1}(1-\mu\xi^{\alpha-2})] \\ &\ge \frac{t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}[(1-s)^{\alpha-2} - (1-s)^{\alpha-1}(1-\mu\xi^{\alpha-2})] \\ &= t^{\alpha-1} G_1(1,s). \end{aligned}$$

Therefore, $G_1(t,s) \geq t^{\alpha-1}G_1(1,s)$, where

$$G_1(1,s) = \frac{1}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-2} - \mu(\xi-s)^{\alpha-2} - (1-s)^{\alpha-1}(1-\mu\xi^{\alpha-2}) \\ \text{for } \ 0 \leq s \leq \xi, \\ (1-s)^{\alpha-2} - (1-s)^{\alpha-1}(1-\mu\xi^{\alpha-2}) \text{ for } \ \xi \leq s \leq 1. \end{cases}$$

Obviously, by **le2.2**, there is $G_1(t,s) \leq \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})}, t,s \in (0,1)$.
That is, the condition (ii) is true. □

Remark 2.1. [10] The Green's function $G_2(t,s)$ has the same properties as $G_1(t,s)$, where

$$G_2(t,s) = \begin{cases} \frac{(1-s)^{\beta-2}t^{\beta-1} - \mu(\xi-s)^{\beta-2}t^{\beta-1} - (t-s)^{\beta-1}(1-\mu\xi^{\beta-2})}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}, 0 \leq s \leq t \leq 1, s \leq \xi, \\ \frac{(1-s)^{\beta-2}t^{\beta-1} - (t-s)^{\beta-1}(1-\mu\xi^{\beta-2})}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}, 0 < \xi \leq s \leq t \leq 1, \\ \frac{(1-s)^{\beta-2}t^{\beta-1} - \mu(\xi-s)^{\beta-2}t^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}, 0 \leq t \leq s \leq \xi < 1, \\ \frac{(1-s)^{\beta-2}t^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}, 0 \leq t \leq s \leq 1, \xi \leq s. \end{cases} \tag{2.2}$$

Let $(E, \|\cdot\|_E)$ be a real Banach space, and it is partially ordered by a cone $P \subset E$. By θ it denotes the zero element of E. For $\forall x, y \in E$, $\theta \leq x \leq y$ and $\exists N > 0$, such that $\|x\|_E \leq N\|y\|_E$, then P is called normal. For $h > \theta$, we define a set $P_h = \{x \in E \mid \lambda h \leq x \leq \mu h, \lambda, \mu > 0\}$. Obviously, $P_h \subset P$. Take $e \in P$ with $\theta \leq e \leq h$, we define $P_{h,e} = \{x \in E | x + e \in P_h\}$. Let a vector $h = (h_1, h_2)$, and $h_1, h_2 \in P$ with $h_1, h_2 \neq \theta$, then $h \in \widetilde{P} := P \times P$. Obviously, if P is normal, then $\widetilde{P}$ is also normal. Take $\theta \leq e_1 \leq h_1, \theta \leq e_2 \leq h_2$, and let $\overline{\theta} = (\theta, \theta)$, $e = (e_1, e_2)$, then $\overline{\theta} = (\theta, \theta) \leq (e_1, e_2) \leq (h_1, h_2) = h$. That is, $\overline{\theta} \leq e \leq h$.

Definition 2.3. [26] Suppose that $T : P_{h,e} \to E$ is an operator and it satisfies: for $x \in P_{h,e}$ and $0 < \tau < 1$, there exists $\varphi(\tau) > \tau$, such that $T(\tau x + (\tau - 1)e) \geq \varphi(\tau)Tx + (\varphi(\tau) - 1)e$. Then T is called a φ-(h,e)-concave operator.

Lemma 2.4. *[26] Let P be normal and T be an increasing $\varphi - (h,e)-$concave operator satisfying $Th \in P_{h,e}$,then T has a unique fixed point $x^* \in P_{h,e}$. For $\omega_0 \in P_{h,e}$, defining a sequence $\omega_n = T\omega_{n-1}, n = 1, 2, \ldots,$ then $\|\omega_n - x^*\| \to 0 (n \to \infty)$.*

Lemma 2.5. *[25] $\widetilde{P}_h = \{(x,y) : x \in P_{h_1}, y \in P_{h_2}\}$, $\widetilde{P}_h = P_{h_1} \times P_{h_2}$.*

Lemma 2.6. *[3] $\widetilde{P}_{h,e} = P_{h_1,e_1} \times P_{h_2,e_2}$.*

3. Main results

We consider a Banach space $E = C[0,1]$, equipped with the norm

$$\|u\| = \max\{| \ u(t) \ | : t \in [0,1]\}.$$

Let $\|(u,v)\|_E = \max\{\|u\|, \|v\|\}$, for $(u,v) \in E \times E$, so $(E \times E, \|(\cdot,\cdot)\|_E)$ is a Banach space. Let P be the cone in E given by

$$P = \{u \in E | u(t) \geq 0, t \in [0,1]\},$$

then

$$\widetilde{P} = \{(u, v) \in E \times E | u(t), v(t) \geq 0, t \in [0, 1]\},$$

is also a cone. Obviously, $\widetilde{P} = P \times P \subset E \times E$ is normal, and we get

$$(u_1, v_1) \leq (u_2, v_2) \Longleftrightarrow u_1(t) \leq u_2(t) \ \text{and} \, v_1(t) \leq v_2(t), t \in [0, 1].$$

By Lemma 2.2 and the reference [10], we can obtain the following results.

Lemma 3.1. *Assume that f, g are continuous, then $(u, v) \in E \times E$ is a solution of system (1.1) if and only if $(u, v) \in E \times E$ is a solution of the following equations:*

$$\begin{cases} u(t) = \int_0^1 G_1(t, s) f(s, v(s), I^\gamma v(s)) ds - a \int_0^1 G_1(t, s) ds, \\ v(t) = \int_0^1 G_2(t, s) g(s, u(s), I^\delta u(s)) ds - b \int_0^1 G_2(t, s) ds. \end{cases}$$

For $(u, v) \in E \times E$, define operators $T_1 : E \to E, T_2 : E \to E$ and $T : E \times E \to E \times E$ as follows:

$$T_1 u(t) = \int_0^1 G_1(t, s) f(s, v(s), I^\gamma v(s)) ds - a \int_0^1 G_1(t, s) ds,$$

$$T_2 v(t) = \int_0^1 G_2(t, s) g(s, u(s), I^\delta u(s)) ds - b \int_0^1 G_2(t, s) ds,$$

$$T(u, v)(t) = (T_1 u(t), T_2 v(t)).$$

Thus, from **le3.1**, the solution of (1.1) is a fixed point of the operator T. Let

$$\begin{aligned} e_1(t) &= a \int_0^1 G_1(t, s) ds, \ e_2(t) &= b \int_0^1 G_2(t, s) ds, \\ h_1(t) &= L_1 t^{\alpha - 1}, \qquad h_2(t) &= L_2 t^{\beta - 1}, \end{aligned} \tag{3.1}$$

with $L_1 \geq \frac{a}{(1 - \mu \xi^{\alpha - 2}) \Gamma(\alpha)(\alpha - 1)}$, $L_2 \geq \frac{b}{(1 - \mu \xi^{\beta - 2}) \Gamma(\beta)(\beta - 1)}$.

Theorem 3.1. *Let $2 < \alpha, \beta \leq 3$, $a, b > 0$ and e_1, e_2, h_1, h_2 be given as in (3.1). Suppose that the following assumptions holds:*
(H_1) $f : [0, 1] \times [-e_2^*, +\infty) \times [-e_2^*, +\infty) \to [0, +\infty)$, $g : [0, 1] \times [-e_1^*, +\infty) \times [-e_1^*, +\infty) \to [0, +\infty)$ are increasing with respect to the second and third variables and continuous, where $e_2^* = \max\{e_2(t) : t \in [0, 1]\}$, $e_1^* = \max\{e_1(t) : t \in [0, 1]\}$;
(H_2) for $0 < \tau < 1$, there exists $\varphi(\tau) > \tau$ such that

$$f(t, \tau x_1 + (\tau - 1) y_1, \tau x_2 + (\tau - 1) y_1) \geq \varphi(\tau) f(t, x_1, x_2),$$

$$g(t, \tau x_1 + (\tau - 1) y_2, \tau x_2 + (\tau - 1) y_2) \geq \varphi(\tau) g(t, x_1, x_2),$$

$t \in [0, 1], x_1, x_2 \in (-\infty, +\infty), y_1 \in [0, e_2^*], y_2 \in [0, e_1^*]$;
(H_3) for $t \in [0, 1]$, $f(t, 0, 0) \geq 0, g(t, 0, 0) \geq 0$ with $f(t, 0, 0) \not\equiv 0$, $g(t, 0, 0) \not\equiv 0$.
Then
(1) system (1.1) has a unique solution $(u^*, v^*) \in \widetilde{P}_{h,e}$, where

$$e(t) = (e_1(t), e_2(t)), h(t) = (h_1(t), h_2(t)), t \in [0, 1];$$

(2) *for any point* $(u_0, v_0) \in \widetilde{P}_{h,e}$, *construct the following sequences:*

$$u_{n+1}(t) = \int_0^1 G_1(t,s)f(s, v_n(s), I^\gamma v_n(s))ds - a\int_0^1 G_1(t,s)ds, n = 0,1,2\cdots,$$

$$v_{n+1}(t) = \int_0^1 G_2(t,s)g(s, u_n(s), I^\delta u_n(s))ds - b\int_0^1 G_2(t,s)ds, n = 0,1,2\cdots,$$

we have

$$\|u_{n+1} - u^*\| \to 0, \|v_{n+1} - v^*\| \to 0, n \to \infty.$$

Proof. By **le2.3**, for $t \in [0,1]$, we have

$$e_1(t) = a\int_0^1 G_1(t,s)ds \ge 0, e_2(t) = b\int_0^1 G_2(t,s)ds \ge 0,$$

and

$$\begin{aligned} e_1(t) &= a\int_0^1 G_1(t,s)ds \le a\int_0^1 \frac{(1-s)^{\alpha-2}t^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}ds \\ &= \frac{at^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-2}ds = \frac{at^{\alpha-1}}{(1-\mu\xi^{\alpha-2})\Gamma(\alpha)(\alpha-1)} \\ &\le L_1 t^{\alpha-1} = h_1(t); \end{aligned}$$

$$\begin{aligned} e_2(t) &= b\int_0^1 G_2(t,s)ds \le b\int_0^1 \frac{(1-s)^{\beta-2}t^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}ds \\ &= \frac{bt^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)}\int_0^1 (1-s)^{\beta-2}ds = \frac{bt^{\beta-1}}{(1-\mu\xi^{\beta-2})\Gamma(\beta)(\beta-1)} \\ &\le L_2 t^{\beta-1} = h_2(t); \end{aligned}$$

So, $0 \le e_1 \le h_1$, $0 \le e_2 \le h_2$.
Firstly, we prove that $T : \widetilde{P}_{h,e} \to E \times E$ is a φ-(h,e)-concave operator.
For $(u,v) \in \widetilde{P}_{h,e}$, $0 < \tau < 1$, we have

$$\begin{aligned} T(\tau(u,v) + (\tau-1)e)(t) &= T(\tau(u,v) + (\tau-1)(e_1,e_2))(t) \\ &= T(\tau u + (\tau-1)e_1, \tau v + (\tau-1)e_2)(t) \\ &= (T_1(\tau u + (\tau-1)e_1), T_2(\tau v + (\tau-1)e_2))(t). \end{aligned}$$

By (H_2) and the continuity of Riemann-Liouville fractional integral I^γ and I^δ,

$$\begin{aligned} T_1(\tau u + (\tau-1)e_1)(t) &= \int_0^1 G_1(t,s)f(s, \tau v(s) + (\tau-1)e_2, I^\gamma(\tau v(s) \\ &\quad + (\tau-1)e_2))ds - e_1(t) \\ &\ge \varphi(\tau)\int_0^1 G_1(t,s)f(s, v(s), I^\gamma v(s))ds - e_1(t) \\ &= \varphi(\tau)[\int_0^1 G_1(t,s)f(s, v(s), I^\gamma v(s))ds - e_1(t)] \\ &\quad + [\varphi(\tau) - 1]e_1(t) \\ &= \varphi(\tau)T_1 u(t) + [\varphi(\tau) - 1]e_1(t), \end{aligned}$$

$$\begin{aligned}T_2(\tau v+(\tau-1)e_2)(t)&=\int_0^1 G_2(t,s)g(s,\tau u(s)+(\tau-1)e_1,I^\delta(\tau u(s)\\&\quad+(\tau-1)e_1))ds-e_2(t)\\&\geq\varphi(\tau)\int_0^1 G_2(t,s)g(s,u(s),I^\delta u(s))ds-e_2(t)\\&=\varphi(\tau)[\int_0^1 G_2(t,s)g(s,u(s),I^\delta u(s))ds-e_2(t)]\\&\quad+[\varphi(\tau)-1]e_2(t)\\&=\varphi(\tau)T_2v(t)+[\varphi(\tau)-1]e_2(t).\end{aligned}$$

So,

$$\begin{aligned}T(\tau(u,v)+(\tau-1)e)(t)&\geq(\varphi(\tau)T_1u(t)+[\varphi(\tau)-1]e_1(t),\varphi(\tau)T_2v(t)\\&\quad+[\varphi(\tau)-1]e_2(t))\\&=(\varphi(\tau)T_1u(t),\varphi(\tau)T_2v(t))\\&\quad+([\varphi(\tau)-1]e_1(t),[\varphi(\tau)-1]e_2(t))\\&=\varphi(\tau)(T_1u(t),T_2v(t))+(\varphi(\tau)-1)(e_1(t),e_2(t))\\&=\varphi(\tau)T(u,v)(t)+(\varphi(\tau)-1)e(t).\end{aligned}$$

That is,

$$T(\tau(u,v)+(\tau-1)e)\geq\varphi(\tau)T(u,v)+(\varphi(\tau)-1)e,(u,v)\in\widetilde{P}_{h,e},\tau\in(0,1).$$

Therefore, $T:\widetilde{P}_{h,e}\to E\times E$ is a φ-(h,e)-concave operator.
Secondly, we prove that $T:\widetilde{P}_{h,e}\to E\times E$ is increasing.
For $(u,v)\in\widetilde{P}_{h,e}$, we have $(u,v)+e\in\widetilde{P}_h$. From **le2.5**, $(u+e_1,u+e_2)\in P_{h_1}\times P_{h_2}$. So, there exists $\tau_1,\tau_2>0$ such that

$$u(t)+e_1(t)\geq\tau_1h_1(t),v(t)+e_2(t)\geq\tau_2h_2(t),t\in[0,1].$$

Thus, $u(t)\geq\tau_1h_1(t)-e_1(t)\geq-e_1(t)\geq-e_1^*$, $v(t)\geq\tau_2h_2(t)-e_2(t)\geq-e_2(t)\geq-e_2^*$. By (H_1) and the monotonicities of I^γ and I^δ, operator T is increasing.
Next, we show that $Th\in\widetilde{P}_{h,e}$, that is $Th+e\in\widetilde{P}_h$. For $t\in[0,1]$,

$$\begin{aligned}Th(t)+e(t)&=T(h_1,h_2)(t)+e(t)\\&=(T_1h_1(t),T_2h_2(t))+(e_1(t),e_2(t))\\&=(T_1h_1(t)+e_1(t),T_2h_2(t)+e_2(t)).\end{aligned}$$

By **le2.3** and (H_1), (H_3),

$$\begin{aligned}T_1h_1(t)+e_1(t)&=\int_0^1 G_1(t,s)f(s,h_2(s),I^\gamma h_2(s))ds\\&\geq\int_0^1 G_1(1,s)t^{\alpha-1}f(s,h_2(s),I^\gamma h_2(s))ds\\&\geq t^{\alpha-1}\int_0^1 G_1(1,s)f(s,L_2t^{\beta-1},I^\gamma L_2t^{\beta-1})ds\\&\geq t^{\alpha-1}\int_0^1 G_1(1,s)f(s,0,0)ds\\&=\frac{1}{L_1}h_1(t)\int_0^1 G_1(1,s)f(s,0,0)ds,\end{aligned}$$

$$\begin{aligned}
T_1h_1(t)+e_1(t) &= \int_0^1 G_1(t,s)f(s,h_2(s),I^\gamma h_2(s))ds\\
&\le \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})} f(s,L_2t^{\beta-1},I^\gamma L_2t^{\beta-1})ds\\
&\le \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})} f(s,L_2,I^\gamma L_2)ds\\
&\le \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})} f(s,L_2,L_2)ds\\
&= \frac{1}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})L_1} h_1(t)\int_0^1 (1-s)^{\alpha-2} f(s,L_2,L_2)ds.
\end{aligned}$$

By (H_1), (H_3) and **le2.3**,

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})} f(s,L_2,L_2)ds \ge \int_0^1 G_1(1,s)f(s,0,0)ds > 0.$$

Let

$$l_1 := \frac{1}{L_1}\int_0^1 G_1(1,s)f(s,0,0)ds,$$

$$l_2 := \frac{1}{\Gamma(\alpha)(1-\mu\xi^{\alpha-2})L_1}\int_0^1 (1-s)^{\alpha-2} f(s,L_2,L_2)ds,$$

and from **le2.3** and (H_3), $0<l_1\le l_2$. That is, $l_1h_1(t)\le T_1h_1(t)+e_1(t)\le l_2h_1(t)$. So, $T_1h_1+e_1\in P_{h_1}$. Similarly, we have $T_2h_2+e_2\in P_{h_2}$.
Therefore, by **le2.6**,

$$Th+e=(T_1h_1+e_1,T_2h_2+e_2)\in \widetilde{P}_h.$$

Finally, by **le2.4**, T has a unique fixed point $(u^*,v^*)\in \widetilde{P}_{h,e}$. That is, (1.1) has a unique solution $(u^*,v^*)\in \widetilde{P}_{h,e}$. In addition, taking any point $(u_0,v_0)\in \widetilde{P}_{h,e}$, define the sequences:

$$u_{n+1}(t)=\int_0^1 G_1(t,s)f(s,v_n(s),I^\gamma v_n(s))ds - a\int_0^1 G_1(t,s)ds, n=0,1,2,\dots,$$

$$v_{n+1}(t)=\int_0^1 G_2(t,s)g(s,u_n(s),I^\delta u_n(s))ds - b\int_0^1 G_2(t,s)ds, n=0,1,2,\dots,$$

we must have

$$\|u_{n+1}-u^*\|\to 0, \|v_{n+1}-v^*\|\to 0, n\to\infty.$$

□

Example 3.1. We consider the following system:

$$\begin{cases}
D^{\frac{5}{2}}u(t)+f(t,v(t),I^{\frac{7}{6}}v(t))=1, t\in(0,1),\\
D^{\frac{7}{3}}v(t)+g(t,u(t),I^{\frac{5}{4}}u(t))=1, t\in(0,1),\\
u(0)=u'(0)=0, u'(1)=\frac{1}{3}u'(\frac{1}{2})dt,\\
v(0)=v'(0)=0, v'(1)=\frac{1}{3}v'(\frac{1}{2})dt,
\end{cases} \tag{3.2}$$

where, $\alpha=\frac{5}{2}$, $\beta=\frac{7}{3}$, $\gamma=\frac{7}{6}$, $\delta=\frac{5}{4}$, $a=b=1$, $\mu=\frac{1}{3}$, $\xi=\frac{1}{2}$ and

$$\begin{aligned} f(t,x_1,x_2)=&\{\frac{1}{\Gamma(\frac{10}{3})}(-Ax_1-1)t^{\frac{7}{3}}+\frac{18\sqrt[3]{2}-3}{(24\sqrt[3]{2}-8)\Gamma(\frac{7}{3})}(Ax_1+1)t^{\frac{4}{3}}\}^{\frac{1}{3}}\\ &+\{\frac{1}{\Gamma(\frac{10}{3})}(-Ax_2-1)t^{\frac{7}{3}}+\frac{1}{\Gamma(\frac{7}{3})}\frac{18\sqrt[3]{2}-3}{24\sqrt[3]{2}-8}(Ax_2+1)t^{\frac{4}{3}}\}^{\frac{1}{3}},\\ g(t,x_1,x_2)=&\{\frac{1}{\Gamma(\frac{7}{2})}(-A_1x_1-1)t^{\frac{5}{2}}+\frac{6\sqrt{2}-1}{(9\sqrt{2}-3)\Gamma(\frac{5}{2})}(A_1x_1+1)t^{\frac{3}{2}}\}^{\frac{1}{3}}\\ &+\{\frac{1}{\Gamma(\frac{7}{2})}(-A_1x_2-1)t^{\frac{5}{2}}+\frac{6\sqrt{2}-1}{(9\sqrt{2}-3)\Gamma(\frac{5}{2})}(A_1x_2+1)t^{\frac{3}{2}}\}^{\frac{1}{3}}, \end{aligned}$$

$A=\frac{1}{\frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}-\frac{1}{\Gamma(\frac{10}{3})}}$, $A_1=\frac{1}{\frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}-\frac{1}{\Gamma(\frac{7}{2})}}$.

$$\begin{aligned} e_1(t)&=\int_0^1 G_1(t,s)ds\\ &=-\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}+\frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}}>0,\\ e_2(t)&=\int_0^1 G_2(t,s)ds\\ &=-\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}}+\frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}}>0,\\ e_1^*&=\max\{e_1(t),t\in[0,1]\}=-\frac{1}{\Gamma(\frac{7}{2})}+\frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)},\\ e_2^*&=\max\{e_2(t),t\in[0,1]\}=-\frac{1}{\Gamma(\frac{10}{3})}+\frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}. \end{aligned}$$

Set $h_1(t)=L_1t^{\frac{3}{2}}$, $h_2(t)=L_2t^{\frac{4}{3}}$, where

$$L_1\geq\frac{2\sqrt{2}}{\Gamma(\frac{5}{2})(3\sqrt{2}-1)},L_2\geq\frac{9\sqrt[3]{2}}{\Gamma(\frac{7}{3})(12\sqrt[3]{2}-4)}.$$

So,

$$\begin{aligned} e_1(t)&=\frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}}-\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}}<\frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}}<\frac{2\sqrt{2}}{\Gamma(\frac{5}{2})(3\sqrt{2}-1)}\\ &=L_1t^{\frac{3}{2}}=h_1(t),\\ e_2(t)&=\frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}}-\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}}<\frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}}<\frac{9\sqrt{2}-3}{\Gamma(\frac{7}{3})(12\sqrt[3]{2}-4)}t^{\frac{4}{3}}\\ &=L_2t^{\frac{4}{3}}=h_2(t), \end{aligned}$$

and

$$\begin{aligned} f(t,0,0)=&\{-\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}}+\frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}}\}^{\frac{1}{3}}\\ &+\{-\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}}+\frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}}\}^{\frac{1}{3}}=2(e_2(t))^{\frac{1}{3}}\geq 0, \end{aligned}$$

$$g(t,0,0) = \{-\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}}\}^{\frac{1}{3}}$$
$$+ \{-\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}}\}^{\frac{1}{3}} = 2(e_1(t))^{\frac{1}{3}} \geq 0,$$

with $f(t,0,0) \not\equiv 0$, $g(t,0,0) \not\equiv 0$. Obviously,

$$f(t,x_1,x_2) = \{Ae_2(t)x_1 + e_2(t)\}^{\frac{1}{3}} + \{Ae_2(t)x_2 + e_2(t)\}^{\frac{1}{3}},$$

$$g(t,x_1,x_2) = \{A_1e_1(t)x_1 + e_1(t)\}^{\frac{1}{3}} + \{A_1e_1(t)x_2 + e_1(t)\}^{\frac{1}{3}}.$$

For $\tau \in (0,1)$, $x_1, x_2 \in (-\infty,+\infty)$, $y_1 \in [0,e_2^*]$, $y_2 \in [0,e_1^*]$, taking $\varphi(\tau) = \tau^{\frac{1}{3}}$, we have

$$\begin{aligned}
&f(t,\tau x_1 + (\tau-1)y_1, \tau x_2 + (\tau-1)y_1)\\
&= \{Ae_2(t)(\tau x_1 + (\tau-1)y_1) + e_2(t)\}^{\frac{1}{3}} + \{Ae_2(t)(\tau x_2 + (\tau-1)y_1) + e_2(t)\}^{\frac{1}{3}}\\
&= \tau^{\frac{1}{3}}\{Ae_2(t)(x_1 + (1-\frac{1}{\tau})y_1) + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} + \tau^{\frac{1}{3}}\{Ae_2(t)(x_2 + (1-\frac{1}{\tau})y_1) + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}}\\
&= \tau^{\frac{1}{3}}\{Ae_2(t)x_1 + Ae_2(t)(\tau-1)y_1 + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} + \tau^{\frac{1}{3}}\{Ae_2(t)x_2\\
&+ Ae_2(t)(\tau-1)y_1 + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}}\\
&\geq \tau^{\frac{1}{3}}\{Ae_2(t)x_1 + (1-\frac{1}{\tau})y_1 + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}} + \tau^{\frac{1}{3}}\{Ae_2(t)x_2 + (1-\frac{1}{\tau})y_1 + \frac{1}{\tau}e_2(t)\}^{\frac{1}{3}}\\
&= \tau^{\frac{1}{3}}f(t,x_1,x_2) = \varphi(\tau)f(t,x_1,x_2),
\end{aligned}$$

Similarly, $g(t,\tau x_1 + (\tau-1)y_2, \tau x_2 + (\tau-1)y_2) \geq \varphi(\tau)g(t,x_1,x_2)$.
By **th3.1**, problem (3.2) has a unique solution $(u^*,v^*) \in \widetilde{P}_{h,e}$, where

$$e(t) = (e_1(t),e_2(t)) = (-\frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} + \frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}}, -\frac{1}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}} + \frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}}),$$

$$h(t) = (h_1(t),h_2(t)) = (L_1t^{\frac{3}{2}}, L_2t^{\frac{4}{3}}), t \in [0,1].$$

For $\forall(u_0,v_0) \in \widetilde{P}_{h,e}$, construct the following two sequences:

$$u_{n+1}(t) = \int_0^1 G_1(t,s)f(s,v_n(s),I^{\frac{7}{6}}v_n(s))ds + \frac{1}{\Gamma(\frac{7}{2})}t^{\frac{5}{2}} - \frac{6\sqrt{2}-1}{\Gamma(\frac{5}{2})(9\sqrt{2}-3)}t^{\frac{3}{2}},$$

$$v_{n+1}(t) = \int_0^1 G_2(t,s)f(s,u_n(s),I^{\frac{5}{4}}u_n(s))ds + \frac{1}{\Gamma(\frac{10}{3})}t^{\frac{7}{3}} - \frac{18\sqrt[3]{2}-3}{\Gamma(\frac{7}{3})(24\sqrt[3]{2}-8)}t^{\frac{4}{3}},$$

$n = 0,1,2,\ldots$. Then,

$$\|u_{n+1} - u^*\| \to 0, \|v_{n+1} - v^*\| \to 0, n \to \infty.$$

References

[1] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1993.

[3] C. Zhai, R. Jiang, *Unique solutions for a new coupled system of fractional differential equations*, Adv. Differ. Equ, 2018, 2018: 1.

[4] C. Yang, C. Zhai, L. Zhang, *Local uniqueness of positive solutions for a coupled system of fractional differential equations with integral boundary conditions*, Adv. Differ. Equ, 2017, 2017: 282.

[5] S. K. Ntouyas, H. H. Al-Sulami, *A study of coupled systems of mixed order fractional differential equations and inclusions with coupled integral fractional boundary conditions*, Adv. Differ Equ., 2020, 2020: 73.

[6] J. Jiang, D. O'Regan, J. Xu, et al, *Positive solutions for a system of nonlinear Hadamard fractional differential equations involving coupled integral boundary conditions*, J. Inequ. Appl, 2019, 2019: 204.

[7] A. Lachouri, A. Ardjouni, A. Djoudi, *Positive solutions of a fractional integro-differential equation with integral boundary conditions*, Commun. Optim. Theory, 2020, 2020(1): 1-9.

[8] R. Sevinik-Adigüzel, ü. Aksoy, E. Karapinar, et al, *Uniqueness of solution for higher-order nonlinear fractional differential equations with multipoint and integral boundary conditions*, Racsam. Rev. R. Acad. A, 2021, 115(3): 155.

[9] A. Salem, F. Alzahrani, M. Alnegga, *Coupled system of nonlinear fractional Langevin equations with multipoint and nonlocal integral boundary conditions*, Math. Probl. Eng, 2020, 2020: 7345658.

[10] H. Feng, X. Zhang, *Existence and uniqueness of positive solutions for a coupled system of fractional differential Equations*, Discrete. Dyn. Nat. Soc, 2022, 2022:7779977.

[11] A. Boutiara, M. Benbachir, *Existence and uniqueness results to a fractional q-difference coupled system with integral boundary conditions via topological degree theory*, Int. J. Nonlinear. Anal, 2022, 13(1): 3197-3211.

[12] M. Awadalla, *Some existence results for a system of nonlinear sequential fractional differential equations with coupled nonseparated boundary conditions*, Complexity, 2022, 2022:8992894.

[13] X. Zhang, L. Liu, Y. Wu, et al, *Existence and uniqueness of solutions for systems of fractional differential equations with Riemann-Stieltjes integral boundary condition*, Adv. Differ. Equ, 2018, 2018:204.

[14] B. A. Ahmed Alsaedi, A. Assolami, S. K. Ntouyas, *On a nonlinear coupled system of differential equations involving Hilfer fractional derivative and Riemann-Liouville mixed operators with nonlocal integro-multi-point boundary conditions*, AIMS Mathematics, 2022, 7(7): 12718-12741.

[15] A. Ghanmi, S. Horrigue, *Existence of positive solutions for a coupled system of nonlinear fractional differential equations*, Ukr. Math. J, 2019, 71: 39-49.

[16] A. Boutiara, J. Alzabut, M. Ghaderi, et al, *On a coupled system of fractional (p, q)-differential equation with Lipschitzian matrix in generalized metric space*,AIMS Mathematics, 2023, 8(1): 1566-1591.

[17] X. Su, *Boundary value problem for a coupled system of nonlinear fractional differential equations*, Appl. Math. Letters, 2009, 22(1): 64-69.

[18] B. Ahmad, J. Nieto, *Existence results for a coupled system of nonlinear fractional differential equations with three-point boundary conditions*, Comput. Math. Appl, 2009, 58(9): 1838-1843.

[19] B. Ahmad, M. Alblewi, S. K. Ntouyas, et al, *Existence results for a coupled system of nonlinear multi-term fractional differential equations with anti-periodic type coupled nonlocal boundary conditions*, Math. Method. Appl. Sci, 2021, 44(11): 8739-8758.

[20] Y. Arioua, M. Titraoui, *Boundary value problem for a coupled system of nonlinear fractional differential equations involving Erdélyi-Kober derivative*, Appl. Math. E-Notes, 2021, 21: 291-306.

[21] D. Zhao, Y. Liu, *Positive solutions for a class of fractional differential coupled system with integral boundary value conditions*, J. Nonlinear Sci. Appl, 2016, 9(5): 2922-2942.

[22] D. Zhao, J. Mao, *Positive solutions for a Class of nonlinear singular fractional differential systems with Riemann-Stieltjes coupled integral boundary value conditions*, Symmetry, 2021, 13(1): 107.

[23] H. Wang, J. Jiang, *Existence and multiplicity of positive solutions for a system of nonlinear fractional multi-point boundary value problems with p-Laplacian operator*, J. Appl. Anal. Comput, 2021, 11: 351-366.

[24] W. Yang, *Three-point boundary value problems for a coupled system of nonlinear fractional differential equations*, J. Appl. Math. Inform, 2012, 30(5-6): 773-785.

[25] C. Zhai., J. Ren, *Some properties of sets, fixed point theorems in ordered product spaces and applications to a nonlinear system of fractional differential equations* , Topol. Methods. Nonlinear Anal, 2017, 49(2): 625-645.

[26] C. Zhai, L. Wang, *φ-(h, e)-concave operators and applications*, J. Math. Anal. Appl, 2017, 454: 571-584..

[27] B. Ahmad, S.K. Ntouyas, *A fully Hadamard type integral boundary value problem of a coupled system of fractional differential equations*, Fract. Calc. Appl. Anal, 2014, 17(2):348-360.

[28] J. Wang, M. Feckan, Y. Zhou, *A survey on impulsive fractional differential equations*, Fract. Calc. Appl. Anal, 2016, 19: 806-831.

[29] J. Wang, M. Feckan, Y. Zhou, *Fractional order differential dwitched dystems with coupled nonlocal initial and impulsive conditions*, B. Sci. Math, 2017, 141: 727-746.

[30] D. Luo, Z. Luo, *Existence and Hyers-Ulam stability results for a class of fractional order delay differential equations with non-instantaneous impulses*, Math. Slovaca, 2020, 70: 1231-1248.

[31] J. Xu, L. Liu, S. Bai, Y. Wu, *Solvability for a system of Hadamard fractional multi-point boundary value problems*, Nonlinear Anal. Model. Control, 2021, 26(3): 502-521.

[32] H. Wang, L. Zhang, *Uniqueness methods for the higher-order coupled fractional differential systems with multi-point boundary conditions*, B. Sci. Math, 2021, 166 : 102935.

[33] W. Wang, *Unique positive solutions for boundary value problem of p-Laplacian fractional differential equation with a sign-changed nonlinearity*, Nonlinear Anal. Model. Control, 2022, 27(6) : 1110-1128.

[34] J.E. Lazreg, S. Abbas, M. Benchohra, E. Karapinar, *Impulsive Caputo-Fabrizio fractional differential equations in b-metric spaces*, Open Mathematics, 2021, 19(1): 363-372.

[35] A. Salim, M. Benchohra, E. Karapinar, et al. *Existence and Ulam stability for impulsive generalized Hilfer-type fractional differential equations*, Adv Differ Equ 2020, 601 (2020).

[36] R.S. Adiguzel, U. Aksoy, E. Karapinar, I.M. Erhan, *On the solution of a boundary value problem associated with a fractional differential equation*, Math. Meth. Appl. Sci., 2020,1-12.

[37] R.S. Adiguzel, U. Aksoy, E. Karapinar, I.M. Erhan, *On the solutions of fractional differential equations via Geraghty type hybrid contractions*, 2021, 20(2): 313-333.

[38] S. Krim, A. Salim, M. Benchohra, *on implicit Caputo tempered fractional boundary value problems with delay*, lett. nonlinear anal. appl., 2023, 1(1):12-29.

[39] V. Roomi, H. Afshari, M. Nosrati, *Existence and uniqueness for a fractional differential equation involving Atangana-Baleanu derivative by using a new contraction*, lett. nonlinear anal. appl., 2023, 1(2):52-56.