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Asymptotic Expression of Eigenvalues for a Class of Fractional Boundary Value Problems*

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Abstract Based on the properties of Mittag-Leffler function, the behavior of eigenvalue and the eigenfunction for a class of fractional differential equations with integral boundary value condition is discussed, and the asymptotic expression of the eigenfunction is given.

Keywords Fractional differential equation, boundary value problem, eigenvalue, eigenfunction, asymptotic estimation

MSC(2010) 34A08, 34L20.

1. Introduction

In recent years, fractional differential equations have become a hot topic, as they can describe problems in optical and thermal systems, rheology and material and mechanical systems, signal processing and system identification, control and robotics, and other applications. Many researchers focus on the existence of solutions for fractional differential equations boundary value problems ([1,2], [4,5], [7–10]). For example, in [5], the author considered the existence of solutions for fractional differential equations with integral boundary conditions. However, the theoretical study of fractional differential equations is very difficult because of the nonlocal and singularity of fractional differential operators.

$$\begin{cases} {}^cD_{0+}^{\alpha}y(t)+f(t,y(t))=0, t\in(0,1), \alpha\in(2,3), & (1.1)\\ y(0)=y'(0)=0, y(1)=\lambda\displaystyle\int_0^1 y(s)ds, & (1.2)\end{cases}$$

where λ is a parameter and ${}^cD_{0+}^{\alpha}y(t)$ is the standard Caputo fractional derivative. Using the Guo-Krasnoselskii fixed point theorem, the author obtained the sufficient conditions on the existence of positive solutions for problem (1.1)-(1.2). On the other hand, some researchers pay attention to the eigenvalue problem of fractional

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differential equations, see for example [2–4], [6], [12]. In [4], the author mainly studied the eigenvalue problems of fractional order differential equations with Dirichlet and Neumann equal boundary conditions.

$$\begin{cases} {}^cD^{\delta}_{0+}u(t) + \lambda u(t) = 0, t \in (0, a), 1 < \delta < 2, & (1.3) \\ u(0) = u(a) = 0. & (1.4) \end{cases}$$

According to the properties of Caputo fractional derivative and Riemann-Liouville fractional integral, using the Laplace transform, the author shows that the eigenvalue of the fractional order differential equations $\lambda > 0$ is the zero of $f(\lambda) = E_{\delta,2}(-\lambda a^{\delta})$ and the eigenfunction is $u(t, \lambda) = tE_{\delta,2}(-\lambda t^{\delta})$. However, it is difficult to obtain the exact zeros of $f(\lambda)$ and their specific distribution, so the further information of the eigenvalues is not clear, and the specific asymptotic expressions about eigenvalues and eigenfunctions become an important problem to be solved, which inspire us to do this work.

Motivated by [4, 5], using the properties of Mittag-Leffler functions ([10, 11]), we give the specific asymptotic expressions of eigenvalues and eigenfunctions of the following problems (1.5)-(1.6).

$$\begin{cases} {}^cD^{\delta}_{0+}u(t) + \lambda u(t) = 0, t \in (0, 1), 1 < \delta < 2, & (1.5) \\ u(0) = 0, u(1) = \displaystyle\int_0^1 u(s)ds, & (1.6) \end{cases}$$

where ${}^cD^{\delta}_{0+}u(t)$ is a Caputo fractional derivative. As in [4], for a given λ, we call that the nonzero solution $u(t)$ of problems (1.5)-(1.6) is the eigenfunction corresponding to λ, and λ is the eigenvalue.

In order to facilitate the readers, we give the following definitions and lemmas.

Definition 1.1 (Definition 1.1.1, [4])**.** The Riemann-Liouville fractional integral of $\alpha > 0$ of function $y : (0, \infty) \to R$ is defined as

$$J^{\alpha}_{0+}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} y(s)ds.$$

Definition 1.2 (Definition 1.1.3, [4])**.** The Caputo fractional derivative of $\alpha > 0$ of function $f : (0, \infty) \to R$ is defined as

$${}^cD^{\alpha}_{0+}f(t) := J^{n-\alpha}_{0^+} D^n(f(t)) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \frac{f^{(n)}(s)}{(t - s)^{\alpha-n+1}} ds.$$

Definition 1.3 ([4], [11])**.** The Mittag-Leffer type function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} (\alpha, \beta > 0).$$

Lemma 1.1 (Lemma 1.1.3, [4])**.** *Semigroup relations of fractional calculus hold* $J^{\alpha}_{0+}J^{\beta}_{0+}\varphi(t) = J^{\alpha+\beta}_{0^+}\varphi(t)$ *in the following circumstances:*

(i) $\beta \geq 0$, $\alpha + \beta \geq 0$, $\varphi(t) \in L^1(0, 1)$*;*

(ii) $\beta \leq 0$, $\alpha \geq 0$, $\varphi(t) \in J^{-\beta}_{0^+}(L^1(0, 1))$*;*

(iii) $\alpha \leq 0$, $\alpha + \beta \leq 0$, $\varphi(t) \in J^{-\alpha-\beta}_{0^+}(L^1(0, 1))$.

Lemma 1.2 (Theorem 2.1.1, [11]). *Let (1)* $\rho > \frac{1}{2}$, $\mu \in \mathbb{C}$, $\mu \neq 1, 0, -1, -2, ...,$ *for* $\rho = 1$, *or (2)* $\rho = 1$ *and* $Re\mu > 3$. *Then all sufficiently large (in modulus) zeros* z_n *of the function* $E_\rho(z;\mu)$ *are simple and the following asymptotic formula hold:*

$$z_n^\rho = 2\pi in - \frac{\tau_\mu}{\rho} ln2\pi in + lnc_\mu + \frac{d_\mu/c_\mu}{(2\pi in)^{1/\rho}} + (\frac{\tau_\mu}{\rho})^2 \frac{ln2\pi in}{2\pi in} - \frac{\tau_\mu}{\rho}\frac{lnc_\mu}{2\pi in} + \alpha_n,$$

as $n \to \pm\infty$, *where*

$$\alpha_n = O\left(\frac{ln|n|}{|n|^{1+1/\rho}}\right) + O\left(\frac{1}{|n|^{2/\rho}}\right) + O\left(\frac{ln^2|n|}{n^2}\right), \text{if } \rho > \frac{1}{2}.$$

Lemma 1.3 (Theorem 1.2.1, [11]). *For any* $\rho > \frac{1}{2}$, $\mu \in \mathbb{C}$, $m \in \mathbb{N}$, *the following asymptotics hold.*

If $|argz| \leq min(\pi, \frac{\pi}{\rho})$, *then*

$$E_\rho(z;\mu) = \rho z^{\rho(1-\mu)} e^{z^\rho} - \sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - \frac{k}{\rho})} + O(|z|^{-m-1}),$$

as $z \to \infty$.

If $\rho > 1$ *and* $\frac{\pi}{\rho} \leq |argz| \leq \pi$, *then*

$$E_\rho(z;\mu) = -\sum_{k=1}^{m} \frac{z^{-k}}{\Gamma(\mu - \frac{k}{\rho})} + O(|z|^{-m-1}),$$

as $z \to \infty$.

By substituting $\alpha = \frac{1}{\rho}$, $\beta = \mu$, $m = 1$ into Definition 1.3 and Lemma 1.3, we can obtain the following lemma.

Lemma 1.4. *Under the conditions of Lemma 1.3, one has*

$$\frac{1}{\rho} z^m E_\rho(z,\mu) = z^{\tau_\mu} e^{z^\rho} - c_\mu - \frac{d_\mu}{z} + \mathrm{O}(\frac{1}{z^2}),$$

where $c_\mu = \frac{1}{\rho\Gamma(\mu - \frac{1}{\rho})}$, $d_\mu = \frac{1}{\rho\Gamma(\mu - \frac{2}{\rho})}$, *and* $\tau_\mu = 1 + \rho(1-\mu)$.

Lemma 1.5. *The eigenvalues of problem (1.5)-(1.6) are zeros of* $E_{\delta,2}(-\lambda) - E_{\delta,3}(-\lambda) = 0$.

Proof. By the Laplace transformation method, the general solution of (1.5) is $u(t) = c_1 t E_{\delta,2}(-\lambda t^\delta) + c_2 E_{\delta,1}(-\lambda t^\delta)$, where $E_{\alpha,\beta}(z)(\alpha = \delta > 0, \beta = 1, 2)$ is the two-parameter Mittag - Leffler function, which together with condition (1.6) can lead to the result by the standard way. □

2. Main results and proofs

In order to facilitate the proof of our results, we introduce the following notation:

$$c_2 = \frac{\delta}{\Gamma(2-\delta)}, c_3 = \frac{\delta}{\Gamma(3-\delta)}, d_3 = \frac{\delta}{\Gamma(3-2\delta)},$$

$$h_1 = (1-\delta)2\pi n \ln(2\pi n) - (1-\delta)\ln(c_2 - c_3)(\frac{\pi}{2} - \frac{\ln(c_2 - c_3)}{2\pi n}),$$

$$h_2 = (1-\delta)\left(\ln(c_2 - c_3)(\ln(2\pi n) + 1) - n\pi^2\right), h_3 = \ln(c_2 - c_3) + \delta - 1.$$

Theorem 2.1. *All sufficiently large(in modulus) eigenvalues λ_n are simple and the following asymptotic formula holds:*

$$\begin{aligned}\lambda_n^{\frac{1}{\delta}} =&(-1)^{\frac{1}{\delta}}\left(\ln(c_2-c_3)+\frac{2h_1n\pi+(h_2+1)h_3}{4n^2\pi^2+h_3^2}\right)\\ &+(-1)^{\frac{1}{\delta}}\left(2n\pi+\frac{h_1h_3-2(h_2+1)n\pi}{4n^2\pi^2+h_3^2}\right)i+\varepsilon_n,\end{aligned}$$

where $\lim\limits_{n\to\infty}\varepsilon_n=0$.

Proof. By Lemma 1.5, the eigenvalues of (1.5)-(1.6) satisfy the following equation

$$E_{\delta,2}(-\lambda)-E_{\delta,3}(-\lambda)=0. \tag{2.1}$$

Set $z=-\lambda$. Then equation (2.1) becomes

$$\delta zE_{\delta,2}(z)-\delta zE_{\delta,3}(z)=0.$$

Further, we have

$$e^{z^{\frac{1}{\delta}}+\ln(z^{\tau_2}-z^{\tau_3})}=c_2-c_3+\frac{d_2-d_3}{z}+O(\frac{1}{z^2}). \tag{2.2}$$

Set $y=z^{\frac{1}{\delta}}$ and

$$z^{\frac{1}{\delta}}+\ln(z^{\tau_2}-z^{\tau_3})=\omega. \tag{2.3}$$

Note that $\tau_2=1-\frac{1}{\delta}$ and $\tau_3=1-\frac{2}{\delta}$. Then (2.3) can be reduced to

$$y+(\delta-2)\ln y+\ln(y-1)=\omega. \tag{2.4}$$

Let $y=\omega+r(\omega)$. Noting that $r(\omega)=o(\omega),\omega\to\infty$, by (2.4), one has

$$\begin{aligned}r(\omega)&=(2-\delta)\ln(\omega+r(\omega))-\ln(\omega+r(\omega)-1)\\ &=(1-\delta)\ln\omega+(2-\delta)\ln(1+\frac{r(\omega)}{\omega})-\ln\left(1+\frac{r(\omega)-1}{\omega}\right),\end{aligned}$$

and by the Taylor formula we have

$$r(\omega)=\frac{(1-\delta)\omega\ln\omega+1}{\omega+\delta-1}+\varepsilon_\omega,$$

where $\varepsilon_\omega=O\left(\frac{(1-\delta)r^2(\omega)+2r(\omega)-1}{\omega^2}\right)$, and $\lim\limits_{\omega\to\infty}\varepsilon_\omega=0$. So

$$y=\omega+\frac{(1-\delta)\omega\ln\omega+1}{\omega+\delta-1}+\varepsilon_\omega, \tag{2.5}$$

which together with (2.2) leads to

$$e^\omega=c_2-c_3+\frac{d_2-d_3}{z}+O(\frac{1}{z^2}),$$

and futher,

$$\begin{aligned}\omega&=2\pi in+\ln(c_2-c_3+\frac{d_2-d_3}{z}+O(\frac{1}{z^2}))\\ &=2\pi in+\ln(c_2-c_3)+\ln\left(1+\frac{d_2-d_3}{c_2-c_3}\frac{1}{z}+O(\frac{1}{z^2})\right)\\ &=2\pi in+\ln(c_2-c_3)+\alpha_n,\end{aligned}$$

where $\alpha_n = \ln\left(1 + \frac{d_2-d_3}{c_2-c_3}\frac{1}{z} + O(\frac{1}{z^2})\right)$, and $\lim_{n\to\infty} \alpha_n = 0$.

Therefore, we obtain

$$\begin{aligned}\ln\omega &= \ln(2\pi in + \ln(c_2-c_3) + \alpha_n)\\ &= \ln(2\pi in) + \ln\left(1 + \frac{\ln(c_2-c_3)}{2\pi in} + \alpha_n\right)\\ &= \ln(2\pi n) + \left(\frac{\pi}{2} - \frac{\ln(c_2-c_3)}{2\pi n}\right) i + \beta_n,\end{aligned}$$

where $\lim_{n\to\infty} \beta_n = 0$. Hence,

$$\begin{aligned}\lambda_n^{\frac{1}{\delta}} =&(-1)^{\frac{1}{\delta}}\left(\frac{(1-\delta)(2\pi in + \ln(c_2-c_3))\left(\ln(2n\pi) + \left(\frac{\pi}{2} - \frac{\ln(c_2-c_3)}{2\pi n}\right)i\right)}{2\pi in + \ln(c_2-c_3) + \delta - 1}\right)\\ &+ (-1)^{\frac{1}{\delta}}\left(2\pi in + \ln(c_2-c_3) + \frac{1}{2\pi in + \ln(c_2-c_3) + \delta - 1}\right) + \varepsilon_n\\ =&(-1)^{\frac{1}{\delta}}\left(2\pi in + \ln(c_2-c_3) + \frac{h_1 i + h_2 + 1}{2\pi in + h_3}\right) + \varepsilon_n\\ =&(-1)^{\frac{1}{\delta}}\left(2n\pi i + \ln(c_2-c_3) + \frac{h_1 i(2n\pi - h_3)}{4n^2\pi^2 + h_3^2} + \frac{(h_2+1)(h_3 - 2n\pi)}{4n^2\pi^2 + h_3^2}\right) + \varepsilon_n\\ =&(-1)^{\frac{1}{\delta}}\left(\ln(c_2-c_3) + \frac{2h_1 n\pi + (h_2+1)h_3}{4n^2\pi^2 + h_3^2} + \left(2n\pi + \frac{h_1 h_3 - 2(h_2+1)n\pi}{4n^2\pi^2 + h_3^2}\right) i\right)\\ &+ \varepsilon_n,\end{aligned}$$

where $\lim_{n\to\infty} \varepsilon_n = 0$.

The proof is complete. □

In order to more intuitively observe the properties of the eigenvalue of problem (1.5)-(1.6), the following Figure 1 and Figure 2 are figures of $\lambda_n^{\frac{1}{\delta}}$ and $|\lambda_n|^{\frac{1}{\delta}}$ of $\delta = 1.05$.

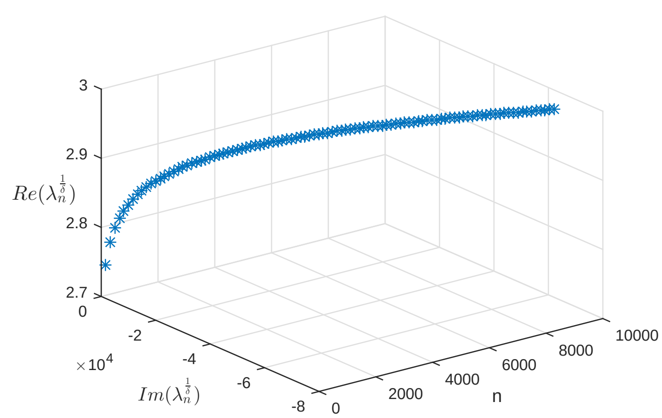

Figure 1. when $\delta = 1.05$, the distribution of $\lambda_n{}^{\frac{1}{\delta}}$

As can be seen from Figure 1, we can clearly see that $\lim\limits_{n\to\infty} \mathrm{Re}\lambda_n^{\frac{1}{\delta}} = \infty$, $\lim\limits_{n\to\infty} \mathrm{Im}\lambda_n^{\frac{1}{\delta}} = \infty$, where $\mathrm{Re}\lambda_n^{\frac{1}{\delta}}$ and $\mathrm{Im}\lambda_n^{\frac{1}{\delta}}$ represent the real and imaginary parts of ${\lambda_n}^{\frac{1}{\delta}}$, respectively.

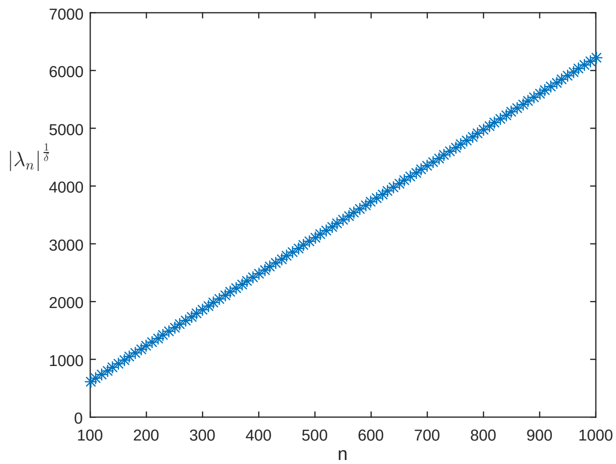

Figure 2. when $\delta = 1.05$, the distribution of $|\lambda_n|^{\frac{1}{\delta}}$

As can be seen from Figure 2, $\lim\limits_{n\to\infty} |\lambda_n|^{\frac{1}{\delta}} = \infty$.

The asymptotic representation and properties of the eigenvalues of problem (1.5)-(1.6) is already clear, so we also hopes to obtain the asymptotic representation of the eigenfunctions, as shown in Theorem 2.2.

Theorem 2.2. *For any $m, M \in \mathbb{N}$, the eigenfunction of problem (1.5)-(1.6) can be approximately expressed as*

$$C_h(t) = \begin{cases} \frac{(-1)^{\frac{1}{\delta}} e^{(-\lambda)^{\frac{1}{\delta}} t}}{\delta \lambda^{\frac{1}{\delta}}} - \sum\limits_{k=1}^{m} \frac{(-\lambda)^{-k} t^{1-\delta k}}{\Gamma(2-\delta k)} + R_m(-\lambda t^{\delta+1}; \frac{1}{\delta}, 2), & 1 < \delta \le \frac{4}{3}, \\ \sum\limits_{|\arg(-\lambda t^{\delta}) + 2\pi n| \le \frac{3\pi}{4}\delta} \frac{(-1)^{\frac{1}{\delta}} e^{(-\lambda)^{\frac{1}{\delta}} t e^{\frac{2\pi}{\delta} in} - \frac{2\pi}{\delta} in}}{\delta \lambda^{\frac{1}{\delta}}} - & \\ \sum\limits_{k=1}^{M} \frac{(-\lambda)^{k} t^{1-\delta k}}{\Gamma(2-\delta k)} + R_M(-\lambda t^{\delta+1}; \frac{1}{\delta}, 2), & \frac{4}{3} \le \delta < 2, \end{cases}$$

where $\left|R_m(-\lambda t^{\delta}; \frac{1}{\delta}, 2)\right| \le \frac{2^{\frac{b+2}{2}} \Gamma(b+1)}{|\lambda|^{m+1} a^{(\delta+1)(m+1)}}$, $\left|R_M(-\lambda t^{\delta+1}; \frac{1}{\delta}, 2)\right| \le \frac{2^{\frac{b+2}{2}} \Gamma(b+\frac{3}{2})}{|\lambda|^{M+1} a^{(\delta+1)(M+1)}}$, $b = \delta(s+1) - 2 \ge 0$, *$s = m$ or M.*

Proof. It is easy to know that the eigenfunction of (1.5)-(1.6) is $C_h(t) = tE_{\delta,2}(-\lambda t^{\delta})$.

Set $\mu, x \in \mathbb{C}$ and $x \neq 0$. By Theorem 1.4.2 and Theorem 1.5.1 of [11], one has

$$E_\rho(x;\mu) = \begin{cases} \rho x^{\rho(1-\mu)} e^{x^\rho} - \sum_{k=1}^{m} \frac{x^{-k}}{\Gamma(\mu - \frac{k}{\rho})} + R_m(x;\rho,\mu), & \rho \in [\frac{3}{4}, 1], \\ \rho \sum_{|\arg x + 2\pi n| \leq \frac{3\pi}{4\rho}} \left(x^\rho e^{2\pi i n \rho}\right)^{1-\mu} e^{(x^\rho(e^{2\pi i n\rho}))} & \\ - \sum_{k=1}^{M} \frac{x^{-k}}{\Gamma(\mu - \frac{k}{\rho})} + R_M(x;\rho,\mu), & \rho \leq \frac{3}{4}, \end{cases}$$

where

$$|R_m(x;\rho,\mu)| \leq \frac{2^{\frac{b+2}{2}}\Gamma(b+1)e^{(\frac{5}{4}\pi|\mathrm{Im}\mu|)}}{|x|^{m+1}}, |R_M(x;\rho,\mu)| \leq \frac{2^{\frac{b+2}{2}}\Gamma(b+\frac{3}{2})e^{(\frac{5}{4}\pi|\mathrm{Im}\mu|)}}{|x|^{M+1}}, b \geq 0.$$

Substituting $\rho = \frac{1}{\delta}$, $\mu = 2$, $x = -\lambda t^\delta$ into (2.5), we can obtain $E_{\delta,2}(-\lambda t^\delta) = E_{\frac{1}{\delta}}(-\lambda t^\delta; 2)$.

So Theorem 2.2 is proved. □

Remark. We only obtain the asymptotic expressions of eigenvalues with sufficient large moduli. For eigenvalues with insufficient modulus, their asymptotic representation will be studied in our next work.

References

[1] A. Ali, N. Khan and S. Israr. *On establishing qualitative theory to nonlinear boundary value problem of fractional differential equations*, Mathematical Sciences, 2021(15): 395-403.

[2] E. Azroul, A. Benkirane and M. Srati. *Problem associated with nonhomogeneous integro-differential operators: Eigenvalue problem*, Journal of Elliptic and Parabolic Equations, 2021(1): 1-18.

[3] A. Bahrouni and K. Ho. *Remarks on eigenvalue problems for fractional $p(\cdot)$-Laplacian*, 2020.
DOI: 10.3233/ASY-201628.

[4] Z. Bai. *Theory and application of boundary value problems for fractional differential equations*, China Science and Technology Press, 2013.(in Chinese)

[5] A. Cabada and G. Wang. *Positive solutions of nonlinear fractional differential equations with integral boundary value conditions*, Journal of Mathematical Analysis and Applications, 2012, 389(1): 403-411.

[6] M. Dehghan and A.B. Mingarell. *Fractional Sturm-Liouville eigenvalue problems*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales-Serie A: Matematicas, April 2020(114): 1-14.

[7] R. Gorenflo, Anatoly A. Kilbas and F. Mainardi, Sergei V. Rogosin. *Mittag-Leffler functions, related topics and applications*, Springer Monographs in Mathematics, 2014.
DOI: 10.1007/978-3-662-43930-2

[8] H. Lv and C. Hou. *Existence of positive solutions for boundary value problems of p-Laplacian fractional differential equations with integral boundary conditions*, Journal of Yanbian University: Natural Science, 2019(2):95-102.(in Chinese)

[9] X. Liu, Y. Wei and Z. Bai. *Existence of positive solutions for fractional differential equations with integral boundary value conditions*, Journal of Shandong University of science and technology: Natural Science, 2019, 038(002):100-105.(in Chinese)

[10] Z. Lou. *Existence of eigenvalues and solutions of fractional differential equations*, Shandong University, 2012. DOI: 10.7666/d.y2182673(in Chinese)

[11] A. Yu. Popov and A. M. Sedletskii. *Distribution of roots of Mittag-Leffler functions*, Journal of Mathematical Sciences, 2013. 190(2): 209-409.

[12] S. Zhang. *Eigenvalue problems for a class of fractional differential equations*, Journal of Lanzhou University: Natural Science, 2005, 41(003): 98-101.(in Chinese)