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Uniqueness for the Semilinear Elliptic Problems*

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Abstract In this paper, we study the positive solutions of the semilinear elliptic equation

$$\begin{cases} Lu + g(x,u)u = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, L is an elliptic operator, B is a general boundary operator and $g(\cdot,\cdot)$ is a continuous function. This is a general problem proposed by Amann [*Arch. Rational Mech. Anal.* 44 (1972)], Cac [*J. London Math. Soc.* 25 (1982)] and Hess [*Math. Z.* 154 (1977)]. We obtain various uniqueness results when the nonlinearity function g satisfies some additional conditions.

Keywords Elliptic, reaction-diffusion equation, uniqueness

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1. Introduction

In this paper, we study the positive solutions of the elliptic problem

$$\begin{cases} Lu + g(x,u)u = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded and smooth domain, the boundary $\partial\Omega$ consists of disjoint open subset Γ_0 and closed subset Γ_1 with finitely many components such that $\partial\Omega = \Gamma_0 \cup \Gamma_1$, and

$$L = -\sum_{i,j=1}^{N} a_{ij} D_i D_j + \sum_{i=1}^{N} a_i(x) D_i + a_0(x)$$

is a second order uniformly elliptic operator with $a_{ij}, a_i, a_0 \in C^\mu(\bar{\Omega})$, $i,j = 1,\cdots,N$. In (1.1), the boundary operator

$$B : C(\Gamma_0) \cup C^1(\Omega \cup \Gamma_1) \to C(\partial\Omega)$$

is given by

$$Bu = \begin{cases} u & \text{on } \Gamma_0, \\ \frac{\partial u}{\partial n} + \beta(x)u & \text{on } \Gamma_1, \end{cases}$$

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where n is the outward unit normal of $\partial\Omega$, and $\beta \geq 0$ is continuous. Note that Γ_0 or Γ_1 can be empty. The operator B includes Dirichlet, Neumann as well as the Robin boundary condition. As far as the function $g(x,u)$, we assume that $g \in C(\Omega \times \mathbb{R}^+; \mathbb{R}^+)$ and there is a positive constant M such that

$$a_0(x) + g(x, M) \geq 0 \text{ in } \bar{\Omega}. \tag{1.2}$$

Throughout the paper, we consider the positive solution $\omega \in C^{2+\mu}(\Omega)$ of (1.1) with $0 \leq \omega(x) \leq M$ and $\omega(x) \not\equiv 0$ in $\bar{\Omega}$.

Note that (1.1) is widely used in the study of various diffusion problems and it is also called the diffusive logistic model [4, 5, 7, 8, 10, 12, 14–16]. In 1977, Hess [13] investigated the uniqueness problem on positive solutions of

$$\begin{cases} \Delta u = g(x,u)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Since then, the positive solution problem (1.3) has attracted much attention, see Cac [9], Allegretto [1] and references therein. In this paper, we shall consider the uniqueness of positive solutions to the general problem (1.1). We shall obtain various uniqueness results under the assumptions that the nonlinear function g satisfies some additional conditions. In the case of (1.3), our results partially improve the classical conclusions of [1, 13]. We also prove the stability of positive solutions to (1.1).

Throughout this paper, let $\lambda[L; B, \Omega]$ be the principal eigenvalue of

$$\begin{cases} Lu = \lambda u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

associated with a positive eigenfunction $\phi(x)$. We then have

$$\frac{\partial \phi}{\partial n} < 0 \text{ for } x \in \Gamma_0,$$

and

$$\phi(x) > 0 \text{ for } x \in \Omega \cup \Gamma_1,$$

see [17, Chapter 8]. Moreover, we have

$$\lambda[L + s; B, \Omega] = \lambda[L; B, \Omega] + s$$

for all $s \in \mathbb{R}$ and

$$\lambda[L + f_1; B, \Omega] < \lambda[L + f_2; B, \Omega]$$

for any bounded functions $f_1 < f_2$ and $f_1 \not\equiv f_2$.

We first state our results on the existence of positive solutions to (1.1).

Theorem 1.1. *If $g(x,0) < -\lambda[L; B, \Omega]$ for $x \in \bar{\Omega}$, then (1.1) admits a positive solution $\omega(x)$ satisfying*

$$\frac{\partial \omega}{\partial \nu} < 0 \textit{ for } x \in \Gamma_0,$$

and

$$\omega(x) > 0 \textit{ for } x \in \Omega \cup \Gamma_1.$$

Theorem 1.2. *If $g(x,s) > g(x,0)$ for $x \in \bar{\Omega}$ and $s \in (0, M]$, then we know that* (1.1) *admits a positive solution if and only if*

$$\lambda[L + g(x,0); B, \Omega] < 0.$$

Our next result is concerning with the uniqueness problem on the positive solutions of (1.1).

Theorem 1.3. *Fix $x \in \bar{\Omega}$. Assume that $g(x,s)$ is increasing with respect to $s > 0$. If there exists a subset Ω_0 of Ω with the positive measure such that $g(x,s)$ is strictly increasing in s for $x \in \Omega_0$, then* (1.1) *has at most one positive solution.*

If $a_0(x)$ satisfies some additional assumptions, we can prove the following result. Note that the uniqueness result holds for general nonlinear function g which may not satisfy the assumption (1.2), see Remark 3.1.

Theorem 1.4. *Suppose that $g(x,s)$ is non-negative and increasing with $s > 0$ for $x \in \bar{\Omega}$ and $a_0(x) \geq 0$ for $x \in \bar{\Omega}$. Then* (1.1) *has at most one positive solution.*

Our last unique result is concerned with the following reaction-diffusion equation

$$\begin{cases} \Delta u - g(x,u)u = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

Theorem 1.5. *Assume that there exists $l > 0$ such that $g(x,s)$ satisfies*

$$g(x,s_1) - g(x,s_2) < l(s_1 - s_2) \quad \text{or} \quad g(x,s_1) - g(x,s_2) > l(s_1 - s_2),$$

where $x \in \bar{\Omega}$ and $s_1 > s_2 \geq 0$. Then (1.5) *has at most one positive solution.*

Finally, we study the dynamical behavior of parabolic problem

$$\begin{cases} u_t + Lu + g(x,u)u = 0 & \text{in } \Omega \times (0,\infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) & \text{on } \bar{\Omega}, \end{cases} \tag{1.6}$$

where $u_0 \in C^{2+\mu}(\bar{\Omega})$ is nontrivial and

$$0 \leq u_0(x) \leq M$$

for $x \in \bar{\Omega}$. It is known that (1.6) admits a unique non-negative solution $u(x,t;u_0)$. Note that a non-negative solution $\omega(x)$ of (1.1) is (globally asymptotically) stable if

$$\lim_{t\to\infty} \|u(\cdot,t;u_0) - \omega(\cdot)\|_{C(\bar{\Omega})} = 0.$$

We are ready to give the longtime behavior of the solutions to (1.6).

Theorem 1.6. *Assume that $g(x,s) > g(x,0)$ for $x \in \bar{\Omega}$ and $s \in (0, M]$.*

(a) *If $\lambda[L + g(x,0); B, \Omega] \geq 0$, then the trivial solution 0 is stable.*

(b) *If $\lambda[L + g(x,0); B, \Omega] < 0$, then* (1.1) *has a positive solution and it is stable.*

The rest of this paper is organized as follows. Section 2 contains the existence problem of (1.1). Then we devote to Section 3 the uniqueness results on the positive solutions of (1.1). In Section 4, we study the parabolic problem (1.6) and prove Theorem 1.6.

2. Existence of positive solutions

In this section, we study the semilinear equation

$$\begin{cases} Lu + g(x,u)u = 0 & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

It is known that various methods are widely used in the study of existence problem to (2.1), e.g. Amann [4], and Crandall and Rabinowitz [11]. Here we prove that there exists a positive solution to (2.1) by employing the upper-lower solutions.

Definition 2.1. The function $u \in C^{2+\mu}(\bar{\Omega})$ is an upper-solution to (2.1) if

$$\begin{cases} Lu + g(x,u)u \geq 0 & \text{in } \Omega, \\ Bu \geq 0 & \text{on } \partial\Omega. \end{cases}$$

The lower-solution is defined similarly by reversing the inequalities.

We then have the following lemma, see [2, Theorem A].

Lemma 2.1. *Let $\hat{v} \geq \bar{v}$ be a pair of upper-lower solutions to* (2.1)*. Then there exist a maximum solution $\hat{u}$ and a minimum solution $\bar{u}$ of* (2.1) *in the sense that, for every solution u of* (2.1)*, the inequality*

$$\bar{u} \leq u \leq \hat{u}$$

holds.

Proposition 2.1. *Suppose that* (1.2) *holds and* $g(x,0) < -\lambda[L;B,\Omega]$*. Then* (1.1) *admits a positive solution ω such that*

$$\omega(x) > 0 \text{ for } x \in \Omega \cup \Gamma_1 \tag{2.2}$$

and

$$\frac{\partial \omega}{\partial \nu} < 0 \text{ for } x \in \Gamma_0. \tag{2.3}$$

Proof. Let $\phi(x)$ be a positive eigenfunction associated with $\lambda[L;B,\Omega]$. Then we know that

$$\phi(x) > 0 \text{ for } x \in \Omega \cup \Gamma_1.$$

Given $\varepsilon > 0$ and set $\hat{u}(x) = \varepsilon\phi(x)$, we can see

$$L\hat{u} + g(x,\hat{u})\hat{u} = (\lambda[L;B,\Omega] + g(x,\varepsilon\phi(x)))\varepsilon\phi(x).$$

But $g(x,0) < -\lambda[L;B,\Omega]$, we can take $\varepsilon_0 > 0$ such that

$$L\hat{u} + g(x,\hat{u})\hat{u} \leq 0,$$

provided $0 < \varepsilon \leq \varepsilon_0$ and $x \in \Omega$. Therefore $\hat{u}$ is a lower-solution.

Note that $a_0(x) + g(x,M) \geq 0$ for $x \in \bar{\Omega}$. We obtain that $v(x) \equiv M$ satisfies

$$Lv + g(x,v)v = a_0(x)M + g(x,M)M \geq 0.$$

This implies that M is an upper-solution of (2.1). Thus we obtain that there is a positive solution by Lemma 2.1.

Now let $\omega(x)$ be a positive solution of (2.1). We get

$$\lambda[L + g(x, \omega); B, \Omega] = 0.$$

This together with Hopf's Lemma implies that (2.2) and (2.3) hold. □

We then give a sufficient and necessary condition on the existence of positive solutions to (2.1).

Proposition 2.2. *Assume that $g(x, s) > g(x, 0)$ for $x \in \bar{\Omega}$ and $s \in (0, M]$. Then* (1.1) *exists a positive solution if and only if*

$$\lambda[L + g(x, 0); B, \Omega] < 0$$

for $x \in \bar{\Omega}$.

Proof. Let $\phi(x) > 0$ be an eigenfunction of $\lambda[L + g(x, 0); B, \Omega]$. We have

$$\phi(x) > 0 \text{ for } x \in \Omega \cup \Gamma_1.$$

For $\varepsilon > 0$, denote $\hat{u}(x) = \varepsilon\phi(x)$, then we get

$$L\hat{u} + g(x, \hat{u})\hat{u} = (\lambda[L + g(x, 0); B, \Omega] - g(x, 0) + g(x, \varepsilon\phi(x)))\varepsilon\phi(x).$$

This implies that we can take $\varepsilon_0 > 0$ such that

$$L\hat{u} + g(x, \hat{u})\hat{u} \leq 0$$

for $0 < \varepsilon \leq \varepsilon_0$ and $x \in \Omega$. We know that $\hat{u}$ is a lower-solution to (2.1). Since M is an upper-solution of (2.1), we obtain that there exists a positive solution u to (2.1) such that

$$u(x) > 0 \text{ for } x \in \Omega \cup \Gamma_1,$$

and

$$\frac{\partial u}{\partial \nu} < 0 \text{ for } x \in \Gamma_0.$$

Note that $u(x)$ is a positive eigenfunction of $\lambda[L + g(x, u); B, \Omega]$. We get

$$0 = \lambda[L + g(x, u); B, \Omega] > \lambda[L + g(x, 0); B, \Omega].$$

The proof is thus complete. □

Our main results Theorems 1.1 and 1.2 are followed by Propositions 2.1-2.2.

3. Uniqueness of positive solution

In this section, we analyze the uniqueness problem on the positive solutions of (1.1). Let us begin to consider the case that $g(x, s)$ is increasing in s.

Proposition 3.1. *Suppose that $g(x, s)$ is increasing with respect to $s \in [0, M]$ for $x \in \bar{\Omega}$. If there exists a subset Ω_0 of Ω with the positive measure such that $g(x, s)$ is strictly increasing with respect to $s \in [0, M]$ for $x \in \Omega_0$, then there exists at most one positive solution to* (1.1).

Proof. Let $u(x)$ and $v(x)$ be two positive solutions of (1.1). We may assume that

$$u(x) \geq v(x)$$

for $x \in \bar{\Omega}$. Otherwise, note that the constant M is an upper-solution, while $\max\{u(x), v(x)\}$ is a lower-solution to (1.1). We can obtain new ordered two solutions. Using the uniqueness of principal eigenvalues, we must have

$$\lambda[L + g(x,u); B, \Omega] = \lambda[L + g(x,v); B, \Omega] = 0.$$

Since $g(x,s)$ is strictly increasing for $s \in [0, M]$ and $x \in \Omega_0$, we get

$$\lambda[L + g(x,u); B, \Omega] > \lambda[L + g(x,v); B, \Omega]$$

if $u(x) \not\equiv v(x)$ for $x \in \Omega$. This also shows the uniqueness of positive solutions. □

Following a similar argument as above, we can show the following result.

Corollary 3.1. *Suppose that $g(x,s)$ is decreasing with respect to $s \in [0, M]$ for every $x \in \bar{\Omega}$. If there exists a subset Ω_0 of Ω with the positive measure such that $g(x,s)$ is strictly decreasing with $s \in [0, M]$ for $x \in \Omega_0$, then there exists at most one positive solution to* (1.1).

The strictly monotone condition of function g is added in the study of uniqueness problem to the Dirichlet problem (1.3) by Hess [13].

Proposition 3.2. *Suppose that $g(x,s)$ is non-negative and increasing with respect to $s > 0$ for every $x \in \bar{\Omega}$. If $a_0(x) \geq 0$ for $x \in \bar{\Omega}$, then there exists at most one positive solution to* (1.1).

Proof. If there exist two positive solutions u and v of (1.1) such that

$$u(x) \not\equiv v(x)$$

for $x \in \bar{\Omega}$. We may assume that

$$\Omega^* = \{x \in \bar{\Omega} : u(x) > v(x)\} \tag{3.1}$$

is not empty. We can see that $u(x) = v(x)$ for $x \in \partial\Omega^*$. Then a direct computation gives that

$$L(u - v) + [g(x,u) - g(x,v)]u + g(x,v)(u - v) = 0$$

for $x \in \Omega$. By the monotone property of g, we have

$$L(u - v) \leq 0$$

for $x \in \Omega^*$. By the maximum principle,

$$u(x) < v(x) \text{ or } u(x) \equiv v(x)$$

for $x \in \Omega^*$. This contradicts (3.1) and the proof follows. □

Remark 3.1. In the proof of Proposition 3.2, we do not need the assumption (1.2). In fact, Proposition 3.2 is a general unique result to the positive solutions of (1.1).

At the end of this section, we investigate the initial equation

$$\begin{cases} \Delta u = g(x,u)u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Proposition 3.3. *Suppose that there exists $l > 0$ such that*

$$g(x, s_1) - g(x, s_2) < l(s_1 - s_2) \tag{3.3}$$

for $x \in \bar{\Omega}$, $s_1 > s_2 \geq 0$. Then (3.2) admits at most one positive solution.

Proof. Let u, v be two different positive solutions of (3.2). We still assume that

$$\Omega^* = \{x \in \bar{\Omega} : u(x) > v(x)\}$$

is not empty and $u(x) = v(x)$ for $x \in \partial\Omega^*$. Note that

$$\begin{cases} \Delta u = g(x, u)u & \text{in } \Omega, \\ Bu = 0 & \text{on } \partial\Omega, \end{cases}$$

and

$$\begin{cases} \Delta v = g(x, v)v & \text{in } \Omega, \\ Bv = 0 & \text{on } \partial\Omega, \end{cases}$$

we obtain

$$\int_{\Omega^*} v(x)[\Delta u + g(x, u)u]dx = \int_{\Omega^*} u(x)[\Delta v + g(x, v)v]dx.$$

This implies that

$$\int_{\Omega^*} [g(x, u) - l]u(x)v(x)dx = \int_{\Omega^*} [g(x, v) - l]u(x)v(x)dx.$$

By (3.3) we get a contradiction. □

Corollary 3.2. *Suppose that there exists $l > 0$ such that $g(x, s)$ satisfies*

$$g(x, s_1) - g(x, s_2) > l(s_1 - s_2)$$

for $x \in \bar{\Omega}$, $s_1 > s_2 \geq 0$. Then (3.2) has at most one positive solution.

By the above conclusions, we know that Theorems 1.3-1.5 hold.

4. Stability of positive solutions

In this section, we study the initial value problem

$$\begin{cases} u_t + Lu + g(x, u)u = 0 & \text{in } \Omega \times (0, \infty), \\ Bu = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{on } \bar{\Omega}, \end{cases} \tag{4.1}$$

where $u_0 \in C^{2+\mu}(\bar{\Omega})$ is nontrivial and

$$0 \leq u_0(x) \leq M \tag{4.2}$$

for $x \in \bar{\Omega}$. If follows that (4.1) has a unique global solution $u(x, t; u_0)$.

Let us begin to show that the trivial stationary solution zero of (4.1) is stable if $\lambda[L + g(x, 0); B, \Omega] \geq 0$.

Lemma 4.1. *Suppose that $g(x,s) > g(x,0)$ for $x \in \bar{\Omega}$ and $s \in (0,M]$. If*

$$\lambda[L + g(x,0); B, \Omega] \geq 0,$$

then zero is globally asymptotically stable.

Proof. Let $\phi(x)$ be a positive eigenfunction associated with $\lambda[L + g(x,0); B, \Omega]$. Since $\lambda[L + g(x,0); B, \Omega] \geq 0$, we can see that $\hat{u}(x) = \varepsilon\phi(x)$ satisfies

$$\hat{u}_t + L\hat{u} + g(x,\hat{u})\hat{u} = (\lambda[L + g(x,0); B, \Omega] - g(x,0) + g(x, \varepsilon\phi(x)))\varepsilon\phi(x) \geq 0$$

for $\varepsilon > 0$ and $x \in \Omega$. We obtain that $\hat{u}$ is an upper-solution of (4.1) provided

$$\varepsilon\phi(x) \geq u_0(x)$$

$x \in \Omega$. Note that $u(x,t;\varepsilon\phi(x))$ is decreasing and converges to a stationary solution of (1.1), see [4]. By Proposition 2.2, the positive solution only exists when

$$\lambda[L + g(x,0); B, \Omega] < 0.$$

Note also that $u(x,t;\varepsilon\phi(x))$ is non-negative, we know that

$$\lim_{t\to\infty} \|u(\cdot,t;u_0)\|_{C(\bar{\Omega})} = 0.$$

□

Lemma 4.2. *Suppose that $g(x,s) > g(x,0)$ for $x \in \bar{\Omega}$ and $s \in (0,M]$. If $\lambda[L + g(x,0); B, \Omega] < 0$, then the unique positive stationary solution ω of (4.1) is globally asymptotically stable.*

Proof. Since we consider the longtime behavior of (4.1), we may assume that $u_0(x) > 0$ for $x \in \bar{\Omega}$. Let $\phi(x)$ be a positive eigenfunction associated with $\lambda[L + g(x,0); B, \Omega]$. For $\varepsilon > 0$, denote $\hat{u}(x) = \varepsilon\phi(x)$. Then

$$\hat{u}_t + L\hat{u} + g(x,\hat{u})\hat{u} = (\lambda[L + g(x,0); B, \Omega] - g(x,0) + g(x, \varepsilon\phi(x)))\varepsilon\phi(x)$$

and we can take $\varepsilon_0 > 0$ such that

$$\hat{u}_t + L\hat{u} + g(x,\hat{u})\hat{u} \leq 0,$$

provided $0 < \varepsilon \leq \varepsilon_0$. In this case, we get $\hat{u}$ is a lower-solution of (2.1). Note that M is an upper-solution to (4.1). We can see that $u(x,t;M)$ is decreasing and $u(x,t;\varepsilon\phi(x))$ is increasing with respect to t, such that

$$0 < u(x,t;\varepsilon\phi(x)) \leq u(x,t;u_0) \leq u(x,t;M) \tag{4.3}$$

for $x \in \Omega$. Since $\lambda[L + g(x,0); B, \Omega] < 0$, we know from (4.3) that

$$\lim_{t\to\infty} \|u(\cdot,t;u_0) - \omega(\cdot)\|_{C(\bar{\Omega})} = 0$$

for some positive solution ω of (1.1). □

According to Lemmas 4.1-4.2, we obtain Theorem 1.6.

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