Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 4, December 2024, 1091–1121 DOI:10.12150/jnma.2024.1091

The Global Dynamics for a Stochastic SIR Epidemic Model with Vaccination

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Abstract A new stochastic SIR epidemic model with vaccination is established and its dynamical behavior is analyzed. Considering the random effects of vaccination rates and mortality in this model, it is demonstrated that the extinction and persistence of the virus is only correlated with the threshold R_0^s. If $R_0^s < 1$, the disease dies out with probability one. And if $R_0^s > 1$, the disease is stochastic persistent in the means with probability one. In addition, the existence and uniqueness of a smooth distribution are proven using the Itôs formula, and the sufficiency criterion is obtained using the Lyapunov function. Finally, the accuracy and efficiency of the stochastic SIR epidemic model with vaccination in predicting disease transmission trends were verified through simulation. Unlike the singularity of stochastic perturbations in existing infectious disease models, the innovation of this paper is in the addition of multiple stochastic perturbations, especially distinguishing the stochastic perturbations of mortality under vaccination, which are used to study the dynamics of the model.

Keywords Vaccination, random disturbance, smooth distribution, Lyapunov function, stability

MSC(2010) 58F15, 53F35.

1. Introduction

It is well known that infectious diseases have been the number one killer threatening human life and health. In particular, the spread of COVID-19 has caused great impact on human life and health in recent years. Thus, the prevention and control of infectious diseases have become urgent problems for all countries. And the analysis of infectious diseases requires the establishment of corresponding mathematical models. Therefore, studying how to develop a realistic mathematical model is of great practical significance. The use of mathematical models for the analysis of infectious diseases has yielded a wealth of results. For example, Wen considered that the disease has temporary immunity and proposed the SIR model [1]. And the author ingeniously constructed Lyapunov function to investigate the stabilization of the equilibrium point. In [1], it proved the stability problem related to the equilibrium point and obtained conclusions that when $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable regardless of time delay. But when $R_0 > 1$, the endemic equilibrium is existent while the disease-free equilibrium gets

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unstable. Peijiang Liu, Din Anwarud and Zenab [2] are devoted to the study of the dengue epidemic model. They used a randomly perturbed dengue fever model and information intervention factors. Qualitative analysis was conducted on the positive solutions of the model based on stochastic differential equations. By constructing Lyapunov functions, a basic recurrence number scheme is introduced to ensure the exponential stability.

With the development of biotechnology, experts found that vaccination can effectively slow down the diffusion of infectious diseases and thus reduce morbidity and mortality [3–5]. Therefore, vaccination has become an effective measure to prevent various infectious diseases. Over time, more experts are considering vaccination as a factor when studying infectious diseases [6–8]. In 2014, Chauhan S, Misra O P and Dhar J introduced vaccination rates in their original SIR model [9]. In this paper, the linear stability and global stability of the two models were addressed, and the existence of disease-free and epidemic equilibrium points of the two models were compared. It was analyzed that the local and global stability of the model is determined by the underlying reproduction number. Afterward, Maurício de Carvalho João P. S [10] introduced logistic growth into the SIR model and combined it with vaccination, studying the bifurcation issues of the model. Finally, a parameter space (R_0, p) was established to evaluate the proportion of vaccinated individuals necessary to eliminate the the disease and to conclude how the vaccination may affect the outcome of the epidemic. In 2023, the paper [11] further considered the reduction of vaccine costs and proposes several options of the stochastic SIR epidemics model with limited treatment. It demonstrated the efficiency of different vaccination strategies. Finally, a method was presented to obtain the optimal vaccination strategy that minimizes the cost functional. Afterwards, Changjin Xu investigated the infectious disease model, using COVID-19 as an example [12]. The study considered factors such as incubation times, vaccine effectiveness, and quarantine periods in the spread of the virus in symptomatically contagious individuals. Additionally, nonlinear analysis was employed to illustrate some findings concerning the ergodic aspect of the stochastic model.

Indeed, regardless of the infectious disease model utilized for analysis, white noise originating from real-life environments will inevitably impact the model to some extent. Din Anwarud developed and analyzed a random hepatitis B model considering propagation coefficient delay and CTL immune response categories through research. He investigated whether the model has a unique global solution and further analyzed the extinction and persistence of the disease. The research proved that the ergodic stationary distribution exists under certain conditions. From this, it can be inferred that white noise plays an important role in controlling infection [13]. Hence, the study of stochastic epidemic models becomes more relevant. Consequently, an increasing number of researchers have incorporated random disturbances into infectious disease models and analyzed various system properties such as stability [14–18]. For instance, in [19], Khan T et al. proposed a stochastic model for novel coronaviruses based on the assumption that only transmission rates fluctuate stochastically, and used Lyapunov functions to find conditions for extinction and persistence. The paper [28] provided insights into the impact of information campaigns on the spread of stochastic hepatitis. In [21], the authors also considered the random fluctuations in infection and recovery rates caused by environmental and weather variations. They proposed a stochastic infectious disease model that incorporates vaccination and saturation treatment. They then applied the random maximum principle and a four-step approach to investigate the stochastic optimal vaccination problem for the SIR model with saturation treatment. Building on the conclusion of [19], the paper [22] demonstrated the conditions for extinction and persistence of SARS-CoV-2 viral disease by combining multiple random sources. Later, the existence of stochastic persistence and smooth distribution of the stochastic epidemic model has been investigated [23–25, 27, 28]. Among them, Yiliang Chen et al. studied the SIQ model on the basis of considering isolation factors and random disturbances, and discussed the extinction and persistence of the model as well as the sufficient conditions for stationary distribution. Based on the characteristics of the disease and the underlying assumptions, Din Anwarud [28] also formulated the associated deterministic model for which the threshold parameter is calculated. The model was further extended to a stochastic model and it is well-justified that the model is both mathematically and biologically feasible by showing that the model solution exists globally, bounded stochastically and is positive. By utilizing the concepts of stochastic theory and by constructing appropriate Lyapunov functions, developed the theory for the extinction and persistence of the disease. Further, it is shown that the model is ergodic and has a unique stationary distribution. The stochastic bifurcation theory is utilized and a detailed bifurcation analysis of the model is presented.

Taking into account the factors mentioned earlier, this paper puts forward a novel approach which incorporates independent Brownian motion to introduce random perturbations to natural mortality and vaccination rate. This aims to enhance the accuracy of predicting the future dynamic behavior of the model.

First, we incorporate the vaccination rate into the general model, and the specific formulation of the model is outlined below,

$$\begin{aligned} S^{'} &= A - \mu S - \beta(1-u)IS - uS + \zeta R, \\ I^{'} &= \beta(1-u)IS - (\mu + \alpha + \varpi)I, \\ R^{'} &= uS + \varpi I - (\mu + \zeta)R. \end{aligned} \tag{1.1}$$

where $S(t)$ stands for a susceptible community, $I(t)$ stands for a infectious community, $R(t)$ stands for a recovery community. A represents the input rate of $S(t)$, while β is the transmission rate between community $S(t)$ and $I(t)$. The natural death rate of $S(t)$, $I(t)$ and $R(t)$ is denoted by μ, α is the disease-related mortality rate of $I(t)$, u is the population vaccination rate, ϖ is the recovery rate of $I(t)$, and ζ is the rate where individuals return to $S(t)$ from $R(t)$. All parameters are non-negative.

At the same time, this paper adds random disturbance to the model and assumes that the mortality rate under vaccination and non vaccination is different, respectively, denoted by α_2 and α_3. Then, we further define $\mu_1 = \mu$, $\mu_2 = \mu + \alpha_2$ for the convenience. Finally, we incorporate randomness into model (1.1), by using $\mu_1 \to \mu_1 + \sigma_1 dW_1(t)$, $\mu_2 \to \mu_2 + \sigma_2 dW_2(t)$, $\mu_3 \to \mu_3 + \sigma_3 dW_3(t)$, $u \to u + \sigma_4 dW_4$ instead of the parameters u and μ_i(i=1,2,3). On some probability space (Ω, F, P), $W_i(t)$(i=1,2,3,4) are independent standard Brownian motion and the intensity of $W_i(t)$ is $\sigma_i > 0$(i=1,2,3,4).

Thus, this paper establishes the following stochastic SIR epidemic model with multi-parameters white noises perturbations and the vaccination. The specific model is as follows.

$$\begin{aligned}dS(t) &= [A-\mu_1 S-\beta(1-u)IS-uS+\zeta R]dt+\sigma_1 SdW_1(t)+\sigma_4(\beta IS-S)dW_4(t),\\ dI(t) &= [\beta(1-u)IS-\mu_3 I-\varpi I]dt-\sigma_4\beta ISdW_4(t)+\sigma_3 IdW_3(t),\\ dR(t) &= (uS-\mu_2 R+\varpi I-\zeta R)dt+\sigma_4 SdW_4(t)+\sigma_2 RdW_2(t).\end{aligned} \tag{1.2}$$

The objective of this paper is to investigate the extinction and persistence and the stationary distribution of model (1.2) applying stochastic theories, formulas, and constructing suitable Liapunov function. This paper establishes a set of adequate preconditions to guarantee the extinction and persistence in the mean of the model with probability one, as well as the existence of a unique stationary distribution for model (1.2).

Apart from other studies, this study is the incorporation of random perturbations in the mortality rates for both vaccinated and unvaccinated populations, along with the inclusion of multiple types of white noise disturbances rather than a single one. This enhances the accuracy of the model for real-life applications. Additionally, the existence of steady-state distributions of the model is investigated, and the variations of compartments under random perturbations are analyzed using MATLAB for various infectious disease scenarios. We sincerely hope you can review it again.The content of this paper is roughly distributed as below. In Part 2, some necessary preliminaries and useful lemmas are mainly given. In Part 3, the conditions on the extinction and persistence in the mean with probability one for model (1.2) are described and shown. In Part 4, the unique stationary distributional properties of the model (1.2) are described and demonstrated. In Part 5, we use MATLAB to simulate the influence of vaccination and random disturbance on the model.

2. Materials and methods

2.1. Preliminaries and lemmas

Lemma 2.1. *For specific model (1.1), let* $R_0=\frac{A\beta(1-u)}{(\mu+\varpi+\alpha)(u+\mu)}$*. There are two conclusions below.*
(1) If $R_0<1$*, there is only one disease-free equilibriumc defined by* $E^0=(\frac{A(u+\zeta)}{u\mu-(u+\mu)(u+\zeta)},0,\frac{uA}{u\zeta-(u+\mu)(u+\zeta)})$ *in model (1.1) and it is globally asymptotically stable.*
(2) If $R_0>1$*,* $2\mu-\zeta>0$ *and* $(2\mu-\varpi)(u-\varpi)-u(\alpha+u-\varpi)<0$*, there is an endemic equilibrium defined by* $E^*=(S^*,I^*,R^*)$ *in the model (1.1) and it is globally asymptotically stable.*

$$\begin{aligned}S^* &= \frac{A}{(u+\mu)}, I^*=[A(1-\frac{1}{R_0})+\frac{(u+\varpi+\alpha)(\mu+\zeta)-\zeta\varpi}{u+\zeta}],\\ R^* &= \frac{1}{\mu+\zeta}[\frac{uA}{R_0(u+\mu)}+\varpi I^*].\end{aligned}$$

Proof. By calculating the Jacobian matrix of the system at the equilibrium E^0, the result indicates that it is locally asymptotically stable if $R_0<1$ and is unstable if $R_0>1$. To prove the global stability of E^0, we construct the Liapunov function

V s.t. $V = I$ and derive that,

$$V^{'} = [\beta(1-u)S - (\mu+\varpi+\alpha)]I \leq [\frac{\beta A(1-u)}{u+\mu} - (\mu+\varpi+\alpha)]I \leq 0.$$

In the above equation, $V = 0$ only when $I = 0$. In this model (1.1), $R^{'}(t) = uS + \varpi I - (\mu+\zeta)R$, $R \to 0$ as $t \to 0$. Thus, $R^{'}(t) = A - \mu S - uS + \zeta R$ can be equated to $R^{'}(t) = A - (\mu+u)S$. Therefore, based on the equation above, it is evident that $S \to \frac{A}{u+\mu}$. It can also be that when $I = 0$, all solutions in this model tend to the disease-free equilibrium E^0. According to the LaSalle Invariant Set Theorem in [29], the disease-free equilibrium $E^0 = (\frac{A}{u+\mu}, 0, 0)$ is globally asymptotically stable.

Consider the Liapunov function

$$\begin{aligned} V = & \frac{u(\alpha+u-\varpi)-(2\mu-\zeta)(u-\varpi)}{\beta u(1-u)}[I - I^* - I^* ln\frac{I}{I^*}] + \frac{2\mu-\zeta}{2u}(R-R^*)^2 \\ & + \frac{u(\alpha+u-\varpi)-u(\alpha+u-\varpi)-(2\mu-\zeta)(u-\varpi)}{2u\alpha}(N-N^*)^2 \\ & + \frac{1}{2}[-(N-N^*)-(R-R^*)]^2. \end{aligned}$$

Obviously, V is a positive definite function. Next, take the derivative of V, we have

$$\begin{aligned} V^{'} = & -\frac{u(\alpha+u-\varpi)-(2\mu-\zeta)(u-\varpi)}{u}(I-I^*)^2 \\ & -[\frac{(2\mu-\zeta)(u+\mu+\zeta)}{u} + (u+\mu+\zeta)](R-R^*)^2 \\ & -[\frac{\mu(2\mu-\zeta)(u-\zeta)}{u\alpha}](N-N^*)^2. \end{aligned}$$

Based on the Liapunov theorem [30], the endemic equilibrium can be obtained globally asymptotically stable, due to the negative qualitative nature of $V^{'}$. □

Lemma 2.2. *When an initial value of* $(S(0), I(0), R(0)) \in R^3_+$ *is arbitrarily given, model (1.2) has a distinctive global positive solution* $(S(t), I(t), R(t))$*. In other words, when* $t \geq 0$*, the solution* $(S(t), I(t), R(t))$ *stays in* R^3_+ *with probability one.*

Proof. Firstly, we adopt the nonlinear incidence rate $IS/f(I)$, where $f(I)$ is continuously differentiable with $f(0) = 1$ and $f^{'}(I) \geq 0$. So we consider the following model.

$$\begin{aligned} dS(t) &= [A - \mu_1 S - \frac{\beta(1-u)IS}{f(I)} - uS + \zeta R]dt + \sigma_1 S dW_1(t) + \sigma_4(\beta IS)dW_4(t), \\ dI(t) &= [\frac{\beta(1-u)IS}{f(I)} - \mu_3 I - \varpi I]dt - \sigma_4 \beta IS dW_4(t) + \sigma_3 I dW_3(t), \\ dR(t) &= (uS - \mu_2 R + \varpi I - \zeta R)dt + \sigma_4 S dW_4(t) + \sigma_2 R dW_2(t). \end{aligned}$$

Since the coefficients of the equation are locally Lipschitz continuous, for any given initial value $(S(0), I(0), R(0)) \in R^3_+$, there is a unique local solution $(S(t), I(t), R(t)) \in R^3_+$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we must demonstrate that $\tau_e = \infty$. Define the stopping time τ° by $\tau^\circ = inf\{t \in [0, \tau_e) : S(t) \leq 0 \ or \ I(t) \leq 0 \ or \ R(t) \leq 0\}$.

ϕ denotes the empty set, and we set $inf\phi = \infty$. We have $\tau^\circ \leq \tau_e$. So, if we can show that $\tau^\circ = \infty$, then for all $t \geq 0$, we have $\tau_e = \infty$ and $(S(t), I(t), R(t)) \in R_+^3$. Assume that $\tau^\circ < \infty$, then there exists a K such that $P\{\tau^\circ < K\} > 0$. Next, define a C^2-function $V(S, I, Q) = lnSIQ$. For all $t \in [0, \tau^\circ)$, according to the $It\hat{o}'s$ formula, we get

$$
\begin{aligned}
dV(S,I,R) = &[\frac{A}{S} - \mu_1 - \frac{\beta I(1-u)}{f(I)} + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_4^2(\beta I - 1)^2]dt + \sigma_1 dW_1 \\
&+ \sigma_4(\beta I - 1)dW_4 + [\frac{\beta(1-u)S}{f(I)} - \mu_3 - \varpi - \frac{1}{2}(\sigma_4\beta S)^2 + \frac{1}{2}\sigma_3^2]dt \\
&- \sigma_4\beta S dW_4 + \sigma_3 dW_3 + [u\frac{S}{R} - \mu_2 + \varpi\frac{I}{R} - \zeta + \frac{1}{2}(\sigma_4\frac{S}{R})^2 \\
&+ \frac{1}{2}\sigma_2^2]dt + \sigma_4\frac{S}{R}dW_4 + \sigma_2 dW_2 \\
= &[\frac{A}{S} - \mu_1 - \frac{\beta I(1-u)}{f(I)} + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_4^2(\beta I - 1)^2 + \frac{\beta(1-u)S}{f(I)} \\
&- \mu_3 - \varpi - \frac{1}{2}(\sigma_4\beta S)^2 + \frac{1}{2}\sigma_3^2 + u\frac{S}{R} - \mu_2 + \varpi\frac{I}{R} - \zeta + \frac{1}{2}(\sigma_4\frac{S^2}{R^2}) \\
&+ \frac{1}{2}\sigma_2^2]dt + \sigma_1 dW_1 + \sigma_4(\beta I - 1)dW_4 - \sigma_4\beta S dW_4 + \sigma_3 dW_3 \\
&+ \sigma_4\frac{S}{R}dW_4 + \sigma_2 dW_2,
\end{aligned}
$$

where $f(I)$ is used, and we have

$$
\begin{aligned}
dV(S,I,R) \geq &[-\beta I(1-u) - \mu_1 - \mu_3 - \varpi - \frac{1}{2}\sigma_4^2\beta^2 S^2 - \mu_2 - \zeta]dt + \sigma_1 dW_1 \\
&+ \sigma_4(\beta I - 1)dW_4 - \sigma_4\beta S dW_4 + \sigma_3 dW_3 + \sigma_4\frac{S}{R}dW_4 + \sigma_2 dW_2.
\end{aligned}
$$

Let $M(S, I, R) = -\beta I(1-u) - \mu_1 - \mu_3 - \varpi - \frac{1}{2}\sigma_4^2\beta^2 S^2 - \mu_2 - \zeta$, then

$$
\begin{aligned}
dV(S,I,R) \geq &M(S,I,Rdt + \sigma_1 dW_1 + \sigma_4(\beta I - 1)dW_4 - \sigma_4\beta S dW_4 + \sigma_3 dW_3 \\
&+ \sigma_4\frac{S}{R}dW_4 + \sigma_2 dW_2.
\end{aligned}
$$

So,

$$
\begin{aligned}
V(S,I,R) \geq &V(S(0), I(0), R(0)) + \int_0^t M(S(u), I(u), R(u))du + \sigma_1 W_1(t) + \sigma_2 W_2(t) \\
&+ \sigma_3 W_3(t) + \sigma_4 \int_0^t [\frac{S(u)}{R(u)} + \beta(u)I(u) - 1]dW_4(u).
\end{aligned}
$$

We know that some components of $(S(\tau^\circ), I(\tau^\circ), R(\tau^\circ))$ equal 0. So $\lim_{t\to\tau^\circ} V(S(t), I(t), R(t)) = -\infty$. Letting $t \to \tau^\circ$, we have

$$
\begin{aligned}
-\infty \geq &V(S(0), I(0), R(0)) + \int_0^{\tau^\circ} M(S(u), I(u), R(u))du + \sigma_1 W_1(t) + \sigma_2 W_2(t) \\
&+ \sigma_3 W_3(t) + \sigma_4 \int_0^t [\frac{S(u)}{R(u)} + \beta(u)I(u) - 1]dW_4(u) > -\infty,
\end{aligned}
$$

which leads to contradictions. So we have $\tau^\circ = \infty$, and the translation certification completes. □

Lemma 2.3. *Let* $(S(t), I(t), R(t))$ *be the solution of model (1.2) that has an initial value* $(S(0), I(0), R(0)) \in R^3_+$. *Then*

$$\lim_{t\to\infty} sup(S(t)+I(t)+R(t)) < \infty. \tag{2.1}$$

Furthermore,

$$\lim_{t\to\infty} sup(S(t)+I(t)+R(t)) \leq \frac{A}{\mu_1}. \tag{2.2}$$

Proof.

$$\begin{aligned} d(S+I+R) = {} & [A-\mu_1(S+I+R)-\alpha_2 R-\alpha_3 I]dt+\sigma_1 S dW_1(t)+\sigma_2 R dW_2(t) \\ & +\sigma_3 R dW_3(t), \end{aligned} \tag{2.3}$$

where $\alpha_3 = \mu_3 - \mu_1 \geq 0$ and $\alpha_2 = \mu_2 - \mu_1 \geq 0$.

By integrating the above equation, we obtain

$$\begin{aligned} S(t)+I(t)+R(t) = {} & \frac{A}{\mu_1}+[S(0)+I(0)+R(0)-\frac{A}{\mu_1}]e^{-\mu_1 t}-2\beta u\int_0^t SIe^{-\mu_1(t-s)}ds \\ & -\alpha_2\int_0^t R(s)e^{-\mu_1(t-s)}ds-\alpha_3\int_0^t I(s)e^{-\mu_1(t-s)}ds \\ & +\sigma_1\int_0^t S(s)e^{-\mu_1(t-s)}dW_1(s) \\ & +\sigma_2\int_0^t R(s)e^{-\mu_1(t-s)}dW_2(s)+\sigma_3\int_0^t I(s)e^{-\mu_1(t-s)}dW_3(s) \\ \leq {} & \frac{A}{\mu_1}+[S(0)+I(0)+R(0)-\frac{A}{\mu_1}]e^{-\mu_1 t}+M(t). \end{aligned} \tag{2.4}$$

$$\begin{aligned} M(t) = {} & \sigma_1\int_0^t S(s)e^{-\mu_1(t-s)}dW_1(s)+\sigma_2\int_0^t R(s)e^{-\mu_1(t-s)}dW_2(s) \\ & +\sigma_3\int_0^t I(s)e^{-\mu_1(t-s)}dW_3(s). \end{aligned}$$

Clearly, $M(t)$ is a continuous local martingale with $M(0) = 0$. Define

$$Y(t) = Y(0)+A(t)-U(t)+M(t),$$

where

$$\begin{aligned} & Y(0) = S(0)+I(0)+R(0), A(t) = \frac{A}{\mu_1}(1-e^{-\mu_1 t}), \\ & U(t) = (S(0)+I(0)+R(0)(1-e^{-\mu_1 t}). \end{aligned}$$

By (2.4), when $t \geq 0$, we have $S(t)+I(t)+R(t) \leq Y(t)$.

Obviously, when $t \geq 0$, $A(t)$ and $U(t)$ are continuous adapted increasing processes, and $A(0) = U(0) = 0$. So according to Theorem 1.1.2 in [31], it holds that $\lim_{t\to\infty} Y(t) < \infty$. Therefore, the result (2.1) is established. Set

$$M_1(t) = \int_0^t S(s)dW_1(s), M_1^*(t) = \int_0^t e^{-\mu_1}(t-s)S(s)dW_1(s),$$

$$M_2(t) = \int_0^t R(s)dW_2(s), M_2^*(t) = \int_0^t e^{-\mu_1}(t-s)R(s)dW_2(s),$$

$$M_3(t) = \int_0^t I(s)dW_3(s), M_3^*(t) = \int_0^t e^{-\mu_1}(t-s)I(s)dW_3(s).$$

Due to the quadratic variations

$$< M_1(t), M_1(t) > = \int_0^t S^2(t)ds \leq (\sup_{t\geq 0} S^2(t))t,$$
$$< M_1^*(t), M_1^*(t) > = \int_0^t e^{-2\mu_1(t-s)} S^2(t)ds \leq (\sup_{t\geq 0} S^2(t))t,$$

and according to the large number theorem for martingales(See [31]), the following two equations hold

$$\lim_{t\to\infty} \frac{1}{t} M_1(t) = 0, \lim_{t\to\infty} \frac{1}{t} M_1^*(t) = 0. \tag{2.5}$$

Similarly, we also have

$$\lim_{t\to\infty} \frac{1}{t} M_2(t) = 0, \lim_{t\to\infty} \frac{1}{t} M_2^*(t) = 0, \lim_{t\to\infty} \frac{1}{t} M_3(t) = 0, \lim_{t\to\infty} \frac{1}{t} M_3^*(t) = 0. \tag{2.6}$$

And because

$$\begin{aligned}
< M(t) > = & \frac{\sigma_1}{t} \int_0^t \int_0^s e^{-\mu_1(s-u)} S(u)dW_1(u)ds \\
& + \frac{\sigma_2}{t} \int_0^t \int_0^s e^{-\mu_1(s-u)} R(u)dW_2(u)ds \\
& + \frac{\sigma_3}{t} \int_0^t \int_0^s e^{-\mu_1(s-u)} I(u)dW_3(u)ds \\
= & \frac{\sigma_1}{\mu_1 t} [\int_0^t S(u)dW_1(u) - \int_0^t e^{-\mu_1(t-u)} S(u)dW_2(u)] \\
& + \frac{\sigma_2}{\mu_1 t} [\int_0^t R(u)dW_2(u) - \int_0^t e^{-\mu_1(t-u)} R(u)dW_2(u)] \\
& + \frac{\sigma_3}{\mu_1 t} [\int_0^t I(u)dW_3(u) - \int_0^t e^{-\mu_1(t-u)} I(u)dW_3(u)],
\end{aligned}$$

from (2.5) and (2.6), we obtain that $\lim_{t\to\infty} < M(t) >= 0$ holds.

From (2.4), we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t (S(0) + I(0) + R(0) - \frac{A}{\mu_1}) e^{-\mu_1 s} ds$$

$$= \lim_{t\to\infty} \frac{1}{\mu_1 t} (S(0) + I(0) + R(0) - \frac{1}{\mu_1 t} (1 - e^{-\mu_1 t}) = 0.$$

The above demonstrates the validity of conclusion (2.2). This concludes the proof. □

Lemma 2.4. *Let $(S(t), I(t), R(t))$ be the solution of model (1.2) and model (1.2) meets the following prerequisites:*
(1)Model (1.2) is with an initial value $(S(0), I(0), R(0)) \in R^3_+$.
(2)$N(t) = S(t) + I(t) + R(t)$.
Then

$$< S(t) >= \frac{(\mu_2 + \zeta)A}{\mu_1(\mu_2 + \zeta) + \mu_2 u} - \frac{\mu_3(\mu_2 + \zeta) + \mu_2 \varpi}{\mu_1(\mu_2 + \zeta) + \mu_2 u} < I(t) > + H(t), \tag{2.7}$$

$$\begin{aligned} < N^2(t) > = \frac{A}{\mu_1} < N(t) > - \frac{\alpha_2}{\mu_1} < R(t)N(t) > - \alpha_3\mu_1 < I(t)N(t) > \\ + \frac{\sigma_1^2}{2\mu_1} < S^2(t) > + \frac{\sigma_3^2}{2\mu_1} < I^2(t) > + B(t), \end{aligned} \tag{2.8}$$

where

$$\begin{aligned} B(t) = &\frac{\sigma_2}{\mu_1 t}\int_0^t R(s)N(s)dW_2(s) + \frac{\sigma_3}{\mu_1 t}\int_0^t I(s)N(s)dW_3(s) \\ &+ \frac{\sigma_1}{\mu_1 t}\int_0^t S(s)N(s)dW_1(s) + \frac{1}{2\mu_1 t}(N^2(t) - N^2(0)), \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} H(t) = &-\frac{\mu_2\sigma_4}{[\mu_1(\mu_2 + \zeta) + \mu_2 u]t}\int_0^t S(s)dW_4(s) + \frac{\sigma_2(\mu_2 + \zeta) - \mu_2\sigma_2}{[\mu_1(\mu_2 + \zeta) + \mu_2 u]t}\int_0^t R(s)dW_2(s) \\ &+ \frac{\mu_2}{[\mu_1(\mu_2 + \zeta) + \mu_2 u]t}[R(t) - R(0)] + \frac{\sigma_1}{t}\frac{\mu_2 + \zeta}{\mu_1(\mu_2 + \zeta) + \mu_2 u}\int_0^t S(s)dW_1(s) \\ &+ \frac{\sigma_3}{t}\frac{\mu_2 + \zeta}{\mu_1(\mu_2 + \zeta) + \mu_2 u}\int_0^t I(s)dW_3(s) \\ &- \frac{\mu_2 + \zeta}{t[\mu_1(\mu_2 + \zeta) + \mu_2 + \zeta u]}[S(t) + I(t) + R(t) - (S(0) + I(0) + R(0))]. \end{aligned} \tag{2.10}$$

Proof. By applying the Itôs formula [32] and utilizing equation (2.3), we establish the validity of the following equation

$$dN^2(t) = LN^2(t)dt + 2N(t)[\sigma_1 S dW_1(t) + \sigma_2 R dW_2(t) + \sigma_3 I dW_3(t)], \tag{2.11}$$

where

$$\begin{aligned} LN^2(t) = &2AN(t) - 2\mu_1 N^2(t) - 2\alpha_2 R(t)N(t) - 2\alpha_3 I(t)N(t) + \sigma_1^2 S^2(t) + \sigma_2^2 R^2(t) \\ &+ \sigma_3^2 I^2(t). \end{aligned}$$

Next, integrating both sides of equation (2.11) from 0 to t, we have

$$\begin{aligned} N^2(t) - N^2(0) = &2A\int_0^t N(s)ds - 2\mu_1\int_0^t N^2(s)ds - 2\alpha_2\int_0^t R(s)N(s)ds \\ &- 2\alpha_3\int_0^t I(s)N(s)ds + \sigma_1^2\int_0^t S^2(s)ds + \sigma_2^2\int_0^t R^2(s)ds \\ &+ \sigma_3^2\int_0^t I^2(s)ds + 2\sigma_1\int_0^t S(s)N(s)dW_1(s) \\ &+ 2\sigma_2\int_0^t R(s)N(s)dW_2(s) + 2\sigma_3\int_0^t I(s)N(s)dW_3(s). \end{aligned} \tag{2.12}$$

Then, both sides of equation (2.12) are divided by t. So the following equation holds

$$< N^2(t) > = \frac{A}{\mu_1} < N(t) > - \frac{\alpha_2}{\mu_1} < R(t)N(t) > - \frac{\alpha_3}{\mu_1} < I(t)N(t) > + \frac{\sigma_1^2}{2\mu_1} < S^2(t) > + \frac{\sigma_2^2}{2\mu_1} < R^2(t) > + \frac{\sigma_3^2}{2\mu_1} < I^2(t) > + B(t),$$

where $B(t)$ is already given in equations (2.9). So from the above analysis, equation (2.8) can be obtained.

Next, by integrating both sides of the third equation in model (1.2) simultaneously, we obtain the following equation

$$\begin{aligned} R(t) - R(0) &= u \int_0^t S(s)ds - \mu_2 \int_0^t R(s)ds + \varpi \int_0^t I(s)ds - \zeta \int_0^t R(s)ds \\ &\quad + \sigma_4 \int_0^t S(s)dW_4(s) + \sigma_2 \int_0^t R(s)dW_2(s) \\ &= u \int_0^t S(s)ds - (\mu_2 + \zeta) \int_0^t R(s)ds + \varpi \int_0^t I(s)ds + \sigma_4 \int_0^t S(s)dW_4(s) \\ &\quad + \sigma_2 \int_0^t R(s)dW_2(s), \end{aligned} \tag{2.13}$$

$$\begin{aligned} < R(t) > &= \frac{u}{\mu_2 + \zeta} < S(t) > + \frac{\varpi}{\mu_2 + \zeta} < I(t) > + \frac{\sigma_4}{(\mu_2 + \zeta)t} \int_0^t S(s)dW_4(s) \\ &\quad + \frac{\sigma_2}{(\mu_2 + \zeta)t} \int_0^t R(s)dW_2(s) - \frac{1}{(\mu_2 + \zeta)t}(R(t) - R(0)). \end{aligned} \tag{2.14}$$

Based on integrating equation (2.3) from 0 to t, and dividing both sides of the equation by t, the following equation is obtained

$$\begin{aligned} \frac{1}{t}[S(t) + I(t) + R(t) - (S(0) + I(0) + R(0))] &= A - \mu_1 < S(t) > - \mu_3 < I(t) > \\ &\quad - \mu_2 < R(t) > \\ &\quad + \sigma_1 \frac{1}{t} \int_0^t S(s)dW_1(s) \\ &\quad + \sigma_2 \frac{1}{t} \int_0^t R(s)dW_2(s) \\ &\quad + \sigma_3 \frac{1}{t} \int_0^t I(s)dW_3(s). \end{aligned}$$

Consequently,

$$\begin{aligned} < S(t) > &= \frac{A}{\mu_1} - \frac{\mu_3}{\mu_1} < I(t) > - \frac{\mu_2}{\mu_1} < R(t) > + \frac{\sigma_1}{\mu_1 t} \int_0^t S(s)dW_1(s) \\ &\quad + \frac{\sigma_2}{\mu_1 t} \int_0^t R(s)dW_2(s) + \frac{\sigma_3}{\mu_1 t} \int_0^t I(s)dW_3(s) \\ &\quad - \frac{1}{\mu_1 t}[S(t) + I(t) + R(t) - (S(0) + I(0) + R(0))]. \end{aligned} \tag{2.15}$$

By substituting (2.14) into (2.15), we obtain

$$< S(t) >= \frac{(\mu_2+\zeta)A}{\mu_1(\mu_2+\zeta)+\mu_2 u} - \frac{\mu_3(\mu_2+\zeta)+\mu_2\varpi}{\mu_1(\mu_2+\zeta)+\mu_2 u} < I(t) > +H(t),$$

where $H(t)$ is already given by equation (2.3) in (2.10), equation (2.7) can be derived. □

Lemma 2.5. *(see [25]) Supposing there exist functions $Y \in C(R_+ \times \Omega, R_+)$ and $Z \in C(R_+ \times \Omega, R_+)$ that satisfy $\lim\limits_{t\to\infty} \frac{Z(t)}{t}$. Moreover, when $t \geq 0$, there are two constants $v_0 > 0$ and $v > 0$ such that $lnY(t) = v_0 t - v\int_0^t Y(s)ds + Z(t)$, then $\lim\limits_{t\to\infty} inf \frac{1}{t}\int_0^t Y(s)ds = \frac{v_0}{v}$.*

3. Results

3.1. Persistence and extinction

In order to investigate the dynamics of the proposed system, firstly, we are concerned about the solution being non-negative. In the following part, we make a proper Lyapunov function to verify the qualitative analysis of globally positive solutions of model (1.2). Define

$$R_0^S = \frac{2A\beta(1-u)(\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2)}{\sigma_4^2\beta^2A^2+\sigma_3^2(\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2)+2(\mu_3+\varpi)(\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2)^2}.$$

Theorem 3.1. *Assume that $\sigma_1 = 0$ and $a_{11}, a_{22} > 0$ (the definition of a_{11}, a_{22} will be given later) in model (1.2). Let $(S(t), I(t), R(t))$ be the solution of system (1.2) with initial value $(S(0), I(0), R(0)) \in R_+^3$. If $R_0^S > 1$, then $\lim\limits_{t\to\infty} inf < S(t) >> 0$, $\lim\limits_{t\to\infty} inf < I(t) >> 0$, $\lim\limits_{t\to\infty} inf < R(t) >> 0$.*

In other words, model (1.2) must be stochastic.

Proof. By using Itôs formula, the following equation holds

$$dlnI(t) = (\beta S - \beta S u - \mu_3 - \varpi + \frac{1}{2}\sigma_4^2\beta^2S^2 - \frac{1}{2}\sigma_3^2)dt - \sigma_4\beta S dW_4(t) + \sigma_3 dW_3(t). \tag{3.1}$$

By integrating equation (3.1) over the interval from 0 to t, and then dividing both ends of the equation by t, we arrive at the following equation

$$\begin{aligned}\frac{lnI(t)-lnI(0)}{t} &= (\beta-\beta u) < S(t) > -(\mu_3+\varpi) + \frac{1}{2}\sigma_4^2\beta^2 < S^2(t) > -\frac{1}{2}\sigma_3^2 \\ &\quad - \sigma_4\beta\frac{1}{t}\int_0^t S(s)dW_4(s) + \sigma_3\frac{1}{t}W_3(t).\end{aligned} \tag{3.2}$$

Further, by means of equation (2.7), we have

$$\begin{aligned}\frac{lnI(t)-lnI(0)}{t}=&\frac{(\mu_2+\zeta)A(\beta-\beta u)}{\mu_1(\mu_2+\zeta)+\mu_2 u}-\frac{(\mu_2+\zeta)\mu_3(\beta-\beta u)+\mu_2(\beta-\beta u\varpi)}{\mu_1(\mu_2+\zeta)+\mu_2 u}\\&<I(t)>+(\beta-\beta u)H(t)-(\mu_3+\varpi)-\frac{1}{2}\sigma_4^2\beta^2<S^2(t)>\\&-\frac{1}{2}\sigma_3^2-\sigma_4\beta\frac{1}{t}\int_0^t S(s)dW_4(s)+\sigma_3\frac{1}{t}W_4(t).\end{aligned}\tag{3.3}$$

From (2.8) in Lemma 2.4, when $\sigma_1=0$,we obtain

$$<N^2(t)>\leq\frac{A}{\mu_1}+\frac{\sigma_2^2}{2\mu_1}<R^2(t)>+\frac{\sigma_3^2}{2\mu_1}<I^2(t)>+B(t).\tag{3.4}$$

Similarly, from Lemma 2.3, for solution $(S(t),I(t),R(t))$ of model (1.2), there exists a constant $M>0$ which makes $I(t)\leq M$ holds. At the same time, $R(t)\leq M$ a.s. for $t\geq 0$. Therefore, the following can be deduced from (3.4),

$$<N^2(t)>\leq\frac{A}{\mu_1}<N(t)>+\frac{M\sigma_2^2}{2\mu_1}<R(t)>+\frac{M\sigma_3^2}{2\mu_1}<I(t)>+B(t).\tag{3.5}$$

Secondly, equation (2.2) has been proved in Lemma 2.3, thus it is possible to obtain that for any small $\varepsilon>0$ there exists $T>0$ such that the following inequality holds

$$<N(t)><\frac{A}{\mu_1}+\varepsilon,\tag{3.6}$$

for all $t\geq T$.

When $t\geq T$, by replacing (2.14) and (3.6) into (3.5), it can be found that the following inequality holds,

$$\begin{aligned}<N^2(t)>\leq&\frac{A}{\mu_1}(\frac{A}{\mu_1}+\varepsilon)+(\frac{M\sigma_3^2}{2\mu_1}+\frac{M\sigma_2^2}{2\mu_1}-\frac{\varpi}{\mu_2+\zeta})<I(t)>\\&+\frac{M\sigma_2^2u}{2\mu_1(\mu_2+\zeta)}<S(t)>+T(t)+B(t),\end{aligned}\tag{3.7}$$

where

$$\begin{aligned}T(t)=&\frac{M\sigma_2^2\sigma_4}{2\mu_1(\mu_2+\zeta)t}\int_0^t S(s)dW_4(s)+\frac{M\sigma_2^2\sigma_2}{2\mu_1(\mu_2+\zeta)t}\int_0^t R(s)dW_2(s)\\&-\frac{M\sigma_2^2}{2\mu_1(\mu_2+\zeta)t}[R(t)-R(0)].\end{aligned}\tag{3.8}$$

From Lemma 2.4, by substituting (2.7) into (3.7), we have

$$\begin{aligned}<N^2(t)>\leq&\frac{A}{\mu_1}(\frac{A}{\mu_1}+\varepsilon)+(\frac{M\sigma_3^2}{2\mu_1}+\frac{M\sigma_2^2\varpi}{2\mu_1(\mu_2+\zeta)})<I(t)>\\&+\frac{M\sigma_2^2u}{2\mu_1}\frac{A}{\mu_1(\mu_2+\zeta)+\mu_2u}\\&-\frac{M\sigma_2^2u}{2\mu_1(\mu_2+\zeta)}\frac{\mu_3(\mu_2+\zeta)+\mu_2\varpi}{\mu_1(\mu_2+\zeta)+\mu_2u}<I(t)>\\&+\frac{M\sigma_2^2u}{2\mu_1(\mu_2+\zeta)}H(t)+T(t)+B(t).\end{aligned}$$

Replacing (3.8) into (3.3), when $t \geq T$, due to $< S^2(t) > \leq < N^2(t) >$, the following inequality can be further obtained

$$\begin{aligned}\frac{lnI(t)-lnI(0)}{t} \geq& \frac{(\mu_2+\zeta)A(\beta-\beta u)}{\mu_1(\mu_2+\zeta)\mu_2 u} - \frac{(\beta-\beta u)[(\mu_2+\zeta)\mu_3+\mu_2\varpi]}{\mu_1(\mu_2+\zeta)+\mu_2 u} < I(t) > \\ &+ (\beta-\beta u)H(t) - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2 + \sigma_4\beta\frac{1}{t}\int_0^t S(s)dW_4(s) \\ &- \frac{1}{2}\sigma_4^2\beta^2\frac{A}{\mu_1}(\frac{A}{\mu_1}+\varepsilon) - \frac{1}{2}\sigma_4^2\beta^2[\frac{M\sigma_3^2}{2\mu_1} + \frac{M\sigma_3^2\varpi}{2\mu_1(\mu_2+\zeta)}] < I(t) > \\ &+ \sigma_3 W_3(t)\frac{1}{t} - \frac{1}{2}\sigma_4^2\beta^2\frac{M\sigma_2^2 u}{2\mu_1}\frac{A}{\mu_1(\mu_1+\zeta)+\mu_2 u} \\ &+ \frac{1}{2}\sigma_4^2\beta^2[\frac{M\sigma_2^2 u}{2\mu_1(\mu_2+\zeta)}\frac{\mu_3(\mu_2+\zeta)+\mu_2\varpi}{\mu_1(\mu_2+\zeta)+\mu_2 u}] < I(t) > \\ &- \frac{1}{2}\sigma_4^2\beta^2(T(t)+B(t)).\end{aligned}$$

Consequently, for all $t \geq T$,

$$\begin{aligned}&[\frac{(\beta-\beta u)[(\mu_2+\zeta)\mu_3+\mu_2\varpi]}{\mu_1(\mu_2+\zeta)+\mu_2 u} + \frac{\sigma_4^2\beta^2}{2}(\frac{M\sigma_3^2}{2\mu_1} + \frac{M\sigma_3^2\varpi}{2\mu_1(\mu_2+\zeta)} \\ &- \frac{\sigma_4^2\beta^2 M\sigma_2^2 u}{4\mu_1(\mu_2+\zeta)}\frac{\mu_3(\mu_2+\zeta)+\mu_2\varpi}{\mu_1(\mu_2+\zeta)+\mu_2 u}] < I(t) > \\ \geq& \frac{(\mu_2+\zeta)A(\beta-\beta u)}{\mu_1(\mu_2+\zeta)+\mu_2 u} - \frac{1}{2}\sigma_3^2 + \sigma_4\beta\frac{1}{t}\int_0^t S(s)dW_4(s) + \sigma_3\frac{1}{t}dW_3(t) \\ &+ (\beta-\beta u)H(t) - (\mu_3+\varpi) - \frac{1}{2}\sigma_4^2\beta^2\frac{A}{\mu_1}(\frac{A}{\mu_1}+\varepsilon) \\ &- \frac{1}{2}\sigma_4^2\beta^2\frac{M\sigma_2^2 u}{2\mu_1}\frac{A}{\mu_1(\mu_2+\zeta)+\mu_2 u} - \frac{1}{2}\sigma_4^2\beta^2\frac{M\sigma_2^2 u}{2\mu_1(\mu_2+\zeta)}H(t) \\ &- \frac{1}{2}\sigma_4^2\beta^2(T(t)+B(t)).\end{aligned} \tag{3.9}$$

According to the large number theorem and Lemmas 2.3 and 2.4 mentioned above, the following equations hold based on (2.9), (2.10) and (3.8)

$$\lim_{t\to\infty} B(t) = 0, \lim_{t\to\infty} H(t) = 0, \lim_{t\to\infty}\frac{1}{t}(lnI(t)-lnI(0)) = 0,$$

$$\lim_{t\to\infty} T(t) = 0, \lim_{t\to\infty}\frac{1}{t}\int_0^t S(s)dW_4(s) = 0, \lim_{t\to\infty} W_3(t) = 0.$$

Therefore, due to the arbitrariness of ε and the fact from (3.9), eventually the following inequality holds,

$$\begin{aligned}\lim_{t\to\infty} < I(t) > \geq& \frac{1}{P}[\frac{(\mu_2+\zeta)A(\beta-\beta u)}{\mu_1(\mu_2+\zeta)+\mu_2 u} - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2 - \frac{1}{2}\sigma_4^2\beta^2\frac{A}{\mu_1}(\frac{A}{\mu_1}+\varepsilon) \\ &- \frac{1}{2}\sigma_4^2\beta^2\frac{M\sigma_2^2 u}{2\mu_1}\frac{A}{\mu_1(\mu_2+\zeta)+\mu_2 u}] \\ =& \frac{X}{P}(R_0^s - 1) > 0,\end{aligned}$$

where

$$P=[\frac{(\beta-\beta u)[(\mu_2+\zeta)\mu_3+\mu_2\varpi]}{\mu_1(\mu_2+\zeta)+\mu_2 u}+\frac{1}{2}\sigma_4^2\beta^2(\frac{M\sigma_3^2}{2\mu_1}+\frac{M\sigma_2^2\varpi}{2\mu_1(\mu_2+\zeta)})$$
$$-\frac{\sigma_4^2\beta^2}{2}\frac{M\sigma_2^2 u}{2\mu_1(\mu_2+\zeta)}\frac{\mu_3(\mu_2+\zeta)+\mu_2\varpi}{\mu_1(\mu_2+\zeta)+\mu_2 u}]>0,$$

$$X=\frac{a_{11}a_{12}}{a_{21}a_{22}},$$

$$a_{11}=2A\mu_1^2(\mu_2+\zeta)(\beta-\beta u)-[\mu_1(\mu_2+\zeta)+\mu_2 u][2\mu_1^2(\mu_3+\varpi)+\sigma_4^2\beta^2A(A+\mu_1 u)],$$

$$a_{12}=\sigma_4^2\beta^2A^2+(\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2)^2[\sigma_3^2+2(\mu_3+\varpi)]>0,$$

$$a_{21}=2\mu_1^2[\mu_1(\mu_2+\zeta)+\mu_2 u]>0,$$

$$a_{22}=(\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2)[A\beta(1-u)-\sigma_3^2(\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2)-2(\mu_3+\varpi)]$$
$$-\sigma_4^2\beta^2A^2.$$

From the first equation of model (1.2), the following equation holds,

$$\begin{aligned}\frac{S(t)-S(0)}{t}&=\frac{1}{t}\int_0^t(A-\mu_1S-\beta IS-\beta uIS-uS+\zeta R)dt\\&\quad+\frac{\sigma_4}{t}\int_0^t(\beta I(s)S(s)-S(s))dW_4(s)\\&\geq A-(\beta M+\mu_1+\beta uM)<S(t)>+\frac{\sigma_4}{t}M_4(s),\end{aligned}\tag{3.10}$$

where

$$M_4(t)=\int_0^t(\beta IS-S)dW_4(s)=\int_0^t\beta ISdW_4(s)-\int_0^t SdW_4(s)=M_4^1(t)+M_4^2(t).$$

The following quadratic variations are made

$$<M_4^1(t),M_4^1(t)>=\int_0^t\beta^2S^2(s)I^2(s)ds\leq\beta^2M^4t,$$
$$<M_4^2(t),M_4^2(t)>=\int_0^t S^2(s)ds\leq L^2t.$$

And by the large number theorem for martingales, it is found that

$$\lim_{t\to\infty}M_4^1(t)=0,\lim_{t\to\infty}M_4^2(t)=0.$$

Then, $\lim_{t\to\infty} \frac{1}{t}M_4(t) = 0$.
Consequently, it follows further from Lemma 2.3 and equation (3.10) that

$$\lim_{t\to\infty} inf < S(t) >\geq \frac{A}{\beta M + \mu_1 + \beta u M} > 0.$$

From the third equation of model (1.2), the following equations hold,

$$\begin{aligned}\frac{R(t)-R(0)}{t} &= u < S(t) > -(\mu_2+\zeta) < R(t) > +\varpi < I(t) > + \frac{\sigma_4}{t}\int_0^t S(s)ds \\ &+ \frac{\sigma_2}{t}\int_0^t R(s)ds,\end{aligned}$$

$$\begin{aligned}\lim_{t\to\infty} inf < R(t) > &= \frac{u}{\mu_2+\zeta}\lim_{t\to\infty} inf < S(t) > + \frac{\varpi}{\mu_2+\zeta}\lim_{t\to\infty} inf < I(t) > \\ &\geq \frac{Au}{(\mu_2+\zeta)(\beta M + \mu_1 + \beta u M)} + \frac{X}{P}(R_0^s - 1) > 0.\end{aligned}$$

This demonstrates that model (1.2) must be persistent. The proof is complete. □

Theorem 3.2. *Assume $\sigma_4 = 0$ in model (1.2). Let $(S(t), I(t), R(t))$ be the solution of system (1.2). The following two conditions are met*
(1)The initial value of system (1.2) is $(S(0), I(0), R(0)) \in R^3_+$,
(2)$R_0^s > 1$.
Then the model exhibits the following properties,

$$\begin{aligned}&\lim_{t\to\infty} < I(t) > \\ &\quad= \frac{2A(\beta-\beta u)(\mu_2+\zeta) - 2(\mu_3+\varpi)\mu_1 - 2u\mu_2(\beta-\beta u)S^* - \sigma_3^2\mu_1(\mu_2+\zeta)}{2\mu_2\varpi(\beta-\beta u)} \\ &\quad:= I^*, \\ &\lim_{t\to\infty} < R(t) >= \frac{u}{\mu_2+\zeta}S^* - \frac{\varpi}{\mu_2+\zeta}I^* := R^*, \\ &\lim_{t\to\infty} < S(t) >= \frac{A(\mu_2+\zeta)}{\mu_1(\mu_2+\zeta)+\mu_2 u} - \frac{\mu_3(\mu_2+\zeta)+\mu_2\varpi}{\mu_1(\mu_2+\zeta)+\mu_2 u}I^* := S^*.\end{aligned}$$

Proof. According to Itôs formula, the following equation holds

$$\begin{aligned}d(lnI + \frac{\beta-\beta u}{\mu_1}N) =& [(\beta-\beta u)S - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2]dt + \sigma_3 dW_3(t) \\ &+ \frac{\beta-\beta u}{\mu_1}(A - \mu_1 S - \mu_3 I - \mu_2 R)dt + \frac{\beta-\beta u}{\mu_1}\sigma_3 I dW_3(t) \\ &+ \frac{\beta-\beta u}{\mu_1}\sigma_2 R dW_2(t) + \frac{\beta-\beta u}{\mu_1}\sigma_1 I dW_1(t) \\ =& [\frac{\beta-\beta u}{\mu_1}A - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2]dt - \frac{\beta-\beta u}{\mu_1}\mu_2 R dt + \sigma_3 dW_3(t) \\ &+ \frac{\beta-\beta u}{\mu_1}\sigma_1 S dW_1(t) + \frac{\beta-\beta u}{\mu_1}\sigma_2 R dW_2(t) + \frac{\beta-\beta u}{\mu_1}\sigma_3 I dW_3(t).\end{aligned} \tag{3.11}$$

Integrate both ends of the above equation (3.11) and divide by t, which has

$$\begin{aligned}
&\frac{1}{t}(lnI(t)+\frac{\beta-\beta u}{\mu_1}N(t))-\frac{1}{t}(lnI(0)+\frac{\beta-\beta u}{\mu_1}N(0))\\
=&\frac{\beta-\beta u}{\mu_1}A-(\mu_3+\varpi)-\frac{1}{2}\sigma_3^2-\frac{\beta-\beta u}{\mu_1}\mu_2<R(t)>\\
&-\frac{\beta-\beta u}{\mu_1}\mu_3<I(t)>+\sigma_3\frac{1}{t}W_3(t)\\
&+\frac{\beta-\beta u}{\mu_1 t}\sigma_1\int_0^t S(s)dW_1(s)+\frac{\beta-\beta u}{\mu_1 t}\sigma_2\int_0^t R(s)dW_2(s)\\
&+\frac{\beta-\beta u}{\mu_1 t}\sigma_3\int_0^t I(s)dW_3(s).
\end{aligned}$$

Meanwhile, deriving from (2.14), the subsequent equation is obtained

$$\begin{aligned}
\frac{1}{t}lnI(t)=&\frac{\beta-\beta u}{\mu_1}A-(\mu_3+\varpi)-\frac{1}{2}\sigma_3^2-\frac{\beta-\beta u}{\mu_1}\mu_2\frac{u}{\mu_2+\zeta}<S(t)>\\
&-\frac{\beta-\beta u}{\mu_1}\mu_2\frac{\varpi}{\mu_2+\zeta}<I(t)>-\frac{\beta-\beta u}{\mu_1}\mu_3<I(t)>+J(t),
\end{aligned}$$

where

$$\begin{aligned}
J(t)=&\frac{1}{t}(lnI(0)+\frac{\beta-\beta u}{\mu_1}N(0)-\frac{\beta-\beta u}{\mu_1}N(t))\\
&-\frac{\beta-\beta u}{\mu_1}\mu_2\frac{\sigma_2}{(\mu_2+\zeta)t}\int_0^t R(s)dW_2(s)+\frac{\beta-\beta u}{\mu_1}\mu_2\frac{1}{(\mu_2+\zeta)t}(R(t)-R(0))\\
&+\sigma_3\frac{1}{t}W_3(t)+\frac{\beta-\beta u}{\mu_1 t}\sigma_1\int_0^t S(s)dW_1(s)+\frac{\beta-\beta u}{\mu_1 t}\sigma_2\int_0^t R(s)dW_2(s)\\
&+\frac{\beta-\beta u}{\mu_1 t}\sigma_3\int_0^t I(s)dW_3(s)-\frac{\beta-\beta u}{\mu_1}\mu_3\frac{\sigma_4}{(\mu_2+\zeta)t}\int_0^t S(s)dW_4(s).
\end{aligned}$$

And according to the large number theorem for martingales and Lemma 2.3 mentioned above, it is clear that $\lim_{t\to\infty}J(t)=0$. Thus, from Lemma 2.5, it can be deduced that

$$\begin{aligned}
&\lim_{t\to\infty}<I(t)>\\
=&\frac{2A(\beta-\beta u)(\mu_2+\zeta)-2(\mu_3+\varpi)\mu_1(\mu_2+\zeta)-2u\mu_2(\beta-\beta u)S^*-\sigma_3^2\mu_1(\mu_2+\zeta)}{2\mu_2\varpi(\beta-\beta u)}\\
:=&I^*.
\end{aligned}$$

Among them, the definition of S^* will be given later.

Furthermore, from equation (2.14) it follows that

$$\lim_{t\to\infty}<R(t)>=\frac{u}{\mu_2+\zeta}S^*-\frac{\varpi}{\mu_2+\zeta}I^*:=R^*,$$

and from (2.7) we further obtain

$$\lim_{t\to\infty}<S(t)>=\frac{A(\mu_2+\zeta)}{\mu_1(\mu_2+\zeta)+\mu_2 u}-\frac{\mu_2(\mu_2+\zeta)+\mu_2\varpi}{\mu_1(\mu_2+\zeta)+\mu_2 u}I^*:=S^*.$$

□

Particularly, when $\sigma_i = 0 (i = 1, 2, 3, 4)$, then the stochastic model (1.2) is degenerated into the specific model (1.1). At the same time, we have

$$R_0^s = R_0 = \frac{A\beta(1-u)}{(\mu+\varpi+\alpha)(u+\mu)}.$$

From Theorem 3, when $R_0 > 1$ for any solution $(S(t), I(t), R(t))$ of model (1.1) with initial value $(S(0), I(0), R(0)) \in R_+^3$, it follows

$$\lim_{t\to\infty} < S(t) >= \frac{A}{(u+\mu)R_0}, \lim_{t\to\infty} < I(t) > \tag{3.8}$$
$$= [A(1-\frac{1}{R_0}) + \frac{(u+\varpi+\alpha)(\mu+\zeta)-\zeta\varpi}{u+\zeta}],$$

$$\lim_{t\to\infty} < R(t) >= (\frac{uA}{(u+\mu)R_0} + \varpi I^*)\frac{1}{\mu+\zeta}.$$

Hence, Theorem 3.2 on the stochastic model (1.2) can be considered as an extension of the conclusion (2) of Lemma 2.1 on the specific model (1.1).

Theorem 3.3. *Let $(S(t), I(t), R(t))$ be the solution of model (1.2), where $(S(0), I(0), R(0)) \in R_+^3$ is the initial value of the model. Assume either of the two criteria below holds*
(A) $b_{11}b_{22} > 0$, $R_0^s < 1$ (the definition of $b_{11}b_{22}$ will be given later);

$$(B)\ \frac{(\beta-\beta u)^2}{2\sigma_4^2\beta^2} - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2 < 0.$$

Then the infectious people $I(t)$ almost certainly dies out exponentially. In other words, if condition (A) is satisfied, then $\lim_{t\to\infty} sup\frac{lnI(t)}{t} \le \chi(R_0^s - 1)$($\chi$ will be given later). And $\lim_{t\to\infty} sup\frac{lnI(t)}{t} \le \frac{(\beta-\beta u)^2}{2\sigma_4^2\beta^2} - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2 < 0$ a.s. if (B) holds.

Proof. If condition (A) is satisfied, and based on equations (2.6) and (3.6), the following inequalities hold,

$$\begin{aligned}\frac{lnI(t)}{t} \le & \frac{(\mu_2+\zeta)A(\beta-\beta u)}{\mu_1(\mu_2+\zeta)+\mu_2 u} - \frac{(\beta-\beta u)[\mu_3(\mu_2+\zeta)+\mu_2\varpi]}{\mu_1(\mu_2+\zeta)+\mu_2 u} < I(t) > \\ & + (\beta-\beta u)H(t) - (\mu_3+\varpi) - \frac{1}{2}\sigma_4^2\beta^2[\frac{(\mu_2+\zeta)A}{\mu_1(\mu_2+\zeta)+\mu_2 u} \\ & - \frac{\mu_3(\mu_2+\zeta)+\mu_2\varpi}{\mu_1(\mu_2+\zeta)+\mu_2 u} < I(t) > +H(t)]^2 \\ & - \frac{1}{2}\sigma_3^2 + \sigma_4\beta\frac{1}{t}\int_0^t S(s)dW_4(s) + \sigma_3 W_3(t)\frac{1}{t} + \frac{lnI(0)}{t} \\ \le & \frac{A(\mu_2+\zeta)(\beta-\beta u)}{\mu_1(\mu_2+\zeta)+\mu_2 u} + (\beta-\beta\beta u)H(t) - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2 \\ & + \sigma_4\beta\frac{1}{t}\int_0^t S(s)dW_4(s) + \sigma_3\frac{1}{t}W_3(s).\end{aligned}$$

According to the large number theorem for martingales and Theorem 3.1, the following inequality holds,

$$\lim_{t\to\infty} sup\frac{lnI(t)}{t} \le \frac{A(\mu_2+\zeta)(\beta-\beta u)}{\mu_1(\mu_2+\zeta)+\mu_2 u} - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2.$$

Thus, it follows that,

$$\lim_{t\to\infty} sup\frac{lnI(t)}{t} \le \chi(R_0^s - 1) < 0,$$

where

$$\chi = \frac{b_{11}b_{12}}{b_{21}b_{22}},$$

$$\begin{aligned}
b_{11} &= A(\mu_2+\zeta)(\beta-\beta u) - (\mu_3+\varpi+\sigma_3^2)[\mu_1+(\mu_2+\zeta)+\mu_2 u],\\
b_{12} &= \sigma_4^2\beta^2A^2 + (\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2)^2[\sigma_3^2+2(\mu_3+\varpi)],\\
b_{21} &= 2[\mu_1(\mu_2+\zeta)+\mu_2 u],\\
b_{22} &= (\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2)[A\beta(1-u) - \sigma_3^2(\mu_1+u+\frac{1}{2}\sigma_1^2+\frac{1}{2}\sigma_4^2) - 2(\mu_3+\varpi)]\\
&\quad - \sigma_4^2\beta^2A^2.
\end{aligned}$$

If condition (B) is valid, the following inequalities hold from (3.2),

$$\begin{aligned}
\frac{lnI(t)}{t} &\le \frac{lnI(0)}{t} + (\beta-\beta u) < S(t) > -(\mu_3+\varpi) - \frac{1}{2}\sigma_4^2\beta^2 < S^2(t) > -\frac{1}{2}\sigma_3^2\\
&\quad + \sigma_4\beta\frac{1}{t}\int_0^t S(s)dW_4(s) + \sigma_4\frac{1}{t}W_3(t)\\
&= \frac{lnI(0)}{t} + \frac{(\beta-\beta u)^2}{2\sigma_4^2\beta^2} - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2 - \frac{1}{2}\sigma_4^2\beta^2(< S(t) > -\frac{\beta-\beta u}{\sigma_4^2\beta^2})^2\\
&\quad + \sigma_4\beta\frac{1}{t}\int_0^t S(s)dW_4(s) + \sigma_4\frac{1}{t}W_3(t)\\
&\le \frac{lnI(0)}{t} + \frac{(\beta-\beta u)^2}{2\sigma_4^2\beta^2} - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2\\
&\quad + \sigma_4\frac{1}{t}\beta\int_0^t S(s)dW_4(s) + \sigma_3\frac{1}{t}W_3(s).
\end{aligned}$$

From the above equation, it is clear that the following holds

$$\lim_{t\to\infty} sup\frac{lnI(t)}{t} \le \frac{(\beta-\beta u)^2}{2\sigma_4^2\beta^2} - (\mu_3+\varpi) - \frac{1}{2}\sigma_3^2 < 0.$$

□

4. Stationary distribution

The following section studies the existence of the unique stationary distribution of model(1.2).

First, let $X(t)$ be an autonomous $Markov$ process in R^d, which can be expressed as the solution of the below stochastic differential equation [33]:

$$dX(t) = b(X)dt + \sum_{r=1}^{k} g_r(X)dB_r(t). \tag{4.1}$$

The diffusion matrix of equation (4.1) is

$$\Lambda(x) = (\lambda_{ij}(x), \lambda_{ij}(x)) = \sum_{r=1}^{k} g_r^i(x)g_r^j(x).$$

Lemma 4.1. *(see [22])This is defined by the existence of a bounded domain $U \in R^d$ with a regular boundary that meets the requirements of (i) and (ii) below.*
(i) In the domain $U \in R^d$ and its vicinity, the minimum eigenvalue of the diffusion matrix $\Lambda(x)$ is bounded far from zero.
(ii)If $x \in R^d \setminus U$, the mean time τ at which a path issuing from x reaches the set U is finite, and $\sup_{x\in K} E_x\tau < \infty$ for every compact subset $K \in R^d$.

Then, the $Markov$ process $X(t)$ of equation(4.1) has a stationary distribution $\mu(\cdot)$ with density in R^d. Assume that $f(\cdot)$ is an integrable function about measure μ. Then for any $x \in R^d$, we have

$$P_x \lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))dt = \int_{R^d} f(x)\mu(dx) = 1.$$

When there is no stochastic perturbation in the model (1.2), that means $\sigma_i = 0, (i = 1, 2, 3, 4)$, then model (1.2) becomes the specific model,

$$\begin{aligned} S^{'} &= A - \beta IS + \beta uIS - (\mu_1 + u)S + \zeta R. \\ I^{'} &= \beta IS - (\mu_3 + \varpi)I. \\ R^{'} &= uS + \varpi I - (\mu_2 + \zeta)R. \end{aligned} \tag{4.2}$$

Let $\tilde{R}_0 = \frac{\beta A}{\beta Au+(u+\mu_1)(\mu_3+\varpi)}$. What can be proved is that model (4.2) has a unique endemic equilibrium (S^*, I^*, R^*) when $\tilde{R}_0 > 1$, where

$$\begin{aligned} S^* &= \frac{\mu_3 + \varpi}{\beta - \beta u}, \\ I^* &= \frac{A(\mu_2 + \zeta) - (\mu_2 + \zeta)(\mu_1 + u)S^* + u\zeta S^*}{\beta(\mu_2 + \zeta)S^* - u(\mu_2 + \zeta)\beta S^* - \zeta\varpi}, \\ R^* &= \frac{-A + (\beta S^* - u\beta S^*)I^* + (\mu_1 + u)S^*}{\zeta}. \end{aligned}$$

Define the constants

$$\begin{aligned} \lambda_1 &= (-1 - a_1)\sigma_2^2 - (a_3 + a_2 I^*)\sigma_4^2 + a_3(\mu_1 + u - \sigma_1^2) + \mu_1, \\ \lambda_2 &= (-1 - a_3)\sigma_3^2 + (a_3 + 1)\mu_3 + a_3\varpi, \\ \lambda_3 &= (-1 - a_1)\sigma_2^2 + a_1(\mu_2 + \zeta) + \mu_2, \\ E &= \frac{\sigma_3^2}{2}I^* + d_1(S^*)^2 + d_2(I^*)^2 + d_3(R^*)^2, \\ d_1 &= 2(\sigma_1^2 + \sigma_4^2) + \sigma_4^2 I^*, d_2 = 2\sigma_3^2, d_3 = 2\sigma_2^2, \end{aligned}$$

where

$$a_1 = \frac{\mu_1 - \mu_3}{u - \varpi},$$
$$a_2 = \frac{[(\mu_2 + \mu_3)(u - \varpi) - \varpi(\mu_1 - \mu_3)](\mu_3 + \varpi + \mu_1 + u) + \zeta(u - \varpi)(\mu_1 + \mu_3)}{\beta\zeta(u - \varpi)(1 - u)},$$
$$a_3 = \frac{(\mu_2 + \mu_3)(u - \varpi) - \varpi(\mu_1 - \mu_3)}{(u - \varpi)\zeta}.$$

Now, we concern the existence and uniqueness of the stationary distribution for model (1.2).

Theorem 4.1. *Let* $\tilde{R}_0 = \frac{\beta A}{\beta A u + (u+\mu_1)(\mu_3+\varpi)} > 1$, *which satisfies the following two inequalities*

$$\lambda_i > 0, \quad E < min\{\lambda_1 S^*, \lambda_2 I^*, \lambda_3 R^*\}. \tag{4.3}$$

Then model (1.2) has a unique stationary distribution and ergodic property.

Proof. First, the definition of the Lyapunov function is as follows [28], $V(S,I,R) = a_1V_1(R) + a_2V_2(I) + a_3V_3(S,I) + V_4(S,I,R)$, where

$$V_1 = \frac{1}{2}(R - R^*)^2, V_2 = I - I^* ln\frac{I}{I^*},$$
$$V_3 = \frac{1}{2}(S + I - S^* - I^*)^2, V_4 = \frac{1}{2}(S + I + R - S^* - I^* - R^*)^2.$$

$$\begin{aligned}
LV_1 &= (R - R^*)[uS - \varpi I - (\mu_2 + \zeta)R] + \frac{1}{2}\sigma_4^2 S^2 + \frac{1}{2}\sigma_2^2 R^2 \\
&\le u(S - S^*)(R - R^*) - \varpi(R - R^*)(I - I^*) - (\mu_2 + \zeta - \sigma_2^2)(R - R^*)^2 \\
&\quad + \sigma_4^2(S^*)^2 + \sigma_2^2(R^*)^2 + \sigma_2^2(S - S^*)^2, \\
LV_2 &= (1 - \frac{I^*}{I})[\beta IS(1 - u) - (\mu_3 + \varpi)I] + \frac{I^*}{2I^2}(\sigma_4^2\beta^2 I^2 S^2 + \sigma_3^2 I^2) \\
&\le \beta(1 - u)(I - I^*)(S - S^*) + \frac{I^*}{2}\sigma_3^2 + I^*\sigma_4^2(S - S^*)^2 + I^*\sigma_4^2(S^*)^2, \\
LV_3 &= (S + I - S^* - I^*)[A - (\mu_1 + u)S - (\mu_3 + \varpi)I + \zeta R] + \frac{1}{2}\sigma_1^2 S^2 \\
&\quad + \frac{1}{2}\sigma_3^2 I^2 - \frac{1}{2}\sigma_4^2 S^2 \\
&\le -(\mu_1 + u - \sigma_1^2 - \sigma_4^2)(S - S^*) - (\mu_3 + \varpi - \sigma_3^2)(I - I^*)^2 \\
&\quad - (\mu_3 + \varpi + \mu_1 + u)(S - S^*)(I - I^*) + \zeta(S - S^*)(R - R^*) \\
&\quad + \zeta(I - I^*)(R - R^*) + (\sigma_1^2 + \sigma_4^2)(S^*)^2 + \sigma_3^2(I^*)^2, \\
LV_4 &= (S + I + R - S^* - I^* - R^*)(A - \mu_2 R - \mu_3 I - \mu_1 S) + \frac{1}{2}\sigma_1^2 S^2 \\
&\quad + \frac{1}{2}\sigma_3^2 I^2 + \frac{1}{2}\sigma_2^2 R^2 \\
&\le -(\mu_1 - \sigma_1^2)(S - S^*)^2 - (\mu_2 - \sigma_2^2)(R - R^*)^2 - (\mu_3 - \sigma_3^2)(I - I^*)^2 \\
&\quad - (\mu_1 + \mu_3)(I - I^*)(S - S^*) - (\mu_1 + \mu_2)(S - S^*)(R - R^*) \\
&\quad - (\mu_2 + \mu_3)(I - I^*)(R - R^*) + \sigma_1^2(S^*)^2 + \sigma_2^2(R^*)^2 + \sigma_3^2(I^*)^2,
\end{aligned}$$

$$\begin{aligned} LV(S,I,R) &= a_1LV_1(R) + a_2LV_2(I) + a_3LV_3(S,I) + LV_4(S,I,R) \\ &\leq -\lambda_1(S-S^*)^2 - \lambda_2(I-I^*)^2 - \lambda_3(R-R^*)^2 + E. \end{aligned}$$

If (4.3) holds, then the episode $\lambda_1(S-S^*)^2 + \lambda_2(I-I^*)^2 + \lambda_3(R-R^*)^2 = E$ lies in the positive zone of R^3_+. Hence, there exists a constant $C > 0$ and a compact set $K \subset R^3_+$ such that for any $x = (S,I,R) \in R^3_+/K$, there are the following inequalities

$$\lambda_1(S-S^*)^2 + \lambda_2(I-I^*)^2 + \lambda_3(R-R^*)^2 \geq E + C.$$

Therefore, the following equation ultimately holds

$$LV(x) \leq -C.$$

This demonstrates the fulfillment of condition (i) in Lemma 4.1. The diffusion matrix related to model (1.2) is

$$\Lambda(x) = (b_{ij}(x))_{3\times 3} = \begin{bmatrix} \sigma_1^2S^2 + \sigma_4^2(\beta IS - S)^2 & -\sigma_4^2\beta IS(\beta IS - S) & 0 \\ -\sigma_4^2\beta IS(\beta IS - S) & \sigma_4^2\beta^2I^2S^2 + \sigma_3^2I^2 & \sigma_4^2\beta IS^2 \\ \sigma_4^2S(\beta IS - S) & 0 & \sigma_4^2S^2 + \sigma_2^2R^2 \end{bmatrix}, \tag{4.4}$$

where $x = (S,I,R)$.

Choose $Q = min_{(S,I,R)\in U}\{\sigma_1^2S^2, \sigma_2^2R^2, \sigma_3^2I^2\}$, so $Q > 0$. From (4.4), for any $x = (S,I,R) \in U$ and $(\eta_1,\eta_2,\eta_3) \in R^3_+$, it is obtained that

$$\begin{aligned} \sum_{i,j=1}^{3} b_{ij}(x)\eta_i\eta_j &= [\sigma_1^2S^2 + \sigma_4^2(\beta IS - S)^2]\eta_1\eta_1 - [\sigma_4^2\beta IS(\beta IS - S)]\eta_1\eta_2 \\ &\quad - [\sigma_4^2\beta IS(\beta IS - S)]\eta_2\eta_1 + (\sigma_4^2\beta^2I^2S^2 + \sigma_3^2I^2)\eta_2\eta_2 \\ &\quad + \sigma_4^2S(\beta IS - S - S)\eta_1\eta_3 + (\sigma_4^2S^2 + \sigma_4^2R^2)\eta_3\eta_3 \\ &= \sigma_1^2S^2\eta_1^2 + \sigma_3^2I^2\eta_2^2 + \sigma_3^2R^2\eta_3^2 \\ &\quad + [\frac{1}{2}\sigma_4(\beta IS - S) + \sigma_4S\eta_3]^2 + [\frac{\sqrt{3}}{2}\sigma_4(\beta IS - S)\eta_1 - \frac{1}{\sqrt{3}}\sigma_4\beta IS\eta_2]^2 \\ &\geq min\sigma_1^2S^2, \sigma_2^2R^2, \sigma_3^2I^2(\eta_1^2 + \eta_2^2 + \eta_3^2) = Q|\eta|^2, \end{aligned}$$

where $|\eta| = (\eta_1^2 + \eta_2^2 + \eta_3^2)^{\frac{1}{2}}$.

Thus, model (1.2) has a unique stationary distribution and the ergodic property. This is completed by proving that.

□

In the following, consider a special situation of model (1.2). In this case, the model (1.2) will take the below form

$$\begin{aligned} dS(t) &= [A - \mu_1S - \beta(1-u)IS - uS + \zeta R]dt + \sigma_1SdW_1(t), \\ dI(t) &= [\beta(1-u)IS - \mu_3I - \varpi I]dt + \sigma_3IdW_3(t), \\ dR(t) &= (uS - \mu_2R + \varpi I - \zeta R)dt + \sigma_2RdW_2(t). \end{aligned} \tag{4.5}$$

A novel result on the existence of unique stationary distribution of model(4.5) is given below. By definition we have

$$\bar{R}_0 = \frac{\beta A}{(\mu_1 + u + \frac{1}{2}\sigma_1^2)(\mu_3 + \varpi + \frac{1}{2}\sigma_3^2)}.$$

Theorem 4.2. *Let $\bar{R}_0 > 1$. Then model (4.5) has a unique stationary distribution and the ergodic property.*

Proof. First let $Z(S, I, R)$ be a C^2-function which takes the specific form

$$Z(S, I, R) = MV_1 + V_2 - lnS - lnR,$$

where $V_1 = -D_1 lnS - -D_1 lnI$, $V_2 = \frac{1}{\partial+1}(S + I + R)^{\partial+1}$.
∂ is a constant satisfying

$$0 < \partial < \frac{2\mu_1}{\sigma_1^2 + \sigma_2^2 + \sigma_3^2},$$

the definition of the constant $M > 0$ will be given later, and

$$D_1 = \frac{2A}{2\mu_1 + 2u + \sigma_1^2}, D_2 = \frac{2A}{2\mu_3 + 2\varpi + \sigma_3^2}.$$

Then, define a nonnegative C^2-function V that can be expressed in the following manner,

$$V(S, I, R) = Z(S, I, R) - Z(S_0, I_0, R_0).$$

According to the Itôs formula, for any solution $(S(t), I(t), R(t))$ of model (1.2), the following equations hold,

$$\begin{aligned}
L(lnS) &= -\frac{A}{S} + \mu_1 + \beta I - \beta S u + u - \zeta\frac{R}{S} + \frac{1}{2}\sigma_1^2,\\
L(lnR) &= -u\frac{S}{R} + \mu_2 - \varpi\frac{I}{R} + \zeta + \frac{1}{2}\sigma_2^2,\\
LV_1 &= -\frac{D_1 A}{S} + D_1\mu_1 + D_1\beta I - D_1\beta I u + D_1 u - D_1\zeta\frac{R}{S} + \frac{1}{2}D_1\sigma_1^2 - D_2(\beta S - \beta S u)\\
&\quad + D_2(\mu_3 + \varpi) + \frac{1}{2}\sigma_3^2 D_2\\
&\le -2[(AD_1D_2\beta)^{\frac{1}{2}} - A] + D_1\beta I - D_1\beta I u - D_1\zeta\frac{R}{S}\\
&= -2A[(\bar{R})^{\frac{1}{2}} - 1] + D_1\beta I - D_1\beta I u - D_1\zeta\frac{R}{S},\\
LV_2 &= (S + I + R)^{\partial}[A - \mu_1(S + I + R) - \alpha_2 R - \alpha_3 I]\\
&\quad + \frac{\partial}{2}(S + I + R)^{\partial-1}(\sigma_1^2 S^2 + \sigma_2^2 R^2 + \sigma_3^2 I^2)\\
&\le A(S + I + R)^{\partial} - [\mu_1 - \frac{\partial}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)](S + I + R)^{\partial+1}\\
&\le -\mu^*(S^{\partial+1} + I^{\partial+1} + R^{\partial+1}) + D,
\end{aligned}$$

where

$$\mu^* = \frac{1}{2}[\mu_1 - \frac{\partial}{2}(\sigma_1^2 \vee \sigma_2^2 \vee \sigma_3^2)] > 0$$

and

$$D = \sup_{(S,I,R)\in R_+^3}\{A(S + I + R)^{\partial} - \mu^*(S + I + R)^{\partial+1}\} < \infty.$$

Thus, the differential operator L that applying to the V yields

$$\begin{aligned} LV \leq & -2AM\psi + D_1 M\beta I - D_1 M\beta I u - D_1 M\zeta \frac{R}{S} - \mu^*(S^{\partial+1} + I^{\partial+1} + R^{\partial+1}) + D \\ & - \frac{A}{S} + \mu_1 + \beta I - \beta S u + u - \zeta \frac{R}{S} + \frac{1}{2}\sigma_1^2 - u\frac{S}{R} + \mu_2 - \varpi \frac{I}{R} + \zeta + \frac{1}{2}\sigma_2^2 \\ \leq & -2AM\psi + D + \mu_1 + \mu_2 + \zeta + u + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 - \mu^*(S^{\partial+1} + I^{\partial+1} + R^{\partial+1}) \\ & + (D_1 M\beta + \beta) - \frac{A}{S} - \varpi \frac{I}{R}, \end{aligned}$$

where $\psi = (\bar{R_0})^{\frac{1}{2}} - 1$. In the following, it is necessary to find that a compact subset Γ s.t. the condition (ii) in Lemma 4.1 is valid. The bounded closed set is defined as follows,

$$\Gamma = \{(S, I, R) : \varepsilon_1 \leq S \leq \frac{1}{\varepsilon_1}, \varepsilon_2 \leq I \leq \frac{1}{\varepsilon_2}, \varepsilon_3 \leq R \leq \frac{1}{\varepsilon_3}\},$$

where $\varepsilon_i (i = 1, 2, 3)$ are sufficiently small positive constants, which will be defined later. To make it easier to calculate, the regions of $R^3_+ \setminus \Gamma$ are divided into six regions.

$$\begin{aligned} &\Gamma_1 = \{(S, I, R) \in R^3_+, S \geq \frac{1}{\varepsilon_1}\}, \quad \Gamma_2 = \{(S, I, R) \in R^3_+, I \geq \frac{1}{\varepsilon_2}\}, \\ &\Gamma_3 = \{(S, I, R) \in R^3_+, S \geq \frac{1}{\varepsilon_3}\}, \quad \Gamma_4 = \{(S, I, R) \in R^3_+, 0 < S < \varepsilon_1\}, \\ &\Gamma_5 = \{(S, I, R) \in R^3_+, 0 < I < \varepsilon_2, S > \varepsilon_1\}, \\ &\Gamma_6 = \{(S, I, R) \in R^3_+, 0 < Q < \varepsilon_3, S \geq \varepsilon_1, I \geq \varepsilon_2\}. \end{aligned}$$

Ultimately, it can be proved that $LV(S, I, R) \leq -\frac{1}{2}$ on $R^3_+ \setminus \Gamma$, which is equivalent to verifying it in the above six regions.

Case1. If $(S, I, R) \in \Gamma_1$, we can obtain

$$LV \leq -\frac{1}{2}\mu^* S^{\partial+1} + F_1 \leq -\frac{1}{2}\mu^*(\frac{1}{\varepsilon})^{\partial+1} + F_1,$$

where

$$\begin{aligned} F_1 = sup\{ & D + \mu_1 + \mu_2 + \zeta + u + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 - \frac{1}{2}\mu^*(S^{\partial+1} + I^{\partial+1} + R^{\partial+1}) \\ & + (D_1 M\beta + \beta)I\}. \end{aligned}$$

We choose a constant $\varepsilon_1 > 0$ small enough such that

$$-\frac{1}{2}\mu^*(\frac{1}{\varepsilon_1})^{\partial+1} + F_1 \leq -\frac{1}{2},$$

then it can be derived that $LV \leq -\frac{1}{2}$ for all $(S, I, R) \in \Gamma_1$.

Case2. If $(S, I, R) \in \Gamma_2$, then the following equation holds

$$LV \leq -\frac{1}{2}\mu^* I^{\partial+1} + F_1 \leq -\frac{1}{2}\mu^*(\frac{1}{\varepsilon_2})^{\partial+1} + F_1.$$

We choose a constant $\varepsilon_1 > 0$ small enough such that

$$-\frac{1}{2}\mu^*(\frac{1}{\varepsilon_2})^{\partial+1} + F_1 \leq -\frac{1}{2},$$

then it follows that $LV \leq -\frac{1}{2}$ for all $(S, I, R) \in \Gamma_2$.
Case3. If $(S, I, R) \in \Gamma_3$, we can obtain

$$LV \leq -\frac{1}{2}\mu^* R^{\partial+1} + F_1 \leq -\frac{1}{2}\mu^*(\frac{1}{\varepsilon_3})^{\partial+1} + F_1.$$

We choose a constant $\varepsilon_1 > 0$ small enough such that

$$-\frac{1}{2}\mu^*(\frac{1}{\varepsilon_3})^{\partial+1} + F_1 \leq -\frac{1}{2},$$

then it follows that $LV \leq -\frac{1}{2}$ for all $(S, I, R) \in \Gamma_3$.
Case4. If $(S, I, R) \in \Gamma_4$, we can obtain

$$LV \leq -\frac{A}{S} + F_1 \leq -\frac{A}{\varepsilon_1} + F_1,$$

We choose a constant $\varepsilon_1 > 0$ sufficiently small to make $-\frac{A}{\varepsilon_1} + F_1 \leq -\frac{1}{2}$, then then there is $LV \leq -\frac{1}{2}$ for all $(S, I, R) \in \Gamma_4$.
Case5. If $(S, I, R) \in \Gamma_5$, we can obtain

$$LV \leq -2AM\psi + (D_1 M\beta)I + F_5 \leq -2AM\psi + (D_1 M\beta)\varepsilon_2 + F_5,$$

where

$$F_5 = \sup\{D + \mu_1 + \mu_2 + \zeta + u + \frac{1}{2}\sigma_1^2 + \frac{1}{2}\sigma_2^2 - \mu^*(S^{\partial+1} + R^{\partial+1})\}.$$

We choose a constant $\varepsilon_1 > 0$ small enough and $M > 0$ large enough such that

$$-2AM\psi + (D_1 M\beta)\varepsilon_2 + F_5 \leq \frac{1}{2},$$

then it follows that $LV \leq -\frac{1}{2}$ for all $(S, I, R) \in \Gamma_5$.
Case6. If $(S, I, R) \in \Gamma_6$, we can obtain

$$LV \leq -\varpi\frac{I}{R} + F_1 \leq -\varpi\frac{\varepsilon_2}{\varepsilon_3} + F_1.$$

We choose a constant $\varepsilon_2 > 0, \varepsilon_3 > 0$ small enough such that

$$-\varpi\frac{\varepsilon_2}{\varepsilon_3} + F_1 \leq -\frac{1}{2}(S, I, R) \in \Gamma_6.$$

It follows that condition (i) needs to be proved in Lemma 4.1. Obviously, the diffusion matrix related to model (1.2) is

$$\Lambda(x) = (b_{ij}(x))_{3\times 3} = \begin{bmatrix} \sigma_1^2 S^2 & 0 & 0 \\ 0 & \sigma_3^2 I^2 & 0 \\ 0 & 0 & \sigma_2^2 R^2 \end{bmatrix},$$

where $x = (S, I, R)$.

Choosing $Q = min_{(S,I,R)\in U}\{\sigma_1^2 S^2, \sigma_2^2 R^2, \sigma_3^2 I^2\}$, we have $Q > 0$. For any $x = (S, I, R) \in U$ and $(\eta_1, \eta_2, \eta_3) \in R^3_+$, we have

$$\sum_{i,j=1}^{3} b_{ij}(x)\eta_i\eta_j = [\sigma_1^2 S^2\eta_1^2 + \sigma_3^2 I^2\eta_2^2 + \sigma_2^2 R^2\eta_3^2]$$
$$\geq min\sigma_1^2 S^2, \sigma_3^2 I^2, \sigma_2^2 R^2(\eta_1^2 + \eta_2^2 + \eta_3^2)$$
$$= M|\eta|^2,$$

where $|\eta| = (\eta_1^2 + \eta_2^2 + \eta_3^2)^{\frac{1}{2}}$.

According to Remark 4.1, it is straightforward to demonstrate that condition (i), as stated in Theorem 4.1, holds true. Hence, the final result is that the model (1.2) has a unique stationary distribution and is ergodic. The proof is complete. □

5. Numerical Simulation

Now, assign certain values to the parameters of $A = 0.4, \beta = 0.1, \varpi = 0.3, \zeta = 0.2, \mu_1 = 0.1, \mu_2 = 0.3, and \mu_3 = 0.2$ in the model (1.2), and give a certain initial proportional value of $S = 0.5, I = 0.4, and R = 0.2$. While keeping the above parameters unchanged, change the value of u to observe how the proportion of the three groups of people in the population changes during the fifty day period.

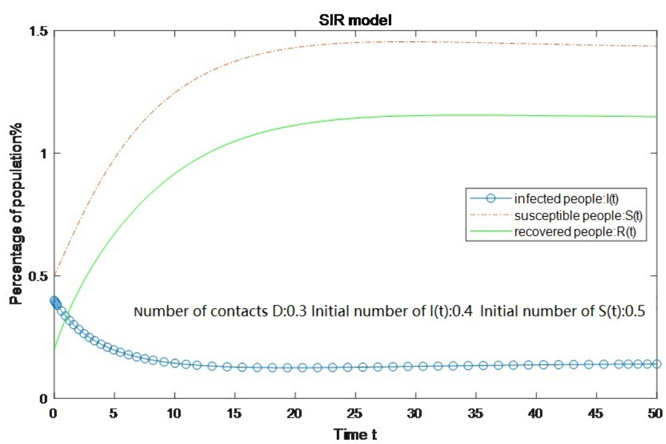

Figure 1. When u is 0.3, the variation in the proportion of population accounted for by SIR groups in the population.

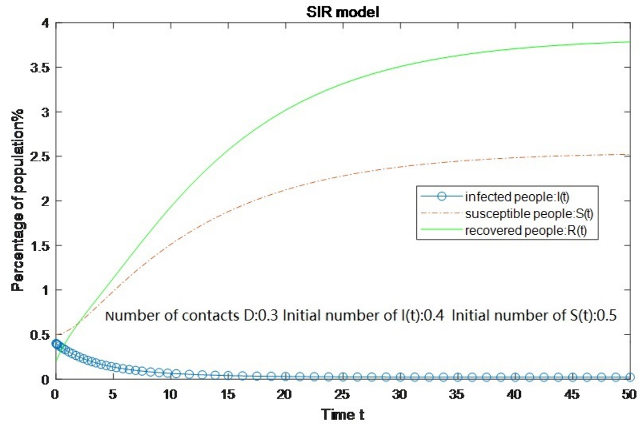

Figure 2. When u is 0.6, the variation in the proportion of population accounted for by SIR groups.

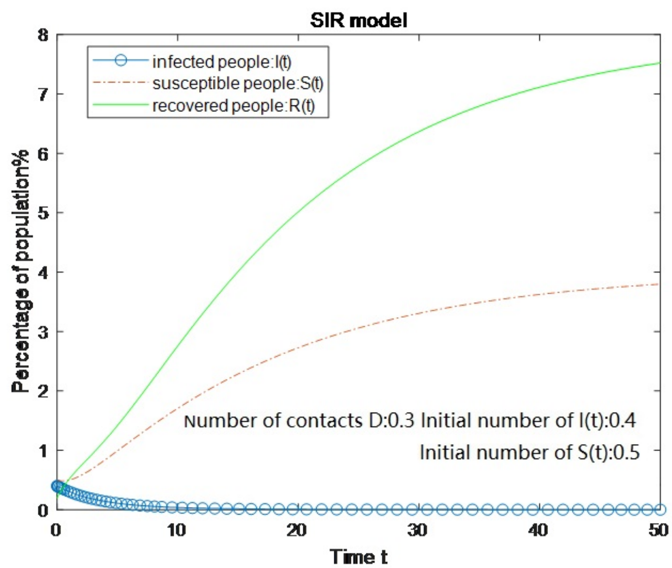

Figure 3. When u is 0.8, the variation in the proportion of population accounted for by SIR groups.

It can be seen from the Figures 1, 2 and 3, that with the increase of the vaccination rate, the number of infections will reduce the time to zero, which indicates that vaccination plays a great role in controlling the spread of infectious diseases. Next, it considers the population proportion of three groups of people under random disturbance.

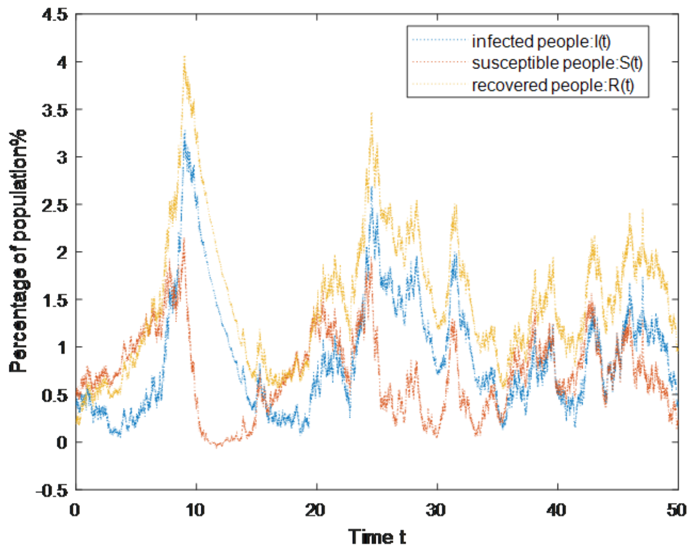

Figure 4. When u=0.3, the change of the proportion of SIR groups in the population under random disturbance.

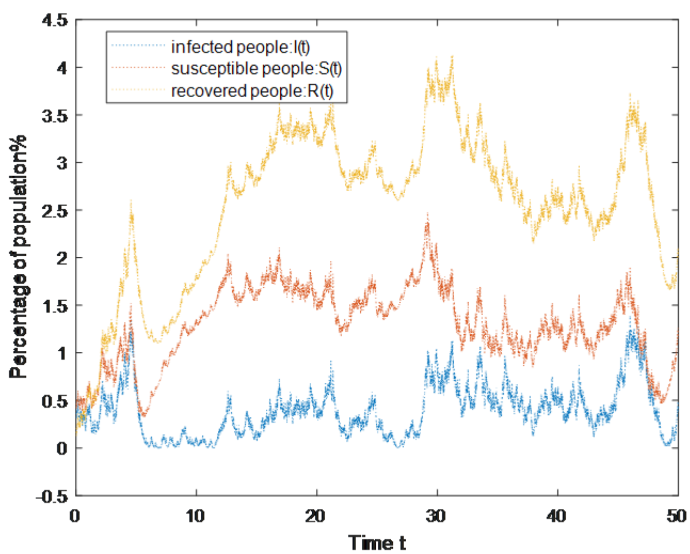

Figure 5. When u=0.6, the change of the proportion of SIR groups in the population under random disturbance.

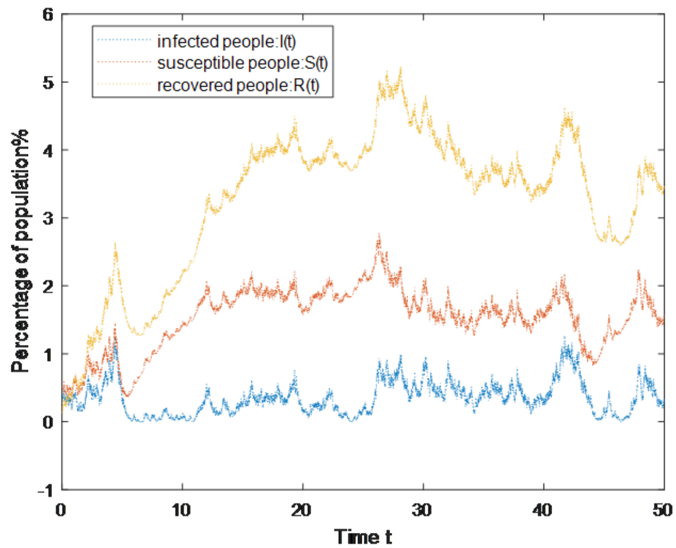

Figure 6. When u=0.8, the change of the proportion of SIR groups in the population under random disturbance.

It can be seen from Figure 6 that when the vaccination rate is large enough, the number of infected people tends to zero soon after a short period of oscillation.

Through the numerical simulation in this article, it can be found that as the vaccination rate gradually increases, the time for the number of infected individuals to approach zero will be shortened, and even under random interference, it will quickly approach zero after a short period of oscillation.

6. Conclusion

This article mainly studies a stochastic SIR infectious disease model with vaccination, and the random effects are assumed to be fluctuations in mortality and vaccination rates. Research has shown that if $R_0^s < 1$, the disease will die out, and if $R_0^s > 1$, the disease is permanent, and a conclusion on the global stability of the model's equilibrium point is given. Importantly, this article obtains the conditions for the existence of a unique stationary distribution in the model by constructing a Lyapunov function.

Acknowledgements

Firstly, the authors would like to express our gratitude to Editor and the anonymous referees for their valuable comments and suggestions which have led to truly significant improvement of the manuscript. Secondly, I would like to express my gratitude to all the literature authors who have been cited and referenced in this article. Their research results have provided a reference and inspiration for this article.

References

[1] L. Wen and X. Yang, *Global stability of a delayed SIRS model with temporary immunity*, Chaos Solitons Fractals, 2006.
DOI: 10.1006/j.chaos.2006.11.010

[2] P. Liu, D. Anwarud, *Impact of information intervention on stochastic dengue epidemic model*, Alexandria Engineering Journal, 2021, 60(6), 5725–5739.
DOI: 10.1016/j.aej.2021.03.068

[3] Z. Wang and T. B. Chris, *Statistical physics of vaccination*, Physics Reports, 2016,664(2016), 1–113.
DOI: 10.1006/j.physrep.2016.10.006

[4] H. M. Elizabeth and B. R. Levin, *Infectious diseases of humans: dynamics and control*, Trends in Microbiology, 1993, 1(5), 202–203.
DOI: 10.1016/0966-842X(93)90094-8

[5] J. J. Jillian and G. R. David, *Statistical physics of human cooperation*, Physics Reports, 2017,687(2017),1–51.
DOI: 10.1006/j.physcrep.2017.05.004

[6] A. Din and T. Khan, *Mathematical analysis of dengue stochastic epidemic model*, Results in Physics, 2021,20(13).
DOI: 10.1006/j.RINP.2020.103719

[7] K. Hattaf and A. A. Mohsen, *Modeling the dynamics of covid-19 with carrier effect and environmental contamination*, International Journal of Modeling, Simulation, and Scientific Computing, 2021,3(12),1–16.
DOI: 10.1142/S1793962321500483

[8] K. Toshikazu, *Prediction of the Epidemic Peak of Coronavirus Disease in Japan*, Journal of Clinical Medicine, 2020, 9(3), 789–789.
DOI: 10.3390/jcm9030789

[9] S, Sinha. *Stability Analysis of SIR Model with Vaccination*, American J. of Computational and Applied Mathematics, 2014, 4(1),17–23.
DOI: 10.5923/j.ajcam.20140401.03

[10] M. Carvalho,J. P. S. Mauricio and A. A. Rodrigues, *SIR Model with Vaccination: Bifurcation Analysis*, Qualitative Theory of Dynamical Systems, 2023,3(22),105–128.
DOI: 10.1007/S12346-023-00802-2

[11] O. Bogdanov, *Variants of the Stochastic Sir Models and Vaccination Strategies*, Cybernetics and Systems Analysis, 2023, 59(2),325–330.
DOI: 10.1007/s10559-023-00566-2

[12] Ch Xu, Z Liu, Y Pang and A Akg, *Stochastic analysis of a COVID-19 model with effects of vaccination and different transition rates: Real data approach*, Chaos, Solitons and Fractals, 2023,170(2023).
DOI: 10.1016/J.CHAOS.2023.113395

[13] D. Anwarud, Y. Li, Y. Abdullahi, *Delayed hepatitis B epidemic model with stochastic analysis*, Chaos, Solitons and Fractals, 2021,146(2021).
DOI: 10.1016/J.CHAOS.2021.110839

[14] D. Jiang and J. Yu, *Asymptotic behavior of global positive solution to a stochastic SIR model*, Journal of Clinical Medicine, 2011, 54(1), 221–232.
DOI: 10.1016/j.mcm.2011.02.004

[15] Y. Lin and D. Jiang, *Stationary distribution of a stochastic SIS epidemic model with vaccination*, Physica A: Statistical Mechanics and its Applications, 2014, 394(2014),187–197.
DOI: 10.1016/j.physa.2013.10.006

[16] Y. Zhao and D. Jiang, *The threshold of a stochastic SIS epidemic model with vaccination*, Applied Mathematics and Computation, 2014,243(2014),718–727.
DOI: 10.1016/j.amc.2014.05.124

[17] Z. Gul and Y. Kang, *Stability analysis and optimal vaccination of an SIR epidemic model*, BioSystems, 2008, 93(3), 240–249.
DOI: 10.1016/j.biosystems.2008.05.004

[18] L. Aadil and O. Lahcen, *Complete global stability for an SIRS epidemic model with generalized non-linear incidence and vaccination*, Applied Mathematics and Computation, 2012, 218(11), 6519–6525.
DOI: 10.1016/j.amc.2011.12.024

[19] T. Khan and G. Zaman, *Modeling the dynamics of novel coronavirus(covid-19) via stochastic epidemic model*, Results in Physics, 2021,2(2021),4121–4130.
DOI: 10.1016/J.RINP.2021.104004

[20] A. Din, Y. Li, et al, *Impact of information intervention on stochastic hepatitis B model and its variable-order fractional network*, The European Physical Journal Special Topics, 2022, 10(231), 1859–1873.
DOI: 10.1140/epjs/s11734-022-00453-5

[21] M Ishikawa, *Optimal Vaccination Strategy under Saturated Treatment using the Stochastic SIR Model*, Control and Information Engineers, 2013, 26(11), 382–388.
DOI: 10.5687/iscie.26.382

[22] T. Khan and R. Ullah, *Correlated stochastic epidemic model for the dynamics of SARS-CoV-2 with vaccination*, BioSystems, 2022, 12(1), 16105–16105.
DOI: 10.1038/s41598-022-20059-0

[23] Y. L. Chen and B. Y. Wen, *The global dynamics for a stochastic SIS epidemic model with isolation*, Physica A: Statistical Mechanics and its Applications, 2018, 492(2018), 1604–1624.
DOI: 10.1016/j.physa.2017.11.085

[24] Q. Liu and D. Q. Jiang, *Stationary distribution and extinction of a stochastic SIR model with nonlinear perturbation*, Applied Mathematics Letters, 2017, 73(2017), 8–15.
DOI: 10.1016/j.aml.2017.04.021

[25] X. Liao, *Stability theory and application of dynamical systems*, National Defense Industry Press, Beijing, 2000.

[26] A. Din and Y. Li, *Stationary distribution extinction and optimal control for the stochastic hepatitis B epidemic model with partial immunity*, Physica Scripta, 2021, 96(2021).
DOI: 10.1088/1402-4896/ABFACC

[27] A. Din and Q. T. Ain, *Stochastic Optimal Control Analysis of a Mathematical Model: Theory and Application to Non-Singular Kernels*, Fractal and Fractional, 2022, 6(5), 279–279.
DOI: 10.3390/fractalfract6050279

[28] A. Din and Y. Li, *Mathematical analysis of a new nonlinear stochastic hepatitis B epidemic model with vaccination effect and a case study*, The European Physical Journal Plus, 2022, 137(5), 558–558.
DOI: 10.1140/epjp/s13360-022-02748-x

[29] Q. Linda and Gao, *Four SEI endemic models with periodicity and separatrices*, Mathematical Biosciences, 1995, 128(1-2), 157–184.
DOI: 10.1016/0025-5564(94)00071-7

[30] C. Ji and D. Jiang, *Stochastic Biological Model and Infectious Disease Model*, Science Press, Beijing, 2018.

[31] K. Rafail, *Stochastic Stability of Differential Equations*, Springer, Berlin, 1980.

[32] X. Mao, *Stochastic Differential Equations and Applications*, Horwood Publishing, Chichester, 1997.

[33] Y. Chen, B. Wen and Z. Teng, *The global dynamics for a stochastic SIS epidemic model with isolation*, Physica A: Statistical Mechanics and its Applications, 2018.
DOI: 10.1016/j.physa.2017.11.085