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Pseudo-Differential Operators and $\mathfrak{T}$- Wigner Function on Locally Compact Communicative Hausdorff Groups

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Abstract In this article, we consider a harmonic analysis of locally compact groups and introduce a generalization of the classical cross-Wigner distribution defined on $G \times \hat{G}$ by

$$W_{\mathfrak{T}}(\psi, \varphi)(g,\ \xi) = \int_G \overline{\xi(h)} \psi(\tau_1(g,\ h)) \overline{\varphi(\tau_2(g,\ h))} d\mu(h).$$

We construct the so-called Weyl-Heisenberg frame on a locally compact communicative Hausdorff group and establish its properties. Thus, we show that assume Λ and Γ are closed cocompact subgroups of G and $\hat{G}$, respectively, then, for a given window $\phi \in L^2(G)$, either both systems $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ and $\{m_\kappa \tau_\upsilon \phi\}_{\kappa\in\Lambda^\perp,\ \upsilon\in\Gamma^\perp}$ are Gabor systems in $L^2(G)$, simultaneously, with the same upper bound, or neither $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ nor $\{m_\kappa \tau_\upsilon \phi\}_{\kappa\in\Lambda^\perp,\ \upsilon\in\Gamma^\perp}$ comprises a Gabor system. Finally, pseudo-differential operators on locally compact groups are studied, we establish that assuming a pseudo-differential operator A_a corresponds to the symbol $a \in W_{\tau,\ 1\circ\iota^{-1}}^{\infty,1}\left(G\times\hat{G}\right)$ then A_a is bounded operator $W_\tau^{p,q}(G) \to W_\tau^{p,q}(G)$.

Keywords Fourier transform, Wigner function, compact group, pseudo-differential operator, symbol, Weyl-Heisenberg frame

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1. Introduction (classical theory)

The concept of the pseudo-differential operator is a generalization of the partial differential operator. The theory of pseudo-differential operators is a fundamental tool of quantum physics and is widely interweaved in partial differential equations. The pioneer works belong to J. Kohn, L. Nirenberg, L. Hormander, E. Stein, and others [18, 26].

There are many articles dedicated to pseudo-differential operators and cross-Wigner functions for some recent issues that the reader may be interested in [1, 2, 11, 17, 21, 22]. In [7], authors consider Wigner analysis of linear operators, replacing standard Wigner function with the A-Wigner distribution with a symplectic matrix and developing a theory of global Hormander wavefront. The Gaussian state is

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considered [3] and a more complex system in [20]. The list of references consists of 26 articles.

The Weyl quantization is a correspondence between the set of pseudo-differential operators A_a closely and densely defined on the Hilbert space and the class of functions $a(x,z)$ mapping on the phase space $R^n \times R^n$. In its classical form [18], Weyl quantization is given by

$$A_a(f)(x) = \int_{R^n} a(x,z) \exp(2\pi i x \cdot z) \hat{f}(z)\, dz,$$

where the Fourier transform $\hat{f}$ of the function f is defined by

$$\hat{f}(x) = \int_{R^n} \exp(-2\pi i x \cdot z) \hat{f}(z)\, dz.$$

The Weyl quantization is intimately connected with the Wigner transform and cross-Wigner pseudo-distribution is defined by

$$W(f,g)(x,\ y) = \int_{R^n} \exp(-2\pi i y \cdot z) f\left(x + \frac{1}{2}z\right) \overline{g\left(x - \frac{1}{2}z\right)} dz$$

for all $f, g \in L^2(R^n)$.

The Weyl transform a_W of the symbol $a \in S'(R^n \times R^n)$ is given by

$$\langle a_W f, g \rangle = \langle a, W(f,g) \rangle$$

for all $f, g \in L^2(R^n)$. Then, the twisted product $a\#b$ of symbols a and b can be defined by

$$a\#b(x,\ y) = \int_{R^{4n}} a(z,u)\, b(v,w) \frac{\exp(4\pi i (x-z)(y-w))}{\exp(4\pi i (x-v)(y-u))} dz du dv dw.$$

Definition 1.1. The modulation space is a set $M_t^{\infty,1}(R^n \times R^n)$ of all functions $\sigma \in S'(R^n \times R^n)$ such that

$$\sup_{(x,p) \in R^n \times R^n} \left| \left(1 + |x|^2 + |p|^2\right)^{\frac{t}{2}} W(\sigma, \upsilon)(x,p,y,q) \right| \in L^1(R^n \times R^n)$$

for every window $\upsilon \in S'(R^n \times R^n)$. The space $M_0^{\infty,1}(R^n \times R^n) \equiv M^{\infty,1}(R^n \times R^n)$ is called a Sjostrand class [17].

Harmonic analysis shows that the modulation spaces $M^{\infty,1}(R^n \times R^n)$ constitute a Banach algebra with respect to the twisted product.

In the present article, we extend the ideas of classical harmonic analysis and the theory of pseudo-differential operators to locally compact Hausdorff groups [18]. To reach this goal we modernize methods of phase-time analysis and employ methods of convex estimations on Banach spaces. We redefine the $\mathfrak{T}$- Wigner pseudo-distribution by

$$W_{\Im}(\psi,\varphi)(g,\ \xi) = \int_G \overline{\xi(h)} \psi(\tau_1(g,\ h)) \overline{\varphi(\tau_2(g,\ h))} d\mu(h)$$

for all $g \in G$ and all $\xi \in \hat{G}$, which satisfies the Plancherel formula $\|W_{\Im}(\psi,\varphi)\|_{L^2(G \times \hat{G})} \leq const \|\psi\|_{L^2(G)} \|\varphi\|_{L^2(G)}$ for all $\psi, \varphi \in L^2(G)$. We develop a theory of modulation spaces $M_m^{p,q}(G)$ on locally compact commutative groups. As an interesting example, we establish that the Richczek operator A_a with the symbol $a \in W_{\tau, 1 \circ \iota^{-1}}^{\infty,1}\left(G \times \hat{G}\right)$ is a bounded linear operator from $W_\tau^{p,q}(G)$ to $W_\tau^{p,q}(G)$.

2. The short-time Fourier transform

Let G be a locally compact commutative Hausdorff group with the Radon measure μ on it. Let $\hat{G}$ be a dual group that consists of all continuous homomorphisms χ form G to the circle group S^1. These homomorphisms $\chi \ : \ G \to S^1$ are called characters.

Definition 2.1. A Fourier transform $\psi \mapsto F(\psi)$ of the function ψ is defined by

$$F(\psi)(\chi) = \hat{\psi}(\chi) = \int_G \psi(g)\,\overline{\chi(g)}\, d\mu(g)$$

for all $\chi \in \hat{G}$.

An inverse Fourier transform of the integrable function$\hat{\psi}$ on $\hat{G}$ is given by

$$\psi(g) = \int_{\hat{G}} \hat{\psi}(\chi)\,\chi(g)\ d\hat{\mu}(\chi)$$

for all $g \in G$.

Definition 2.2. The short-time Fourier transform V_ϕ with respect to the window ϕ is given by the formula

$$V_\phi\psi(g,\ \chi) = \int_G \overline{\chi(h)}\psi(h)\,\overline{\phi(g^{-1}h)}\, d\mu(h)$$

for all $g \in G$ and $\xi \in \hat{G}$.

We introduce a translation operator τ_h given by $\tau_h\psi(g) = \psi\left(h^{-1}g\right)$ and the modulation operator m_χ by the identity $m_\chi\psi(g) = \psi(g)\,\chi(g)$. We write

$$\begin{aligned} V_\phi\psi(g,\ \chi) &= \textstyle\int_G \psi(h)\, m_\chi\tau_g\phi(h)\, d\mu(h) \\ &= \textstyle\int_{\hat{G}} \hat{\psi}(\xi)\,\tau_\chi m_{g^{-1}}\hat{\phi}(\xi)\, d\hat{\mu}(\xi) \\ &= \overline{\chi(g)}\, V_{\hat{\phi}}\hat{\psi}\left(\chi,\ g^{-1}\right) \end{aligned}$$

since there is a commutation equality $\tau_g m_\chi = \overline{\chi(g)} m_\chi\tau_g$.

Theorem 2.1. *Let G be a locally compact Abel-Hausdorff group. For all $\psi,\ \varphi \in L^2(G)$, then we have the Cauchy–Schwarz inequality*

$$\|\, V_\phi\psi\|_{C_0\left(G\times\hat{G}\right)} \le \|\psi\|_{L^2(G)}\, \|\phi\|_{L^2(G)}$$

and the Moyal equality

$$\|\, V_\phi\psi\|_{L^2\left(G\times\hat{G}\right)} = \|\psi\|_{L^2(G)}\, \|\phi\|_{L^2(G)}\,.$$

Proof. The Moyal equality follows from the Plancherel theory and the first inequality can be obtained from the Cauchy-Schwarz theorem with an application of the limit

$$\lim_{(g,\ \chi)\to(e,\ e)} \|m_\chi\tau_g\psi - \psi\| = 0$$

that holds for all $\psi \in L^2(G)$.

□

3. The Weyl-Heisenberg frame

Let G be a locally compact Abel-Hausdorff group with the Radon measure μ. According to Pontryagin's duality theorem, there is an isomorphism between G and its double dual group $\hat{\hat{G}}$. The Fourier transform of measure is given by $\hat{\mu}(\chi) = \int_G \overline{\chi(g)}\, d\mu(g)$.

Let H be a separable Hilbert space.

Definition 3.1. Let $(\Omega,\ \Sigma,\ \mu_\Omega)$ be a locally compact measurable space where Σ is a σ-algebra of all μ_Ω-measurable subsets of Ω. The frame is a set of elements $\{\varphi_k\}_{k\in\Omega}$ that satisfy the condition: for all elements $\psi \in H$, the mapping $k \mapsto \langle \psi,\ \varphi_k\rangle_H$ is measurable, and the inequality

$$A\,\|\psi\|_H^2 \le \int_\Omega |\langle \psi,\ \varphi_k\rangle_H|^2\, d\mu_\Omega(k) \le C\,\|\psi\|_H^2$$

holds for some positive constants c and C, which are called a frame lower and upper bounds.

If the lower and upper bounds coincide and equal one, the frame is called a Parseval frame.

Definition 3.2. The Weyl-Heisenberg frame for $L^2(G)$ generated by the window ϕ is a system of functions $\varphi(\lambda,\ \gamma)$ given by $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ where $\Lambda \subseteq G$ and $\Gamma \subseteq \hat{G}$.

The analysis operator $A_\phi\ :\ L^2(G) \to L^2\left(G \times \hat{G}\right)$ is defined by

$$(A_\phi \psi)(\lambda, \gamma) = \langle \psi(\cdot),\ m_\gamma \tau_\lambda \phi(\cdot)\rangle_{L^2}.$$

The annihilator $K^\perp$ of normal $K \subset G$ is a closed subgroup of $\hat{G}$ given by

$$K^\perp = \left\{\chi \in \hat{G}\ :\ \chi(g) = 1 \in S^1 \quad \forall g \in K\right\}.$$

Theorem 3.1. *Let Γ be a closed subgroup of $\hat{G}$ such that the quotient group $\hat{G}/\Gamma$ is a compact metrizable group. Let σ-finite measurable space $(\Lambda,\ \Sigma,\ \mu_\Lambda)$ satisfy the conditions: the mapping $h \mapsto \phi_h$ and $(h,\ s) \mapsto \phi_h(s)$ are measurable. Assume that for a pair of systems $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ and $\{m_\gamma \tau_\lambda \vartheta\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$, the inequalities*

$$\int_{\Lambda\times\Gamma} \left|\langle \psi,\ m_\gamma \tau_\lambda \phi\rangle_{L^2}\right|^2 d\mu_{\Lambda\times\Gamma}(\gamma, \lambda) \le C\,\|\psi\|_{L^2}^2$$

and

$$\int_{\Lambda\times\Gamma} \left|\langle \psi,\ m_\gamma \tau_\lambda \vartheta\rangle_{L^2}\right|^2 d\mu_{\Lambda\times\Gamma}(\gamma, \lambda) \le C\,\|\psi\|_{L^2}^2$$

hold for all functions $\psi \in L^2(G)$ with the positive constant C.

Then, for all functions $\psi_1,\ \psi_2 \in L^2(G)$, the equality

$$\langle \psi_1,\ \psi_2\rangle_{L^2} = \int_{\Lambda\times\Gamma} \langle \psi_1,\ m_\gamma \tau_\lambda \phi\rangle_{L^2} \langle m_\gamma \tau_\lambda \vartheta,\ \psi_2\rangle_{L^2}\, d\mu_{\Lambda\times\Gamma}(\gamma, \lambda)$$

holds if and only if the equality

$$\delta_{\varsigma,e} = \int_\Lambda \overline{\phi(g\lambda^{-1})}\vartheta\left(g\lambda^{-1}\varsigma\right) d\mu_\Lambda(\lambda)$$

holds for almost every $g \in G$ and each $\varsigma \in \Gamma^\perp$.

Proof. The system $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ is not a generalized translation invariant system. However for the system $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ to be a frame, it is necessary and sufficient that the system $\{\tau_\lambda m_\gamma \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ be a frame since $\tau_g m_\chi = \overline{\chi(g)} m_\chi \tau_g$. Now, we must show that if $\{\tau_\lambda m_\gamma \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ and $\{\tau_\lambda m_\gamma \vartheta\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ are two Bessel systems, then, for all functions $\psi_1,\ \psi_2 \in L^2(G)$, the equality

$$\langle \psi_1,\ \psi_2\rangle_{L^2} = \int_{\Lambda\times\Gamma} \langle \psi_1,\ \tau_\lambda m_\gamma \phi\rangle_{L^2} \langle \tau_\lambda m_\gamma \vartheta,\ \psi_2\rangle_{L^2}\, d\mu_{\Lambda\times\Gamma}(\gamma,\lambda)$$

hold if and only if the equality

$$\delta_{\beta,e} = \int_\Gamma \overline{\widehat{m_\gamma\phi}(\chi)}\,\widehat{m_\gamma\vartheta}(\beta\chi)\, d\mu_\Gamma(\gamma)$$

holds for almost every $\chi \in \hat{G}$ and each $\beta \in \Lambda^\perp$. The convergence and correctness of integrals follow from the Cauchy-Schwarz inequality.

To make our considerations completely rigorous, we need the following statement. Let H_1 and H_2 be a pair of separable Hilbert spaces and let $U\ :\ H_2 \to H_1$ be a unitary mapping; let $(\Omega_1,\ \Sigma_1,\ \mu_{\Omega_1})$ and $(\Omega_2,\ \Sigma_2,\ \mu_{\Omega_2})$ be two measurable spaces; let $\{\psi_k\}_{k\in\Omega_1}$ and $\left\{\tilde{\psi}_k\right\}_{k\in\Omega_1}$ be two systems in H_1 and let $\{\varphi_k\}_{k\in\Omega_2}$ and $\{\tilde{\varphi}_k\}_{k\in\Omega_2}$ be two systems in H_2 such that $\psi_k = a_k U\varphi_{\pi(k)}$ and $\tilde{\psi}_k = a_k U\tilde{\varphi}_{\pi(k)}$ where mapping $\pi\ :\ \Omega_1 \to \Omega_2$ is a pointwise isomorphism and the mapping $\Omega_1 \to C\ :\ k \mapsto a_k$ so that $|a_k| = 1$ for all $k \in \Omega_1$. Then for systems $\{\psi_k\}_{k\in\Omega_1}$ and $\left\{\tilde{\psi}_k\right\}_{k\in\Omega_1}$ to be dual with respect to $(\Omega_1,\ \Sigma_1,\ \mu_{\Omega_1})$, it is necessary and sufficient that the system $\{\varphi_k\}_{k\in\Omega_2}$ and $\{\tilde{\varphi}_k\}_{k\in\Omega_2}$ are dual with respect to $(\Omega_2,\ \Sigma_2,\ \mu_{\Omega_2})$.

Indeed, let $\{\psi_k\}_{k\in\Omega_1}$ and $\left\{\tilde{\psi}_k\right\}_{k\in\Omega_1}$ be dual frames. Then we write

$$\begin{aligned}\langle \psi, \varphi\rangle &= \textstyle\int_{\Omega_1} \left\langle U\psi, \tilde{\psi}_k\right\rangle \langle \psi_k, \varphi\rangle\, d\mu_{\Omega_1}(k)\\ &= \textstyle\int_{\Omega_1} \left\langle U\psi, a_k U\tilde{\varphi}_{\pi(k)}\right\rangle \left\langle a_k U\varphi_{\pi(k)}, \varphi\right\rangle d\mu_{\Omega_1}(k)\\ &= \textstyle\int_{\Omega_1} \left\langle \psi, \tilde{\varphi}_{\pi(k)}\right\rangle \left\langle \varphi_{\pi(k)}, U^*\varphi\right\rangle d\mu_{\Omega_1}(k)\\ &= \textstyle\int_{\Omega_2} \langle \psi, \tilde{\varphi}_k\rangle \langle \varphi_k, U^*\varphi\rangle\, d\mu_{\Omega_2}(k),\end{aligned}$$

where U^* is the adjoint of U.

Thus, for the system $\{m_\gamma \tau_\lambda \phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ to be a frame, it is necessary and sufficient that the system $\left\{\tau_\gamma F^{-1} \tau_\lambda \phi\right\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ is a frame.

Weil's lemma establishes the possibility of normalization of the measure $\mu_{G/k}$ so that the equality

$$\int_G \psi(g)\, d\mu_G(g) = \int_{G/K}\int_K \psi(gh)\, d\mu_K(h)\, d\mu_{G/K}(\dot{g})$$

holds for a given measure μ_K and μ_G.

By application of the Plancherel theorem and the Weil lemma, we obtain

$$\begin{aligned}&\textstyle\int_{\Lambda\times\Gamma} \langle \psi_1,\ \tau_\lambda m_\gamma \phi\rangle_{L^2} \langle \tau_\lambda m_\gamma \vartheta,\ \psi_2\rangle_{L^2}\, d\mu_{\Lambda\times\Gamma}(\lambda,\gamma)\\ &= \textstyle\int_\Gamma \int_{\hat{G}} \int_{\Lambda^\perp} \hat{\psi}_1(\chi)\, \overline{\hat{\psi}_2(\chi\beta)\widehat{m_\gamma\phi}(\chi)}\, \widehat{m_\gamma\vartheta}(\chi\beta)\, d\mu_{\Lambda^\perp}(\beta)\, d\mu_{\hat{G}}(\chi)\, d\mu_\Gamma(\gamma).\end{aligned}$$

Therefore, we define a mapping $\Upsilon(\psi) : G \to C$ by

$$\Upsilon(\psi)(g) = \int_{\Lambda\times\Gamma} \langle \tau_g\psi,\ \tau_\lambda m_\gamma \phi\rangle_{L^2} \langle \tau_\lambda m_\gamma \vartheta,\ \tau_g\psi\rangle_{L^2} d\mu_{\Lambda\times\Gamma}(\lambda,\gamma).$$

Since $\widehat{\tau_\lambda\psi}(\chi)\overline{\widehat{\tau_\lambda\psi}(\chi\beta)} = \beta(\lambda)\widehat{\psi}(\chi)\overline{\widehat{\psi}(\chi\beta)}$, we have

$$\Upsilon(\psi)(g)$$

$$= \int_\Gamma\int_{\hat{G}}\int_{\Lambda^\perp} \beta(g)\hat{\psi}(\chi)\overline{\hat{\psi}(\chi\beta)\widehat{m_\gamma\phi}(\chi)}\widehat{m_\gamma\vartheta}(\chi\beta)\, d\mu_{\Lambda^\perp}(\beta)\, d\mu_{\hat{G}}(\chi)\, d\mu_\Gamma(\gamma).$$

Using Fubini's theorem and Lebesgue's dominated convergence theorem, we obtain

$$\Upsilon(\psi)(g) = \int_{\Lambda^\perp} \beta(g)\widehat{\Upsilon}(\beta)\, d\mu_{\Lambda^\perp}(\beta),$$

where we denote

$$\widehat{\Upsilon}(\beta) = \int_\Gamma\int_{\hat{G}} \hat{\psi}(\chi)\overline{\hat{\psi}(\chi\beta)\widehat{m_\gamma\phi}(\chi)}\widehat{m_\gamma\vartheta}(\chi\beta)\, d\mu_{\hat{G}}(\chi)\, d\mu_\Gamma(\gamma).$$

The sufficiency follows from

$$\begin{aligned}\Upsilon(\psi)(e) &= \textstyle\int_{\Lambda\times\Gamma} \langle \psi,\ \tau_\lambda m_\gamma\phi\rangle_{L^2} \langle \tau_\lambda m_\gamma\vartheta,\ \psi\rangle_{L^2} d\mu_{\Lambda\times\Gamma}(\lambda,\gamma)\\ &= \textstyle\int_{\hat{G}}\int_{\Lambda^\perp} \beta(e)\hat{\psi}(\chi)\overline{\hat{\psi}(\chi\beta)}\delta_{\beta,e} d\mu_{\Lambda^\perp}(\beta)\, d\mu_{\hat{G}}(\chi) = \langle\psi,\psi\rangle.\end{aligned}$$

Now, we are going to show the necessity of the theorem's conditions. We assume that equality

$$\Upsilon(\psi)(g) = \int_{\Lambda\times\Gamma} \langle \tau_g\psi,\ \tau_\lambda m_\gamma\phi\rangle_{L^2} \langle \tau_\lambda m_\gamma\vartheta,\ \tau_g\psi\rangle_{L^2} d\mu_{\Lambda\times\Gamma}(\lambda,\gamma) = \langle\psi,\psi\rangle$$

holds for all $\psi \in \left\{L^2(G) \quad \textit{such that } \hat{\psi} \in L^\infty\left(\hat{G}\right)\right\}$ with the compact support. We denote the continuous function $\Xi(g) = \Upsilon(\psi)(g) - \langle\psi,\psi\rangle$ that is identical to zero. We have $\Xi(g) = \int_{\Lambda^\perp} \beta(g)\widehat{\Xi}(\beta)\, d\mu_{\Lambda^\perp}(\beta)$ where we denote

$$\Xi(g) = \begin{cases} \int_{\hat{G}}\int_\Gamma \left|\widehat{\psi}(\chi)\right|^2 \overline{\widehat{m_\gamma\phi}(\chi)}\widehat{m_\gamma\vartheta}(\chi)\, d\mu_\Gamma(\gamma)\, d\mu_{\hat{G}}(\chi) - \|\psi\|^2 & ,\beta = e,\\ \int_{\hat{G}}\int_\Gamma \widehat{\psi}(\chi)\overline{\widehat{\psi}(\beta\chi)\widehat{m_\gamma\phi}(\chi)}\widehat{m_\gamma\vartheta}(\beta\chi)\, d\mu_\Gamma(\gamma)\, d\mu_{\hat{G}}(\chi) & ,\beta \neq e.\end{cases}$$

Thus, we see that $\int_\Gamma \overline{\widehat{m_\gamma\phi}(\chi)}\widehat{m_\gamma\vartheta}(\chi)\, d\mu_\Gamma(\gamma) = 1$ for almost all $\chi \in \hat{G}$ and $\int_\Gamma \overline{\widehat{m_\gamma\phi}(\chi)}\widehat{m_\gamma\vartheta}(\beta\chi)\, d\mu_\Gamma(\gamma) = 0$ for all $\beta \neq e$. □

So, we also have Theorem 3.2.

Theorem 3.2. *The conditions of Theorem 2.1 hold if and only if the equality*

$$\delta_{\theta,e} = \int_\Gamma \overline{\hat{\phi}(\chi\gamma)}\hat{\vartheta}(\chi\gamma\theta)\, d\mu_\Gamma(\lambda)$$

holds for almost all $\chi \in \hat{G}$ and for each $\theta \in \Lambda^\perp$, namely, the equality

$$\langle\psi_1,\ \psi_2\rangle_{L^2} = \int_{\Lambda\times\Gamma} \langle\psi_1,\ m_\gamma\tau_\lambda\phi\rangle_{L^2} \langle m_\gamma\tau_\lambda\vartheta,\ \psi_2\rangle_{L^2} d\mu_{\Lambda\times\Gamma}(\gamma,\lambda)$$

holds for all functions $\psi_1,\ \psi_2 \in L^2(G)$.

4. Wigner and ambiguity functions

For a given window ϕ, the short-time Fourier transform V_ϕ can be rewritten in terms of translation and modulo as follows

$$V_\phi \psi (g,\ \chi) = \int_G \psi (h)\, m_\chi \tau_g \phi (h)\ d\mu (h)$$

for all $g \in G$ and $\xi \in \hat{G}$.

Let $\Im\ : G \times G \to G \times G$ be a measure preserving continuous transformation given by

$$u = \tau_1 (g,\ h),$$
$$v = \tau_2 (g,\ h)$$

for all $g, h \in G$, and it satisfies the conditions: first, the equalities $\tau_1 (g,\ e) = g, \quad \tau_2 (g,\ e) = g$ hold for all $g \in G$; second, for all $g \in G$, we have $h = uv^{-1}$ for all $u, v \in G$; the inverse transformation $\Im^{-1}\ : G \times G \to G \times G$ is given by

$$g = \tau (g,\ h),$$
$$h = uv^{-1},$$

where τ continuously maps $G \times G \to G$. The mappings $\tau_1,\ \tau_2$ and τ are continuous.

Definition 4.1. The $\mathfrak{T}$ - Wigner function is defined by the formula

$$W_\Im (\psi, \varphi) (g,\ \xi) = \int_G \overline{\xi (h)} \psi (\tau_1 (g,\ h))\, \overline{\varphi (\tau_2 (g,\ h))} d\mu (h)$$

for all $g \in G$ and all $\xi \in \hat{G}$.

The $\mathfrak{T}$ -ambiguity function is defined by the formula

$$Am_\Im (\psi, \varphi) (h,\ \xi) = \int_G \overline{\xi (g)} \psi (\tau_1 (g,\ h))\, \overline{\varphi (\tau_2 (g,\ h))} d\mu (g)$$

for all $h \in G$ and all $\xi \in \hat{G}$.

Lemma 4.1. *Let G be a locally compact commutative group. Then, the $\mathfrak{T}$ - Wigner and $\mathfrak{T}$-ambiguity functions satisfy the following inequality*

$$\|W_\Im (\psi, \varphi)\|_{L^\infty \left(G \times \hat{G}\right)} \le \|\psi\|_{L^2(G)} \|\varphi\|_{L^2(G)}$$

and

$$\|Am_\Im (\psi, \varphi)\|_{L^\infty \left(G \times \hat{G}\right)} \le \|\psi\|_{L^2(G)} \|\varphi\|_{L^2(G)},$$

where all functions $\psi, \varphi \in L^2 (G)$.

Now, we suppose that G is a locally compact commutative Hausdorff group and $Am_\Im (\psi, \varphi) \in L^2 \left(G \times \hat{G}\right)$, then the equality $\|Am_\Im (\psi, \varphi)\|_{L^2} = \|W_\Im (\psi, \varphi)\|_{L^2}$ holds for any pair of functions $\psi, \varphi \in L^2 (G)$. Indeed, we write the equality

$$\begin{aligned}
&\|W_\Im (\psi, \varphi)\|_{L^2} \\
&= \int_G \int_G \left| \overline{\xi (h)} \psi (\tau_1 (g,\ h))\, \overline{\varphi (\tau_2 (g,\ h))} \right|^2 d\mu (h)\, d\mu (g) \\
&= \int_G \int_G \left| \overline{\xi (h)} \psi (\tau_1 (g,\ h))\, \overline{\varphi (\tau_2 (g,\ h))} \right|^2 d\mu (g)\, d\mu (h) \\
&= \|Am_\Im (\psi, \varphi)\|_{L^2}
\end{aligned}$$

which holds for all $\psi, \varphi \in L^2(G)$.

For $\mathfrak{T}$-Wigner and $\mathfrak{T}$-ambiguity functions, the exact marginals are correct so that equalities

$$\int_{\hat{G}} W_{\mathfrak{T}}(\psi,\varphi)(g,\xi)\, d\hat{\mu}(\xi) = \psi(g)\overline{\varphi(g)}$$

and

$$\int_{G} W_{\mathfrak{T}}(\psi,\varphi)(g,\xi)\, d\mu(g) = F(\psi)(\xi)\overline{F(\varphi)(\xi)}$$

hold $\psi, \varphi \in L^2(G)$ and for all $g \in G$ and $\xi \in \hat{G}$. The proof is similar to the classical theorem. Next, in $L^2\left(G \times \hat{G}\right)$ we have $\widehat{Am_{\mathfrak{T}}}(\psi,\varphi) = W_{\mathfrak{T}}(\psi,\varphi)$for all $\psi, \varphi \in L^2(G)$.

5. The Feichtinger algebra on the locally compact groups

In 1979, the Feichtiger algebra $S_0(G)$, which is a special case of the broader concept of modulation spaces, was introduced by H.G. Feichtinger. The modulation space $M_m^{p,q}(G)$ on the locally compact Abelian group is a Banach space with a norm defined by

$$M_m^{p,q}(G) = \{\psi \ : \ G \to C \ : \quad \psi \in S'(G) \ :$$

$$\left.\left(\int_{\hat{G}}\left(\int_{G}|V_{\phi}\psi(h,\ \chi)|^p\, m(h,\ \chi)^p\ d\mu(h)\right)^{\frac{q}{p}} d\hat{\mu}(\chi)\right)^{\frac{1}{q}} < \infty\right\},$$

where the function ϕ is the window, the function m is a non-negative function on $G \times \hat{G}$ and the numbers $1 \le p, q \le \infty$.

Definition 5.1. Assume that window $\phi \in C_C(G)$ is a non-zero function such that $\hat{\phi} \in L^1\left(\hat{G}\right)$. The Feichtinger algebra $S_0(G)$ is a set of all elements $\psi \in L^1(G)$ that satisfy the inequality $\int_{\hat{G}}\int_{G}|V_{\phi}\psi(h,\ \chi)|\, d\mu(h)\, d\hat{\mu}(\chi) < \infty$.

Lemma 5.1. *The space $S_0(G)$ is topologically embedded in $L^1(G)$.*

Indeed, straightforward calculation shows that there exists a constant c dependent on the non-zero window ϕ such that for all $\psi \in S_0(G)$, we have

$$\begin{aligned}\|\psi\|_{L^1(G)} &= \int_{G}|\psi(g)|\ d\mu(g)\\ &\le c\int_{\hat{G}}\int_{G}\left|\int_{G}\psi(g)\overline{\chi(h)}\phi\left(gh^{-1}\right)\ d\mu(g)\right| d\mu(h)\, d\hat{\mu}(\chi)\\ &= c\int_{\hat{G}}\int_{G}|V_{\phi}\psi(h,\ \chi)|\, d\mu(h)\, d\hat{\mu}(\chi) = c\,\|\psi\|_{S_0(G)}.\end{aligned}$$

So function ψ belongs to $L^1(G)$.

The next theorem determines the class of functions that produce Gabor systems.

Theorem 5.1. *Let Λ and Γ be closed subgroups of G and $\hat{G}$, respectively. Then for any window $\phi \in S_0(G)$, the set $\{m_{\gamma}\tau_{\lambda}\phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ constitutes a cocompact Gabor system in $L^2(G)$ with the bound $c(\Lambda,\Gamma)\,\|\phi\|_{S_0(G)}^2$.*

Proof. Since the functional set $C_C(G)$ is dense in the space $L^2(G)$, it is sufficient to show that

$$\int_{\Lambda\times\Gamma} |\langle\psi,\ m_\gamma\tau_\lambda\phi\rangle|^2\, d\mu_{\Lambda\times\Gamma}(\lambda,\gamma) \le c(\Lambda,\Gamma)\,\|\phi\|^2_{S_0(G)}\,\|\psi\|^2_{L^2(G)}$$

holds for all functions $\psi \in C_C(G)$ and each window $\phi \in S_0(G)$. Therefore, assume that $\psi \in C_C(G)$ and then we estimate

$$\begin{aligned}
&\textstyle\int_{\Lambda\times\Gamma} |\langle\psi,\ m_\gamma\tau_\lambda\phi\rangle|^2\, d\mu_{\Lambda\times\Gamma}(\lambda,\gamma)\\
&\textstyle\le \int_\Lambda\int_{\Gamma^\perp}\int_{\Gamma^\perp}\int_{G/\Gamma^\perp} \overline{\psi(gh^{-1})}\psi\left(g\tilde{h}^{-1}\right)\phi\left(g\lambda^{-1}h^{-1}\right)\\
&\overline{\phi\left(g\lambda^{-1}\tilde{h}^{-1}\right)} d\mu_{G/\Gamma^\perp}(\dot{g})\,d\mu_{\Gamma^\perp}(h)\,d\mu_{\Gamma^\perp}\left(\tilde{h}\right)d\mu_\Lambda(\lambda)\\
&\textstyle\le \int_{G/\Gamma^\perp}\left(\int_{\Gamma^\perp}\left|\psi\left(gh^{-1}\right)\right|^2\int_\Lambda\int_{\Gamma^\perp}\left|\phi\left(g\lambda^{-1}h^{-1}\right)\phi\left(g\lambda^{-1}\tilde{h}^{-1}\right)\right|\right.\\
&\left. d\mu_{\Gamma^\perp}\left(\tilde{h}\right)d\mu_\Lambda(\lambda)\,d\mu_{\Gamma^\perp}(h)\right)^{\frac{1}{2}}\\
&\textstyle\left(\int_{\Gamma^\perp}\left|\psi\left(g\tilde{h}^{-1}\right)\right|^2\int_\Lambda\int_{\Gamma^\perp}\left|\phi\left(g\lambda^{-1}h^{-1}\right)\phi\left(g\lambda^{-1}\tilde{h}^{-1}\right)\right|\right.\\
&\left. d\mu_{\Gamma^\perp}(h)\,d\mu_\Lambda(\lambda)\,d\mu_{\Gamma^\perp}\left(\tilde{h}\right)\right)^{\frac{1}{2}} d\mu_{G/\Gamma^\perp}(\dot{g}).
\end{aligned}$$

By the Fubini theorem

$$\begin{aligned}
&\textstyle\int_\Lambda\int_{\Gamma^\perp}\left|\phi\left(g\lambda^{-1}h^{-1}\right)\phi\left(g\lambda^{-1}\tilde{h}^{-1}\right)\right| d\mu_{\Gamma^\perp}(h)\,d\mu_\Lambda(\lambda)\\
&\textstyle= \int_\Lambda\left|\phi\left(g\lambda^{-1}\tilde{h}^{-1}\right)\right|\int_{\Gamma^\perp}\left|\phi\left(g\lambda^{-1}h^{-1}\right)\right| d\mu_{\Gamma^\perp}(h)\,d\mu_\Lambda(\lambda)\\
&\textstyle\le \|\phi\|_{S_0(G)}\,\tilde{c}(\Gamma)\int_\Lambda\left|\phi\left(g\lambda^{-1}\tilde{h}^{-1}\right)\right| d\mu_\Lambda(\lambda) \le c(\Lambda,\Gamma)\,\|\phi\|^2_{S_0(G)},
\end{aligned}$$

thus, we have

$$\begin{aligned}
&\textstyle\int_{\Lambda\times\Gamma} |\langle\psi,\ m_\gamma\tau_\lambda\phi\rangle|^2\, d\mu_{\Lambda\times\Gamma}(\lambda,\gamma)\\
&\textstyle\le \int_{G/\Gamma^\perp}\left(\int_{\Gamma^\perp}\left|\psi\left(gh^{-1}\right)\right|^2 c(\Lambda,\Gamma)\,\|\phi\|^2_{S_0(G)}\,d\mu_{\Gamma^\perp}(h)\right)^{\frac{1}{2}}\\
&\textstyle\left(\int_{\Gamma^\perp}\left|\psi\left(g\tilde{h}^{-1}\right)\right|^2 c(\Lambda,\Gamma)\,\|\phi\|^2_{S_0(G)}\,d\mu_{\Gamma^\perp}\left(\tilde{h}\right)\right)^{\frac{1}{2}} d\mu_{G/\Gamma^\perp}(\dot{g})\\
&\le c(\Lambda,\Gamma)\,\|\phi\|^2_{S_0(G)}\,\|\psi\|^2_{L^2(G)}.
\end{aligned}$$

For any Gabor system $\{m_\gamma\tau_\lambda\phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$, there exists a dual Gabor system $\{m_\kappa\tau_\upsilon\phi\}_{\kappa\in\Lambda^\perp,\ \upsilon\in\Gamma^\perp}$ where $\Lambda^\perp \equiv \widehat{G/\Lambda}$ and $\Gamma^\perp \equiv \widehat{\hat{G}/\Gamma}$ where we employ the Pontryagin duality theory. □

Theorem 5.2. *Let Λ and Γ be closed cocompact subgroups of G and $\hat{G}$, respectively. Then for a given window $\phi \in L^2(G)$, both systems $\{m_\gamma\tau_\lambda\phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ and $\{m_\kappa\tau_\upsilon\phi\}_{\kappa\in\Lambda^\perp,\ \upsilon\in\Gamma^\perp}$ are Gabor systems in $L^2(G)$, simultaneously, with the same upper bound or neither $\{m_\gamma\tau_\lambda\phi\}_{\lambda\in\Lambda,\ \gamma\in\Gamma}$ nor $\{m_\kappa\tau_\upsilon\phi\}_{\kappa\in\Lambda^\perp,\ \upsilon\in\Gamma^\perp}$ comprises Gabor system.*

6. Realization of pseudo-differential operators on locally compact groups

Weyl's correspondence is a correlation between the pseudo-differential operators and their symbols. Pseudo-differential operators can be considered as a generalization of partial differential operators on the locally compact commutative Hausdorff groups, which are presented in the integral form.

We formally define a pseudo-differential operator A_a with the symbol a by

$$(A_a\psi)(g) = \int_{\hat{G}} \xi(g)\, a(g,\ \xi)\, \hat{\psi}(\xi)\, d\hat{\mu}_{\hat{G}}(\xi).$$

Let $\psi \in L^2(G)$. Then we calculate

$$\begin{aligned}
&(A_a\psi)(g)\\
&= \textstyle\int_{\hat{G}} \xi(g)\, a(g,\ \xi)\, \hat{\psi}(\xi)\, d\hat{\mu}_{\hat{G}}(\xi)\\
&= \textstyle\int_{\hat{G}} \int_G \xi(g)\, a(g,\ \xi)\, \overline{\xi(h)}\psi(h)\, d\mu_G(h)\, d\hat{\mu}_{\hat{G}}(\xi)\\
&= \textstyle\int_{\hat{G}} \int_G \int_G \int_{\hat{G}} \hat{a}(\chi,\ t)\, \chi(g)\, \overline{\xi(t)}\xi(g)\, \overline{\xi(h)}\psi(h)\, d\hat{\mu}_{\hat{G}}(\chi)\, d\mu_G(t)\, d\mu_G(h)\, d\hat{\mu}_{\hat{G}}(\xi)\\
&= \textstyle\int_{\hat{G}} \int_G \chi(g)\, \hat{a}(\chi,\ t)\, \psi(gt)\, d\mu_G(t)\, d\hat{\mu}_{\hat{G}}(\chi)\\
&= \textstyle\int_{\hat{G}} \int_G \hat{a}(\chi,\ t)\, m_\chi \tau_{t^{-1}} \psi(g)\, d\mu_G(t)\, d\hat{\mu}_{\hat{G}}(\chi)
\end{aligned}$$

for all $g \in G$.

We assume the symbol a has a compact support and denote

$$\hat{a}_1(\chi,\ \xi) = \int_G \overline{\xi(h)} a(h,\ \xi)\, d\mu_G(h).$$

Next we write

$$\begin{aligned}
A_a(\psi)(g) &= \textstyle\int_{\hat{G}} \xi(g)\, a(g,\ \xi)\, \hat{\psi}(\xi)\, d\hat{\mu}_{\hat{G}}(\xi)\\
&= \textstyle\int_{\hat{G}} \int_{\hat{G}} \chi(g)\, \xi(g)\, \hat{a}_1(\chi,\ \xi)\, \hat{\psi}(\xi)\, d\hat{\mu}_{\hat{G}}(\xi)\, d\hat{\mu}_{\hat{G}}(\chi)\\
&= \textstyle\int_{\hat{G}} A^\chi(\psi)(g)\, d\hat{\mu}_{\hat{G}}(\chi),
\end{aligned}$$

where the multiplier operator A^χ is given by

$$\begin{aligned}
A^\chi(\psi)(g) &= \chi(g) \textstyle\int_{\hat{G}} \xi(g)\, \hat{a}_1(\chi,\ \xi)\, \hat{\psi}(\xi)\, d\hat{\mu}_{\hat{G}}(\xi)\\
&= \chi(g)\, A_{\hat{a}_1}(\psi)(g).
\end{aligned}$$

Now, we have $\|A_{\hat{a}_1}(\psi)\|_{L^2} \leq \sup\limits_{\xi \in \hat{G}} |\hat{a}_1(\chi,\ \xi)|\, \|\psi\|_{L^2}$ and we assume the estimation $\sup\limits_{\xi \in \hat{G}} |\hat{a}_1(\chi,\ \xi)| \leq \Theta(\chi)$ holds for some positive function of $L^1\left(\hat{G}\right)$. Then

$$\|A_a\| \leq \int_{\hat{G}} \Theta(\chi)\, d\hat{\mu}_{\hat{G}}(\xi) < \infty.$$

Thus, we conclude that the pseudo-differential operator A_a is bounded $L^2(G) \to L^2(G)$.

Definition 6.1. The twisted product $\hat{a}\#\hat{b}$ of symbols $\hat{a},\ \hat{b} \in L^1\left(\hat{G}\times G\right)$ is defined by

$$\hat{a}\#\hat{b}\,(\xi,\ g) = \int_{G\times\hat{G}} \xi\,(h)\,\overline{\chi\,(h)}\hat{a}\,(\chi, h)\,\hat{b}\left(\chi^{-1}\xi, h^{-1}g\right) d\mu_{G\times\hat{G}}\,(h\times\chi)\,.$$

Straightforward calculations yield the following theorem.

Theorem 6.1. *Let operators A_a and A_b be corresponded to the symbols $\hat{a},\ \hat{b} \in L^1\left(\hat{G}\times G\right)$ then the product A_aA_b corresponds to the symbol $F^{-1}\left(\hat{a}\#\hat{b}\right)$ so that there exists an operator $C_{F^{-1}\left(\hat{a}\#\hat{b}\right)}$ such that $C_{F^{-1}\left(\hat{a}\#\hat{b}\right)} = A_aA_b$.*

7. The Richczek operator

In recent years, new methods based on nonlinear representations have found wide applications for descriptions of nonstationary systems such as cognitive functions of the brain and the characterization of interactions between cognizing systems. The synchronicity of interactions correlates with the estimation of the instantaneous phase of a certain frequency by time-frequency analysis methods. We consider a representation of the signal defined by the Richaczek distribution.

Natural spaces on which Richaczek distributions perform are modulation spaces $M_m^{p,q}\,(G)$ with the norm defined by

$$\|\psi\|_{M_m^{p,q}(G)} = \left(\int_{\hat{G}}\left(\int_G |\,V_\phi\psi\,(h,\ \chi)|^p\, m\,(h,\ \chi)^p\ \ d\mu\,(h)\right)^{\frac{q}{p}} d\hat{\mu}\,(\chi)\right)^{\frac{1}{q}}.$$

Similarly, we define the $\mathfrak{T}$-modulation space defining the $\mathfrak{T}$- Wigner norm.

Definition 7.1. A $\mathfrak{T}$-modulation space $W_\tau^{p,q}\,(G)$ is the completion of $S_C\,(G)$ space with respect to the norm

$$\|\psi\|_{W_\tau^{p,q}(G)} = \left(\int_{\hat{G}}\left(\int_G \chi\,(h)^p\,|W_{\mathfrak{I}}\,(\psi,\varphi)\,(h,\ \chi)|^p\ \ d\mu\,(h)\right)^{\frac{q}{p}} d\hat{\mu}\,(\chi)\right)^{\frac{1}{q}}$$

for each fixed window $\varphi \in S_C\,(G)$.

A Richaczek distribution $\Re\,(\psi,\ \varphi)$ is given by

$$\Re\,(\psi,\ \varphi)\,(g,\ \xi) = \overline{\xi\,(g)}\psi\,(g)\,\overline{\hat{\varphi}\,(\xi)}.$$

Definition 7.2. A class of admissible functions consists of all nonnegative functions ϖ defined on $G\times\hat{G}$, which satisfy the following conditions:

1. each admissible function ϖ is continuous, even in all coordinates;
2. the inequality $\varpi\,((g,\chi)\cdot(h,\xi)) \le \varpi\,(g,\chi)\,\varpi\,(h,\xi)$ holds for all $(g,\chi)\,,\ (h,\xi) \in G\times\hat{G}$, and $\varpi\,(e,e) = 1$;
3. admissible functions ϖ satisfy the equality

$$\lim_{n\to\infty} \varpi\,((g,\chi)^n)^{\frac{1}{n}} = 1$$

for all $(g,\chi) \in G\times\hat{G}$.

We define an antisymmetry operator ι by $\iota(g,\chi)=\left(\chi^{-1},g\right)$ for all $(g,\chi)\in G\times\hat{G}$.

Definition 7.3. An admissible weight m is a positive function on $G\times\hat{G}$ that satisfies the following condition

$$\sup_{(g,\chi)\in G\times\hat{G}} m\left((g,\chi)\cdot(h,\xi)\right)\le cm(g,\chi)\varpi(h,\xi)$$

for all $(h,\xi)\in G\times\hat{G}$.

Theorem 7.1. *Let ϖ be an admissible function then the inequality*

$$\|\Re(\psi,\ \varphi)\|_{M^{1,\infty}_{\varpi^{-1}\circ\iota^{-1}}}\le c\|\psi\|_{M^{p_1,q_1}_m}\|\varphi\|_{M^{p_2,q_2}_{m^{-1}}}$$

holds for all $\psi\in M^{p_1,q_1}_m(G)$, $\varphi\in M^{p_2,q_2}_{m^{-1}}(G)$ and all admissible weights m where $1\le p_i,\quad q_i<\infty$ and $p_1+p_2=p_1p_2$, $q_1+q_2=q_1q_2$.

Proof. Let $\psi\in M^{p_1,q_1}_m(G)$ and $\varphi\in M^{p_2,q_2}_{m^{-1}}(G)$. By Holder inequality calculate

$$\begin{aligned}
&\|\Re(\psi,\ \varphi)\|_{M^{1,\infty}_{\varpi^{-1}\circ\iota^{-1}}}\\
&=\sup_{(\xi,\ h)\in\hat{G}\times G}\varpi\left(\iota^{-1}(\xi,\ h)\right)^{-1}\\
&\int_{\hat{G}}\int_G\left|V_{\Re(\phi,\ \phi)}\Re(\psi,\ \varphi)\left((g,\ \chi),\ (\xi,\ h)\right)\right|d\mu_G(g)\,d\hat{\mu}_{\hat{G}}(\chi)\\
&=\sup_{(\xi,\ h)\in\hat{G}\times G}\varpi\left(h,\ \xi^{-1}\right)^{-1}\int_{\hat{G}}\int_G|V_\phi(\psi)(gh,\ \chi)|\,|V_\phi(\varphi)(g,\ \chi\xi)|\,d\mu_G(g)\,d\hat{\mu}_{\hat{G}}(\chi)\\
&=c\sup_{(\xi,\ h)\in\hat{G}\times G}\varpi\left(h,\ \xi^{-1}\right)^{-1}\int_{\hat{G}}\int_G\begin{array}{l}m(gh,\chi)\,|V_\phi(\psi)(gh,\ \chi)|\\|V_\phi(\varphi)(g,\ \chi\xi)|\,m(g,\chi\xi)^{-1}d\mu_G(g)\,d\hat{\mu}_{\hat{G}}(\chi)\end{array}\\
&\le c\|\psi\|_{M^{p_1,q_1}_m}\|\varphi\|_{M^{p_2,q_2}_{m^{-1}}}.
\end{aligned}$$

□

Remark 7.1. Theorem 7.1 is a consequence of more general and symmetric equality

$$V_{\Re(\alpha,\ \beta)}\left(\Re(\psi,\ \varphi)\right)\left((g,\ \chi),\ (\xi,\ h)\right)=\overline{\chi(h)}\,V_\alpha(\varphi)(g,\chi\xi)\,\overline{V_\beta(\psi)(gh,\chi)},$$

which holds for all $\alpha,\ \beta,\ \psi,\ \varphi\in L^2(G)$ and all $(g,\chi)\in G\times\hat{G}$, $(h,\xi)\in G\times\hat{G}$.

Indeed, let $\alpha,\ \beta,\ \psi,\ \varphi\in L^2(G)$. Then equalities

$$\begin{aligned}
&V_{\Re(\alpha,\ \beta)}\left(\Re(\psi,\ \varphi)\right)\left((g,\ \chi),\ (\xi,\ h)\right)\\
&=\int_{\hat{G}}\int_G\varphi(s)\,\overline{\hat{\psi}(\varsigma)}\\
&\overline{\varsigma(s)}\hat{\beta}\left(\varsigma\chi^{-1}\right)\overline{\alpha\left(sg^{-1}\right)}\,\varsigma\left(sg^{-1}\right)\overline{\chi\left(sg^{-1}\right)\xi(s)\,\varsigma(h)}d\mu_G(s)\,d\hat{\mu}_{\hat{G}}(\varsigma)\\
&=\chi(g)\int_{\hat{G}}\overline{\hat{\psi}(\varsigma)}\hat{\beta}\left(\varsigma\chi^{-1}\right)\overline{\varsigma(gs)}\,d\hat{\mu}_{\hat{G}}(\varsigma)\int_G\varphi(s)\,\overline{\xi(s)\,\chi(s)\alpha\left(sg^{-1}\right)}d\mu_G(s)\\
&=\overline{\chi(h)}\,V_\alpha(\varphi)(g,\chi\xi)\,\overline{V_\beta(\psi)(gh,\chi)}
\end{aligned}$$

hold for all $(g,\chi)\in G\times\hat{G}$, $(h,\xi)\in G\times\hat{G}$.

Theorem 7.2. *For all symbols $a\in M^{\infty,1}_{\varpi^{-1}\circ\iota^{-1}}\left(G\times\hat{G}\right)$, the pseudo-differential operators $A_a:\quad M^{p,q}_\varpi(G)\to M^{p,q}_\varpi(G)$ are linear bounded integral operators for all $p,\quad q\in(1,\ \infty)$.*

Proof. Indeed, for all functions $\psi \in M_{\varpi}^{p_1,q_1}(G)$, $\varphi \in M_{\varpi^{-1}}^{p_2,q_2}(G)$, we estimate

$$|\langle A_a\psi,\varphi\rangle| = \left|\int_{\hat{G}}\int_G a(h,\xi)\,\Re(\psi,\ \varphi)(h,\xi)\,d\mu_G(h)\,d\hat{\mu}_{\hat{G}}(\xi)\right| \\ \leq c\,\|a\|_{M^{1,\infty}_{\varpi^{-1}\circ\iota^{-1}}}\,\|\psi\|_{M_{\varpi}^{p_1,q_1}}\,\|\varphi\|_{M_{\varpi^{-1}}^{p_2,q_2}}\,.$$

□

Analogous results are true in $\mathfrak{T}$-modulation space $W_{\tau}^{p,q}(G)$.

Lemma 7.1. *Let functions $\psi \in W_{\tau}^{p_1,q_1}(G)$ and $\varphi \in W_{\tau}^{p_2,q_2}(G)$ where $1 \leq p,\ q,\ p_i,\ q_i < \infty$. Then, the $\mathfrak{T}$-Wigner function satisfies the estimation*

$$\|W_{\mathfrak{I}}(\psi,\varphi)\|_{W_{\tau}^{p,q}(G)} \leq \tilde{c}\,\|\psi\|_{W_{\tau}^{p_1,q_1}(G)}\,\|\varphi\|_{W_{\tau}^{p_2,q_2}(G)}$$

for all $p_i,\ q_i < q, \quad i = 1,\ 2$ such that ${p_1}^{-1} + {p_2}^{-1} \geq p^{-1} + q^{-1}$ and ${q_1}^{-1} + {q_2}^{-1} \geq p^{-1} + q^{-1}$.

Theorem 7.3. *Let an integral operator A_a corresponds to the symbol $a \in W_{\tau,1\circ\iota^{-1}}^{\infty,1}\left(G \times \hat{G}\right)$ then A_a is a bounded linear operator from $W_{\tau}^{p,q}(G)$ to $W_{\tau}^{p,q}(G)$.*

Proof. The idea of the proof is similar to theorem 7.2, so, we have

$$|\langle A_a\psi,\varphi\rangle| = \left|\int_{\hat{G}}\int_G a(h,\xi)\,W_{\mathfrak{I}}(\psi,\varphi)(h,\xi)\,d\mu_G(h)\,d\hat{\mu}_{\hat{G}}(\xi)\right| \\ \leq c\,\|a\|_{W_{\tau,1\circ\iota^{-1}}^{\infty,1}}\,\|\psi\|_{W_{\tau}^{p_1,q_1}}\,\|\varphi\|_{W_{\tau}^{p_2,q_2}}\,,$$

where we applied Lemma 7.1.

□

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