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Complex Dynamical Behaviors of a Leslie-Gower Predator-Prey Model with Herd Behavior*

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Abstract In this paper, we consider a Leslie-Gower predator-prey model with a square root functional response while prey forms a herd as a form of group defense. We show that the solution of the system is non-negative and bounded. By applying the blow-up technique, it can be deduced that the origin displays instability. Moreover, employing the proof-by-contradiction approach, we demonstrate that the unique equilibrium point can be globally asymptotically stable under certain conditions. The sufficient conditions for the occurrence, stability, and direction of Hopf bifurcation are obtained. We further explore the conditions for the existence and uniqueness of the limit cycle. Theoretical results are validated through numerical simulations. Thus, our findings reveal that herd behavior has an important impact on the Leslie-Gower prey-predator system.

Keywords Leslie-Gower predator-prey model, herd behavior, stability, Hopf bifurcation, limit cycle

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1. Introduction

The dynamical behavior between predator and prey plays a crucial role in the fields of biology, mathematics, and ecology. Establishing mathematical models is the preferred method for scholars to understand the dynamical behavior of the predator-prey system. With the advancement of ecology, the development of ecological models has become increasingly sophisticated. Among the models, the Leslie-Gower model [1] can be applied to describe the interaction and evolution process of two species in the ecosystem, illustrating that both prey and predator have their upper limit on growth rates. This interesting formulation for predator dynamics has been discussed by Leslie and Gower in [2] and by Pielou in [3]. The general form of the

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Leslie-Gower model is as follows

$$\begin{cases} \dot{x} = r_1 x(1 - \frac{x}{k}) - H(x)y, \\ \dot{y} = r_2 y(1 - \frac{y}{px}), \end{cases} \tag{1.1}$$

where $\frac{y}{px}$ in the second equation is called the Leslie-Gower term and p means the food quality of prey for conversion into predator's growth.

Since the model (1.1) was proposed, the functional response $H(x)$ describes the rate at which a predator consumes prey according to the density of prey and has been widely used in ecological models. However, in many real-world situations, predators may come together to form groups while searching for prey. This adds an additional layer of complexity that must be taken into account in the model. To address these complexities, functional responses that depend on both predator and prey characteristics have been developed. Integrating complex functional responses is crucial in comprehending the dynamics of predator-prey interactions and making precise predictions about ecosystem stability and biodiversity. There are two main categories for the exact forms of the functional response: prey-dependent and ratio-dependent (predator-dependent). This subject has been extensively studied by scholars of various disciplines [4–11]. Ding and Huang [8] classified the global dynamics of a ratio-dependent predator-prey model. The functional response takes the form $H(y/x)$, which depends on the ratio of predator and prey populations. Many results have been obtained in the study of limit cycles, including dissipativeness and permanence, local and global stability, periodic orbits, and various kinds of bifurcations. Limit cycles are a crucial area of research. It is a periodic oscillation in predator-prey systems that allows coexistence [12]. This dynamic mechanism has fascinated many researchers, not only in integer-order differential dynamical systems but also in fractional-order differential dynamical systems[13–15].

Group defense refers to the herd behavior of prey species gathering in groups to reduce the probability of being captured by predators. With appropriate assumptions about the form and type of prey's functional responses, the concept of group defense was considered and modeled in general terms [16]. Freedman and Wolkowicz [16] also considered Holling type IV functional response. If there is mutual interference among predators, it can improve their chances of survival. Ajraldi et al. [17] accounted for all types of interactions between populations, symbiosis, competition, and predator-prey interactions. They introduced a non-linear term, the square root of population density, which took into account the assumption that interactions occur along the boundaries of the populations. They assumed that x represents the total quantity of prey inhabiting a specific regular surface, such as circles, and then the predator at the group boundary consumed prey quantities proportional to $\sqrt{x}$. For example, this may be entirely appropriate for herbivores in a large savanna and their large predators. Based on their findings, the predator-prey model in [17] shows stable limit cycles and Hopf bifurcation, which are unique features compared to other predator-prey systems.

The integration of herd behavior of prey into a prey-predator system has also been studied in reference [18–28]. Braza [18] demonstrated the dynamical behavior of the origin. Xu et al. [19] proposed the conditions for periodic orbits and the existence and uniqueness of limit cycles. Bulai et al.[20] proposed replacing the general exponent α $(0 < \alpha < 1)$ with the exponent $\frac{1}{2}$. The stable solution attained by the populations is independent of the shape of the herd. In addition, it is important to study how prey species protect themselves and their offspring when predator cooperation becomes stronger. Many researchers [21–23] assumed that adult prey exhibit group defense as an anti-predator strategy when they are sensitive to predation and developed a stage-structured predator-prey model dividing the prey population into two discrete stages (juvenile and adult). Various other classes of models such as discrete models, as well as multi-scale predator-prey models have also been investigated [24–28].

Furthermore, the time delay has been and continues to be the subject of much recent research, for instance, the papers [33–36]. A delayed predator-prey model with a social behavior for the prey population has been investigated in [36]. Djilali studied the effect of the herd shape for the prey population on the prey and predator equilibrium densities. The spatial spread of species is widespread in the real world. Diffusion patterns in some predator-prey systems are a topic of interest among scholars [37–39]. Cooperating with Leslie Gower, Souna et al. [37] determined the effect of prey social behavior interaction on the temporal behavior of the solutions. Mezouaghi et al. [38] highlighted that harvesting generates significant spatial patterns. Furthermore, Guin et al. [39] showed that cross-diffusion manifests that the prey species resolve a self-defense mechanism to secure themselves from any attack of the predators.

Based on the motivation from the above articles, we are still using the square root response function in this paper to describe herd behavior. He and Li [40] gave the following Leslie-Gower predator-prey model with square root functional response

$$\begin{cases} \dot{x} = r_1 x \left(1 - \frac{x}{k}\right) - a\sqrt{x}y, \\ \dot{y} = r_2 y \left(1 - \frac{y}{px}\right), \end{cases} \tag{1.2}$$

where r_1 and r_2 refer to the intrinsic growth rates of prey and predator species, respectively, while k is the carrying capacity of the natural environment, and p represents the food conversion efficiency from prey to predator growth. a denotes the predator's search efficiency for prey.

The above works indicate that the study of square root functional response is of great practical significance to the predator-prey system. Due to the herd behavior of prey being taken into account, not all prey populations contribute to the predator's natural environment's carrying capacity. Only the prey located at the boundary contributes to the predator's carrying capacity. Therefore, we modify the second equation of the model (1.2),

$$\begin{cases} \dot{x} = r_1 x \left(1 - \frac{x}{k}\right) - a\sqrt{x}y, \\ \dot{y} = r_2 y \left(1 - \frac{y}{p\sqrt{x}}\right). \end{cases} \tag{1.3}$$

All parameters r_1, r_2, k, a and p are positive constants in model (1.3).

This paper is organized as follows: in Section 2, we present preliminary results that include simplification of the model, analysis of equilibria, and boundedness of the system. Furthermore, the method of proof by contradiction provides adequate conditions for achieving global stability. In Section 3, we demonstrate that the system exhibits Hopf bifurcation. Moreover, we demonstrate the existence of a unique stable limit cycle under specific conditions. In Section 4, we perform numerical simulations using MATLAB software to validate previously established theoretical findings. The conclusion is given in Section 5.

2. Stability analysis

Letting $\bar{x} = \frac{x}{k}$, $\bar{y} = \frac{ay}{\sqrt{k r_1}}$, $\bar{t} = r_1 t$, $s = \frac{r_2}{r_1}$, $\bar{p} = \frac{ap}{r_1}$ and dropping the bars, we obtain the following system

$$\begin{cases} \dot{x} = x(1-x) - \sqrt{x}y, \\ \dot{y} = sy\left(1 - \frac{y}{p\sqrt{x}}\right). \end{cases} \tag{2.1}$$

In this section, we establish the boundedness of the solution and examine the stability of the equilibria in the simplified system (2.1).

2.1. Positivity and boundedness of the solution

To satisfy the biological meaning, in the following, we consider system (2.1) in phase space

$$\Omega = \{(x(t), y(t)) \in R^2 | x > 0, y \geq 0\}.$$

We note that all solutions of system (2.1) are positive if the initial value is selected in the interval of R^2_+. For system (2.1) with initial conditions $x(0) > 0$, $y(0) > 0$, we have

$$x(t) = x(0) \exp\left\{\int_0^t [1 - x(u) - \frac{y(u)}{\sqrt{x(u)}}]\, \mathrm{d}u\right\},$$

$$y(t) = y(0) \exp\left\{\int_0^t [s(1 - \frac{y(u)}{p\sqrt{x(u)}})]\, \mathrm{d}u\right\},$$

which shows that all the solutions of system (2.1) are non-negative.

Theorem 2.1. *The solutions of system* (2.1) *are ultimately bounded with an initial value which satisfies* $(x_0, y_0) \in \Omega$ *for all* $t > 0$.

Proof. From the first equation of system (2.1), we can obtain that

$$\dot{x} \leq x(1-x),$$

as the initial values are positive. By applying Lemma 2 in [41], we have

$$\lim_{t \to +\infty} \sup x(t) \leq 1,$$

which implies that $x(t)$ will be limited in the interval (0,1].

For any $\epsilon > 0$, there exists $T_1 > 0$ such that $x(t) < 1 + \epsilon$. This, combined with the second equation of the system (2.1), yields

$$\dot{y} \leq sy\left(1 - \frac{y}{p\sqrt{1+\epsilon}}\right) \quad \text{for } t \geq T_1.$$

We obtain $\lim\limits_{t \to +\infty} \sup y(t) \leq p\sqrt{1+\epsilon}$, letting $\epsilon \to 0$ gives $\lim\limits_{t \to +\infty} \sup y(t) \leq p$.

Hence, the solutions of system (2.1) are bounded in Ω for all $t > 0$. □

2.2. Stability of equilibria

In the following, we will analyze the equilibrium of system (2.1). Compared to classical predator-prey systems, system (2.1) is not well defined at $(0,0)$, so this point is not an equilibrium. It seems that system (2.1) always has a boundary equilibrium $E_1(1,0)$. The Jacobian matrix $J(E_1)$ at $E_1(1,0)$ takes the form as

$$J(E_1) = \begin{bmatrix} -1 & -1 \\ 0 & s \end{bmatrix},$$

as $Det(J(E_1)) = -s < 0$, E_1 is a saddle point with the x-axis as its two stable manifolds. There exists a unique unstable manifold of E_1 in the interior of R_+^2. This suggests that there is a localized region of equilibrium in the first quadrant, as the initial values of the predator and prey are located in this area. Even if the predator population is small, it will not become extinct.

Next, we focus on a unique positive equilibrium $E^*(x^*, y^*)$ of system (2.1) with $x^* = 1-p$ and $y^* = p\sqrt{1-p}$, where $0 < p < 1$. The Jacobian matrix of system (2.1) at the positive equilibrium $E^*(x^*, y^*)$ is given by

$$J(E^*) = \begin{bmatrix} \dfrac{3p}{2} - 1 & -\sqrt{1-p} \\ \dfrac{sp}{2\sqrt{1-p}} & -s \end{bmatrix}.$$

The determinant and the trace of matrix $J(E^*)$ are

$$DetJ((E^*)) = s(1-p)$$

and

$$Tr(J(E^*)) = \frac{3}{2}p - 1 - s.$$

As $0 < p < 1$, $Det(J(E^*))$ is always a positive value, but $Tr(J(E^*))$ is indefinite.

It follows that E^* is locally asymptotically stable when $p \in (0, \frac{2}{3})$ or $\frac{3p}{2} - 1 < s$, $p \in (\frac{2}{3}, 1)$; and unstable when $\frac{3p}{2} - 1 > s$, $p \in (\frac{2}{3}, 1)$.

Due to the presence of the square root functional response, system (2.1) exhibits distinct dynamical behaviors from the previous system (1.1). Combining the above results, we obtain the following theorem.

Theorem 2.2. *For system* (2.1), *we have the following conclusions:*

(i) If $p \in (0, \frac{2}{3})$ or $\frac{3p}{2} - 1 < s$ and $p \in (\frac{2}{3}, 1)$, the unique positive equilibrium $E^*(x^*, y^*)$ of system (2.1) *is locally asymoptotically stable;*

(ii) If $\frac{3p}{2} - 1 > s$ and $p \in (\frac{2}{3}, 1)$, $E^*(x^*, y^*)$ is unstable.

As previously stated, since there is no defined origin, we must utilize the transformation $dt = p\sqrt{x}d\tau$ and continue referring to τ as t to convert system (2.1) into an equivalent topological system as follows

$$\begin{cases} \dot{x} = px^{\frac{3}{2}}(1-x) - pxy, \\ \dot{y} = sy\left(p\sqrt{x} - y\right). \end{cases} \tag{2.2}$$

Theorem 2.3. *The origin is unstable for system* (2.1).

Proof. Since the origin is not defined, we cannot determine the qualitative property directly. Based on the system's characteristics, we replace $x = 0$ with system (2.2). With simple calculation, we get $\dot{x} = 0$ and $\dot{y} = -sy^2 < 0$. In other words, $x = 0$ can be considered as the invariant line of system (2.1). To analyze the qualitative behavior at $(0, 0)$, a more efficient approach is to perform a Briot-Bouquet transformation [42].

Set $x = \bar{x}$, $y = \bar{x}\bar{y}$, $dt = \bar{x}^{\frac{1}{2}} d\tau$ (still denote τ by t). A straightforward calculation shows that system (2.2) is converted into the system

$$\begin{cases} \dfrac{d\bar{x}}{dt} = p\bar{x}\left(1 - \bar{x} - \sqrt{\bar{x}}\bar{y}\right), \\ \dfrac{d\bar{y}}{dt} = \bar{y}\left(p(\bar{x} + s - 1) + (p - s)\sqrt{\bar{x}}\bar{y}\right). \end{cases} \tag{2.3}$$

And letting $u = \sqrt{\bar{x}}$, $v = \bar{y}$, system (2.3) is rewritten as

$$\begin{cases} \dfrac{du}{dt} = \dfrac{1}{2}u\left(p(1 - u^2) - puv\right), \\ \dfrac{dv}{dt} = v\left(p(u^2 + s - 1) + (p - s)uv\right), \end{cases} \tag{2.4}$$

which is well-defined for all $u \geq 0$ and $v \geq 0$. System (2.4) has two equilibrium points at its boundaries which are $A(0, 0)$ and $B(1, 0)$. The Jacobian matrices for points A and B respectively are

$$J_A = \begin{bmatrix} \frac{1}{2}p & 0 \\ 0 & p(s-1) \end{bmatrix}, \quad J_B = \begin{bmatrix} -p & -\frac{1}{2}p \\ 0 & sp \end{bmatrix}.$$

The determinant of J_B is negative, indicating that B is a saddle point. When $s < 1$, A is a saddle point; when $s > 1$, it is an unstable node; when $s = 1$, A is a repelling saddle-node known from the center manifold theorem. We then use the inverse Briot-Bouquet transformation, resulting in the conclusion that the origin is unstable. □

2.3. Globally asymptotic stability of E^*

In this subsection, we give sufficient conditions for the global stability of E^*, based on Theorem 2.2 (i). Before illustrating the globally asymptotic stability of E^*, we need to show that system (2.1) has no periodic orbits in the region Ω through a proof-by-contradiction approach. By using a similar approach introduced in [43], we can obtain the following result.

Lemma 2.1. *Suppose that $Tr(J(E^*)) < 0$. If a periodic solution exists, the orbit will be locally asymptotically (orbital) stable.*

Proof. Let us assume that the system (2.1) has a positive periodic solution $\Gamma(t) = (x(t), y(t))$ in R^2_+ with the period $T > 0$. According to the divergence criterion [44], we introduce that this periodic solution is locally asymptotically (orbital) stable if

$$\int_0^T Tr(J_\Gamma)\mathrm{d}t < 0.$$

We note that

$$\int_0^T \frac{\dot{x}}{x}\mathrm{d}t = \int_0^T \left(1 - x - \frac{y}{\sqrt{y}}\right)\mathrm{d}t = 0, \qquad \int_0^T \frac{\dot{y}}{y}\mathrm{d}t = \int_0^T s\left(1 - \frac{y}{p\sqrt{x}}\right)\mathrm{d}t = 0,$$

and

$$\begin{aligned} Tr(J_\Gamma) =& 1 - 2x - \frac{y}{2\sqrt{x}} + s - \frac{2sy}{p\sqrt{x}} \\ =& \frac{1}{2}\frac{\dot{x}}{x} + 2\frac{\dot{y}}{y} - s + \frac{1-3x}{2} \\ =& \frac{1}{2}\frac{\dot{x}}{x} + 2\frac{\dot{y}}{y} - s + \frac{1-3x^*}{2} + \frac{3}{2}(x^* - x) \\ =& \frac{1}{2}\frac{\dot{x}}{x} + 2\frac{\dot{y}}{y} - s - 1 + \frac{3p}{2} + \frac{3}{2}(x^* - x). \end{aligned}$$

Hence we have

$$\begin{aligned} \int_0^T Tr(J_\Gamma)\mathrm{d}t =& \int_0^T (-s - 1 + \frac{3p}{2})\mathrm{d}t + \frac{3}{2}\int_0^T (x^* - x)\mathrm{d}t \\ =& \int_0^T Tr(J(E^*))\mathrm{d}t + + \frac{3}{2}\int_0^T (x^* - x)\mathrm{d}t. \end{aligned}$$

Considering $E^* = (x^*, y^*)$ is a positive equilibrium of system (2.1), we substitute E^* into the system (2.1) and obtain the equation $y^* = \sqrt{x^*}(1 - x^*)$ and $y^* = p\sqrt{x^*}$. So we can rewrite system (2.1) as

$$\begin{aligned} \frac{\dot{x}}{x} = 1 - x + x^* - x^* - \frac{y}{\sqrt{x}} =& -(x - x^*) + \frac{y^*}{\sqrt{x^*}} - \frac{y}{\sqrt{x}} \\ =& -(x - x^*) + \frac{y^*(x - x^*)}{\sqrt{x^* x}(\sqrt{x} + \sqrt{x^*})} - \frac{y - y^*}{\sqrt{x}}, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \frac{\dot{y}}{y} = s - \frac{s(y - y^*)}{p\sqrt{x}} - \frac{sy^*}{p\sqrt{x}} =& -\frac{s(y - y^*)}{p\sqrt{x}} + s\left(\frac{y^*}{p\sqrt{x^*}} - \frac{y^*}{p\sqrt{x}}\right) \\ =& -s\frac{y - y^*}{p\sqrt{x}} + s\frac{(x - x^*)}{\sqrt{x}(\sqrt{x} + \sqrt{x^*})}. \end{aligned} \tag{2.6}$$

Solving equations (2.5) and (2.6) results in the following expression:

$$x - x^* = -\frac{\dot{x}}{x} + \frac{p}{s}\frac{\dot{y}}{y}.$$

From Green's Formula we obtain that, $\int_0^T (x - x^*)dt = 0$.

Recall that $Tr(J(E^*)) < 0$. This implies that

$$\int_0^T Tr(J_\Gamma)\mathrm{d}t = \int_0^T (-s - 1 + \frac{3p}{2})\mathrm{d}t + \frac{3}{2}\int_0^T (x^* - x)\mathrm{d}t < 0,$$

where Ω is a bounded region surrounded by Γ. This proves Lemma 2.1. □

Theorem 2.4. *The unique positive equilibrium $E^*(x^*, y^*)$ of system* (2.1) *is globally asymptotically stable, if $p \in (0, \frac{2}{3})$ or $\frac{3p}{2} - 1 < s$ and $p \in (\frac{2}{3}, 1)$.*

Proof. If there exists a periodic solution, the orbit will be stable with $Tr(J(E^*)) < 0$ as stated in Lemma 2.1. However, this contradicts the stability of E^* which is asymptotically stable. Therefore, system (2.1) does not exhibit any non-trivial periodic solutions.

Based on Theorem 2.2 (i) and combined with the instability of E_1 and the origin, it can be deduced that the unique positive equilibrium E^* is locally asymptotically stable which is equivalent to globally asymptotically stable. This leads to the proof of Theorem 2.4. □

3. Bifurcation analysis

In this subsection, we focus on certain conditions for the existence of Hopf bifurcation and provide proof for the direction of the Hopf bifurcation of the system (2.1). Furthermore, we investigate the system (2.1) under specific conditions, where it can only generate a unique limit cycle.

3.1. Hopf bifurcation

We first introduce the following lemma.

Lemma 3.1 ([44]). *For a general planar analytic system*

$$\begin{cases} \dot{x} = a_{10}x + a_{01}y + p(x,y), \\ \dot{y} = b_{10}x + b_{01}y + q(x,y), \end{cases}$$

with $\Delta = a_{10}b_{01} - a_{01}b_{10} > 0$, $a_{10} + b_{01} = 0$, $p(x,y) = \sum_{i+j\geq 2} a_{ij}x^i y^j$, *and* $q(x,y) = \sum_{i+j\geq 2} b_{ij}x^i y^j$, *and the matrix*

$$Df(0) = \begin{bmatrix} a_{10} & a_{01} \\ b_{10} & b_{01} \end{bmatrix}$$

has a pair of imaginary eigenvalues and the origin will be a weak focus. The Lyapunov number σ *is then given by the formula*

$$\begin{aligned} \sigma = &-\frac{3\pi}{2a_{01}\Delta^{\frac{3}{2}}}\big\{a_{10}b_{10}(a_{11}^2 + a_{11}b_{02} + a_{02}b_{11}) + a_{10}a_{01}(b_{11}^2 + a_{20}b_{11} + a_{11}b_{02}) \\ &+ b_{10}^2(a_{11}a_{02} + 2a_{02}b_{02}) - 2a_{10}b_{10}(b_{02}^2 - a_{20}a_{02}) - 2a_{10}a_{01}(a_{20}^2 - b_{20}b_{02}) \\ &- a_{01}^2(2a_{20}b_{20} + b_{11}b_{20}) + (a_{01}b_{10} - 2a_{10}^2)(b_{11} - b_{02} - a_{11}a_{20}) \\ &+ (a_{10}^2 + a_{01}b_{10})\big(3(b_{10}b_{03} - a_{01}a_{30}) + 2a_{10}(a_{21} + b_{12}) + (b_{10}a_{12} - a_{01}b_{21})\big)\big\}. \end{aligned}$$

The periodic solution is subcritical or supercritical in nature of the value of $\sigma > 0$ *and* $\sigma < 0$*, respectively.*

Next, we will first prove the transversality condition for the occurrence of a Hopf bifurcation. Then, we calculate the first Lyapunov coefficient σ at the positive equilibrium point E^* of system (2.1) to predict the stability of the limit cycle emerging through a Hopf bifurcation.

Theorem 3.1. *If $\frac{3p}{2} - 1 = s$ and $p \in (\frac{2}{3}, 1)$, the system* (2.1) *undergoes a supercritical Hopf bifurcation at the positive equilibrium $E^*(x^*, y^*)$.*

Proof. Selecting s as the hopf bifurcation parameter when $s = \frac{3}{2}p - 1 \triangleq s_h$, it will change the stability of E^* when the parameter passes from one side of $s = s_h$ to the other side.

The Jacobian matrix at the positive equilibrium point E^* corresponds to the following characteristic equations

$$\lambda^2 - \beta_1 \lambda + \beta_2 = 0, \tag{3.1}$$

where $\beta_1 = Tr(J(E^*))$ and $\beta_2 = Det(J(E^*))$.

When $s = s_h$ and $\beta_1 = 0$, equation (3.1) turns into

$$\lambda^2 + \beta_2 = 0.$$

Then, we can obtain a pair of purely imaginary roots as $\lambda_1 = i\sqrt{\beta_2}$, $\lambda_2 = -i\sqrt{\beta_2}$. Differentiating both sides of equation (3.1) with respect to s results in

$$2\lambda \frac{d\lambda}{ds} - \lambda \frac{d\beta_1}{ds} - \beta_1 \frac{d\lambda}{ds} + \frac{d\beta_2}{ds} = 0. \tag{3.2}$$

By simplifying expression (3.2), we obtain

$$\frac{d\lambda}{ds} = \frac{\lambda \dfrac{d\beta_1}{ds} - \dfrac{d\beta_2}{ds}}{2\lambda - \beta_1}. \tag{3.3}$$

Substituting $\lambda = i\sqrt{\beta_2}$ into equation (3.3),

$$\left.\frac{d\lambda}{ds}\right|_{\lambda = i\sqrt{\beta_2}} = \frac{i\sqrt{\beta_2}\dot{\beta}_1 - \dot{\beta}_2}{2i\sqrt{\beta_2} - \beta_1} = \frac{2\dot{\beta}_1\beta_2 + \beta_1\dot{\beta}_2}{4\beta_2 + \beta_1^2} + i\frac{2\sqrt{\beta_2} + \dot{\beta}_2 - \beta_1\sqrt{\beta_2}}{4\beta_2 + \beta_1^2},$$

consequently,

$$Re\left(\frac{d\lambda}{ds}\right)\Bigg|_{\lambda = i\sqrt{\beta_2}} = \frac{2\dot{\beta}_1\beta_2 + \beta_1\dot{\beta}_2}{4\beta_2 + \beta_1^2}. \tag{3.4}$$

Substituting $\beta_1 = \frac{3}{2}p - 1 - s$ and $\beta_2 = s\left(\frac{p^2 - 3p + 2}{2}\right)$ into (3.4), we can get

$$Re\left(\frac{d\lambda}{ds}\right)\Bigg|_{\lambda = i\sqrt{\beta_2}} = -\frac{1}{8} \neq 0.$$

In summary, we have verified the transversality condition of the Hopf bifurcation. Based on Lemma 3.1, now we proceed to calculate the first Lyapunov coefficient σ. We convert E^* to the origin by $\bar{x} = x - x^*$, $\bar{y} = y - y^*$ and Taylor expand system (2.1) near the origin. Then dropping the bars, the system (2.1) is transformed into

$$\begin{aligned} \dot{x} &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 \\ &\quad + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + o(|x,y|^4), \\ \dot{y} &= b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 \\ &\quad + b_{21}x^2y + b_{12}xy^2 + b_{03}y^3 + o(|x,y|^4). \end{aligned} \tag{3.5}$$

The coefficients of (3.5) are listed as follows.

$$
\begin{aligned}
a_{10} &= \frac{3p}{2} - 1, & a_{11} &= -\frac{1}{2\sqrt{1-p}}, & a_{12} &= 0,\\
a_{01} &= -\sqrt{1-p}, & a_{20} &= \frac{8-9p}{4(p-1)}, & a_{02} &= 0,\\
a_{21} &= \frac{1}{4(1-p)^{\frac{3}{2}}}, & a_{30} &= \frac{8-9p}{4(p-1)}, & a_{03} &= 0,\\
b_{10} &= \frac{p(3p-2)}{4\sqrt{1-p}}, & b_{11} &= \frac{3p-2}{2(1-p)}, & b_{12} &= \frac{3p-2}{2(1-p)^{\frac{3}{2}}p},\\
b_{01} &= \frac{1}{2}(2-3p), & b_{20} &= -\frac{3p(3p-2)}{8(1-p)^{\frac{3}{2}}}, & b_{02} &= \frac{2-3p}{p\sqrt{1-p}},\\
b_{21} &= -\frac{3(3p-2)}{4(1-p)^2}, & b_{30} &= \frac{15p(3p-2)}{16(1-p)^{\frac{5}{2}}}, & b_{03} &= 0.
\end{aligned}
$$

Hence, using the formula of the first Lyapunov number $\sigma = -\dfrac{3\pi L}{2a_{01}\Delta^{\frac{3}{2}}}$ and the condition $\frac{3}{2}p - 1 = s$, we have

$$
\sigma = \frac{3\pi\Big(-12 + 8\left(\sqrt{1-p}+5\right)p - 2\left(13\sqrt{1-p}+23\right)p^2 + 21\left(\sqrt{1-p}+1\right)p^3\Big)}{4(p-1)^3 p\sqrt{-6p^2+10p-4}}.
$$

Focus on the numerator of the σ

$$
l \triangleq (-12 + 8(\sqrt{1-p}+5)p - 2(13\sqrt{1-p}+23)p^2 + 21(\sqrt{1-p}+1)p^3). \qquad (3.6)
$$

Letting $\sqrt{1-p}$ as $\hat{p}$, (3.6) is turned to

$$
\begin{aligned}
l &= 21(\hat{p}+1)p^3 - 2(13\bar{p}+23)p^2 + 8(\hat{p}+5)p - 12\\
&= 21p^3 + 40p - 12 + \hat{p}(21p^2 - 26p + 8)p\\
&= l_1 + \hat{p}l_2,
\end{aligned}
$$

where $l_1 = 21p^3 + 40p - 12$, and $l_2 = (21p^2 - 26p + 8)p$. Under the condition of $p \in (\frac{2}{3}, 1)$, both $\hat{p}$ and l_2 are positive. By simple calculation, we can also see that the derivative of l_1 with respect to p is always positive over the range $p \in (\frac{2}{3}, 1)$.

$$
l_1 = 21p^3 + 40p - 12,
$$

it follows that

$$
\frac{dl_1}{dp} = 63p^2 + 40 > 0,
$$

and

$$
l_1\big|_{p=\frac{2}{3}} = \frac{28}{9}.
$$

Therefore, it follows that $l_1 > 0$.

However, the denominator of σ is negative for $p \in (\frac{2}{3}, 1)$,

$$
4(p-1)^3 p\sqrt{-6p^2+10p-4} = 4(p-1)^3 p\sqrt{-2(3p-2)(p-1)} \leq 0,
$$

indicating that $\sigma < 0$. Therefore, we can infer that system (2.1) undergoes a supercritical Hopf bifurcation, leading to the emergence of a stable limit cycle from the equilibrium point E^* when s passes through s_h. Thus, we have demonstrated the validity of Theorem 3.2. $\square$

3.2. The conditions of generating one limit cycle

In this subsection, we give the following lemmas leading to the proof of the existence of a unique limit cycle under certain conditions.

Lemma 3.2. *System (2.1) has at least one limit cycle if* $\frac{3p}{2} - 1 > s$ *and* $p \in (\frac{2}{3}, 1)$ *hold.*

Proof. In the previous discussion, we obtained system (2.2), which is topologically equivalent to the system (2.1), implying that both systems exhibit identical dynamical behaviors. Under the conditions $\frac{3p}{2} - 1 > s$ and $p \in (\frac{2}{3}, 1)$, E^* is unstable.

It's noticed that $\dot{y}|_{x=0} = -sy^2 < 0$ and $\dot{x}|_{y=0} = px^{\frac{3}{2}}(1-x) > 0$ $(x < 1)$.

Let $L_1 : x - 1 = 0$, $L_2 : y - p = 0$, then

$$\left.\frac{dx}{dt}\right|_{x=1} = -py < 0,$$
$$\left.\frac{dy}{dt}\right|_{y=p} = sp^2(\sqrt{x} - 1) < 0 \quad (x < 1).$$

Using the Poincaré–Bendixson Theroem [45], system(2.1) has at least one limit cycle if $\frac{3p}{2} - 1 > s$ and $p \in (\frac{2}{3}, 1)$ hold. □

Lemma 3.3. *If the conditions* $\frac{3}{2}p - 1 > s$ *and* $p \in (\frac{2}{3}, 1)$ *hold, system (2.1) has at most one closed orbit in the interior of the first quadrant. Moreover, the closed orbit is stable if it exists.*

Proof. Next, we will apply transformations to the system(2.2) whose method is similar to that used in [46]. This will simplify the system (2.2) into the form of a generalized Liénard system, then utilize the modified Z.F. Zhang's theorem proposed in [47]. Z.F. Zhang's theorem was first proposed in [48].

Denote $F_0(x) = px^{\frac{3}{2}}(1-x)$, $F_1(x) = px$, $G_1(x) = sp\sqrt{x}$, $G_2(x) = -s$, and $E(x) = \dfrac{(G_2(x) - F_1'(x))}{F_1(x)} = \frac{-s-p}{px}$. From system (2.2), we can obtain equations: $y^* = p\sqrt{x^*} = \sqrt{x^*}(1 - x^*)$. Subsequently, we employ the transformation shown in [46] to eliminate the cross-term xy,

$$\begin{aligned}
&x = x, \quad \widetilde{y} = F_0(x) - F_1(x)y;\\
&x = x, \quad u = \widetilde{y}\exp\left(\int_{x^*}^{x} E(w)\mathrm{d}w\right), \quad \mathrm{d}\widetilde{t} = \exp\left(-\int_{x^*}^{x} E(w)\mathrm{d}w\right)\mathrm{d}t;\\
&x = x, \quad v = u - F_0(x)\exp\left(\int_{x^*}^{x} E(w)\mathrm{d}w\right) + F_0(x^*);\\
&x = \hat{x}, \quad \hat{y} = \ln\left(1 - \frac{v}{F_0(x^*)}\right), \quad \mathrm{d}\hat{t} = -F_0(x^*)\mathrm{d}\widetilde{t},
\end{aligned}$$

still denoting $\hat{x}$, $\hat{y}$ and $\hat{t}$ as x, y, and t, respectively. With this notation, we have transformed system (2.2) into a Liénard system as follows

$$\begin{aligned}
\dot{x} &= \varPhi(y) - F(x),\\
\dot{y} &= -g(x),
\end{aligned} \tag{3.7}$$

where

$$\Phi(y) = e^y, \quad F(x) = \frac{F_0(x)}{F_0(x^*)} exp\left(\int_{x^*}^{x} E(w)\mathrm{d}w\right),$$
$$g(x) = \frac{F_1(x)G_1(x) + F_0(x)G_2(x)}{F_0(x^*)F_1(x)} exp\left(\int_{x^*}^{x} E(w)\mathrm{d}w\right)$$
$$\triangleq \frac{Q(x)}{F_0(x^*)} exp\left(\int_{x^*}^{x} E(w)\mathrm{d}w\right).$$

Here

$$Q(x) = G_1(x) + \frac{F_0(x)G_2(x)}{F_1(x)} = sp\sqrt{x} - s\sqrt{x}(1-x),$$

where $F_0(x^*) = p^2(x^*)^{\frac{3}{2}} > 0$, $Q(x^*) = sy^* - s\frac{y^*}{\sqrt{x^*}}\sqrt{x^*} = 0$. Note that all transformations are one-to-one correspondences on R_+. As a result, the systems described in (3.7) and (2.2) exhibit topological equivalence. Next, we will discuss the conditions under which system (3.7) has only one limit cycle. After the transformation, $E^*(x^*, y^*)$ is mapped to $(x^*, 0)$ of system (3.7). Obviously, we can know that $\Phi'(y) = e^y > 0$, $g(x^*) = 0$ and $(x-x^*)g(x) > 0$ for $x > 0$ and $x \neq x^*$. In conclusion, we have satisfied conditions (i) and (ii) in Theorem 1.1 of [47].

Then, it can be determined that

$$f(x) = F'(x) = \frac{f_1(x)}{F_0(x^*)} \exp\left(\int_{x^*}^{x} E(w)\mathrm{d}w\right),$$
$$f_1(x) = F_0'(x) + F_0(x)E(x) = \frac{3}{2}p(1-x)\sqrt{x} + (p+s)(x-1)\sqrt{x} - px^{\frac{3}{2}}$$
$$= \frac{1}{2}\sqrt{x}\left(p + 2s(x-1) - 3px\right),$$
$$f_1(x^*) = \frac{1}{2}p\sqrt{1-p}(3p-2-2s) > 0 \quad (\frac{3}{2}p - 1 > s, \quad p \in (\frac{2}{3}, 1)).$$

The above calculation indicates that $f(x^*) > 0$. On the contrary

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{p + 2s(x-1) - 3px}{2s(x+p-1)} \triangleq h(x),$$

and $\frac{3}{2}p - 1 > s$ implies $2s - 3p + 2 < 0$, so that

$$h'(x) = \frac{p(2-3p+2s)}{2s(x+p-1)^2} < 0, h(0) = \frac{2s-p}{2s(1-p)} < \frac{3p-p-2}{2s(1-p)} = \frac{2(p-1)}{2s(1-p)} < 0.$$

The root of $h(x)$ is $x_0 = \frac{2s-p}{2s-3p} < \frac{2p-2}{2s-3p} < 0$. So we can obtain that $\frac{d}{dx}(\frac{f(x)}{g(x)}) < 0$ for $0 < x \neq x^*$. Therefore, the assumption (iii) of Theorem 1.1 of [20] is satisfied. This proves Lemma 3.3. $\square$

Theorem 3.2. *If $\frac{3p}{2} - 1 > s$ and $p \in (\frac{2}{3}, 1)$, $E^*(x^*, y^*)$ is unstable and system (2.1) has a unique stable limit cycle.*

Proof. Lemma 3.2 and Lemma 3.3 provide the conditions for system (2.1) to have only one stable limit cycle, which are $\frac{3p}{2} - 1 > s$ and $p \in (\frac{2}{3}, 1)$. This proves Theorem 3.2. $\square$

Therefore, system (2.1) has at most one closed orbit if it exists. Furthermore, the conditions above imply that the equilibrium E^* is unstable. Hence, the limit cycle is stable if it exists.

4. Numerical simulations

In this section, we will validate our previous theoretical analysis through numerical simulations conducted in MATLAB R2016a. To accomplish this, we will measure the following parameter set for system (2.1).

Example 4.1. We select specific values for the parameters, which are presented below.

In Figure 1,

$$s = 0.0575, p = 0.635, E^*(0.23, 0.36928),$$

with the following three initial conditions $(x(0), y(0))$ respectively:

- Initial 1: $(0.215, 0.35)$,
- Initial 2: $(0.3, 0.33)$,
- Initial 3: $(0.5, 0.2)$.

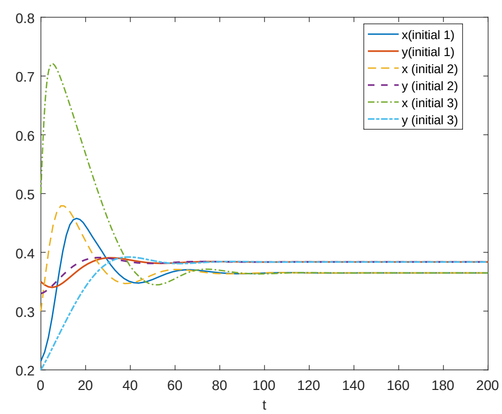

Figure 1. The equilibrium point $E^*(0.23, 0.36928)$ is locally asymptotically stable for $s = 0.0575$, $p = 0.635 \in (0, \frac{3}{2})$.

In Figure 2,

$$s = 0.2, p = 0.7777, E^*(0.2223, 0.36668),$$

with the same three initial conditions $(x(0), y(0))$ in Fig.1.

The time series diagram illustrated in Figure 1 and Figure 2 effectively captures the properties of stability and long-term trends around the E^* of the model (2.1) over an extended time. According to Theorem 2.4, the parameter p lies in $(0, \frac{2}{3})$, and the only positive equilibrium E^* is globally asymptotically stable in Figure 1. Next, in Figure 2, the parameter satisfies the conditions $\frac{3p}{2} - 1 < s$ and $p \in (\frac{2}{3}, 1)$, which enables us to achieve a similar outcome of global asymptotic stability of E^*.

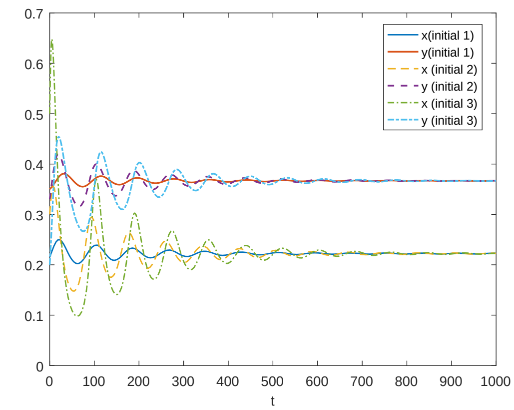

Figure 2. The equilibrium point $E^*(0.2223, 0.36768)$ is locally asymptotically stable for $s = 0.2 > s_h = 0.16655$, $p = 0.7777 \in (\frac{3}{2}, 1)$.

Example 4.2. For a numerical simulation of Hopf bifurcation, we consider the following values of parameters: $s = 0.166549$ and $p = 0.7777$, with an initial condition $(x, y) = (0.215, 0.35)$ near the E^* in Figure 3.

As previously discussed, we have calculated that the Hopf bifurcation occurs at a critical value of s_h equal to 0.16655 through simple calculations. By selecting different values of s around the critical value and performing numerical simulations, we obtain the following results: for $s = 0.166549 < 0.16655$, the equilibrium point $E^* = (0.2223, 0.36668)$ is an unstable focus with a stable limit cycle with $\sigma \approx -960.642 < 0$. Next, the time series diagram is depicted in Figure 3.

Our numerical simulations verify the validity of Theorem 3.2.

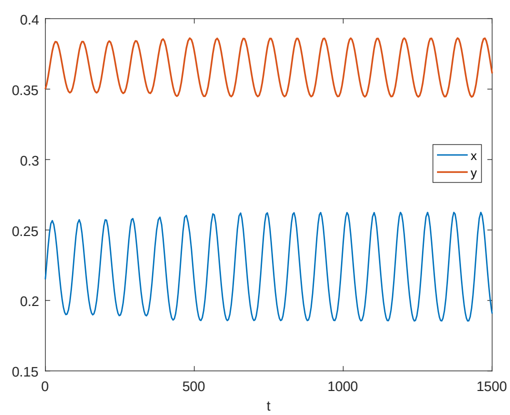

Figure 3. The equilibrium point $E^*(0.2223, 0.36768)$ becomes unstable and a Hopf bifurcation occurs for $s = 0.166549 < 0.16655 = s_h$, $p = 0.7777 \in (\frac{3}{2}, 1)$.

5. Conclusion

In this paper, we investigate the dynamics of a Leslie-Gower predator-prey model with herd behavior. Compared to the system (1.1) without herd behavior, the proposed system exhibits more complex behaviors. We first show that the solution of the proposed system (2.1) is non-negative and bounded. Secondly, by analyzing the Jacobian matrices of each equilibrium point, the stability of the equilibrium points of the system (2.1) is analyzed in detail. Subsequently, we deduce that the origin exhibits instability with the blow-up technique. Then, the proof-by-contradiction approach provides conditions for global stability of positive equilibrium E^*. Thirdly, we obtain the conditions for the occurrence of Hopf bifurcation and the existence of a unique stable limit cycle. Finally, we validate our previous theoretical analysis through numerical simulations.

In Section 2, by using dimensionless variables and parameters, the system (1.3) transforms into the system (2.1). Notably, we substitute the parameters $\bar{p} = \frac{ap}{r_1}$ and $s = \frac{r_2}{r_1}$ into Theorem 2.4 and Theorem 3.2, which can further derive the conditions and establish the equivalence between system (2.1) and system (1.3). From a biological perspective, we provide further explanation of our model (1.3).

(i) The conditions in Theorem 2.4

$$p \in (0, \frac{2}{3}) \text{ or } \frac{3p}{2} - 1 < s \text{ and } p \in (\frac{2}{3}, 1) \text{ in system (2.1)}$$
$$\iff \frac{3ap}{2} < r_1 + r_2 \text{ in the system (1.3).}$$

The following scenarios may occur when the prey gather together in herds and adopt a group defense strategy.

When the growth rate of prey surpasses that of the predator, the predator's ability to capture prey decreases due to herd behavior but still allows the predator to catch more prey. The predator's carrying capacity is related to the prey, thereby limiting the prey's population's growth indefinitely.

On the other hand, if the intrinsic growth rate of the prey is relatively smaller than that of the predator, then the predator may capture less prey. The herd behavior of prey prevents the extinction of prey.

In other words, the situations mentioned above, the food conversion efficiency or the predator's search efficiency is small. In addition, we demonstrate that as long as the sum of the intrinsic growth rates of the prey and the predator is sufficiently large, both species can survive and their populations can remain stable near the equilibrium point $E^*(\frac{r_1-ap}{r_1}, \frac{ap}{r_1}\sqrt{\frac{r_1-ap}{r_1}})$ of system (1.3).

(ii) The conditions in Theorem 3.2

$$\frac{3p}{2} - 1 > s \text{ and } p \in (\frac{2}{3}, 1) \text{ in system (2.1)}$$
$$\iff \frac{3ap}{2} > r_1 + r_2 \text{ in system (1.3).}$$

Conversely, the sum of the intrinsic growth rates of the prey and the predator is relatively small, while the predator's search ability or food conversion efficiency improved. In this case, the system (1.3) generates a unique and stable limit cycle. This implies that the predator and prey can oscillate periodically to survive together.

When comparing our results with those in [40], we observe both commonalities and differences between the models (1.2) and (1.3). The dynamical behavior of the two models exhibits similar results with the stability and unique limit cycle of the systems, even under different conditions. Different from our model (1.3), the condition in model (1.2) is that the growth rates of predator and prey respectively correspond to a threshold, and the sizes of r_1 and r_2 need to be discussed respectively. However, for model (1.3), we only need to consider the size of $r_1 + r_2$ and the threshold value $\frac{3ap}{2}$ to obtain similar results. Furthermore, by exploring the various values for the predator's search efficiency for prey a and the food conversion efficiency from prey to predator growth p, we can also observe that different a and p have an important impact on the dynamical behavior of the system (1.3).

For further study, we will consider herd behavior not only in prey but also in other terms of the system through different modeling approaches. Such a model would be more realistic and provide more information on the interactions between predators and prey. Inspired by reference [49, 50], we can introduce the term diffusion, exploring its significant impact on system dynamics. Additionally, imprecise parameters [51] can also be considered in biological models to provide valuable insights into various spatial patterns.

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