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Thermo-Electro-Elastic Friction Problem with Modified Signorini Contact Conditions

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Abstract The purpose of this paper is to investigate a frictional contact problem between a thermo-piezoelectric body and an obstacle (such as a foundation). The thermo-piezoelectric constitutive law is assumed to be nonlinear. Modified Signorini's contact conditions are used to describe the contact, and these are adjusted to account for temperature-dependent unilateral conditions, which are associated with a nonlocal Coulomb friction law. The problem is formulated as a coupled system of displacement field, electric potential, and temperature, which is solved using a variational approach. The existence of a weak solution is established through the utilization of elliptic quasi-variational inequalities, strongly monotone operators, and the fixed point method. Finally, an iterative method is suggested to solve the coupled system, and a convergence analysis is established under appropriate conditions.

Keywords Thermo-piezoelectric body, foundation, Signorini's modified contact conditions, Coulomb friction law, variational approach, elliptic quasi-variational inequalities, fixed point, iterative method

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1. Introduction

A notable set of challenges in engineering applications and technology pertains to the interaction between a deformable piezoelectric body and a conductive foundation. Practical instances of these challenges are prevalent in various sectors, including railways, automotive, civil engineering and aeronautics, among others. During these interactions, energy dissipation due to friction is a common phenomenon, resulting in the heating of the material. Moreover, specific piezoelectric structures display a pyroelectric effect, signifying their responsiveness to temperature variations, wherein exposure to different temperatures induces the generation of electric charge or voltage. Consequently, addressing a temperature load in a piezoelectric material requires a thorough consideration of interconnected thermo-electro-mechanical

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fields.

When studying friction contact models, the coupling of piezoelectric and thermal effects can create complications. The complex physical mechanisms of forces and heat on the contact interfaces are responsible for frictional effects and heating at the contact area. The dissipation of energy or production of heat can lead to dilation, which may increase the pressure field and modify the contact conditions, leading to instability. A brief history of the classical theory of thermo-piezoelectricity has been introduced by Mindlin in [1], while the governing equations and physical laws for thermo-piezoelectric materials have been explored by Nowacki in [2]. Primarily, owing to the intrinsic coupling between mechanical, electrical and thermal fields, significant research efforts have been devoted to addressing contact problems involving both piezoelectric and thermo-piezoelectric materials. The literature on electro-mechanical and thermo-electro-mechanical properties of piezoelectric materials is very interesting and extensive. We refer to [3–16] and the references therein.

This paper focuses on a new mathematical model, different from existing works, that describes the static contact with friction between a thermo-piezoelectric body and a thermally conductive foundation. Unlike previous references, this model differs in the way it models the contact, energy equation, thermal, frictional, and conductivity conditions. The contact is modeled using a modified Signorini's condition (refer to [17, Chapter 3, p. 147] for the linear elasticity case) and a version of Coulomb's friction law with a slip-dependent coefficient of friction. The heat flux is assumed to be unilateral from the foundation to the body, so the body temperature does not exceed the temperature of the foundation on the contact part. The model demonstrates strong coupling not only in the constitutive relations but also in the equilibrium equations and boundary conditions at the contact surface. From a mathematical point of view, the resulting model is well-posed, and weak solvability is established under appropriate assumptions on the problem's data. The proof relies on an abstract result on elliptic quasi-variational inequalities and Banach's fixed point.

The article is structured as follows: Firstly, in Section 2 we present some preliminary notations, definitions, and formulas that are necessary for the rest of the paper. Then, in Section 3 we state the mechanical problem and outline the assumptions on the data, followed by the derivation of a variational model. Section 4 is devoted to proving an existence and uniqueness result, utilizing quasi-variational inequalities and Banach's fixed point theorem. Lastly, in Section 5 we introduce an iterative approach for solving the resulting variational coupled system, which converges given certain conditions.

2. Notation and preliminaries

In this section, we will provide fundamental definitions, notations, and preliminary results that will be used in the subsequent sections. For further details, we refer the reader to references [19–21]. To this end, the summation convention over repeated indices is used, and all indices take values in $1, \dots, d$. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$. We define the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$, that is $\forall u,\ v \in \mathbb{R}^d$ and $\forall \sigma,\ \tau \in \mathbb{S}^d$.

$$u \cdot v = u_i v_i, \quad \|v\| = (v \cdot v)^{\frac{1}{2}} \quad \text{and} \quad \sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}.$$

Let the open $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ denote the domain occupied by the body and Γ the boundary of Ω, which is assumed to be Lipschitz, and is divided into three disjoint measurable parts Γ_D, Γ_N and Γ_C, such that $meas(\Gamma_D) > 0$. The subsets Γ_C, Γ_a and Γ_b, where $\Gamma_D \cup \Gamma_N = \Gamma_a \cup \Gamma_b$, are relatively open with mutually disjoint closures, such that $meas(\Gamma_a) > 0$. Throughout the paper, we adopt the following notation: $u : \Omega \to \mathbb{R}^d$ for the displacement field, $\sigma : \Omega \to \mathbb{S}^d$ for the stress tensor, $D : \Omega \to \mathbb{R}^d$ for the electric displacements field, $\theta : \Omega \to \mathbb{R}$ for the temperature field and $q : \Omega \to \mathbb{R}^d$ for the heat flux. Moreover, let $\boldsymbol{\varepsilon}(u) = (\varepsilon_{ij}(u))$ denote the linearized strain tensor given by $\varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i})$ and $E(\varphi) = -\nabla\varphi$ represent the electric vector field, where $\varphi : \Omega \to \mathbb{R}$ is an electrical potential. We also denote by $\operatorname{Div}\sigma = (\sigma_{ij,j})$ and $\operatorname{div} D = (D_{j,j})$ the divergence operator for tensor and vector valued functions, respectively. If ν is the unit exterior normal on Γ, then the normal and the tangential components of the displacement vector u and the stress field σ on Γ are

$$u_\nu = u \cdot \nu, \quad u_\tau = u - u_\nu \nu; \qquad \sigma_\nu = \sigma\nu \cdot \nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu.$$

We also define the positive and the negative part of v_ν by $v_\nu^+ = \max(0, v_\nu)$, $v_\nu^- = \max(-v_\nu, 0)$, respectively.

We use standard notation for L^p and Sobolev spaces. Therefore, we consider the following real Hilbert spaces

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d, \quad \mathcal{H} = \{\, \sigma = (\sigma_{ij})\,;\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$

$$\mathcal{H}_1 = \{\sigma \in \mathcal{H}\,;\ \operatorname{Div}\sigma \in H\}, \quad \mathcal{W} = \{D = (D_i) \in L^2(\Omega)^d\,;\ \operatorname{div} D \in L^2(\Omega)\},$$

endowed with the inner products

$$(u,\, v)_H = \int_\Omega u_i\, v_i\, dx,\ (\sigma,\, \tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\, \tau_{ij}\, dx,\ (u,\, v)_{H_1} = (u,\, v)_H + (\varepsilon(u),\, \varepsilon(v))_{\mathcal{H}},$$

$$(\sigma,\, \tau)_{\mathcal{H}_1} = (\sigma,\, \tau)_{\mathcal{H}} + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_H, \quad (D, E)_{\mathcal{W}} = (D, E)_H + (\operatorname{div} D, \operatorname{div} E)_{L^2(\Omega)},$$

and the associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\mathcal{H}_1}$, $\|\cdot\|_{\mathcal{W}}$, respectively.

Let $H_\Gamma = H^{\frac{1}{2}}(\Gamma)^d$ and let $\gamma : H_1 \to H_\Gamma$ be the trace mapping. For every element $v \in H_1$, we write v for the trace γv of v on Γ and let H'_Γ be the dual of H_Γ. The Green's formula is expressed for a regular tensor field σ (assuming that σ is continuously differentiable on $\overline{\Omega}$) as follows

$$\int_\Gamma \sigma\nu \cdot \gamma v\ da = (\sigma, \varepsilon(v))_{\mathcal{H}} + (\operatorname{Div}\sigma, v)_H, \quad \forall v \in H_1, \tag{2.1}$$

where da is the surface measure element.

Let us now consider the closed subspaces V of H_1 and W of $H^1(\Omega)$ defined by

$$V = \big\{v \in H_1\,;\ v = 0 \text{ on } \Gamma_D\big\}, \quad W = \big\{\xi \in H^1(\Omega)\,;\ \xi = 0 \text{ on } \Gamma_a\big\}.$$

On the spaces V and W, we consider the inner products and the corresponding norms given by

$$(u,\, v)_V = (\varepsilon(u),\, \varepsilon(v))_{\mathcal{H}}, \qquad \|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}}, \qquad \forall u, v \in V, \tag{2.2}$$

$$(\varphi, \xi)_W = (\nabla\varphi, \nabla\xi)_H, \qquad \|\xi\|_W = \|\nabla\xi\|_H, \qquad \forall \varphi, \xi \in W. \tag{2.3}$$

Since $meas(\Gamma_D) > 0$, $meas(\Gamma_a) > 0$ and Γ is Lipschitz, Korn's and Friedrichs-Poincaré inequalities hold,

$$\|\varepsilon(v)\|_{\mathcal{H}} \geq c_K \|v\|_{H_1}, \quad \forall v \in V, \tag{2.4}$$

$$\|\nabla \xi\|_H \geq c_F \|\xi\|_{H^1(\Omega)}, \quad \forall \xi \in W, \tag{2.5}$$

where c_K and c_F are positive constants depending on Ω, Γ_D and Γ_a.

It follows from (2.2) and (2.4) that $\|\cdot\|_{H_1}$ and $\|\cdot\|_V$ are equivalent norms on V. Thus, $(V; \|\cdot\|_V)$ is a real Hilbert space. Similarly, it is straightforward from (2.5) and (2.3) that $\|\cdot\|_{H^1(\Omega)}$ and $\|\cdot\|_W$ are equivalent norms on W and thus $(W, \|\cdot\|_W)$ is a real Hilbert space. By Sobolev's trace theorem, there exist two positive constants c_0 and c_1 which depend only on Ω, Γ_C, Γ_D and Γ_a such that

$$\|v\|_{L^2(\Gamma_C)^d} \leq c_0 \|v\|_V, \quad \forall v \in V,$$
$$\|\xi\|_{L^2(\Gamma_C)} \leq c_1 \|\xi\|_W, \quad \forall \xi \in W.$$

We also introduce the following closed subspace of $H^1(\Omega)$ defined by

$$Q = \{\eta \in H^1(\Omega)\,;\ \eta = 0 \text{ on } \Gamma_D \cup \Gamma_N\}.$$

On Q we consider the inner product and the corresponding norm given by

$$(\theta, \eta)_Q = (\theta, \eta)_{H^1(\Omega)}, \qquad \|\eta\|_Q = \|\eta\|_{H^1(\Omega)}, \qquad \forall \theta, \eta \in Q.$$

Since Γ_a is on nonzero measure, it follows from the Friedrich-Poincaré inequalities that $(Q, \|.\|_Q)$ is a real Hilbert space. Moreover, using Sobolev trace theorem and $c_\eta > 0$ such that

$$\|\eta\|_{L^2(\Gamma_C)} \leq c_\eta \|\eta\|_Q, \quad \forall \eta \in Q.$$

For a regular vector field q, the following Green type formula hold

$$(q, \nabla \eta)_{L^2(\Omega)^d} + (\mathrm{div} q, \eta)_{L^2(\Omega)} = \int_\Gamma q \cdot \nu \ \eta \ da, \quad \forall \eta \in H_1.$$

Moreover, we consider the set of admissible temperature defined by

$$Q_{ad} = \{\eta \in Q\,;\ \eta - \theta_F \leqslant 0 \text{ on } \Gamma_C\},$$

where θ_F is the foundation's temperature, and denote by $\mathcal{E}$ the piezoelectric tensor, and by $\mathcal{E}^*$ is its transpose, such that

$$\mathcal{E} = (e_{ijk}), \quad \mathcal{E}^* = (e^*_{ijk}), \quad \text{where } e^*_{ijk} = e_{kij},$$

$$\mathcal{E}\sigma \cdot v = \sigma \cdot \mathcal{E}^* v, \quad \forall \sigma \in \mathbb{S}^d, v \in \mathbb{R}^d. \tag{2.6}$$

Finally, we use the space of normal traces on Γ_C defined by

$$H^{\frac{1}{2}}(\Gamma_C) = \{v_\nu \in L^2(\Gamma_C)\,;\ \exists v \in H_1,\ v_\nu = \gamma v \cdot \nu\},$$

and its dual $H^{-\frac{1}{2}}$, then $\langle \cdot, \cdot \rangle_{\Gamma_C}$ denote the duality pairing between $H^{-\frac{1}{2}}(\Gamma_C)$ and $H^{\frac{1}{2}}(\Gamma_C)$. The corresponding norms are given by

$$\|v_\nu\|_{H^{\frac{1}{2}}(\Gamma_C)} = \inf_{v \in H_1} \{\|v\|_{H_1}\,;\ v_\nu = \gamma v.\nu,\ \forall v_\nu \in H^{\frac{1}{2}}(\Gamma_C)\},$$

$$\|\sigma_\nu\|_{H^{-\frac{1}{2}}(\Gamma_C)} = \sup_{\substack{v_\nu \in H^{\frac{1}{2}}(\Gamma_C),\\ v_\nu \neq 0_{H^{\frac{1}{2}}(\Gamma_C)}}} \left\{ \frac{\langle \sigma_\nu, v_\nu \rangle_{\Gamma_C}}{\|v_\nu\|_{H^{\frac{1}{2}}(\Gamma_C)}} \right\}, \quad \forall \sigma_\nu \in H^{-\frac{1}{2}}(\Gamma_C). \tag{2.7}$$

Since for any v_ν in $H^{\frac{1}{2}}(\Gamma_C)$, there exists $v \in H_1$ and a constant $c_\nu > 0$ (see [22]) such that

$$\|v_\nu\|_{H^{\frac{1}{2}}(\Gamma_C)} \geq c_\nu \|v\|_V. \tag{2.8}$$

When no confusion is likely to appear, the sub-indices denoting the particular spaces will be omitted.

3. Problem statement and variational formulation

In the initial part of this section, we will describe a process in which a thermo-piezo-elastic body responds to a combination of factors, including volume forces, volume electric charges, surface tractions, and surface electric charges. This body is in a frictional contact with a thermally conductive foundation, resulting in temperature variations and associated deformations that lead to changes in the material parameters, which are dependent on temperature. Our interest is in modeling this evolution. We assume that the process is static and satisfies the small deformation hypothesis. The body is fixed at Γ_D, where the displacement field vanishes and the electrical potential is zero on Γ_a. Additionally, we assume that the temperature on $\Gamma_a \cup \Gamma_b$ is zero. The body experiences a volume force of density f_0, a volume electric charge of density ϕ_0, and a heat source q_0 in Ω. Surface tractions of density f_N act on Γ_N and a surface electric charge of density ϕ_b is prescribed on Γ_b. The solid is in frictional contact with a fixed foundation on Γ_C. We assume that the foundation's temperature is maintained at θ_F. Thus, the frictional contact problem in thermo-electro-elasticity is described as follows.

Problem (P). *Find a displacement field $u : \Omega \to \mathbb{R}^d$, a stress field $\sigma : \Omega \to \mathbb{S}^d$, an electric potential $\varphi : \Omega \to \mathbb{R}$, an electric displacement field $D : \Omega \to \mathbb{R}^d$, a temperature field $\theta : \Omega \to \mathbb{R}$ and the heat flux $q : \Omega \to \mathbb{R}^d$ such that*

$$\sigma = \mathfrak{F}(\varepsilon(u)) - \mathcal{E}^* E(\varphi) - \mathcal{M}\theta, \qquad \text{in } \Omega, \tag{3.1}$$

$$D = \mathcal{E}\varepsilon(u) + \boldsymbol{\beta} E(\varphi) + \mathcal{P}\theta, \qquad \text{in } \Omega, \tag{3.2}$$

$$q = -\mathcal{K}\nabla\theta, \qquad \text{in } \Omega, \tag{3.3}$$

$$\mathrm{Div}\sigma + f_0 = 0, \qquad \text{in } \Omega, \tag{3.4}$$

$$\mathrm{div}D = \phi_0, \qquad \text{in } \Omega, \tag{3.5}$$

$$\mathrm{div}q = q_0, \qquad \text{in } \Omega, \tag{3.6}$$

$$u = 0, \qquad \text{on } \Gamma_D, \tag{3.7}$$

$$\sigma\nu = f_N, \qquad \text{on } \Gamma_N, \tag{3.8}$$

$$\left.\begin{array}{l} g_1(\|u\|) \leq \sigma_\nu(u,\varphi,\theta) \leq g_2(\|u\|), \\ g_1(\|u\|) < \sigma_\nu(u,\varphi,\theta) < g_2(\|u\|) \Rightarrow u_\nu = 0, \\ \sigma_\nu = g_1(\|u\|) \Rightarrow u_\nu \geq 0, \\ \sigma_\nu = g_2(\|u\|) \Rightarrow u_\nu \leq 0, \end{array}\right\} \qquad \text{on } \Gamma_C, \tag{3.9}$$

$$q_\nu(u,\varphi,\theta) \leqslant 0,\ (\theta-\theta_F) \leqslant 0,\ q_\nu(u,\varphi,\theta)(\theta-\theta_F)=0, \quad \text{in } \Gamma_C, \tag{3.10}$$

$$\left.\begin{aligned} &\|\sigma_\tau\| \le \mu(\|u_\tau\|)|\mathrm{R}\sigma_\nu(u,\varphi,\theta)|,\\ &\|\sigma_\tau\| < \mu(\|u_\tau\|)|\sigma_\nu(u,\varphi,\theta)| \Rightarrow u_\tau = 0,\\ &\sigma_\tau = -\mu(\|u_\tau\|)|\mathrm{R}\sigma_\nu(u,\varphi,\theta)|\frac{u_\tau}{\|u_\tau\|} \Rightarrow u_\tau \neq 0, \end{aligned}\right\} \quad \text{on } \Gamma_C, \tag{3.11}$$

$$\varphi = 0, \quad \text{on } \Gamma_a, \tag{3.12}$$

$$D\cdot\nu = \phi_b, \quad \text{on } \Gamma_b, \tag{3.13}$$

$$\theta = 0, \quad \text{on } \Gamma_D \cup \Gamma_N. \tag{3.14}$$

The following is a brief overview of the equations and boundary conditions present in **Problem** (P). Equations (3.1), (3.2) and (3.3) represent the thermo-electro-elastic constitutive laws of the material in which $\mathfrak{F}$ is the nonlinear elasticity operator, $\boldsymbol{\beta}$ is the electric permittivity tensor, $\mathcal{M} = (m_{ij})$ and $\mathcal{P} = (p_i)$ are thermal expansion and pyroelectric tensors and $\mathcal{K} = (k_{ij})$ is the thermal conductivity tensor. Equations (3.4)-(3.6) represent the equilibrium equations for the stress, electric displacement, and heat flux fields. Equations (3.7)-(3.8) and (3.12)-(3.14) represent the mechanical, electrical, and thermal boundary conditions. Equation (3.9) embody Signorini's modified contact law (see [17, Chapter 3, p. 147]), wherein the thresholds g_1 and g_2 delineate critical limits for surface contact pressure, which must not be exceeded to prevent localized crushing of the material due to excessive pressure in the contact zone. The condition $g_1(\|u\|) < \sigma_\nu < g_2(\|u\|)$ signifies that as long as the normal stress (contact pressure) remains within the specified range, the contact pressure avoids surpassing the critical thresholds and consequently the normal displacement is zero $u_\nu = 0$. If the normal stress reaches the upper threshold $\sigma_\nu = g_2(\|u\|)$, which is positive, consequently a displacement normal to the contact surface results which is negative $u_\nu \le 0$. Conversely, if the contact pressure reaches the lower threshold $\sigma_\nu = g_1(\|u\|)$, which is negative, consequently a displacement normal to the contact surface results which is positive $u_\nu \ge 0$. Equation (3.10) represents Signorini type thermal conductivity conditions (see [18]). Condition (3.11) is a version of Coulomb's friction law in which μ is the coefficient of friction, and R is a regularization operator.

We will now give a variational formulation for **Problem** (P). To this end, we make the following assumptions on the data.

($\mathcal{H}_1$) The elasticity operator $\mathfrak{F} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

- (a) there exists $M_{\mathfrak{F}} > 0$ such that $\|\mathfrak{F}(x,\xi_1) - \mathfrak{F}(x,\xi_2)\| \le M_{\mathfrak{F}}\|\xi_1-\xi_2\|,\ \forall \xi_1,\xi_2 \in \mathbb{S}^d,$ a.e. $x \in \Omega$,
- (b) there exists $m_{\mathfrak{F}} > 0$ such that $(\mathfrak{F}(x,\xi_1) - \mathfrak{F}(x,\xi_2))(\xi_1-\xi_2) \ge m_{\mathfrak{F}}\|\xi_1-\xi_2\|^2,\ \forall \xi_1,\xi_2 \in \mathbb{S}^d,$ a.e.$x \in \Omega$,
- (c) the mapping $x \to \mathfrak{F}(x,\xi)$ is Lebesgue measurable on $\Omega,\ \forall \xi \in \mathbb{S}^d$,
- (d) the mapping $x \to \mathfrak{F}(x,0)$ belongs to $\mathcal{H}$.

($\mathcal{H}_2$) The piezoelectric tensor $\mathcal{E} = (e_{ijk})$, the thermal stress tensor $\mathcal{M}$ and the pyroelectric tensor $\mathcal{P}$, satisfy

$$e_{ijk} = e_{ikj} \in L^\infty(\Omega), \quad m_{ij} = m_{ji} \in L^\infty(\Omega), \quad p_i \in L^\infty(\Omega).$$

Notice that the above conditions allow us to define

$$M_{\mathcal{E}} = \sup_{ij} \|e_{ijk}\|_{L^\infty(\Omega)}, \quad M_{\mathcal{M}} = \sup_{ij} \|m_{ij}\|_{L^\infty(\Omega)}, \quad M_{\mathcal{P}} = \sup_{i} \|p_i\|_{L^\infty(\Omega)}.$$

($\mathcal{H}_3$) The electric permittivity tensor $\beta=(\beta_{ij})$ and the thermal conductivity tensor $\mathcal{K}=(k_{ij})$, satisfy: for all $\xi\in\mathbb{R}^d$, there exists $m_\beta>0$ and $m_\mathcal{K}>0$ such that

(a) $\beta_{ij}=\beta_{ji}\in L^\infty(\Omega),\quad \beta_{ij}\xi_i\xi_j\geq m_{\boldsymbol{\beta}}\|\xi\|^2$,

(b) $k_{ij}=k_{ji}\in L^\infty(\Omega),\quad k_{ij}\xi_i\xi_j\geq m_\mathcal{K}\|\xi\|^2$.

($\mathcal{H}_4$) The coefficient of friction $\mu:\Gamma_C\times\mathbb{R}_+\to\mathbb{R}_+$ satisfies

(a) The function $x\to\mu(x,w)$ is measurable on Γ_C, for all $w\in\mathbb{R}_+$,

(b) $\exists\, L_\mu>0$, such that $|\mu(.,w_1)-\mu(.,w_2)|\leq L_\mu|w_1-w_2|,\ \forall w_1,w_2\in\mathbb{R}_+$.

(c) $\exists\ \mu^*>0$, such that $\mu(x,w)\leq\mu^*,\forall w\in\mathbb{R}_+,$ a.e. $x\in\Gamma_C$.

($\mathcal{H}_5$) The functions $g_1:\Gamma_C\times\mathbb{R}_+\to\mathbb{R}_-$ and $g_2:\Gamma_C\times\mathbb{R}_+\to\mathbb{R}_+$ satisfy for $g_i,\ i=1,2$:

(a) $\exists\ M_{g_i}>0$, such that $|g_i(x,w)|\leq M_{g_i},\ \forall w\in\mathbb{R}_+,$ a.e.$x\in\Gamma_C$.

(b) $\exists\ L_{g_i}>0$, such that $|g_i(x,w_1)-g_i(x,w_2)|\leq L_{g_i}|w_1-w_2|,\ \forall w_1,w_2\in\mathbb{R}_+,$ a.e. $x\in\Gamma_C$.

(c) The mapping $x\to g_i(x,w)$ is measurable on Γ_C, for all $w\in\mathbb{R}_+$.

($\mathcal{H}_6$) The forces, tractions, charges, heat source densities, and the foundation's temperature are assumed to satisfy

$$f_0\in L^2(\Omega)^d,\quad f_N\in L^2(\Gamma_N)^d,\quad \phi_0,q_0\in L^2(\Omega),\quad \phi_b\in L^2(\Gamma_b),\quad \theta_F\in L^2(\Gamma_C).$$

($\mathcal{H}_7$) The mapping $\mathrm{R}:H'_{\Gamma_C}\to L^\infty(\Gamma_C)$ is linear and continuous with $\|R\|=c_R$.

Next, Using Riesz's representation theorem, we define the elements $f\in V$, $\Phi\in W$ and $\Theta\in Q$ by

$$(f,v)_V=\int_\Omega f_0\cdot v\ dx+\int_{\Gamma_N} f_N\cdot v\ da,\quad \forall v\in V, \tag{3.15}$$

$$(\Phi,\xi)_W=\int_\Omega \phi_0\xi\ dx-\int_{\Gamma_b}\phi_b\,\xi da,\quad \forall\xi\in W, \tag{3.16}$$

$$(\Theta,\eta)_Q=\int_\Omega q_0\,\eta da,\quad \forall\eta\in Q, \tag{3.17}$$

and letting $X=V\times W\times Q$, we define the following functional $J:X\times V\to\mathbb{R}$ given by

$$\begin{aligned}J((u,\varphi,\theta),v)=&\int_{\Gamma_C}\mu(\|u_\tau\|)|R\sigma_\nu(u,\varphi,\theta)|\ \|v_\tau\|\ da+\int_{\Gamma_C}g_2(\|u\|)v_\nu^-da\\&-\int_{\Gamma_C}g_1(\|u\|)v_\nu^+da.\end{aligned} \tag{3.18}$$

Using the above notation and a standard procedure based on Green's formulas, we can express the variational formulation of **Problem** (P) in terms of displacement, electric potential, and temperature fields as follows.

Problem (PV). *Find a displacement field $u\in V$, an electric potential $\varphi\in W$ and a temperature $\theta\in Q_{ad}$* such that

$$\begin{aligned}&(\mathfrak{F}(\varepsilon(u)),\varepsilon(v)-\varepsilon(u))_\mathcal{H}+(\mathcal{E}^*\nabla\varphi,\varepsilon(v)-\varepsilon(u))_H-(\mathcal{M}\theta,\varepsilon(v)-\varepsilon(u))_H\\&+J((u,\varphi,\theta),v)-J((u,\varphi,\theta),u)\geq(f,v-u)_V,\quad\forall v\in V,\end{aligned} \tag{3.19}$$

$$(\boldsymbol{\beta}\nabla\varphi,\nabla\xi)_H-(\mathcal{E}\varepsilon(u),\nabla\xi)_H-(\mathcal{P}\theta,\nabla\xi)_H=(\Phi,\xi)_W,\quad \forall\xi\in W, \tag{3.20}$$

$$(\mathcal{K}\nabla\theta,\nabla(\eta-\theta))_H\geqslant(\Theta,\eta-\theta)_{L^2(\Omega)},\quad \forall\eta\in Q_{ad}. \tag{3.21}$$

Our main result concerning the existence and uniqueness will be established in the following section.

4. Existence and uniqueness of solutions

We have the following existence and uniqueness result.

Theorem 4.1. *Suppose that the assumptions* $(\mathcal{H}_1)$-$(\mathcal{H}_7)$ *hold. If there exists a positive constant* L^* *such that* $L_\mu+\mu^*+L_{g_1}+L_{g_2}<L^*$*, then* **Problem** (PV) *admits a unique solution.*

The proof of Theorem 4.1 relies on well-established results in the theory of elliptic quasi-variational inequalities and utilizes fixed point arguments. The proof is conducted in multiple steps. We assume that $(\mathcal{H}_1)$-$(\mathcal{H}_7)$ hold, and then consider the product spaces $X=V\times W\times Q$ and $Y=(L^2(\Gamma_C))^d\times(L^2(\Gamma_C))^2$ equipped with the following inner products

$$\begin{aligned}\big(x,y\big)_X&=(u,v)_V+(\varphi,\xi)_W+(\theta,\eta)_Q,\\ \big(z,z'\big)_Y&=(z_1,z_1')_{L^2(\Gamma_C)^d}+(z_2,z_2')_{L^2(\Gamma_C)}+(z_3,z_3')_{L^2(\Gamma_C)},\end{aligned} \tag{4.1}$$

for all $x=(u,\varphi,\theta),y=(v,\xi,\eta)\in X,\ z=(z_1,z_2,z_3),y=(z_1',z_2',z_3')\in Y$, and the associated norms $\|\ .\ \|_X$ and $\|\ .\ \|_Y$, respectively. We define the operators $A:X\to X$ and $B:X\to X$, the functional $\widetilde{J}$ on $X\times X$, and the element $F\in X$ as follows

$$\begin{aligned}(Ax,y)_X=&(\mathfrak{F}(\varepsilon(u)),\varepsilon(v))_{\mathcal{H}}+(\boldsymbol{\beta}\nabla\varphi,\nabla\xi)_H+(\mathcal{E}^*\nabla\varphi,\varepsilon(v))_H\\ &-(\mathcal{E}\varepsilon(u),\nabla\xi)_H+(\mathcal{K}\nabla\theta,\nabla\eta)_H,\ \forall x=(u,\varphi,\theta),\ y=(v,\xi,\eta)\in X,\end{aligned} \tag{4.2}$$

$$(Bx,y)_X=-(\mathcal{M}\theta,\varepsilon(v))_H-(\mathcal{P}\theta,\nabla\xi)_H,\ \forall x=(u,\varphi,\theta),\ y=(v,\xi,\eta)\in X, \tag{4.3}$$

$$\widetilde{J}(x,y)=J((u,\varphi,\theta),v),\ \forall x=(u,\varphi,\theta),\ y=(v,\xi,\eta)\in X, \tag{4.4}$$

$$F=(f,\Phi,\Theta)\in X. \tag{4.5}$$

The first step of the proof involves establishing the following equivalence result.

Lemma 4.1. *The triplet* $x=(u,\varphi,\theta)$ *is a solution of* **Problem** (PV) *if and only if the following equation holds for all* $y=(v,\xi,\eta)\in V\times W\times Q_{ad}$

$$(Ax,y-x)_X+(Bx,y-x)_X+\widetilde{J}(x,y)-\widetilde{J}(x,x)\geq(F,y-x)_X. \tag{4.6}$$

Proof. Let $x=(u,\varphi,\theta)\in V\times W\times Q_{ad}$ be a solution to **Problem** (PV), and let $y=(v,\xi,\eta)\in V\times W\times Q_{ad}$. By choosing $(\xi-\varphi)$ as a test function in (3.20), adding the corresponding inequality to (3.19) and (3.21), and utilizing (4.2)-(4.5), we can derive (4.6). This shows the "if" direction of the lemma. Conversely, assume that $x=(u,\varphi,\theta)\in V\times W\times Q_{ad}$ satisfies the elliptic variational inequality

(4.6). By selecting $y = (v, \varphi, \theta)$ in (4.6), where v is an arbitrary element of V, we obtain (3.19). Furthermore, for any $\xi \in W$, we can choose $y = (v, \varphi + \xi, \eta)$ and $y = (v, \varphi - \xi, \eta)$ successively in (4.6) to obtain (3.20). Similarly, by taking $y = (u, \varphi, \eta)$ in (4.6), where η is an arbitrary element of Q_{ad}, we obtain (3.21). Hence, we have established the "only if" direction of the lemma. This completes the proof of Lemma 4.1. □

To solve the problem (4.6), we consider the following auxiliary one with a given $\zeta = (\zeta_1, \zeta_2) \in V \times W$.

Problem (PV$_\zeta$). *Find $x_\zeta = (u_\zeta, \varphi_\zeta, \theta_\zeta) \in V \times W \times Q_{ad}$ such that for all $y \in V \times W \times Q_{ad}$, the following holds*

$$(Ax_\zeta, y - x_\zeta)_X + \widetilde{J}(x_\zeta, y) - \widetilde{J}(x_\zeta, x_\zeta) \geq (F_\zeta, y - x_\zeta)_X, \tag{4.7}$$

where the element $F_\zeta \in X$ is defined by

$$F_\zeta = (f_\zeta, \Phi_\zeta, \Theta), \tag{4.8}$$

and the functions f_ζ and Φ_ζ are given by

$$(f_\zeta, v)_V = (f, v)_V + (\zeta_1, \varepsilon(v))_H, \quad \forall v \in V, \tag{4.9}$$

$$(\Phi_\zeta, \xi)_W = (\Phi, \xi)_W + (\zeta_2, \nabla\xi)_H, \quad \forall \xi \in W. \tag{4.10}$$

We will prove in what follows the unique solvability of the auxiliary **Problem** (PV$_\zeta$).

Lemma 4.2. *For any $\zeta \in V \times W$,* **Problem** (PV$_\zeta$) *has a unique solution given by $x_\zeta = (u_\zeta, \varphi_\zeta, \theta_\zeta) \in V \times W \times Q_{ad}$.*

Proof. The proof of Lemma 4.2 is based on standard results for quasi-variational inequalities. To establish this lemma, we prove that

(a) the operator $A : X \to X$ is strongly monotone and Lipschitz continuous;

(b) the functional $\widetilde{J}$ is proper, convex and lower semicontinuous, and there exists $\alpha > 0$ such that $\forall x_1, x_2, y_1, y_2 \in X$

$$\widetilde{J}(x_1, y_2) - \widetilde{J}(x_1, y_1) + \widetilde{J}(x_2, y_1) - \widetilde{J}(x_2, y_2) \leq \alpha \|x_1 - x_2\|_X \|y_1 - y_2\|_X. \tag{4.11}$$

Considering $x_1 = (u_1, \varphi_1, \theta_1)$ and $x_2 = (u_2, \varphi_2, \theta_2)$ in the space X, we can employ equation (4.2) along with the assumptions $(\mathcal{H}_1)$, $(\mathcal{H}_3)$, and (2.6) to obtain the following result

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_{\mathfrak{F}} \|u_1 - u_2\|_V^2 + m_\beta \|\varphi_1 - \varphi_2\|_W^2 + m_{\mathcal{K}} \|\theta_1 - \theta_2\|_Q^2.$$

Therefore, based on equation (4.1), there exists a positive constant $m_A = \min(m_{\mathfrak{F}}, m_{\boldsymbol{\beta}}, m_{\mathcal{K}})$, dependent on $\mathfrak{F}$, β, $\mathcal{K}$, Ω, and Γ_a, such that the following inequality holds

$$(Ax_1 - Ax_2, x_1 - x_2)_X \geq m_A \|x_1 - x_2\|_X^2. \tag{4.12}$$

In the same way, taking $y = Ax_1 - Ax_2$ in (4.2) and using the assumptions $(\mathcal{H}_1) - (\mathcal{H}_3)$, and (4.1), after some calculation it follows that there exists $M_A = 5\max(M_{\mathfrak{F}}, M_\beta, M_{\mathcal{K}}, M_{\mathcal{E}})$ which depends only on $\mathfrak{F}$, β, $\mathcal{K}$ and $\mathcal{E}$ such that

$$\|Ax_1 - Ax_2\|_X \leq M_A \|x_1 - x_2\|_X. \tag{4.13}$$

The result stated in (a) is now a consequence of equations (4.12) and (4.13).

We proceed to examine the properties of the functional $\widetilde{J}$, as indicated in (b). Let $x_1, x_2, y_1, y_2 \in X$ be taken from (3.18) and (4.4). We can then observe the following

$$\begin{aligned}
&\widetilde{J}(x_1, y_2) - \widetilde{J}(x_1, y_1) + \widetilde{J}(x_2, y_1) - \widetilde{J}(x_2, y_2)\\
=&\int_{\Gamma_C} \Big(\mu(\|u_{1,\tau}\|)|\mathrm{R}\sigma_\nu(u_1, \varphi_1, \theta_1)| - \mu(\|u_{2,\tau}\|)|\mathrm{R}\sigma_\nu(u_2, \varphi_2, \theta_2)|\Big)\Big(\|v_{2,\tau}\| - \|v_{1,\tau}\|\Big)da\\
&+ \int_{\Gamma_C} \Big(g_2(\|u_1\|) - g_2(\|u_2\|)\Big)\Big(v_{2,\nu}^- - v_{1,\nu}^-\Big)da\\
&+ \int_{\Gamma_C} \Big(g_1(\|u_1\|) - g_1(\|u_2\|)\Big)\Big(v_{1,\nu}^+ - v_{2,\nu}^+\Big)da\\
=&\int_{\Gamma_C} \Big((\mu(\|u_{1,\tau}\|) - \mu(\|u_{2,\tau}\|)\Big)|\mathrm{R}\sigma_\nu(u_1, \varphi_1, \theta_1)|\Big(\|v_{2,\tau}\| - \|v_{1,\tau}\|\Big)da\\
&+ \int_{\Gamma_C} \mu(\|u_{2,\tau}\|)\Big(|\mathrm{R}\sigma_\nu(u_1, \varphi_1, \theta_1)| - |\mathrm{R}\sigma_\nu(u_2, \varphi_2, \theta_2)|\Big)\Big(\|v_{2,\tau}\| - \|v_{1,\tau}\|\Big)da\\
&+ \int_{\Gamma_C} \Big(g_2(\|u_1\|) - g_2(\|u_2\|)\Big)\Big(v_{2,\nu}^- - v_{1,\nu}^-\Big)da\\
&+ \int_{\Gamma_C} \Big(g_1(\|u_1\|) - g_1(\|u_2\|)\Big)\Big(v_{1,\nu}^+ - v_{2,\nu}^+\Big)da.
\end{aligned}$$

Now, using the hypotheses $(\mathcal{H}_4)$-$(\mathcal{H}_5)$, and $(\mathcal{H}_7)$ in the previous inequality, we obtain

$$\begin{aligned}
&\widetilde{J}(x_1, y_2) - \widetilde{J}(x_1, y_1) + \widetilde{J}(x_2, y_1) - \widetilde{J}(x_2, y_2)\\
\le& L_\mu \|\mathrm{R}\sigma_\nu(u_1, \varphi_1, \theta_1)\|_{L^\infty(\Gamma_C)} \|u_{1,\tau} - u_{2,\tau}\|_{L^2(\Gamma_C)^d} \|v_{1,\tau} - v_{2,\tau}\|_{L^2(\Gamma_C)^d}\\
&+ \mu_* mes(\Gamma_C)^{\frac{1}{2}} \|\mathrm{R}\sigma_\nu(u_1, \varphi_1, \theta_1) - \mathrm{R}\sigma_\nu(u_2, \varphi_2, \theta_2)\|_{L^\infty(\Gamma_C)} \|v_{1,\tau} - v_{2,\tau}\|_{L^2(\Gamma_C)}\\
&+ L_{g_1} \|u_{1,\nu} - u_{2,\nu}\|_{L^2(\Gamma_C)} \|v_{1,\nu} - v_{2,\nu}\|_{L^2(\Gamma_C)}\\
&+ L_{g_2} \|u_{1,\nu} - u_{2,\nu}\|_{L^2(\Gamma_C)} \|v_{1,\nu} - v_{2,\nu}\|_{L^2(\Gamma_C)^d}\\
\le& \Big(L_\mu \|\mathrm{R}\sigma_\nu(u_1, \varphi_1, \theta_1)\|_{L^\infty(\Gamma_C)} + (L_{g_1} + L_{g_2})\Big) \|u_1 - u_2\|_{L^2(\Gamma_C)^d} \|v_1 - v_2\|_{L^2(\Gamma_C)^d}\\
&+ \mu_* mes(\Gamma_C)^{\frac{1}{2}} c_R \|\sigma_\nu(u_1, \varphi_1, \theta_1) - \sigma_\nu(u_2, \varphi_2, \theta_2)\|_{H^{\frac{-1}{2}}(\Gamma_C)} \|v_1 - v_2\|_{L^2(\Gamma_C)^d}.
\end{aligned}$$

Furthermore, noting that from (2.1), (2.7), (3.1) and (2.8), one has

$$\begin{aligned}
&\|\sigma_\nu(u_1, \varphi_1, \theta_1) - \sigma_\nu(u_2, \varphi_2, \theta_2)\|_{H^{-\frac{1}{2}}(\Gamma_C)}\\
=& \sup_{v \in H^{\frac{1}{2}}(\Gamma_C)} \frac{\langle \sigma_\nu(u_1, \varphi_1, \theta_1) - \sigma_\nu(u_2, \varphi_2, \theta_2), v_\nu \rangle_{\Gamma_C}}{\|v_\nu\|_{H^{\frac{1}{2}}(\Gamma_C)}}\\
=& \sup_{v \in H^{\frac{1}{2}}(\Gamma_C)} \frac{(\mathfrak{F}(\varepsilon(u_1)) - \mathfrak{F}(\varepsilon(u_2)), \varepsilon(v))_{\mathcal{H}} + (\mathcal{E}^*\nabla(\varphi_1 - \varphi_2), \varepsilon(v))_{\mathcal{H}}}{\|v_\nu\|_{H^{\frac{1}{2}}(\Gamma_C)}}
\end{aligned}$$

$$+\sup_{v\in H^{\frac{1}{2}}(\Gamma_C)}\frac{(\mathcal{M}\theta_2-\mathcal{M}\theta_1,\varepsilon(v))_{\mathcal{H}}}{\|v_\nu\|_{H^{\frac{1}{2}}(\Gamma_C)}}$$

$$\leq\frac{1}{c_\nu}\sup_{v\in V}\frac{\Big(\|\mathfrak{F}(\varepsilon(u_1))-(\mathfrak{F}(u_2))\|_{\mathcal{H}}+\|\mathcal{E}^*\nabla(\varphi_1-\varphi_2)\|_{\mathcal{H}}+\|\mathcal{M}(\theta_2-\theta_1)\|_{\mathcal{H}}\Big)\|\varepsilon(v)\|_{\mathcal{H}}}{\|v\|_V}$$

$$\leq\frac{1}{c_\nu}\sup_{v\in V}\big(M_{\mathfrak{F}}\|u_1-u_2\|_V+M_{\mathcal{E}}\|\varphi_1-\varphi_2\|_W+M_{\mathcal{M}}\|\theta_1-\theta_2\|_Q\big)\frac{\|v\|_V}{\|v\|_V}$$

$$\leq c_*\Big(\|u_1-u_2\|_V+\|\varphi_1-\varphi_2\|_W+\|\theta_1-\theta_2\|_Q\Big).$$

Hence, from this inequality, (2.3) and (4.1), it follows that

$$\begin{aligned}
&\widetilde{J}(x_1,y_2)-\widetilde{J}(x_1,y_1)+\widetilde{J}(x_2,y_1)-\widetilde{J}(x_2,y_2)\\
\leq&L_\mu c_0^2\|\mathrm{R}\sigma_\nu(u_1,\varphi_1,\theta_1)\|_{L^\infty(\Gamma_C)}\|u_1-u_2\|_V\|v_1-v_2\|_V\\
&+c_0^2(L_{g_1}+L_{g_2})\|u_1-u_2\|_V\|v_1-v_2\|_V\\
&+\mu_* mes(\Gamma_C)^{\frac{1}{2}}c_Rc_*c_0\left(\|u_1-u_2\|_V+\|\varphi_1-\varphi_2\|_W\right)\|v_1-v_2\|_V\\
&+\mu_* mes(\Gamma_C)^{\frac{1}{2}}c_Rc_*c_0\|\theta_1-\theta_2\|_Q\|v_1-v_2\|_V\\
\leq&L_\mu c_0^2\|\mathrm{R}\sigma_\nu(u_2,\varphi_2,\theta_2)\|_{L^\infty(\Gamma_C)}\|x_1-x_2\|_X\|y_1-y_2\|_X\\
&+c_0^2(L_{g_1}+L_{g_2})\|x_1-x_2\|_X\|y_1-y_2\|_X\\
&+\mu_* mes(\Gamma_C)^{\frac{1}{2}}c_Rc_*c_0\sqrt{3}\|x_1-x_2\|_X\|y_1-y_2\|_X\\
\leq&\alpha_0(L_\mu+\mu_*+L_{g_1}+L_{g_2})\|x_2-x_1\|_X\|y_2-y_1\|_X,
\end{aligned}$$

where $\alpha_0=max\left(c_0^2\|R\sigma_\nu(u_2,\varphi_2,\theta_2)\|_{L^\infty(\Gamma_C)},mes(\Gamma_C)^{\frac{1}{2}}c_Rc_*c_0\sqrt{3},c_0^2\right)$.

Consequently, we obtain with $\alpha=\alpha_0(L_\mu+\mu_*+L_{g_1}+L_{g_2})$ that

$$\widetilde{J}(x_1,y_2)-\widetilde{J}(x_1,y_1)+\widetilde{J}(x_2,y_1)-\widetilde{J}(x_2,y_2)\leq\alpha\|x_2-x_1\|_X\|y_2-y_1\|_X.$$

In conclusion, by combining the aforementioned results $(a)-(b)$, Lemma 4.1, and standard results for quasi-variational inequalities, we can deduce the existence of a unique solution for **Problem** (PV_ζ). □

To complete the proof of Theorem 4.1, we will now examine the operator $\Lambda : V\times W\to V\times W$, which is defined as follows

$$\Lambda\zeta=(\mathcal{M}\theta_\zeta,\mathcal{P}\theta_{\boldsymbol{\zeta}}),\quad\forall\zeta=(\zeta_1,\zeta_2)\in V\times W.\tag{4.14}$$

We show that this operator has a unique fixed point $\zeta=(\zeta_1,\zeta_2)\in V\times W$. To this end, let $\zeta=(\zeta_1,\zeta_2),\ \zeta'=(\zeta_1',\zeta_2')\in V\times W$. From the definition of the operator Λ, we get

$$\begin{aligned}
\|\Lambda\zeta-\Lambda\zeta'\|_{V\times W}&\leq\|\mathcal{M}\theta_\zeta-\mathcal{M}\theta_{\zeta'}\|_V+\|\mathcal{P}\theta_\zeta-\mathcal{P}\theta_{\zeta'}\|_W\\
&\leq M_{\mathcal{M}}\|\theta_\zeta-\theta_{\zeta'}\|_Q+M_{\mathcal{P}}\|\theta_\zeta-\theta_{\zeta'}\|_Q\\
&\leq 2max(M_{\mathcal{M}},M_P)\|\theta_\zeta-\theta_{\zeta'}\|_Q,
\end{aligned}$$

and using (4.1), we have

$$\|\Lambda\zeta - \Lambda\zeta'\|_{V\times W} \leq 2max(M_{\mathcal{M}}, M_{\mathcal{P}})\|x_\zeta - x_{\zeta'}\|_X. \tag{4.15}$$

On the other hand, using (4.7), (4.8), and (4.11), one has

$$\begin{aligned}&(Ax_\zeta - Ax_{\zeta'}, x_\zeta - x_{\zeta'})_X\\ &\leqslant (F_\zeta - F_{\zeta'}, x_\zeta - x_{\zeta'})_X + \widetilde{J}(x_\zeta, x_{\zeta'}) - \widetilde{J}(x_\zeta, x_\zeta) + \widetilde{J}(x_{\zeta'}, x_\zeta) - \widetilde{J}(x_{\zeta'}, x_{\zeta'})\\ &\leqslant \|\zeta - \zeta'\|_{V\times W}\|x_\zeta - x_{\zeta'}\|_X + \alpha\|x_\zeta - x_{\zeta'}\|_X^2.\end{aligned}$$

Keeping in mind (4.12), we get

$$m_A\|x_\zeta - x_{\zeta'}\|_X^2 \leqslant \|\zeta - \zeta'\|_{V\times W}\|x_\zeta - x_{\zeta'}\|_X + \alpha\|x_\zeta - x_{\zeta'}\|_X^2.$$

Therefore,

$$(m_A - \alpha)\,\|x_\zeta - x_{\zeta'}\|_X \leqslant \|\zeta - \zeta'\|_{V\times W},$$

and, taking into account (4.15), it follows that

$$\|\Lambda\zeta - \Lambda\zeta'\|_{V\times W} \leqslant \frac{2max(M_{\mathcal{M}}, M_{\mathcal{P}})}{m_A - \alpha}\|\zeta - \zeta'\|_{V\times W}.$$

This shows that for $\alpha < m_A - 2max(M_{\mathcal{M}}, M_{\mathcal{P}})$ the operator Λ is a contraction in $V\times W$. Thus, there exists a unique element $\zeta = (\zeta_1, \zeta_2) \in V\times W$ such that $\Lambda\zeta = \zeta$ and ζ is also the unique fixed point of Λ.

In conclusion, let $x_{\zeta^*} = (u_{\zeta^*}, \varphi_{\zeta^*}, \theta_{\zeta^*}) \in V\times W\times Q_{ad}$ be the solution of **Problem** (PV$_\zeta$). By setting $\zeta^* = \zeta$, based on the definition of Λ and **Problem** (PV$_\zeta$), we can conclude that x_{ζ^*} is the unique solution of **Problem** (PV).

5. Iteration method

In this section, we present an iterative method for efficiently solving the nonlinear variational **Problem** (PV). The proposed method is based on the fixed-point iteration technique, which involves a sequence of linear equations to effectively solve the derived nonlinear system of equations from the variational problems (3.19)-(3.21). The iteration method follows the procedure outlined below.

Given an initial guess $x_0 = (u_0, \varphi_0, \theta_0)$, *we define a sequence*

$x_{n+1} = (u_{n+1}, \varphi_{n+1}, \theta_{n+1}) \in X = V\times W\times Q_{ad}$, *for all* $n \in \mathbb{N}$ *recursively by*

$$\begin{aligned}&(x_{n+1}, y - x_{n+1})_X + \rho\widetilde{J}(x_n, y) - \rho\widetilde{J}(x_n, x_{n+1})\\ &\geq (x_n, y - x_{n+1})_X - \rho(Ax_n - F, y - x_{n+1})_X - \rho(Bx_n, y - x_{n+1})_X,\ \forall y \in X,\end{aligned} \tag{5.1}$$

where $\rho > 0$.

Now, we present a simple corollary that follows from the classical existence and uniqueness result for the solution of problem (5.1).

Corollary 5.1. *There exists a unique solution* $x_{n+1} \in X$, *satisfying* (5.1).

Proof. Let $x_{n+1} = z$ and denote by

$$b(z, y-z) = (x_{n+1}, y - x_{n+1})_X, \quad \phi(y) = \rho\widetilde{J}(x_n, y), \quad \phi(z) = \rho\widetilde{J}(x_n, x_{n+1}),$$
$$(G, y-z)_X = (x_n, y - x_{n+1})_X - \rho(Ax_n - F, y - x_{n+1})_X - \rho(Bx_n, y - x_{n+1})_X.$$

Using the previous notation, we deduce from (5.1) that

$$\begin{cases} \textit{Find } z \in X \textit{ such that} \\ b(z, y-z) + \phi(y) - \phi(z) \geq (G, y-z)_X, \quad \forall y \in X. \end{cases} \tag{5.2}$$

It is straightforward to observe that $b(z, y)$ is a continuous and X-elliptic bilinear form, $\phi(z)$ is a proper, convex, and lower semi-continuous function, and G is a linear and continuous functional. By applying a standard result on elliptic variational inequalities, we can deduce the existence of a unique element $z \in X$ that satisfies (5.2). □

The main convergence result we establish in this section is as follows.

Theorem 5.1. *Let x and x_{n+1} be solutions of* **Problem** (PV) *and the iterative problem* (5.1)*, respectively. Assuming that the conditions of Theorem 4.1 hold, with the same value of L^*. Then, for $0 < \rho < \dfrac{2(m_A - \beta)}{M_A^2 - \beta^2}$, the sequence (x_{n+1}) converges strongly to x in X as n tends to infinity.*

Proof. The proof of Theorem 5.1 will be presented in multiple steps. Assuming that $(\mathcal{H}_1)$-$(\mathcal{H}_7)$ hold, the first step is to establish the weak convergence of the solution x_{n+1} of (5.1) to x, the solution of **Problem** (PV). Considering x_{n+1} and x_{n+2} as two consecutive solutions of the variational inequality (5.1), we obtain

$$\begin{aligned} &(x_{n+1}, y - x_{n+1})_X + \rho\widetilde{J}(x_n, y) - \rho\widetilde{J}(x_n, x_{n+1}) \\ \geq &(x_n, y - x_{n+1})_X - \rho(Ax_n - F, y - x_{n+1})_X - \rho(Bx_n, y - x_{n+1})_X, \end{aligned} \tag{5.3}$$

$$\begin{aligned} &(x_{n+2}, y - x_{n+2})_X + \rho\widetilde{J}(x_{n+1}, y) - \rho\widetilde{J}(x_{n+1}, x_{n+2}) \\ \geq &(x_{n+1}, y - x_{n+2})_X - \rho(Ax_{n+1} - F, y - x_{n+2})_X - \rho(Bx_{n+1}, y - x_{n+2})_X. \end{aligned} \tag{5.4}$$

By substituting $y = x_{n+2}$ into (5.3), $y = x_{n+1}$ into (5.4), and subsequently adding the resulting inequalities, we obtain

$$\begin{aligned} &(x_{n+2} - x_{n+1}, x_{n+2} - x_{n+1})_X \\ \leq &\rho[\widetilde{J}(x_n, x_{n+2}) - \widetilde{J}(x_n, x_{n+1}) + \widetilde{J}(x_{n+1}, x_{n+1}) - \widetilde{J}(x_{n+1}, x_{n+2})] \\ &+ \rho(Bx_n - Bx_{n+1}, x_{n+2} - x_{n+1})_X \\ &+ [(x_{n+1} - x_n - \rho(Ax_{n+1} - Ax_n), x_{n+2} - x_{n+1})_X]. \end{aligned}$$

Thus

$$\|x_{n+2} - x_{n+1}\|_X^2 \leq \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{5.5}$$

where the quantities $\mathcal{I}_1$, $\mathcal{I}_2$, and $\mathcal{I}_3$ are given by

$$\begin{aligned} \mathcal{I}_1 &= \rho[\widetilde{J}(x_n, x_{n+2}) - \widetilde{J}(x_n, x_{n+1}) + \widetilde{J}(x_{n+1}, x_{n+1}) - \widetilde{J}(x_{n+1}, x_{n+2})], \\ \mathcal{I}_2 &= \rho(Bx_n - Bx_{n+1}, x_{n+2} - x_{n+1})_X, \\ \mathcal{I}_3 &= [(x_{n+1} - x_n - \rho(Ax_{n+1} - Ax_n), x_{n+2} - x_{n+1})_X]. \end{aligned}$$

By employing algebraic manipulations similar to those employed earlier and utilizing the Cauchy-Schwarz inequality, we derive the following expressions

$$\mathcal{I}_1 \le \rho\alpha\|x_{n+2}-x_{n+1}\|_X\|x_{n+1}-x_n\|_X, \tag{5.6}$$
$$\mathcal{I}_2 \le 2\rho max(M_{\mathcal{M}}, M_{\mathcal{P}})\|x_{n+2}-x_{n+1}\|_X\|x_{n+1}-x_n\|_X, \tag{5.7}$$
$$\mathcal{I}_3 \le \sqrt{1-2\rho m_A+\rho^2 M_A^2}\,\|x_{n+2}-x_{n+1}\|_X\|x_{n+1}-x_n\|_X. \tag{5.8}$$

Combining the previous inequalities (5.5) to (5.8), we obtain the following result

$$\|x_{n+2}-x_{n+1}\|_X \le \lambda(\rho)\|x_{n+1}-x_n\|_X, \tag{5.9}$$

where $\lambda(\rho)=\rho\big(\alpha+2max(M_{\mathcal{M}}, M_{\mathcal{P}})\big)+\sqrt{1-2\rho m_A+\rho^2 M_A^2}$.

We can deduce from the previous inequalities that

$$\|x_{n+1}-x_n\|_X \le (\lambda(\rho))^{n+1}\|x_1-x_0\|_X. \tag{5.10}$$

By choosing ρ such that $0<\rho<\dfrac{2\big(m_A-(\alpha+2max(M_{\mathcal{M}}, M_{\mathcal{P}}))\big)}{M_A^2-(\alpha+2max(M_{\mathcal{M}}, M_{\mathcal{P}}))^2}$, we ensure that $\lambda(\rho)<1$. Consequently, we can conclude that the sequence (x_n) is a Cauchy sequence. As a result, (x_n) is bounded in X (since X is a Banach space), which implies the existence of $x^* \in X$. Furthermore, there exists a subsequence, still denoted by (x_n), such that

$$x_n \rightharpoonup x^* \text{ weakly in } X \text{ as } n \longrightarrow +\infty. \tag{5.11}$$

In the second step, we will demonstrate that x^* is a solution of (PV).

Since the trace map $\gamma : X \longrightarrow L^2(\Gamma_C)^d \times L^2(\Gamma_C) \times L^2(\Gamma_C)$ is a compact operator, the weak convergence $x_n \rightharpoonup x^*$ in X implies the strong convergence $x_n \longrightarrow x^*$ in $L^2(\Gamma_C)^d \times L^2(\Gamma_C) \times L^2(\Gamma_C)$. From (5.1), we have

$$\begin{aligned}&\rho(Ax_n, y-x_{n+1})_X+\rho(Bx_n, y-x_{n+1})_X+\rho\widetilde{J}(x_n, y)-\rho\widetilde{J}(x_n, x_{n+1})\\ \ge&(x_n, y-x_{n+1})_X-(x_{n+1}, y-x_{n+1})_X+\rho(F, y-x_{n+1})_X.\end{aligned}$$

By utilizing (4.3)-(4.5), (5.11), $(\mathcal{H}_2)$, $(\mathcal{H}_5)$, and the properties of R, we can conclude that as n tends to infinity, the following holds

$$\begin{aligned}&(Bx_n, y-x_{n+1})_X \longrightarrow (Bx^*, y-x^*)_X,\\ &\widetilde{J}(x_n, y)-\widetilde{J}(x_n, x_{n+1}) \longrightarrow \widetilde{J}(x^*, y)-\widetilde{J}(x^*, x^*),\\ &(x_n, y-x_{n+1})_X \longrightarrow (x^*, y-x^*)_X,\\ &(x_{n+1}, y-x_{n+1})_X \longrightarrow (x^*, y-x^*)_X,\\ &(F, y-x_{n+1})_X \longrightarrow (F, y-x^*)_X,\end{aligned}$$

which implies that

$$\limsup_{n\longrightarrow+\infty}\rho(Ax_n, x_{n+1}-y)_X \le \rho(F, x^*-y)_X+\rho(Bx^*, y-x^*)_X+\rho\widetilde{J}(x^*, y)-\rho\widetilde{J}(x^*, x^*).$$

On the other hand, we note that for all $y \in X$, we have

$$\limsup_{n\longrightarrow+\infty}\rho(Ax_n, x_{n+1}-x^*)_X$$

$$
\begin{aligned}
&= \limsup_{n\longrightarrow+\infty} \rho(Ax_n, x_{n+1}-y)_X + \limsup_{n\longrightarrow+\infty} \rho(Ax_n, y-x^*)_X \\
&\le \limsup_{n\longrightarrow+\infty} \rho((Ax_n, x_{n+1}-y)_X + \rho\|Ax_n\|_X\|y-x^*\|_X),
\end{aligned}
$$

and, we find that

$$
\begin{aligned}
&\limsup_{n\longrightarrow+\infty} \rho(Ax_n, x_{n+1}-x^*)_X \\
\le&\rho(F, x^*-y)_X + \rho\widetilde{J}(x^*, y) - \rho\widetilde{J}(x^*, x^*) + \rho(Bx^*, y-x^*)_X \\
&+ \limsup_{n\longrightarrow+\infty} \rho(\|x_n\|_X\|y-x^*\|_X).
\end{aligned}
$$

Next, it is worth noting that $(\|Ax_n\|)$ is bounded. By substituting $y = x^*$ into the previous inequality, we obtain

$$
\limsup_{n\longrightarrow+\infty} \rho(Ax_n, x_{n+1}-x^*)_X \le 0.
$$

Also, by utilizing the pseudo-monotonicity property of the operator A, we can conclude that

$$
\rho(Ax^*, y-x^*)_X \le \liminf_{n\longrightarrow+\infty} \rho(Ax_n, x_{n+1}-y)_X.
$$

Then, we have $\forall x^* \in X$,

$$
(Ax^*, y-x^*)_X + (Bx^*, y-x^*)_X + \widetilde{J}(x^*, y) - \widetilde{J}(x^*, x^*) \ge (F, y-x^*)_X. \quad (5.12)
$$

From (5.12), we can deduce that x^* is a solution of **Problem** (PV), and due to the uniqueness of the solution for this variational inequality, we obtain $x^* = x$. Therefore, we can conclude that $x = (u, \varphi, \theta)$ is the unique weak limit in $X = V \times W \times Q_{ad}$ of any subsequence of the sequence (x_n). Consequently, the whole sequence (x_n) converges weakly to the element x.

The last step in the proof of the theorem is the following.

(i) The couple $x = (u, \varphi, \theta)$ is a solution of (PV) if and only if

$$
(Ax, y-x)_X + (Bx, y-x)_X + \widetilde{J}(x, y) - \widetilde{J}(x, x) \ge (F, y-x)_X, \ \forall y = (v, \xi, \eta) \in X. \quad (5.13)
$$

(ii) The couple $x_{n+1} = (u_{n+1}, \varphi_{n+1}, \theta_{n+1})$ is a solution of (5.1) if and only if

$$
\begin{aligned}
&(x_{n+1}, y-x_{n+1})_X + \rho\widetilde{J}(x_n, y) - \rho\widetilde{J}(x_n, x_{n+1}) \\
\ge&(x_n, y-x_{n+1})_X - \rho(Ax_n - F, y-x_{n+1})_X - \rho(Bx_n, y-x_{n+1})_X, \ \forall y \in X.
\end{aligned} \quad (5.14)
$$

By multiplying both sides of the inequality (5.13) by ρ, and then taking $y = x_{n+1}$ in (5.13) and $y = x$ in (5.14), we can add the resulting inequalities to obtain

$$
\begin{aligned}
&(x_{n+1}-x_n, x-x_{n+1})_X + \rho(Ax - Ax_n, x_{n+1}-x)_X + \rho(Bx - Bx_n, x_{n+1}-x)_X \\
&+ \rho\left[\widetilde{J}(x_n, x) - \widetilde{J}(x_n, x_{n+1}) + \widetilde{J}(x, x_{n+1}) - \widetilde{J}(x, x)\right] \ge 0.
\end{aligned} \quad (5.15)
$$

The inequality (5.15) can be rewritten as follows

$$(x_{n+1} - x, x_{n+1} - x)_X \leq \mathcal{G}_1 + \mathcal{G}_2, \tag{5.16}$$

where

$$\mathcal{G}_1 = \rho[\widetilde{J}(x_n, x) - \widetilde{J}(x_n, x_{n+1}) + \widetilde{J}(x, x_{n+1}) - \widetilde{J}(x, x) + (Bx - Bx_n, x_{n+1} - x)_X],$$
$$\mathcal{G}_2 = \big(x - x_n - \rho(Ax - Ax_n), x - x_{n+1}\big)_X.$$

Proceeding in the same way as in (5.6) and (5.7), we get

$$\mathcal{G}_1 \leq \rho(\alpha + 2max(M_{\mathcal{M}}, M_{\mathcal{P}}))\|x_{n+1} - x\|_X\|x_n - x\|_X \leq \rho\beta\|x_{n+1} - x\|_X\|x_n - x\|_X, \tag{5.17}$$

where $\beta = \alpha + 2max(M_{\mathcal{M}}, M_{\mathcal{P}})$.
Moreover, it follows from (4.12), (4.13) and Cauchy Schwartz inequality that

$$\mathcal{G}_2 \leq \sqrt{1 - 2\rho m_A + \rho^2 M_A^2}\|x_{n+1} - x\|_X\|x_n - x\|_X. \tag{5.18}$$

Then, in virtue of (5.15), (5.16) and (5.17), we get

$$\|x_{n+1} - x\|_X^2 \leq \lambda(\rho)\|x_{n+1} - x\|_X\|x_n - x\|_X, \tag{5.19}$$

where $\lambda(\rho) = \rho\beta + \sqrt{1 - 2\rho m_A + \rho^2 M_A^2}$.

Next, we use the triangular inequality to conclude that

$$\begin{aligned}\|x_{n+1} - x\|_X^2 &\leq \lambda(\rho)\|x_{n+1} - x\|_X\,(\|x_n - x_{n+1}\|_X + \|x_{n+1} - x\|_X) \\ &\leq \lambda(\rho)\|x_{n+1} - x\|_X^2 + \lambda(\rho)\|x_n - x_{n+1}\|_X\|x_{n+1} - x\|_X.\end{aligned}$$

Hence, we have

$$\|x_{n+1} - x\|_X \leq \frac{\lambda(\rho)}{1 - \lambda(\rho)}\|x_{n+1} - x_n\|_X. \tag{5.20}$$

Finally, from (5.20), letting $n \longrightarrow +\infty$, we obtain $x_n \longrightarrow x$. □

6. Conclusion

In this paper, we have presented a model addressing the static frictional contact process between a thermo-piezoelectric body and a conductive foundation. The constitutive law, incorporating thermo-electro-elastic effects, has been assumed to be nonlinear. Our approach has employed Signorini's modified contact conditions for the displacement field, with Signorini type thermal conductivity conditions. Coulomb's friction law, alongside the electrical conductivity condition, has also been considered. The existence of a unique weak solution has been established using variational inequalities and a fixed-point theorem. Additionally, we have proposed an iterative numerical solution method, and its convergence has been established. A numerical validation of the convergence results included in this method will be provided in a forthcoming paper.

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