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On Controllability of Solution for Nonlinear Neutral Fuzzy Integro-Differential Equations

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Abstract This paper delves into an examination of the existence, uniqueness and controllability results concerning impulsive functional fuzzy nonlinear neutral integro-differential equations with non-local conditions. Additionally, we explore the fuzzy solution within the context of normal, convex, upper semi-continuous and compactly supported interval fuzzy numbers. Our findings are derived through the application of the Banach fixed point theorem. Also, we illustrate the result with an example.

Keywords Nonlinear neutral integro-differential equations, controllability, fuzzy solution, impulsive functional, nonlocal condition, fixed point

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1. Introduction

Many topics in studies relating to elasticity and viscosity have been modeled in several domains of physics and engineering due to their widespread applications. These problems are expressed using integral equations, differential equations, and integro-differential equations. In particular, neutral differential equations appear in many fields of applied mathematics, which is why they have received so much attention in recent decades [8, 13–16, 19, 20].

The significance of fuzzy differential equations within fuzzy analysis is supported by an extensive body of literature in the field, as referenced in [10, 11]. Over recent years, impulsive differential equations have emerged as a focal point of research. Additionally, incorporating a delay in the fuzzy model enables the exploration of more comprehensive situations, as discussed in [7, 17]. Fuzzy theory is a model used for uncertainty. Fuzzy integro-differential equations play the most important role in the analysis of phenomena with memory where imprecision is inherent. Chalishajar et al. [4] studied the existence of fuzzy solutions for nonlocal impulsive neutral functional differential equations. Najat et al. [12] studied the existence of impulsive fuzzy nonlinear integro-differential equations with the nonlocal condition by using the Leray-Schauder alternative fixed point theorem.

Controllability plays an important role in examining and suggesting control systems. It means the presence of a control function that directs the system's solution

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from its initial condition to the intended end state. Recently, there have been a few researchers who are interested in studying the controllability of fuzzy integro-differential systems. Chalishaja et al. [3] developed controllability for impulsive fuzzy neutral functional integro-differential equations using the Banach fixed point theorem. Kumar et al. [1] established the existence and total controllability result of a fuzzy delay differential equation with non-instantaneous impulses. Radhakrishan et al. [2] studied the controllability results for nonlinear impulsive fuzzy neutral integro-differential evolution systems.

In [3] and [5], the authors studied controllability for impulsive fuzzy neutral functional integro-differential equations of the type:

$$\begin{aligned}
&\frac{d}{d\kappa}[\rho(\kappa) - \mathfrak{z}(\kappa, \rho_\kappa)] = A\rho(\kappa) + \mathfrak{W}(\kappa, \rho_\kappa, \int_0^\kappa \mathcal{J}(\kappa, \eta, \rho_\eta) d\eta) + \varsigma(\kappa),\ \kappa \in [0, \mathcal{K}] = \mathcal{L},\\
&\Delta\rho(\kappa_n) = \mathcal{I}_n \rho(\kappa_n^-),\ \kappa \neq \kappa_n,\ n = 1, 2, \ldots, k,\\
&\rho(0) = \psi \in \mathbb{X}^d,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{d}{d\kappa}[\rho(\kappa) - \mathfrak{z}(\kappa, \rho_\kappa)] = A\rho(\kappa) + \mathfrak{W}(\kappa, \rho_\kappa) + \varsigma(\kappa),\ \kappa \in [0, \mathcal{K}] = \mathcal{L},\\
&\Delta\rho(\kappa_n) = \mathcal{I}_n \rho(\kappa_n^-),\ \kappa \neq \kappa_n,\ n = 1, 2, \ldots, k,\\
&\psi(\kappa) = \rho(\kappa) + \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa),\ \kappa \in [-\mathfrak{r}, 0],
\end{aligned}$$

respectively by using Banach fixed point theorem.

Motivated by the above mentioned work, here we consider the controllability of fuzzy solutions for impulsive functional nonlinear neutral integro-differential equations with nonlocal conditions of the following type:

$$\begin{aligned}
&\frac{d}{d\kappa}[\rho(\kappa) - \mathfrak{z}(\kappa, \rho_\kappa)] = A\rho(\kappa) + \mathfrak{W}(\kappa, \rho_\kappa, \int_0^\kappa \mathcal{J}(\kappa, \eta, \rho_\eta) d\eta) + \varsigma(\kappa),\ \kappa \in [0, \mathcal{K}] = \mathcal{L},\\
&\Delta\rho(\kappa_n) = \mathcal{I}_n \rho(\kappa_n^-),\ \kappa \neq \kappa_n,\ n = 1, 2, \ldots, k, \qquad (1.1)\\
&\psi(\kappa) = \rho(\kappa) + \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\kappa),\ \kappa \in [-\mathfrak{r}, 0],
\end{aligned}$$

where $A : \mathcal{L} \to \mathbb{X}^d$ is the fuzzy coefficient, $\mathbb{X}^d$ is the set of all convex, upper semicontinuous, and normal fuzzy numbers with bounded α-levels, $\mathfrak{W} : \mathcal{L} \times \mathbb{X}^d \times \mathbb{X}^d \to \mathbb{X}^d$, $\mathcal{J} : \mathcal{L} \times \mathcal{L} \times \mathbb{X}^d \to \mathbb{X}^d$ and $\mathfrak{h} : (C[-\mathfrak{r}, 0], \mathbb{X}^d)^q \to \mathbb{X}^d$ are regular fuzzy nonlinear functions, $\mathfrak{z} : \mathcal{L} \times C([-\mathfrak{r}, 0], \mathbb{X}^d) \to \mathbb{X}^d$ are continuous nonlinear functions, $\varsigma : \mathcal{L} \to \mathbb{X}^d$ is an admissible control function, $\psi : [-\mathfrak{r}, 0] \to \mathbb{X}^d$, and $\mathcal{I}_n \in C(\mathbb{X}^d, \mathbb{X}^d)$ are bounded functions. $\Delta\rho(\kappa_n) = \rho(\kappa_n^+) - \rho(\kappa_n^-)$, represents the left and right limits of $\rho(\kappa)$ at $\kappa = \kappa_n$, respectively, $n = 1, 2, \ldots, k$. $\rho(\kappa_n^+) = \lim_{\mathfrak{h} \to 0^+} \rho(\kappa_n + \mathfrak{h})$ and $\rho(\kappa_n^-) = \lim_{\mathfrak{h} \to 0^-} \rho(\kappa_n - \mathfrak{h})$. Moreover, $\rho_\kappa(.)$ represents the history where $\rho_\kappa = \rho(\kappa + \mathfrak{w})$; $\mathfrak{w} \in [-\mathfrak{r}, 0]$.

In this study, we generalize the results mentioned in [3] and [5], while also employing the Banach fixed point theorem to derive the existence, uniqueness, and controllability results for nonlinear impulsive fuzzy neutral integro-differential equations with nonlocal conditions.

This paper is organized as follows. We present preliminaries in section 2. We prove the existence, uniqueness and controllability of nonlinear fuzzy solution for neutral impulsive functional integro-differential equations with nonlocal condition in sections 3 and 4. The theorem is demonstrated using an example in section 5. Finally we conclude the study in section 6.

2. Preliminaries

Definition 2.1. [5]. Let Y be a nonempty set. A fuzzy set $\tilde{A}$ in Y is characterized by its membership function $\mu_{\tilde{A}} : Y \to [0,1]$ and $\mu_{\tilde{A}}(\nu)$ is interpreted as the degree of membership of element ν in fuzzy set $\tilde{A}$ for each $\nu \in Y$. It is clear that a fuzzy set is defined by $\tilde{A} = \{(\nu, \mu_{\tilde{A}}(\nu)) : \nu \in Y, \mu_{\tilde{A}}(\nu) \in [0,1]\}$.

Let $K_r(\mathfrak{R}^d)$ be the family consisting of all nonempty, convex, and compact subsets of $\mathfrak{R}^d$. Denote by $\mathbb{X}^d = \{\vartheta : \mathfrak{R}^d \to [0,1]$ such that ϑ satisfies $(1)-(4)$ as follows

(1) ϑ is normal, that is, there exists a $\mathfrak{w}_0 \in \mathfrak{R}^d$ such that $\vartheta(\mathfrak{w}_0) = 1$.
(2) ϑ is fuzzy convex, that is, for $\mathfrak{w}, \mathfrak{z} \in \mathfrak{R}^d$ and $0 < \lambda \leq 1$, $\vartheta(\lambda\mathfrak{w} + (1-\lambda)\mathfrak{z}) \geq \min\{\vartheta(\mathfrak{w}), \vartheta(\mathfrak{z})\}$.
(3) ϑ is upper semicontinuous.
(4) $[\vartheta]^0 = \overline{\{\mathfrak{w} \in \mathfrak{R}^d : \vartheta(\mathfrak{w}) > 0\}}$, is compact.

For $0 < \alpha \leq 1$, $[\vartheta]^\alpha = \{\rho \in \mathfrak{R}^n : \vartheta(\rho) \geq \alpha\}$. Then from (1)–(4), it follows that the α- level sets $[\vartheta]^\alpha \in K_r(\mathfrak{R}^d)$. If $\mathfrak{h} : \mathfrak{R}^d \times \mathfrak{R}^d \to \mathfrak{R}^d$ is a function, then by using Zadeh's extension principle, we can extend $\mathfrak{h}$ to $\mathbb{X}^d \times \mathbb{X}^d \to \mathbb{X}^d$ by the equation $[\mathfrak{h}(\vartheta,\sigma)(w)] = \sup\limits_{w=\mathfrak{h}(\rho,\nu)} \min\{\vartheta(\rho), \sigma(\nu)\}$. It is well known that $[\mathfrak{h}(\vartheta,\sigma)]^\alpha = \mathfrak{h}([\vartheta]^\alpha, [\sigma]^\alpha)$, $\forall \vartheta, \sigma \in \mathbb{X}^d$,$0 \leq \alpha \leq 1$ and the function $\mathfrak{h}$ is a continuous. In addition, we have

$$[\vartheta + \sigma]^\alpha = [\vartheta]^\alpha + [\sigma]^\alpha, \quad [\mathfrak{a}\vartheta]^\alpha = \mathfrak{a}[\vartheta]^\alpha,$$

where

$$\vartheta, \sigma \in \mathbb{X}^d, \ 0 \leq \alpha \leq 1, \ \mathfrak{a} \in \mathfrak{R}.$$

Let C_1 and C_2 be two nonempty bounded subsets of $\mathfrak{R}^d$. The distance between C_1 and C_2 is determined by using the Hausdorff metric

$$\mathcal{H}_\varrho(C_1, C_2) = \max\left\{\sup_{c_1\in C_1}\inf_{c_2\in C_2}\|c_1 - c_2\|, \sup_{c_2\in C_2}\inf_{c_1\in C_1}\|c_1 - c_2\|\right\},$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathfrak{R}^d$. Then $(K(\mathfrak{R}^d), \mathcal{H}_\varrho)$ is a separable and complete metric space [9].

Definition 2.2. [6]. The complete metric ϱ_∞ on $\mathbb{X}^d$ is defined by

$$\varrho_\infty(\vartheta, z) = \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\vartheta]^\alpha, [z]^\alpha) = \sup_{0<\alpha\leq 1} [\vartheta_l^\alpha - z_l^\alpha, \vartheta_r^\alpha - z_r^\alpha].$$

For any $\vartheta, z, \mu \in \mathbb{X}^d$, which satisfies $\mathcal{H}_\varrho(\vartheta + \mu, z + \mu) = \mathcal{H}_\varrho(\vartheta, z)$. Hence $(\mathbb{X}^d, \varrho_\infty)$ is a complete metric space.

Definition 2.3. [6]. The supremum metric $\mathcal{H}_1$ on $C(\mathcal{L}, \mathbb{X}^d)$ is defined by

$$\mathcal{H}_1(\vartheta, z) = \sup_{0 \le \kappa \le \mathcal{K}} \varrho_\infty(\vartheta(\kappa), z(\kappa)).$$

Hence $(C(\mathcal{L}, \mathbb{X}^d), \mathcal{H}_1)$ is a complete metric space.

Definition 2.4. [6]. We define the derivative $\rho'(\kappa)$ of a fuzzy process $\rho \in \mathbb{X}^d$ by

$$[\rho'(\kappa)]^\alpha = [(\rho_l^\alpha)'(\kappa), (\rho_r^\alpha)'(\kappa)],$$

provided that the equation defines a fuzzy set $\rho'(\kappa) \in \mathbb{X}^d$.

Definition 2.5. [6]. We define the fuzzy integral $\int_c^d \rho(\kappa) d\kappa$, $c, d \in [0, \mathcal{K}]$ by

$$\left[\int_c^d \rho(\kappa) d\kappa\right]^\alpha = \left[\int_c^d \rho_l^\alpha(\kappa) d\kappa, \int_c^d \rho_r^\alpha(\kappa) d\kappa\right],$$

provided that the Lebesgue integrals on the right-hand side exist.

3. Existence and uniqueness result

Let $\mathbb{X}^d$ be the space of all fuzzy subsets u of $\mathfrak{R}^d$ which are normal, convex and upper semicontinuous fuzzy sets with bounded supports. Let Υ be the space defined by $\Upsilon = \{\rho | \rho : \mathcal{L} \to \mathbb{X}^d$ is continuous $\}$. In addition, there exists $\rho(\kappa_q^+), \rho(\kappa_q^-)$ where $q = 1, 2, \ldots, k$, with $\rho(\kappa_q^-) = \rho(\kappa_q)$, $\Upsilon' = \Upsilon \cap C(\mathcal{L}, \mathbb{X}^d)$.

Let us introduce the following hypotheses.

$\mathcal{B}_1$) $\mathfrak{D}(\kappa)$ is a fuzzy number, where $[\mathfrak{D}(\kappa)]^\alpha = [\mathfrak{D}_l^\alpha(\kappa), \mathfrak{D}_r^\alpha(\kappa)]$, $\mathfrak{D}(0) = \mathbb{I}$ and $\mathfrak{D}_m^\alpha(\kappa), (m = l, r)$ is continuous with $|\mathfrak{D}_m(\kappa)| \le \mathcal{N}, \mathcal{N} > 0, |A\mathfrak{D}(\kappa)| \le \mathcal{N}_1$, $\forall \kappa \in [0, \mathcal{K}] = \mathcal{L}$.

$\mathcal{B}_2$) The nonlinear function $\mathfrak{z} : \mathcal{L} \times \mathbb{X}^d \to \mathbb{X}^d$ is continuous and there exists a constant $\mathfrak{b}_1 > 0$ satisfying global Lipschitz condition such that

$$\mathcal{H}_\varrho([\mathfrak{z}(\kappa, \rho)]^\alpha, [\mathfrak{z}(\kappa, \nu)]^\alpha) \le \mathfrak{b}_1 \mathcal{H}_\varrho([\rho_\kappa(\mathfrak{w})]^\alpha, [\nu_\kappa(\mathfrak{w})]^\alpha),$$

$\forall \kappa \in \mathcal{L}$ and $\rho, \nu \in \mathbb{X}^d$.

$\mathcal{B}_3$) The nonlinear function $\mathfrak{W} : \mathcal{L} \times \mathbb{X}^d \times \mathbb{X}^d \to \mathbb{X}^d$ is continuous and there exists constants $\mathfrak{b}_2 > 0$ and $\mathfrak{b}_3 > 0$ satisfying global Lipschitz condition such that

$$\begin{aligned} &\mathcal{H}_\varrho([\mathfrak{W}(\eta, \rho_1(\eta), \nu_1(\eta))]^\alpha, [\mathfrak{W}(\eta, \rho_2(\eta), \nu_2(\eta))]^\alpha) \le \\ &\mathfrak{b}_2 \mathcal{H}_\varrho([\rho_1(\mathfrak{w})]^\alpha, [\rho_2(\mathfrak{w})]^\alpha) + \mathfrak{b}_3 \mathcal{H}_\varrho([\nu_1(\mathfrak{w})]^\alpha, [\nu_2(\mathfrak{w})]^\alpha), \end{aligned}$$

where $\rho_n(\eta), \nu_n(\eta) \in \mathbb{X}^d, n = 1, 2$.

$\mathcal{B}_4$) If $\mathfrak{h}$ is continuous and there exists constants $\mathcal{Q}_n$, $n = 1, 2, \ldots, q$ such that

$$\begin{aligned} &\mathcal{H}_\varrho([\mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(\eta)]^\alpha, [\mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(\eta)]^\alpha) \\ &\le \sum_{n=1}^{q} \mathcal{Q}_n \mathcal{H}_\varrho([\rho_{\sigma_n}(\eta)]^\alpha, [\nu_{\sigma_n}(\eta)]^\alpha), \end{aligned}$$

$\forall \eta \in [-\mathfrak{r}, 0]$, and all $\rho_{\sigma_n}, \nu_{\sigma_n} \in C([-\mathfrak{r}, 0], \mathbb{X}^d)$, $n = 1, 2, \ldots, q$.

$\mathcal{B}_5$) The nonlinear function $k : \mathcal{L} \times \mathcal{L} \times \mathbb{X}^d \to \mathbb{X}^d$ is continuous and there exists constant $\mathfrak{b}_4 > 0$ satisfying global Lipschitz condition such that

$$\mathcal{H}_\varrho[\mathcal{J}(\kappa, \eta, \rho_1(\eta))]^\alpha, [\mathcal{J}(\kappa, \eta, \rho_2(\eta))]^\alpha) \leq \mathfrak{b}_4 \mathcal{H}_\varrho([\rho_1(\mathfrak{w})]^\alpha, [\rho_2(\mathfrak{w})]^\alpha),$$

where $\rho_n(\kappa) \in \mathbb{X}^d, n = 1, 2$.

$\mathcal{B}_6$) There exists a constant $\mathfrak{b}_5$ such that

$$\mathcal{H}_\varrho([\mathcal{I}_n \rho(\kappa_n^-)]^\alpha, [\mathcal{I}_n \nu(\kappa_n^-)]^\alpha) \leq \mathfrak{b}_5 \mathcal{H}_\varrho([\rho(\kappa)]^\alpha, [\nu(\kappa)]^\alpha),$$

where $\rho(\kappa), \nu(\kappa) \in \Upsilon'$.

$\mathcal{B}_7$) If

$$\sum_{n=1}^{q} \mathcal{Q}_n + \mathfrak{b}_1(1 + \mathcal{N}\mathcal{N}_1\mathcal{K}) + \mathfrak{b}_2\mathcal{N}\frac{\mathcal{K}^2}{2} + \mathfrak{b}_3\mathfrak{b}_4\mathcal{N}\frac{\mathcal{K}^3}{2} + \mathfrak{b}_5\mathcal{N} < 1,$$

then the problem (1.1) has a unique fuzzy solution.

Definition 3.1. If ρ is an integral solution of the problem (1.1) ($\varsigma \equiv 0$), then ρ is given by

$$\begin{aligned}\rho(\kappa) =& \mathfrak{D}(\kappa)\Big[\psi(0) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0) - \mathfrak{z}(0, \psi)\Big] + \mathfrak{z}(\kappa, \rho_\kappa) \\ &+ \int_0^\kappa A\mathfrak{D}(\kappa - \eta)\mathfrak{z}(\eta, \rho_\eta)d\eta + \int_0^\kappa \mathfrak{D}(\kappa - \eta)\mathfrak{W}\Big(\eta, \rho_\eta, \\ &\int_0^\eta \mathcal{J}(\eta, \kappa, \rho_\kappa)d\kappa\Big)d\eta + \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa - \kappa_n)\mathcal{I}_n(\rho(\kappa_n^-)).\end{aligned} \tag{3.1}$$

Theorem 3.1. *Suppose that the hypotheses* $(\mathcal{B}_1) - (\mathcal{B}_7)$ *hold, then for all* $\mathcal{K} > 0$, *the problem* (3.1) ($\varsigma \equiv 0$) *has a unique fuzzy solution on* $\mathcal{L}$.

Proof. For each $\rho \in \Upsilon$ and $\kappa \in \mathcal{L}$, define $\Theta\rho \in \Upsilon$ by

$$\begin{aligned}\Theta\rho(\kappa) =& \mathcal{D}(\kappa)\Big[\psi(0) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0) - \mathfrak{z}(0, \psi)\Big] + \mathfrak{z}(\kappa, \rho_\kappa) \\ &+ \int_0^\kappa A\mathfrak{D}(\kappa - \eta)\mathfrak{z}(\eta, \rho_\eta)dy + \int_0^\kappa \mathfrak{D}(\kappa - \eta)\mathfrak{W}\Big(\eta, \rho_\eta, \\ &\int_0^\eta \mathcal{J}(\eta, \kappa, \rho_\kappa)d\kappa\Big)d\eta + \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa - \kappa_n)\mathcal{I}_n(\rho(\kappa_n^-))\end{aligned}$$

Similarly,

$$\begin{aligned}\Theta\nu(\kappa) =& \mathcal{D}(\kappa)\Big[\psi(0) - \mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(0) - \mathfrak{z}(0, \psi)\Big] + \mathfrak{z}(\kappa, \nu_\kappa) \\ &+ \int_0^\kappa A\mathfrak{D}(\kappa - \eta)\mathfrak{z}(\eta, \nu_\eta)d\eta + \int_0^\kappa \mathfrak{D}(\kappa - \eta)\mathfrak{W}\Big(\eta, \nu_\eta, \\ &\int_0^\kappa \mathcal{J}(\eta, \kappa, \nu_\kappa)d\kappa\Big)d\eta + \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa - \kappa_n)\mathcal{I}_n(\nu(\kappa_n^-)).\end{aligned}$$

Hence $\Theta\rho : \mathcal{L} \to \Upsilon$ is continuous, so Θ is a mapping from Υ into itself. By hypotheses $(\mathcal{B}_1) - (\mathcal{B}_6)$ we have the following inequalities. Now for $\rho, \nu \in \Upsilon$ we have

$$\begin{aligned}
&\mathcal{H}_\varrho([\Theta\rho(\kappa)]^\alpha, [\Theta\nu(\kappa)]^\alpha)\\
\leq&\mathcal{H}_\varrho\bigg(\bigg[\mathfrak{D}(\kappa)\big[\psi(0) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \dots, \rho_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big] + \mathfrak{z} \qquad (3.2)\\
&(\kappa, \rho_\kappa) + \int_0^\kappa A\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\eta,\rho_\eta)d\eta + \int_0^\kappa \mathfrak{D}(\kappa-\eta)\mathfrak{W}(\eta,\\
&\rho_\eta, \int_0^\eta \mathcal{J}(\eta,\kappa,\rho_\kappa)d\kappa))d\eta, \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa-\kappa_n)\mathcal{I}_n(\rho(\kappa_n^-))\bigg]^\alpha,\\
&\bigg[\mathfrak{D}(\kappa)\big[\psi(0) - \mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2}, \dots, \nu_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big] + \mathfrak{z}(\kappa,\nu_\kappa)\\
+&\int_0^\kappa A\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\eta,\nu_\eta)d\eta + \int_0^\kappa \mathfrak{D}(\kappa-\eta)\mathfrak{W}(\eta,\nu_\eta, \int_0^\eta \mathcal{J}\\
&(\eta,\kappa,\nu_\kappa)d\kappa))d\eta + \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa-\kappa_n)\mathcal{I}_n(\nu(\kappa_n^-))\bigg]^\alpha\bigg)\\
\leq&\mathcal{H}_\varrho([\mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \dots, \rho_{\sigma_q})(0)]^\alpha, [\mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2}, \dots, \nu_{\sigma_q})(0)]^\alpha)\\
+&\mathcal{H}_\varrho([\mathfrak{z}(\kappa,\rho_\kappa)]^\alpha, \mathfrak{z}(\kappa,\nu_\kappa)]^\alpha) + \mathcal{H}_\varrho(\big[\int_0^\kappa A\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\eta,\rho_\eta)\\
&d\eta\big]^\alpha, \big[\int_0^\kappa A\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\kappa,\nu_\kappa)d\kappa\big]^\alpha) + \mathcal{H}_\varrho(\big[\int_0^\kappa \mathfrak{D}(\kappa-\eta)\mathfrak{W}\\
&(\eta,\rho_\eta, \int_0^\eta \mathcal{J}(\eta,\kappa,\rho_\kappa)d\kappa)d\eta)\big]^\alpha, \big[\int_0^\kappa \mathfrak{D}(\kappa-\eta)\\
&\mathfrak{W}(\eta,\nu_\eta, \int_0^\eta \mathcal{J}(\eta,\kappa,\nu_\kappa)d\kappa)d\eta\big]^\alpha) + \mathcal{H}_\varrho([\sum_{0<\kappa n<\kappa} \mathfrak{D}(\kappa-\kappa_n)\\
&\mathcal{I}_n(\rho(\kappa_n^-))]^\alpha, [\sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa-\kappa_n)\mathcal{I}_n(\nu(\kappa_n^-))]^\alpha)\\
\leq&\sum_{n=1}^q \mathcal{Q}_n\mathcal{H}_\varrho([\rho_{\sigma_n}(0)]^\alpha, [\nu_{\sigma_n}(0)]^\alpha) + \mathfrak{b}_1\mathcal{H}_\varrho([\rho(\kappa+\mathfrak{w})]^\alpha,\\
&[\nu(\kappa+\mathfrak{w})]^\alpha) + \mathcal{N}\mathcal{N}_1\int_0^\kappa \mathfrak{b}_1\mathcal{H}_\varrho([\rho(\eta+\mathfrak{w})]^\alpha, [\nu(\eta\\
+&\mathfrak{w})]^\alpha)d\eta + \mathcal{N}\int_0^\kappa \bigg(\mathfrak{b}_2\mathcal{H}_\varrho([\rho(\eta+\mathfrak{w})]^\alpha, [\nu(\eta+\mathfrak{w})]^\alpha)\\
+&\mathfrak{b}_3\mathfrak{b}_4\int_0^\eta \mathcal{H}_\varrho([\rho(\sigma+\mathfrak{w})]^\alpha, [\nu(\sigma+\mathfrak{w})]^\alpha)d\sigma\bigg)d\eta\\
+&\mathcal{N}\mathfrak{b}_5\mathcal{H}_\varrho([\rho(\kappa)]^\alpha, [\nu(\kappa)]^\alpha).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\varrho_\infty(\Theta\rho(\kappa), \Theta\nu(\kappa)) = \sup_{0<\alpha\leq1} \mathcal{H}_\varrho([\Theta\rho(\kappa)]^\alpha, [\Theta\nu(\kappa)]^\alpha) \leq \sum_{n=1}^q \mathcal{Q}_n \sup_{0<\alpha\leq1} \mathcal{H}_\varrho([\rho_{\sigma_n}(0)]^\alpha,\\
[\nu_{\sigma_n}(0)]^\alpha) + \mathfrak{b}_1 \sup_{0<\alpha\leq1} \mathcal{H}_\varrho([\rho(\kappa+\mathfrak{w})]^\alpha, [\nu(\kappa+\mathfrak{w})]^\alpha) + \mathcal{N}\mathcal{N}_1
\end{aligned}$$

$$\int_0^{\kappa} \mathfrak{b}_1 \sup_{0<\alpha\leq 1} \mathcal{H}_{\varrho}([\rho(\eta+\mathfrak{w})]^{\alpha}, [\nu(\eta+\mathfrak{w})]^{\alpha}) d\eta + \mathcal{N} \int_0^{\eta} \Big[\mathfrak{b}_2 \sup_{0<\alpha\leq 1} \mathcal{H}_{\varrho}([\rho(\eta+\mathfrak{w})]^{\alpha}, [\nu(\eta+\mathfrak{w})]^{\alpha}) + \mathfrak{b}_3\mathfrak{b}_4 \int_0^{\eta} \sup_{0<\alpha\leq 1} \mathcal{H}_{\varrho}([\rho(\sigma +\mathfrak{w})]^{\alpha}, [\nu(\sigma+\mathfrak{w})]^{\alpha}) d\sigma \Big] d\eta + \mathcal{N}\mathfrak{b}_5 \sup_{0<\alpha\leq 1} \mathcal{H}_{\varrho}([\rho(\kappa)]^{\alpha}, [\nu(\kappa)]^{\alpha}).$$

Hence,

$$\begin{aligned}\mathcal{H}_1(\Theta\rho,\Theta\nu) &= \sup_{0\leq\kappa\leq\mathcal{K}} \varrho_{\infty}(\Theta\rho(\kappa),\Theta\nu(\kappa)) \leq \sum_{n=1}^{q} \mathcal{Q}_n \sup_{0\leq\kappa\leq\mathcal{K}} \varrho_{\infty}(\rho_{\sigma_n},\nu_{\sigma_n}) \\ &+\mathfrak{b}_1 \sup_{0\leq\kappa\leq\mathcal{K}} \varrho_{\infty}(\rho(\kappa+\mathfrak{w}),\nu(\kappa+\mathfrak{w})) + \mathcal{N}\mathcal{N}_1 \int_0^{\kappa} \mathfrak{b}_1 \sup_{0\leq\kappa\leq\mathcal{K}} \varrho_{\infty} \\ &(\rho(\eta+\mathfrak{w}),\nu(\eta+\mathfrak{w}))d\eta + \mathcal{N}\int_0^{\eta} \Big(\mathfrak{b}_2 \sup_{0\leq\kappa\leq\mathcal{K}} \varrho_{\infty}(\rho(\eta+\mathfrak{w}), \\ &\nu(\eta+\mathfrak{w})) + \mathfrak{b}_3\mathfrak{b}_4 \int_0^{\eta} \sup_{0\leq\kappa\leq\mathcal{K}} \varrho_{\infty}(\rho(\sigma+\mathfrak{w}),\nu(\sigma+\mathfrak{w})d\sigma \Big) d\eta \\ &+\mathcal{N}\mathfrak{b}_5 \sup_{0\leq\kappa\leq\mathcal{K}} \varrho_{\infty}(\rho(\kappa),\nu(\kappa)).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{H}_1(\Theta\rho,\Theta\nu) &\leq \sum_{n=1}^{q} \mathcal{Q}_n\mathcal{H}_1(\rho,\nu) + \mathfrak{b}_1\mathcal{H}_1(\rho,\nu) + \mathcal{N}\mathcal{N}_1 \int_0^{\mathcal{K}} \mathfrak{b}_1\mathcal{H}_1(\rho,\nu)d\eta + \mathcal{N} \\ &\int_0^{\mathcal{K}} \Big(\mathfrak{b}_2\mathcal{H}_1(\rho,\nu) + \mathfrak{b}_3\mathfrak{b}_4 \int_0^{\eta} \mathcal{H}_1(\rho,\nu)d\sigma \Big) d\eta + \mathcal{N}\mathfrak{b}_5\mathcal{H}_1(\rho,\nu) \\ &\leq \Big(\sum_{n=1}^{q} \mathcal{Q}_n + \mathfrak{b}_1(1+\mathcal{N}\mathcal{N}_1\mathcal{K}) + \mathfrak{b}_2\mathcal{N}\frac{\mathcal{K}^2}{2} + \mathfrak{b}_3\mathfrak{b}_4\mathcal{N}\frac{\mathcal{K}^3}{2} \\ &+ \mathfrak{b}_5\mathcal{N} \Big) \mathcal{H}_1(\rho,\nu).\end{aligned}$$

By the hypothesis $\mathcal{B}_7$, Θ is a contraction mapping . By applying the Banach fixed point theorem, it is concluded that (3.1) has a unique fixed point. □

4. Controllability result

Definition 4.1. If $\rho(\kappa)$ is an integral solution of the problem (1.1), then $\rho(\kappa)$ is given by

$$\begin{aligned}\rho(\kappa) =&\mathfrak{D}(\kappa)\Big[\psi(0) - \mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\ldots,\rho_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\Big] + \mathfrak{z}(\kappa,\rho_{\kappa}) \\ &+ \int_0^{\kappa} A\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\eta,\rho_{\eta})d\eta + \int_0^{\kappa} \mathfrak{D}(\kappa-\eta)\mathfrak{W}\Big(\eta,\rho_{\eta},\int_0^{\eta}\mathcal{J}(\eta,\kappa, \\ &\rho_{\kappa})d\kappa)d\eta\Big) + \int_0^{\kappa} \mathfrak{D}(\kappa-\eta)\varsigma(\eta)d\eta + \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa-\kappa_n)\mathcal{I}_n(\rho(\kappa_n^-)),\end{aligned} \tag{4.1}$$

where $\kappa\in\mathcal{L}, \kappa\neq\kappa_n, (n=1,2,\ldots,k)$ and $\mathfrak{D}(\kappa)$ satisfies $\mathcal{B}_1$.

Lemma 4.1. *The fuzzy solution $\rho(\kappa)$ for equation* (4.1) *is controllable and satisfies $\rho(\mathcal{K}) = \rho^1$ i.e. $[\rho(\mathcal{K})]^\alpha = [\rho^1]^\alpha$ where $\rho^1 \in \mathbb{X}^d$ is a target set.*

Proof. Define the α-level set of fuzzy mapping $\mathcal{Q} : P(\mathfrak{R}) \to \mathbb{X}^d$ by

$$\mathcal{Q}^\alpha(\varsigma) = \begin{cases} \int_0^{\mathcal{K}} \mathfrak{D}^\alpha(\mathcal{K}-\eta)\varsigma(\eta)d\eta & ; \qquad \varsigma \subset Z_\varsigma \\ 0 & ; \qquad otherwise, \end{cases}$$

where Z_ς is the closure of support ς. In [18], the support Z_ς of a fuzzy number υ is defined as a special case of the level set by $Z_\varsigma = \{\rho : \xi_\varsigma(\rho) > 0\}$, then there exists $\mathcal{Q}^\alpha_m(m = l, r)$ such that:

$$\mathcal{Q}^\alpha_l(\upsilon_l) = \int_0^{\mathcal{K}} \mathfrak{D}^\alpha_l(\mathcal{K}-\eta)\Omega_l(\eta)d\eta, \quad \Omega_l(\eta) \in [\varsigma^\alpha_l(\eta), \varsigma^1(\eta)],$$
$$\mathcal{Q}^\alpha_r(\upsilon_r) = \int_0^{\mathcal{K}} \mathfrak{D}^\alpha_r(\mathcal{K}-\eta)\Omega_r(\eta)d\eta, \quad \Omega_r(\eta) \in [\varsigma^\alpha_r(\eta), \varsigma^1(\eta)].$$

Consider that $\mathcal{Q}^\alpha_l, \mathcal{Q}^\alpha_r$ are bijective mapping. Introduce α-level set of $\varsigma(\eta)$ for (4.1) and we get

$$\begin{aligned}
[\varsigma(\eta)]^\alpha =& [\varsigma^\alpha_l(\eta), \varsigma^\alpha_r(\eta)] \\
=& (\mathcal{Q}^\alpha_l)^{-1}\Bigg((\rho^1)^\alpha_l - \mathfrak{D}^\alpha_l(\mathcal{K})\Big[\psi^\alpha_l(0) - \mathfrak{h}^\alpha_l(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0) - \mathfrak{z}^\alpha_l(0,\psi)\Big] \\
& - \mathfrak{z}^\alpha_l(\mathcal{K}, \rho^\alpha_{\mathcal{K}_l}) - \int_0^{\mathcal{K}} A^\alpha_l \mathfrak{D}^\alpha_l(\mathcal{K}-\eta)\mathfrak{z}^\alpha_l(\eta, \rho^\alpha_{\eta_l})d\eta - \int_0^{\mathcal{K}} \mathfrak{D}^\alpha_l(\mathcal{K}-\eta)\mathfrak{W}^\alpha_l(\eta, \rho^\alpha_{\eta_l}, \\
& \int_0^\eta \mathcal{J}^\alpha_l(\eta, \kappa, \rho^\alpha_{\kappa_l})d\kappa)d\eta - \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}^\alpha_l(\mathcal{K}-\kappa_n)\mathcal{I}^\alpha_{n_l}(\rho^\alpha_l(\kappa^-_n))\Bigg), (\mathcal{Q}^\alpha_r)^{-1} \\
& \Bigg((\rho^1)^\alpha_r - \mathfrak{D}^\alpha_r(\mathcal{K})\Big[\psi^\alpha_r(0) - \mathfrak{h}^\alpha_r(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0) - \mathfrak{z}^\alpha_r(0,\psi)\Big] - \mathfrak{z}^\alpha_r(\mathcal{K}, \\
& \rho^\alpha_{\mathcal{K}_r}) - \int_0^{\mathcal{K}} A^\alpha_r \mathfrak{D}^\alpha_r(\mathcal{K}-\eta)\mathfrak{z}^\alpha_r(\eta, \rho^\alpha_{\eta_r})d\eta - \int_0^{\mathcal{K}} \mathfrak{D}^\alpha_r(\mathcal{K}-\eta)\mathfrak{W}^\alpha_r(\eta, \rho^\alpha_{\eta_r}, \int_0^\eta \\
& \mathcal{J}^\alpha_r(\eta, \kappa, \rho^\alpha_{\kappa_r})d\kappa))d\eta - \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}^\alpha_r(\mathcal{K}-\kappa_n)\mathcal{I}^\alpha_{n_r}(\rho^\alpha_r(\kappa^-_n))\Bigg).
\end{aligned}$$

Substituting this in Equation (4.1), we get α-level set of $\rho(\mathcal{K})$ as

$$\begin{aligned}
[\rho(\mathcal{K})]^\alpha =& \Bigg(\mathfrak{D}^\alpha_l(\mathcal{K})\big[\psi^\alpha_l(0) - \mathfrak{h}^\alpha_l(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0) - \mathfrak{z}^\alpha_l(0,\psi)\big] + \mathfrak{z}^\alpha_l(\mathcal{K}, \rho^\alpha_{\mathcal{K}_l}) \\
& + \int_0^{\mathcal{K}} A^\alpha_l \mathfrak{D}^\alpha_l(\mathcal{K}-\eta)\mathfrak{z}^\alpha_l(\eta, \rho^\alpha_{\eta_l})d\eta + \int_0^{\mathcal{K}} \mathfrak{D}^\alpha_l(\mathcal{K}-\eta)\mathfrak{W}^\alpha_l(\eta, \rho^\alpha_{\eta_l}, \int_0^\eta \mathcal{J}^\alpha_l \\
& (\eta, \kappa, \rho^\alpha_{\kappa_l})d\kappa)d\eta + \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}^\alpha_l(\mathcal{K}-\kappa_n)\mathcal{I}^\alpha_{n_l}(\rho^\alpha_l(\kappa^-_n)) + \int_0^{\mathcal{K}} \mathfrak{D}^\alpha_l(\mathcal{K}-\eta) \\
& (\mathcal{Q}^\alpha_l)^{-1}\Bigg((\rho^1)^\alpha_l - \mathfrak{D}^\alpha_l(\mathcal{K})\big[\psi^\alpha_l(0) - \mathfrak{h}^\alpha_l(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0) - \mathfrak{z}^\alpha_l(0,\psi)\big] \\
& - \mathfrak{z}^\alpha_l(\mathcal{K}, \rho^\alpha_{\mathcal{K}_l}) - \int_0^{\mathcal{K}} A^\alpha_l \mathfrak{D}^\alpha_l(\mathcal{K}-\eta)\mathfrak{z}^\alpha_l(\eta, \rho^\alpha_{\eta_l})d\eta - \int_0^{\mathcal{K}} \mathfrak{D}^\alpha_l(\mathcal{K}-\eta)\mathfrak{W}^\alpha_l(\eta,
\end{aligned}$$

$$\kappa^{\alpha}_{\eta l}, \int_0^{\eta} \mathcal{J}^{\alpha}_l(\eta,\kappa,\rho^{\alpha}_{\kappa l})d\kappa)d\eta) - \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}^{\alpha}_l(\mathcal{K}-\kappa_n)\mathcal{I}^{\alpha}_{nl}(\rho^{\alpha}_l(\kappa^-_n))\Big)d\eta,$$

$$\Big(\mathfrak{D}^{\alpha}_r(\mathcal{K})\big[\psi^{\alpha}_r(0) - \mathfrak{h}^{\alpha}_r(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(0) - \mathfrak{z}^{\alpha}_r(0,\psi)\big] + \mathfrak{z}^{\alpha}_r(\mathcal{K},\rho^{\alpha}_{\mathcal{K}_r})$$
$$+ \int_0^{\mathcal{K}} A^{\alpha}_r\mathfrak{D}^{\alpha}_r(\mathcal{K}-\eta)\mathfrak{z}^{\alpha}_r(\eta,\rho^{\alpha}_{\eta_r})d\eta + \int_0^{\mathcal{K}} \mathfrak{D}^{\alpha}_r(\mathcal{K}-\eta)\mathfrak{W}^{\alpha}_r(\eta,\rho^{\alpha}_{\eta_r}, \int_0^{\eta} \mathcal{J}^{\alpha}_r(\eta,$$
$$\kappa,\rho^{\alpha}_{\kappa_r})d\kappa)d\eta + \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}^{\alpha}_r(\mathcal{K}-\kappa_n)\mathcal{I}^{\alpha}_{n_r}(\rho^{\alpha}_r(\kappa^-_n)) + \int_0^{\mathcal{K}} \mathfrak{D}^{\alpha}_r(\mathcal{K}-\eta)$$
$$(\mathcal{Q}^{\alpha}_r)^{-1}\Big((\rho^1)^{\alpha}_r - \mathfrak{D}^{\alpha}_r(\mathcal{K})\big[\psi^{\alpha}_r(0) - \mathfrak{h}^{\alpha}_r(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(0) - \mathfrak{z}^{\alpha}_r(0,\psi)\big]$$
$$- \mathfrak{z}^{\alpha}_r(\mathcal{K},\rho^{\alpha}_{\mathcal{K}_r}) - \int_0^{\mathcal{K}} A^{\alpha}_r\mathfrak{D}^{\alpha}_r(\mathcal{K}-\eta)\mathfrak{z}^{\alpha}_r(\eta,\rho^{\alpha}_{\eta_r})d\eta - \int_0^{\mathcal{K}} \mathfrak{D}^{\alpha}_r(\mathcal{K}-\eta)\mathfrak{W}^{\alpha}_r(\eta,$$
$$\rho^{\alpha}_{\eta_r}, \int_0^{\eta} \mathcal{J}^{\alpha}_r(\eta,\kappa,\rho^{\alpha}_{\kappa_r})d\kappa)d\eta) - \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}^{\alpha}_r(\mathcal{K}-\kappa_n)\mathcal{I}^{\alpha}_{n_r}(\rho^{\alpha}_r(\kappa^-_n))\Big)d\kappa.$$

Therefore,

$$[(\rho^1)^{\alpha}_l, (\rho^1)^{\alpha}_r] = [\rho^1]^{\alpha}.$$

Hence, the fuzzy solution $\rho(\kappa)$ for Equation (4.1) satisfies $[\rho(\mathcal{K})]^{\alpha} = [\rho^1]^{\alpha}$. □

Now we define,

$$\begin{aligned}\Theta(\rho(\kappa)) =& \mathfrak{D}(\kappa)[\psi(0) - \mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(0) - \mathfrak{z}(0,\psi)] + \mathfrak{z}(\kappa,\rho_\kappa) + \int_0^{\kappa} A \\ &\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\eta,\rho_\eta)d\eta + \int_0^{\eta} \mathfrak{D}(\kappa-\eta)\mathfrak{W}(\eta,\rho_\eta, \int_0^{\eta} \mathcal{J}(\eta,\kappa,\rho_\kappa)d\kappa)d\eta \\ &+ \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa-\kappa_n)\mathcal{I}_n(\rho(\kappa^-_n)) + \int_0^{\kappa} \mathfrak{D}(\kappa-\eta)(\mathcal{Q}^{-1})\Big((\rho^1) - \mathfrak{D}(\mathcal{K}) \\ &\big[\psi(0) - \mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\dots,\rho_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big] - \mathfrak{z}(\mathcal{K},\rho_{\mathcal{K}}) - \int_0^{\mathcal{K}} A\mathfrak{D}(\mathcal{K} \\ &- \eta)\mathfrak{z}(\eta,\rho_\eta)d\eta - \int_0^{\mathcal{K}} \mathfrak{D}(\mathcal{K}-\eta)\mathfrak{W}(\eta,\rho_\eta, \int_0^{\eta} \mathcal{J}(\eta,\kappa,\rho_\kappa)d\kappa)d\eta) \qquad (4.2) \\ &- \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}(\mathcal{K}-\kappa_n)\mathcal{I}_n(\rho(\kappa^-_n))\Big)d\eta,\end{aligned}$$

where $(\mathcal{Q})^{-1}$ satisfies the previous statements.

Observe $\Theta(\rho(\kappa)) = [\rho^1]$ which represents that the control $\varsigma(\kappa)$ steers (4.2) from the arbitrary stage to ρ^1 in time $\mathcal{K}$, given that there must exist a fixed point of the nonlinear operator Θ.

Similarly,

$$\begin{aligned}\Theta(\nu(\kappa)) =& \mathfrak{D}(\kappa)[\psi(0) - \mathfrak{h}(\nu_{\sigma_1},\nu_{\sigma_2},\dots,\nu_{\sigma_q})(0) - \mathfrak{z}(0,\psi)] + \mathfrak{z}(\kappa,\nu_\kappa) + \int_0^{\kappa} A\mathfrak{D}(\kappa-\eta) \\ &\mathfrak{z}(\eta,\nu_\eta)d\eta + \int_0^{\kappa} \mathfrak{D}(\kappa-\eta)\mathfrak{W}(\eta,\nu_\eta, \int_0^{\eta} \mathcal{J}(\eta,\kappa,\nu_\kappa)d\kappa))d\eta + \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa\end{aligned}$$

$$- \kappa_n)\mathcal{I}_n(\nu(\kappa_n^-)) + \int_0^\kappa \mathfrak{D}(\kappa-\eta)(\mathcal{Q}^{-1})\Big((\nu^1) - \mathfrak{D}(\mathcal{K})\big[\psi(0) - \mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2},$$

$$\ldots, \nu_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big] - \mathfrak{z}(\mathcal{K}, \nu_{\mathcal{K}}) - \int_0^{\mathcal{K}} A\mathfrak{D}(\mathcal{K}-\eta)\mathfrak{z}(\eta,\nu_\eta)d\eta - \int_0^{\mathcal{K}} \mathfrak{D}(\mathcal{K}$$

$$- \eta)\mathfrak{W}(\eta, \nu_\eta, \int_0^\eta \mathcal{J}(\eta,\kappa,\nu_\kappa)d\kappa)d\eta) - \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}(\mathcal{K}-\kappa_n)\mathcal{I}_n(\nu(\kappa_n^-))\Big)d\eta.$$

The following theorem discusses the controllability of the nonlocal impulsive functional for nonlinear fuzzy solutions neutral integro-differential equation.

Theorem 4.1. *If the hypotheses* $(\mathcal{B}_1) - (\mathcal{B}_6)$ *are satisfied and* $\Big(\sum_{n=1}^{q} \mathcal{Q}_n(1+\mathcal{N}\mathcal{K}) + \mathfrak{b}_1(1+\mathcal{N}\mathcal{N}_1\mathcal{K}+\mathcal{N}\mathcal{K}) + \mathfrak{b}_2(\mathcal{N}\mathcal{K}+\mathcal{N}^2\mathcal{K}^2) + \mathfrak{b}_3\mathfrak{b}_4(\mathcal{N}\frac{\mathcal{K}^2}{2} + \mathcal{N}^2\frac{\mathcal{K}^3}{2}) + \mathfrak{b}_5(\mathcal{N}+\mathcal{N}\mathcal{K}) + \mathcal{N}^2\mathcal{N}_1\mathcal{K}^2\Big) < 1$*, then System* (4.2) *is controllable on* $\mathcal{L}$.

Proof. For $\rho, \nu \in \Upsilon'$,

$$\begin{aligned}
\mathcal{H}_\varrho([\Theta\rho(\kappa)]^\alpha, [\Theta\nu(\kappa)]^\alpha) = &\mathcal{H}_\varrho\Bigg(\Bigg[\mathfrak{D}(\kappa)\big[\psi(0) - \mathfrak{h}(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big] \\
&+ \mathfrak{z}(\kappa,\rho_\kappa) + \int_0^\kappa A\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\eta,\rho_\eta)d\eta + \int_0^\kappa \mathfrak{D}(\kappa-\eta) \\
&\mathfrak{W}(\eta,\rho_\eta, \int_0^\eta \mathcal{J}(\eta,\kappa,\rho_\kappa)d\kappa))d\eta + \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa-\kappa_n)\mathcal{I}_n \\
&(\rho(\kappa_n^-)) + \int_0^\kappa \mathfrak{D}(\kappa-\eta)(\mathcal{Q}^{-1})\Big((\rho^1) - \mathfrak{D}(\mathcal{K})\big[\psi(0) - \mathfrak{h} \\
&(\rho_{\sigma_1}, \rho_{\sigma_2}, \ldots, \rho_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big] - \mathfrak{z}(\mathcal{K},\rho_{\mathcal{K}}) - \int_0^{\mathcal{K}} A\mathfrak{D} \\
&(\mathcal{K}-\eta)\mathfrak{z}(\eta,\rho_\eta)d\eta - \int_0^{\mathcal{K}} \mathfrak{D}(\mathcal{K}-\eta)\mathfrak{W}(\eta,\rho_\eta, \int_0^\eta \mathcal{J}(\eta,\kappa, \\
&\rho_\kappa)d\kappa)d\eta - \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}(\mathcal{K}-\kappa_n)\mathcal{I}_n(\rho(\kappa_n^-))\Big)d\eta\Bigg]^\alpha, \\
&\Bigg[\mathfrak{D}(\kappa)[\psi(0) - \mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \nu_{\sigma_q})(0) - \mathfrak{z}(0,\psi)] + \mathfrak{z}(\kappa, \\
&\nu_\kappa) + \int_0^\kappa A\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\eta,\nu_\eta)d\eta + \int_0^\kappa \mathfrak{D}(\kappa-\eta)\mathfrak{W}(\kappa, \\
&\nu_\eta, \int_0^\eta \mathcal{J}(\eta,\kappa,\nu_\kappa)d\kappa))d\eta + \sum_{0<\kappa_n<\kappa} \mathfrak{D}(\kappa-\kappa_n)\mathcal{I}_n(\nu(\kappa_n^-)) \\
&+ \int_0^\kappa \mathfrak{D}(\kappa-\eta)(\mathcal{Q}^{-1})\Big((\nu^1) - \mathfrak{D}(\mathcal{K})\big[\psi(0) - \mathfrak{h}(\nu_{\sigma_1}, \nu_{\sigma_2}, \ldots, \\
&\nu_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big] - \mathfrak{z}(\mathcal{K},\nu_{\mathcal{K}}) - \int_0^{\mathcal{K}} A\mathfrak{D}(\mathcal{K}-\eta)\mathfrak{z}(\eta,\nu_\eta)d\eta \\
&- \int_0^{\mathcal{K}} \mathfrak{D}(\mathcal{K}-\eta)\mathfrak{W}(\eta,\nu_\eta, \int_0^\eta \mathcal{J}(\eta,\kappa,\nu_\kappa)d\kappa))d\eta
\end{aligned}$$

$$- \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}(\mathcal{K}-\kappa_n)\mathcal{I}_n(\nu(\kappa_n^-))\Big)d\eta\Big]^{\alpha}\Big)$$

$$\le \mathcal{H}_{\varrho}([\mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\ldots,\rho_{\sigma_q})(0)]^{\alpha},[\mathfrak{h}(\nu_{\sigma_1},\nu_{\sigma_2},\ldots,\nu_{\sigma_q})$$

$$(0)]^{\alpha}) + \mathcal{H}_{\varrho}([\mathfrak{z}(\kappa,\rho_\kappa)]^{\alpha},\mathfrak{z}(\kappa,\nu_\kappa)]^{\alpha}) + \mathcal{H}_{\varrho}(\big[\int_0^{\kappa} A\mathfrak{D}(\kappa$$

$$-\eta)\mathfrak{z}(\eta,\rho_\eta)d\eta\big]^{\alpha},\big[\int_0^{\kappa} A\mathfrak{D}(\kappa-\eta)\mathfrak{z}(\eta,\nu_\eta)d\eta\big]^{\alpha})$$

$$+ \mathcal{H}_{\varrho}(\big[\int_0^{\kappa}\mathfrak{D}(\kappa-\eta)\mathfrak{W}(\eta,\rho_\eta,\int_0^{\eta}\mathcal{J}(\eta,\kappa,\rho_\kappa)d\kappa))d\eta\big]^{\alpha},$$

$$\big[\int_0^{\kappa}\mathfrak{D}(\kappa-\eta)\mathfrak{W}(\eta,\nu_\eta,\int_0^{\eta}\mathcal{J}(\eta,\kappa,\nu_\kappa)d\kappa)d\eta\big]^{\alpha})$$

$$+ \mathcal{H}_{\varrho}([\sum_{0<\kappa_n<\kappa}\mathfrak{D}(\kappa-\kappa_n)\mathcal{I}_n(\rho(\kappa_n^-))]^{\alpha}),[\sum_{0<\kappa_n<\kappa}\mathfrak{D}(\kappa$$

$$-\kappa_n)\mathcal{I}_n(\nu(\kappa_n^-))]^{\alpha}) + \mathcal{H}_{\varrho}(\Big[\int_0^{\kappa}\mathfrak{D}(\kappa-\eta)(\mathcal{Q}^{-1})$$

$$\Big((\rho^1) - \mathfrak{D}(\mathcal{K})\big[\psi(0) - \mathfrak{h}(\rho_{\sigma_1},\rho_{\sigma_2},\ldots,\rho_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big]$$

$$-\mathfrak{z}(\mathcal{K},\rho_{\mathcal{K}}) - \int_0^{\mathcal{K}} A\mathfrak{D}(\mathcal{K}-\eta)\mathfrak{z}(\eta,\rho_\eta)d\eta - \int_0^{\mathcal{K}}\mathfrak{D}(\mathcal{K}-\eta)$$

$$\mathfrak{W}(\eta,\rho_\eta,\int_0^{\eta}\mathcal{J}(\eta,\kappa,\rho_\kappa)d\kappa)d\eta) - \sum_{0<\kappa_n<\mathcal{K}}\mathfrak{D}(\mathcal{K}-\kappa_n)$$

$$\mathcal{I}_n(\rho(\kappa_n^-))d\eta\Big]^{\alpha},\big[\int_0^{\kappa}\mathfrak{D}(\kappa-\eta)(\mathcal{Q}^{-1})\big((\nu^1) - \mathfrak{D}(\mathcal{K})\big[\psi(0)$$

$$-\mathfrak{h}(\nu_{\sigma_1},\nu_{\sigma_2},\ldots,\nu_{\sigma_q})(0) - \mathfrak{z}(0,\psi)\big] - \mathfrak{z}(\mathcal{K},\nu_{\mathcal{K}}) - \int_0^{\mathcal{K}} A\mathfrak{D}$$

$$(\mathcal{K}-\eta)\mathfrak{z}(\eta,\nu_\eta)d\eta - \int_0^{\mathcal{K}}\mathfrak{D}(\mathcal{K}-\eta)\mathfrak{W}(\eta,\nu_\eta,\int_0^{\eta}\mathcal{J}(\eta,\kappa,$$

$$\nu_\kappa)d\kappa))d\eta - \sum_{0<\kappa_n<\mathcal{K}}\mathfrak{D}(\mathcal{K}-\kappa_n)\mathcal{I}_n(\nu(\kappa_n^-))d\eta]^{\alpha})$$

$$\le \sum_{n=1}^{q}\mathcal{Q}_n\mathcal{H}_{\varrho}([\rho_{\sigma n}(0)]^{\alpha},[\nu_{\sigma_n}(0)]^{\alpha}) + \mathfrak{b}_1\mathcal{H}_{\varrho}([\rho(\kappa+\mathfrak{w})]^{\alpha},[\nu$$

$$(\kappa+\mathfrak{w})]^{\alpha}) + \mathcal{N}\mathcal{N}_1\int_0^{\kappa}\mathfrak{b}_1\mathcal{H}_{\varrho}([\rho(\eta+\mathfrak{w})]^{\alpha},[\nu(\eta+\mathfrak{w})]^{\alpha})d\eta$$

$$+\mathcal{N}\int_0^{\eta}\Big(\mathfrak{b}_2\mathcal{H}_{\varrho}([\rho(\eta+\mathfrak{w})]^{\alpha},[\nu(\eta+\mathfrak{w})]^{\alpha}) + \mathfrak{b}_3\mathfrak{b}_4\int_0^{\eta}\mathcal{H}_{\varrho}$$

$$([\rho(\sigma+\mathfrak{w})]^{\alpha},[\nu(\sigma+\mathfrak{w})]^{\alpha})d\kappa\Big)d\eta + \mathcal{N}\mathfrak{b}_5\mathcal{H}_{\varrho}([\rho(\kappa)]^{\alpha},$$

$$[\nu(\kappa)]^{\alpha}) + \mathcal{N}\int_0^{\mathcal{K}}\Big\{\sum_{n=1}^{q}\mathcal{Q}_n\mathcal{H}_{\varrho}([\rho_{\sigma_n}(0)]^{\alpha},[\nu_{\sigma_n}(0)]^{\alpha}) + \mathfrak{b}_1$$

$$\mathcal{H}_{\varrho}([\rho(\mathcal{K}+\mathfrak{w})]^{\alpha},\nu(\mathcal{K}+\mathfrak{w})]^{\alpha}) + \mathcal{N}\mathcal{N}_1\int_0^{\mathcal{K}}\mathfrak{b}_1\mathcal{H}_{\varrho}([\rho(\eta$$

$$+ \mathfrak{w})]^\alpha, [\nu(\eta + \mathfrak{w})]^\alpha) d\eta + \mathcal{N} \int_0^{\mathcal{K}} \bigg(\mathfrak{b}_2 \mathcal{H}_\varrho([\rho(\eta + \mathfrak{w})]^\alpha, [\nu(\eta + \mathfrak{w})]^\alpha) + \mathfrak{b}_3 \mathfrak{b}_4 \int_0^\eta \mathcal{H}_\varrho([\rho(\sigma + \mathfrak{w})]^\alpha, [\nu(\sigma + \mathfrak{w})]^\alpha) d\sigma \bigg) d\eta + \mathcal{N} \mathfrak{b}_5 \mathcal{H}_d([\rho(\kappa)]^\alpha, [\nu(\kappa)]^\alpha) \bigg\} d\eta.$$

Therefore,

$$\begin{aligned}
\varrho_\infty(\Theta\rho(\kappa), \Theta\nu(\kappa)) &= \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\Theta\rho(\kappa)]^\alpha, [\Theta\nu(\kappa)]^\alpha) \leq \sum_{n=1}^{q} \mathcal{Q}_n \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho_{\sigma_n}(0)]^\alpha \\
&\quad + \mathfrak{b}_1 \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho(\kappa + \mathfrak{w})]^\alpha, , [\nu(\eta + \mathfrak{w})]^\alpha) + \mathcal{N}\mathcal{N}_1 \int_0^\kappa \mathfrak{b}_1 \\
&\quad \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho(\eta + \mathfrak{w})]^\alpha, [\nu(\eta + \mathfrak{w})]^\alpha) d\eta + \mathcal{N} \int_0^\kappa \bigg(\mathfrak{b}_2 \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho \\
&\quad ([\rho(\eta + \mathfrak{w})]^\alpha, [\nu(\eta + \mathfrak{w})]^\alpha) + \mathfrak{b}_3 \mathfrak{b}_4 \int_0^\eta \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho(\sigma + \mathfrak{w})]^\alpha, \\
&\quad [\nu(\sigma + \mathfrak{w})]^\alpha) d\sigma \bigg) d\eta + \mathcal{N} \mathfrak{b}_5 \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho(\kappa)]^\alpha, [\nu(\kappa)]^\alpha) \\
&\quad + \mathcal{N} \int_0^{\mathcal{K}} \bigg\{ \sum_{n=1}^{q} \mathcal{Q}_n \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho_{\sigma_n}(0)]^\alpha, [\nu_{\sigma_n}(0)]^\alpha) + \mathfrak{b}_1 \sup_{0<\alpha\leq 1} \\
&\quad \mathcal{H}_\varrho([\rho(\mathcal{K} + \mathfrak{w})]^\alpha, [\nu(\mathcal{K} + \mathfrak{w})]^\alpha) + \mathcal{N}\mathcal{N}_1 \int_0^{\mathcal{K}} \mathfrak{b}_1 \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho(\eta \\
&\quad + \mathfrak{w})]^\alpha, [\nu(\eta + \mathfrak{w})]^\alpha) d\eta + \mathcal{N} \int_0^{\mathcal{K}} \bigg(\mathfrak{b}_2 \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho(\eta \\
&\quad + \mathfrak{w})]^\alpha, [\nu(\eta + \mathfrak{w})]^\alpha) + \mathfrak{b}_3 \mathfrak{b}_4 \int_0^\eta \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho(\sigma + \mathfrak{w})]^\alpha, \\
&\quad [\nu(\sigma + \mathfrak{w})]^\alpha d\sigma \bigg) d\eta + \mathcal{N} \mathfrak{b}_5 \sup_{0<\alpha\leq 1} \mathcal{H}_\varrho([\rho(\eta)]^\alpha, [\nu(\eta)]^\alpha) \bigg\} d\eta \\
&\leq \sum_{n=1}^{q} \mathcal{Q}_n \varrho_\infty(\rho_{\sigma_n}, \nu_{\sigma_n}) + \mathfrak{b}_1 \varrho_\infty(\rho(\kappa + \mathfrak{w}), \nu(\kappa + \mathfrak{w})) + \mathcal{N}\mathcal{N}_1 \\
&\quad \int_0^\kappa \mathfrak{b}_1 \varrho_\infty(\rho(\eta + \mathfrak{w}), \nu(\eta + \mathfrak{w})) d\eta + \mathcal{N} \int_0^\kappa \bigg(\mathfrak{b}_2 \varrho_\infty(\rho(\kappa + \mathfrak{w}), \\
&\quad \nu(\kappa + \mathfrak{w})) + \mathfrak{b}_3 \mathfrak{b}_4 \int_0^\kappa \varrho_\infty(\rho(\sigma + \mathfrak{w}), \nu(\sigma + \mathfrak{w})) d\sigma \bigg) d\eta \\
&\quad + \mathcal{N} \mathfrak{b}_5 \varrho_\infty(\rho(\kappa), \nu(\kappa)) + \mathcal{N} \int_0^{\mathcal{K}} \bigg\{ \sum_{n=1}^{q} \mathcal{Q}_n \varrho_\infty(\rho_{\sigma_n}(0), \nu_{\sigma_n}(0)) \\
&\quad + \mathfrak{b}_1 \varrho_\infty(\rho(\mathcal{K} + \mathfrak{w}), \nu(\mathcal{K} + \mathfrak{w})) + \mathcal{N}\mathcal{N}_1 \int_0^{\mathcal{K}} \mathfrak{b}_1 \varrho_\infty(\rho(\eta + \mathfrak{w}), \\
&\quad \nu(\eta + \mathfrak{w})) d\eta + \mathcal{N} \int_0^{\mathcal{K}} \bigg(\mathfrak{b}_2 \varrho_\infty(\rho(\eta + \mathfrak{w}), \nu(\eta + \mathfrak{w})) + \mathfrak{b}_3 \mathfrak{b}_4 \int_0^\eta
\end{aligned}$$

$$\varrho_{\infty}(\rho(\sigma+\mathfrak{w}),\nu(\sigma+\mathfrak{w})d\sigma\Big)d\eta+\mathcal{N}\mathfrak{b}_5\varrho_{\infty}(\rho(\eta),\nu(\eta))\Big\}d\eta.$$

Hence,

$$\begin{aligned}\mathcal{H}_1(\Theta\rho,\Theta\nu)=\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\Theta\rho(\kappa),\Theta\nu(\kappa))\le\sum_{n=1}^{q}\mathcal{Q}_n\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho_{\sigma_n},\nu_{\sigma_n})+\mathfrak{b}_1\sup_{0\le\kappa\le\mathcal{K}}\\
\varrho_{\infty}(\rho(\kappa+\mathfrak{w}),\nu(\kappa+\mathfrak{w}))+\mathcal{N}\mathcal{N}_1\int_0^{\kappa}\mathfrak{b}_1\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\eta+\mathfrak{w}),\nu(\eta\\
+\mathfrak{w}))d\eta+\mathcal{N}\int_0^{\kappa}\Big(\mathfrak{b}_2\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\eta+\mathfrak{w}),\nu(\eta+\mathfrak{w}))+\mathfrak{b}_3\mathfrak{b}_4\int_0^{\eta}\\
\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\sigma+\mathfrak{w}),\nu(\sigma+\mathfrak{w})d\sigma\Big)d\eta+\mathcal{N}\mathfrak{b}_5\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\kappa),\nu(\kappa))\\
+\mathcal{N}\int_0^{\mathcal{K}}\Big\{\sum_{n=1}^{q}\mathcal{Q}_n\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\sigma_n),\nu(\sigma_n))+\mathfrak{b}_1\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\mathcal{K}\\
+\mathfrak{w}),\nu(\mathcal{K}+\mathfrak{w}))+\int_0^{\mathcal{K}}\mathcal{N}\mathcal{N}_1\mathfrak{b}_1\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\eta+\mathfrak{w}),[\nu(\eta+\mathfrak{w})])d\eta\\
+\mathcal{N}\int_0^{\mathcal{K}}\Big(\mathfrak{b}_2\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\eta+\mathfrak{w}),\nu(\eta+\mathfrak{w}))+\mathfrak{b}_3\mathfrak{b}_4\int_0^{\eta}\sup_{0\le\kappa\le\mathcal{K}}\\
\varrho_{\infty}(\rho(\sigma+\mathfrak{w}),\nu(\sigma+\mathfrak{w})d\sigma\Big)d\eta+\mathcal{N}\mathfrak{b}_5\sup_{0\le\kappa\le\mathcal{K}}\varrho_{\infty}(\rho(\eta),\nu(\eta))\Big\}d\eta.\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{H}_1(\Theta\rho,\Theta\nu)\le&\sum_{n=1}^{q}\mathcal{Q}_n\mathcal{H}_1(\rho,\nu)+\mathfrak{b}_1\mathcal{H}_1(\rho,\nu)+\mathcal{N}\mathcal{N}_1\int_0^{\mathcal{K}}\mathfrak{b}_1\mathcal{H}_1(\rho,\nu)d\eta\\
&+\mathcal{N}\int_0^{\mathcal{K}}\Big(\mathfrak{b}_2\mathcal{H}_1(\rho,\nu)+\mathfrak{b}_3\mathfrak{b}_4\int_0^{\eta}\mathcal{H}_1(\rho,\nu)d\sigma\Big)d\eta+\mathcal{N}\mathfrak{b}_5\mathcal{H}_1(\rho,\nu)\\
&+\mathcal{N}\int_0^{\mathcal{K}}\Big\{\sum_{n=1}^{q}\mathcal{Q}_n\mathcal{H}_1(\rho,\nu)d\eta+\mathfrak{b}_1\mathcal{H}_1(\rho,\nu)+\mathcal{N}\mathcal{N}_1\int_0^{\mathcal{K}}\mathcal{H}_1(\rho,\nu)d\eta\\
&+\mathcal{N}\int_0^{\mathcal{K}}(\mathfrak{b}_2\mathcal{H}_1(\rho,\nu)+\mathfrak{b}_3\mathfrak{b}_4\int_0^{\eta}\mathcal{H}_1(\rho,\nu)d\sigma)d\eta+\mathcal{N}\mathfrak{b}_5\mathcal{H}_1(\rho,\nu)\Big\}d\eta\\
\le&\Big(\sum_{n=1}^{q}\mathcal{Q}_n(1+\mathcal{N}\mathcal{K})+\mathfrak{b}_1(1+\mathcal{N}\mathcal{N}_1\mathcal{K}+\mathcal{N}\mathcal{K})+\mathfrak{b}_2(\mathcal{N}\mathcal{K}+\mathcal{N}^2\mathcal{K}^2)\\
&+\mathfrak{b}_3\mathfrak{b}_4(\mathcal{N}\frac{\mathcal{K}^2}{2}+\mathcal{N}^2\frac{\mathcal{K}^3}{2})+\mathfrak{b}_5(\mathcal{N}+\mathcal{N}\mathcal{K})+\mathcal{N}^2\mathcal{N}_1\mathcal{K}^2\Big)\mathcal{H}_1(\rho,\nu).\end{aligned}$$

Hence, Θ is a contraction mapping. By applying the Banach fixed point theorem, Equation (4.2) has a unique fixed point $\rho\in\Upsilon$. □

5. Example

Consider the fuzzy solution of the nonlinear fuzzy neutral integrodifferential equation of the form:

$$\begin{aligned}
&\frac{d}{d\kappa}[\rho(\kappa)-\kappa\rho(\kappa+h)^2] = \rho(\kappa)+2\kappa\rho(\kappa+h)^2+3\kappa\rho(\kappa+h)^2+\varsigma(\kappa),\ \kappa\in\mathcal{L},\\
&\rho(0)+\sum_{n=1}^{q} a_n\rho(\kappa_n) = 0\in\mathbb{X}^n, \qquad (5.1)\\
&\mathcal{I}_n(\rho(\kappa_n^-)) = \frac{1}{1+\rho(\kappa_n)},
\end{aligned}$$

where ρ^1 is a target set, and the $\alpha-$ level sets of $0,1,2$ and 3 are $[0]^\alpha=[\alpha-1,1-\alpha],[1]^\alpha=[\alpha,2-\alpha],[2]^\alpha=[\alpha+1,3-\alpha]$ and $[3]^\alpha=[\alpha+2,4-\alpha]$, for $\alpha\in[0,1]$.

Let $\mathfrak{W}(\kappa,\rho_\kappa)=2\kappa\rho(\kappa+h)^2, \mathfrak{z}(\kappa,\rho_\kappa)=\kappa\rho(\kappa+h)^2$ and $\int_0^\kappa \mathcal{J}(\kappa,\eta,\rho_\eta)d\eta = 3\kappa\rho(\kappa+h)^2$. The $\alpha-$ level set of $\mathfrak{h}(\rho)=\sum\limits_{n=1}^{q} a_n\rho(\kappa_n)$ is

$$[\mathfrak{h}(\rho)]^\alpha=[\sum_{n=1}^{q} a_n\rho(\kappa_n)]^\alpha=\left[\sum_{n=1}^{q} a_n\rho_\imath^\alpha(\kappa_n),\sum_{n=1}^{q} a_n\rho_\jmath^\alpha(\kappa_n)\right],$$

and

$$\begin{aligned}
\mathcal{H}_\varrho([\mathfrak{h}(\rho)]^\alpha,[\mathfrak{h}(\nu)]^\alpha) &= \mathcal{H}_d\left([\sum_{n=1}^{q} a_n\rho(\kappa_n)]^\alpha,[\sum_{n=1}^{q} a_n\nu(\kappa_n)]^\alpha\right)\\
&\le \sum_{n=1}^{q}\mathcal{Q}_n\mathcal{H}_\varrho([\rho_{\kappa_n}(\eta)]^\alpha,[\nu_{\kappa_n}(\eta)]^\alpha).
\end{aligned}$$

Similarly, the $\alpha-$ level set of $\mathfrak{W}(\kappa,\rho_\kappa),\mathfrak{z}(\kappa,\rho_\kappa)$ and $\mathcal{I}_n(\rho(\kappa_n)$ is

$$[\textstyle\int_0^\kappa \mathcal{J}(\kappa,\eta,\rho_\eta)d\eta]^\alpha=[3\kappa\rho(\kappa+h)^2]^\alpha=\kappa[(\alpha+2)\rho_\imath^\alpha(\kappa+h)^2,(4-\alpha)\rho_\jmath^\alpha(\kappa+h)^2],$$

$$[\mathfrak{W}(\kappa,\rho_\kappa)]^\alpha=[2\kappa\rho(\kappa+h)^2]^\alpha=\kappa[(\alpha+1)\rho_\imath^\alpha(\kappa+h)^2,(3-\alpha)\rho_\jmath^\alpha(\kappa+h)^2)],$$

$$[\mathfrak{z}(\kappa,\rho_\kappa)]^\alpha=\kappa[(\alpha)\rho_\imath^\alpha(\kappa+h)^2,(2-\alpha)\rho_\jmath^\alpha(\kappa+h)^2],$$

$$[\mathcal{I}_n(\rho(\kappa_n)]^\alpha=\left[\frac{1}{1+\rho(\kappa_n)}\right]^\alpha=\left[\frac{1}{1+\rho_\imath^\alpha(\kappa_n)},\frac{1}{1+\rho_\jmath^\alpha(\kappa_n)}\right],$$

and

$$\mathcal{H}_\varrho[\textstyle\int_0^x \mathcal{J}(\kappa,\eta,\rho_\eta)d\eta]^\alpha,[\int_0^\kappa \mathcal{J}(\kappa,\eta,\nu_\eta)d\eta]^\alpha\le\mathfrak{b}_4\mathcal{H}_\varrho([\rho(\eta+h)]^\alpha,[\nu(\eta+h)]^\alpha)$$

$$\mathcal{H}_\varrho([\mathfrak{W}(\kappa,\rho_\kappa)]^\alpha,[\mathfrak{W}(\kappa,\nu_\kappa)]^\alpha)\le\mathfrak{b}_2\mathcal{H}_\varrho([\rho(\kappa+h)]^\alpha,[\nu(\kappa+h)]^\alpha),$$

where

$$\mathfrak{b}_2=3\ell[\rho_\imath^\alpha(\kappa+h),\nu_\imath^\alpha(\kappa+h)],\mathfrak{b}_4=4\ell[\rho_\imath^\alpha(\kappa+h),\nu_\imath^\alpha(\kappa+h)].$$

Similarly,

$$\mathcal{H}_\varrho([\mathfrak{z}(\kappa,\rho_\kappa)]^\alpha,[\mathfrak{z}(\kappa,\nu_\kappa)]^\alpha)\le\mathfrak{b}_1\mathcal{H}_\varrho([\rho(\kappa+h)]^\alpha,[\nu(\kappa+h)]^\alpha)$$

$$\mathcal{H}_\varrho(\mathcal{I}_n[(\rho(x_n)]^\alpha,\mathcal{I}_n[(\nu(\kappa_n)]^\alpha)\le\mathfrak{b}_5\mathcal{H}_\varrho([\rho(\kappa+h)]^\alpha,[\nu(\kappa+h)]^\alpha),$$

where

$\mathfrak{b}_1 = 2\ell[\rho_\imath^\alpha(\kappa+h), \nu_\imath^\alpha(\kappa+h)], \mathfrak{b}_5 = \frac{1}{(1+|\rho_\imath^\alpha(\kappa_n)|)(1+|\nu_\jmath^\alpha(\kappa_n)|)}$.

Hence the unique fuzzy solution is obtained by choosing $\mathfrak{b} \to 0$. Assuming $\rho^1 = 2$ we prove the controllability

$$\begin{aligned}
[\varsigma(\eta)]^\alpha =&[\varsigma_\imath^\alpha(\eta), \varsigma_\jmath^\alpha(\eta)]\\
=&(\mathcal{Q}_\imath^\alpha)^{-1}\bigg((\rho^1)_\imath^\alpha - \mathfrak{D}_\imath^\alpha(\mathcal{K})\bigg[\psi_\imath^\alpha(0) - \mathfrak{h}_\imath^\alpha(\rho_{\sigma_1}, \rho_{\sigma_2}, \dots, \rho_{\sigma_q})(0) - \mathfrak{z}_\imath^\alpha(0,\psi)\bigg]\\
&- \mathfrak{z}_\imath^\alpha(\mathcal{K}, \rho_{\mathcal{K}\imath}^\alpha) - \int_0^{\mathcal{K}} A_\imath^\alpha \mathfrak{D}_\imath^\alpha(\mathcal{K}-\eta)\mathfrak{z}_\imath^\alpha(\eta, \rho_{\eta\imath}^\alpha)d\eta - \int_0^{\mathcal{K}} \mathfrak{D}_\imath^\alpha(\mathcal{K}-\eta)\mathfrak{W}_\imath^\alpha(\eta, \rho_{\eta\imath}^\alpha,\\
&\int_0^\eta \mathcal{J}_\imath^\alpha(\eta,\kappa,\rho_{\kappa\imath}^\alpha)d\kappa)dy) - \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}_\imath^\alpha(\mathcal{K}-\kappa_n)\mathcal{I}_{n\imath}^\alpha(\rho_\imath^\alpha(\kappa_n^-))\bigg),\\
&(\mathcal{Q}_\jmath^\alpha)^{-1}\bigg((\rho^1)_\jmath^\alpha - \mathfrak{D}_\jmath^\alpha(\mathcal{K})\bigg[\psi_\jmath^\alpha(0) - \mathfrak{h}_\jmath^\alpha(\rho_{\sigma_1}, \rho_{\sigma_2}, \dots, \rho_{\sigma_q})(0)\\
&- \mathfrak{z}_\jmath^\alpha(0,\psi)\bigg] - \mathfrak{z}_\jmath^\alpha(\mathcal{K}, \rho_{\mathcal{K}\jmath}^\alpha) - \int_0^{\mathcal{K}} A_\jmath^\alpha \mathfrak{D}_\jmath^\alpha(\mathcal{K}-\eta)\mathfrak{z}_\jmath^\alpha(\eta, \rho_{\eta\jmath}^\alpha)d\eta\\
&- \int_0^{\mathcal{K}} \mathfrak{D}_\jmath^\alpha(\mathcal{K}-\eta)\mathfrak{W}_\jmath^\alpha(\eta, \rho_{\eta\jmath}^\alpha, \int_0^\eta \mathcal{J}_\jmath^\alpha(\eta,\kappa,\rho_{\kappa\jmath}^\alpha)d\kappa)d\eta)\\
&- \sum_{0<\kappa_n<\mathcal{K}} \mathfrak{D}_\jmath^\alpha(\mathcal{K}-\kappa_n)\mathcal{I}_{n\jmath}^\alpha(\rho_\jmath^\alpha(\kappa_n^-))\bigg).
\end{aligned}$$

Substituting the above derived values into the integral system with respect to (1.1) yields an $\alpha-$ level set of $\rho(\mathcal{K})$ as $[\rho(\mathcal{K})]^\alpha = 2 = [\rho^1]$. So the system (1.1) is controllability on $[0, \mathcal{K}]$.

6. Conclusions

We used the Banach fixed point theorem to prove the existence, uniqueness, and controllability of first-order nonlocal impulsive fuzzy neutral nonlinear integro-differential equations. Sufficient conditions were also determined to get the desired result. The future study is to extend this work to find the existence, uniqueness, and controllability of fuzzy solutions of fractional impulsive neutral functional integrodifferential differential equations with nonlocal conditions by applying a suitable fixed point theorem.

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References

[1] A. Kumar, M. Malik, and K. S. Nisar, *Existence and total controllability results of fuzzy delay differential equation with non-instantaneous impulses,* Alexandria Engineering Journal, 2021, 60(6), 6001-6012.

[2] B. Radhakrishan, P. Anukokila, and T. Sathya, *Controllability results for nonlinear impulsive fuzzy neutral integrodifferential evolution systems,* International Journal of Pure and Applied Mathematics, 2017, 114, 61-76.

[3] D. N. Chalishajar, and R. Ramesh, *Controllability for impulsive fuzzy neutral functional integrodifferential equations,* In AIP Conference Proceedings, 2019, vol. 2159 (1), p. 030007. AIP Publishing LLC.

[4] D. N. Chalishajar, R. Ramesh, S. Vengataasalam, and K. Karthikeyan, *Existence of fuzzy solutions for nonlocal impulsive neutral functional differential equations,* Journal Nonlinear Analysis and Application, 2017, 2017, 1-12.

[5] F. Acharya, V. Kushawaha, J. Panchal, and D. Chalishajar, *Controllability of fuzzy solutions for neutral impulsive functional differential equations with nonlocal conditions,* Axioms, 2021, 10(2), Article ID 84, 14 pages.

[6] G. Wang, Y. Li, and C. Wen, *On fuzzy n-cell numbers and n-dimension fuzzy vectors,* Fuzzy Sets and Systems, 2007, 158(1), 71-84.

[7] G. Kokila, and V. Govindan, *Impulsive neutral mixed integro-differential equations with infinite delay,* In AIP Conference Proceedings, 2022, vol. 2529(1) p. 020011. AIP Publishing LLC.

[8] K. Balachandran, and E. R. Anandhi, *Controllability of neutral functional integrodifferential infinite delay systems in Banach spaces,* Taiwanese Journal of Mathematics, 2004, 8(4), 687-702.

[9] M. L. Puri, D. A. Ralescu, and L. Zadeh, *Fuzzy random variables,* Readings in fuzzy sets for intelligent systems, Morgan Kaufmann, 1993, 265-271.

[10] M. Shahidi, and E. Esmi, *On the existence of approximate solutions to fuzzy delay differential equations under the metric derivative,* Computational and Applied Mathematics, 2022, 41(8), Article ID 412.

[11] M. Mazandarani and M. Najariyan, *Fuzzy differential equations: conceptual interpretations,* Evolutionary Intelligence, 2022, 1-16.

[12] N. H. M. Qumami, R. S. Jain, and B. S. Reddy, *On the existence for impulsive fuzzy nonlinear integro-differential equations with nonlocal condition,* Journal of Mathematics and Computer Science, (2024), 32(1), 13-24.

[13] N. H. M. Qumami, R. S. Jain, and B. S. Reddy, *On Impulsive Nonlocal Nonlinear Fuzzy Integro-Differential Equations in Banach Space,* Contemporary Mathematics, 2024, 4399-4413.

[14] R. S. Jain, and M. B. Dhakne, *On existence of solutions of impulsive nonlinear functional neutral integro-differential equations with nonlocal condition,* Demonstratio Mathematica, 2015, 48(3), 413-423.

[15] R. S. Jain, and M. B. Dhakne, *On global existence of solutions for abstract nonlinear functional neutral integro-differential equations with nonlocal condition,* Thai Journal of Mathematics, 2018, 16(3), 789-799.

[16] R. S. Jain, S. Reddy, and S. D. Kadam, *Approximate solutions of impulsive integro-differential equations,* Arabian Journal of Mathematics, 2018, 7, 273-279.

[17] R. Liu, J. Wang and D. O'Regan, *Existence of solutions to nonlinear impulsive fuzzy differential equations,* Filomat, 2023, 37(4), 1223-1240.

[18] S. Arora, S. Singh, J. Dabas, and M. T. Mohan, *Approximate controllability of semilinear impulsive functional differential systems with non-local conditions,* IMA Journal of Mathematical Control and Information, 2020, 37(4), 1070-1088.

[19] S. D. Kadam, B. S. Reddy, R. Menon, and R. S. Jain, *Existence and controllability of mild solution of impulsive integro-differential equations inclusions*, Nonlinear Functional Analysis and Applications, 2020, 5, 657-670.

[20] T. Gunasekar, J. Thiravidarani, M. Mahdal, P. Raghavendran, A. Venkatesan, and M. Elangovan, *Study of non-Linear impulsive neutral fuzzy delay differential equations with non-Local conditions,* Mathematics, 2023, 11(17), Article ID 3734, 16 pages.