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On Dual K-g-Bessel Sequences and K-g-Orthonormal Bases*

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Abstract In Hilbert spaces, K-g-frames are an advanced version of g-frames that enable the reconstruction of objects from the range of a bounded linear operator K. This research investigates K-g-frames in Hilbert space. Firstly, using the g-preframe operators, we characterize the dual K-g-Bessel sequence of a K-g frame. We provide additional requirements that must be met for the sum of a given K-g-frame and its dual K-g-Bessel sequence to be a K-g-frame. At the end of this paper, we present the concept of K-g-orthonormal bases and explain their link to g-orthonormal bases in Hilbert space. We also provide an alternative definition of K-g-Riesz bases using K-g-orthonormal bases. This gives a better understanding of the concept.

Keywords K-g-frames, dual K-g-Bessel sequences, K-g-orthonormal bases, K-g-Riesz bases

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1. Introduction

In 1952 [9], Duffin and Schaeffer first introduced frames for Hilbert spaces to study problems in the nonharmonic Fourier series. Nowadays, frame theory has become widely used in various fields, such as filter theory [4], signal and image processing [5], encoding and transmission [15] and so on. For further information on frame theory, please refer to the literature [1, 6, 14].

The use of frame theory in Hilbert spaces has led to the emergence of many generalized frames. In 2012, Găvruta [11] introduced K-frames to study atomic systems. Unlike general frames, K-frames are limited to the range of a specific bounded linear operator K, making them more practical and flexible. Furthermore, studying bounded linear operators offers a new research approach. In 2006, Sun [19] introduced the concepts of g-Riesz bases and g-frames. Later, Xiao et al. [22] put forward the concept of K-g-frame, which was limited to the range of a bounded linear operator in Hilbert space and has gained greater flexibility in practical application relative to g-frame (see [2, 17]). K-g-frame, as a more general frame than

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g-frame and K-frame in Hilbert space, has become one of the most active fields in frame theory in recent years. K-g-frames are a generalization of g-frames, and many of their properties are similar to those of g-frames. However, there are some differences, which have led to K-g-frames becoming one of the most active fields in frame theory in recent years. Many studies have been conducted in this area, such as [8, 16, 18, 23]. No one has discussed g-orthonormal bases of range K. Moreover, many problems of K-g-frames, such as how to find the dual, have not been studied. Since the frame operator may not be invertible, there is no classical canonical dual for a K-g-frame, which partly causes the fact that there are few results on the duals of a K-g-frame. This motivates us in this paper to examine the duals of K-g-frames in greater detail and provide characterizations of K-g-orthonormal bases, which we have defined.

This article is divided into four sections, each with its own outline. Section 2 will review essential results related to g-frames and K-g-frames for Hilbert spaces. Moving on to Section 3, we will characterize the dual K-g-Bessel sequence of a K-g-frame by using the g-preframe operators. We will also give some conditions under which the sum of a given K-g-frame and its dual K-g-Bessel sequence is a K-g-frame, taking into account the g-preframe operators. In the final section, Section 4, we will study K-g-orthonormal bases and explore their relationship with g-orthonormal bases. Additionally, we will provide an equivalent characterization of K-g-Riesz bases by K-g-orthonormal bases and discuss their properties.

Throughout this paper, $\mathcal{M}$ and $\mathcal{N}$ are separable Hilbert spaces, and I is the identity operator on $\mathcal{M}$. $\mathbb{L}$ represents a countable index set. ONB denotes the orthonormal basis. Let $\mathcal{B}(\mathcal{M},\mathcal{N})$ be the space of all the bounded linear operators from $\mathcal{M}$ to $\mathcal{N}$ and write $\mathcal{B}(\mathcal{M}) = \mathcal{B}(\mathcal{M},\mathcal{M})$. For an operator $S \in \mathcal{B}(\mathcal{M},\mathcal{N})$, let $\text{ran}S$, $\ker S$ and S^* be the range space, the nullspace and the adjoint of S, respectively. For a sequence of Hilbert spaces $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$, $\left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2}$ is defined by

$$\left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2} = \left\{ \{\xi_l\}_{l\in\mathbb{L}} : \xi_l \in \mathcal{M}_l, \sum_{l\in\mathbb{L}} \|\xi_l\|^2 < \infty \right\}.$$

2. Preliminaries

Here, we will review some key definitions and lemmas that will be required later.

Definition 2.1 ([19]). A sequence $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M},\mathcal{M}_l)\}_{l\in\mathbb{L}}$ is called a g-frame for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$, if there exist two positive constants a and b such that

$$a\|\xi\|^2 \le \sum_{l\in\mathbb{L}} \|\mathcal{D}_l \xi\|^2 \le b\|\xi\|^2, \quad \xi \in \mathcal{M}. \tag{2.1}$$

We call a and b the lower and upper g-frame bounds, respectively. If only the right hand inequality of (2.1) holds, we call $\mathcal{D}$ a g-Bessel sequence for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$ with Bessel bound b. If $a = b$, we call $\mathcal{D}$ a tight g-frame for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$, and if $a = b = 1$, we call $\mathcal{D}$ a Parseval g-frame for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$.

For a g-Bessel sequence $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M},\mathcal{M}_l)\}_{l\in\mathbb{L}}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$, $T_{\mathcal{D}} : \mathcal{M} \to \left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2}$ defines a bounded linear operator,

$$T_{\mathcal{D}}\xi = \{\mathcal{D}_l \xi\}_{l\in\mathbb{L}}, \forall \xi \in \mathcal{M}. \tag{2.2}$$

The adjoint operator $T^*_{\mathcal{D}} : \left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2} \to \mathcal{M}$ is given by

$$T^*_{\mathcal{D}}\{\xi_l\}_{l\in\mathbb{L}} = \sum_{l\in\mathbb{L}} \mathcal{D}^*_l\xi_l, \forall\{\xi_l\}_{l\in\mathbb{L}} \in \left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}. \tag{2.3}$$

The series converges unconditionally in $\mathcal{M}$. We call $T_{\mathcal{D}}$ and $T^*_{\mathcal{D}}$ the analysis operator and synthesis operator of $\mathcal{D}$. Also, the g-frame operator of $\mathcal{D}$ is defined as $S_{\mathcal{D}} = T^*_{\mathcal{D}}T_{\mathcal{D}}$.

Let $\{e_{lm}\}_{m\in\mathbb{M}_l}$ be an ONB for $\mathcal{M}_l, l \in \mathbb{L}$, where $\mathbb{M}_l$ is a subset of integers $\mathbb{Z}$. Define $\widetilde{e}_{lm} = \{\delta_{lj}e_{jm}\}_{l\in\mathbb{L}}$ for all $l \in \mathbb{L}, m \in \mathbb{M}_l$, where δ_{lj} is the Kronecker delta. Then $\{\widetilde{e}_{lm}\}_{l\in\mathbb{L},m\in\mathbb{M}_l}$ is an ONB for $\left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}$.

In [24] and [25], Y. Zhou and Y. C. Zhu studied K-g-frame in Hilbert spaces.

Definition 2.2. Let $K \in \mathcal{B}(\mathcal{M})$. A sequence $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is said to be a K-g-frame for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$ if there are constants $0 < a \le b < \infty$ such that

$$a\|K^*\xi\|^2 \le \sum_{l\in\mathbb{L}} \|\mathcal{D}_l\xi\|^2 \le b\|\xi\|^2, \quad \forall\xi \in \mathcal{M}. \tag{2.4}$$

The lower and upper bounds of a K-g-frame are denoted as a and b, respectively.

If there is no confusion, we use K-g-frame for $\mathcal{M}$ instead of K-g-frame for $\mathcal{M}$ with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$.

It can be observed that every K-g-frame acts as a g-Bessel sequence. The operations of analysis and synthesis are defined by equations (2.2) and (2.3). However, it should be noted that the frame operator $S_{\mathcal{D}}$ is not invertible on $\mathcal{M}$. Nonetheless, $S_{\mathcal{D}} : \text{ran}K \to S_{\mathcal{D}}(\text{ran}K)$ can be made invertible if the operator K has the closed range.

Definition 2.3 ([19]). A sequence $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is said to be a g-orthonormal basis (g-ONB) for $\mathcal{M}$ if it is a g-biorthonormal with itself, i.e. $\langle\mathcal{E}^*_l\eta_l, \mathcal{E}^*_j\eta_j\rangle = \delta_{lj}\langle\eta_l, \eta_j\rangle, l, j \in \mathbb{L}, \eta_l \in \mathcal{M}_l, \eta_j \in \mathcal{M}_j$, and for any $\xi \in \mathcal{M}$, one has $\sum_{l\in\mathbb{L}} \|\mathcal{E}_l\xi\|^2 = \|\xi\|^2$.

Definition 2.4 ([20]). Suppose that $K \in \mathcal{B}(\mathcal{M})$ and $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-frame for $\mathcal{M}$. A g-Bessel sequence $\Gamma = \{\Gamma_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ for $\mathcal{M}$ is said to be a dual K-g-Bessel sequence of $\mathcal{D}$ if

$$K\xi = \sum_{l\in\mathbb{L}} \mathcal{D}^*_l\Gamma_l\xi, \quad \xi \in \mathcal{M}.$$

Lemma 2.1 ([20]). *Suppose $K \in \mathcal{B}(\mathcal{M})$. Then every K-g-frame admits a dual K-g-Bessel sequence.*

Definition 2.5 ([25]). Suppose $K \in \mathcal{B}(\mathcal{M})$. A family of operators $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is called a K-g-Riesz basis for $\mathcal{M}$, if $\mathcal{E}$ is ran$K$-g-complete in $\mathcal{M}$, i.e., $\{\xi \in \mathcal{M} : \mathcal{E}_l\xi = 0, \forall l \in \mathbb{L}\} \subset (\text{ran}K)^\perp$, and there exist $a, b > 0$ such that for any finite set $\mathbb{J} \subset \mathbb{L}, \eta_l \in \mathcal{M}_l, l \in \mathbb{J}$,

$$a\sum_{l\in\mathbb{J}} \|\eta_l\|^2 \le \left\|\sum_{l\in\mathbb{J}} \mathcal{E}^*_l\eta_l\right\|^2 \le b\sum_{l\in\mathbb{J}} \|\eta_l\|^2. \tag{2.5}$$

Lemma 2.2 ([21]). *Suppose that $K \in \mathcal{B}(\mathcal{M})$ and that $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-frame for $\mathcal{M}$. Then $\mathcal{D}$ is* ran$K$*-g-complete in $\mathcal{M}$.*

Lemma 2.3 ([13]). *Suppose that $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-ONB for $\mathcal{M}$. Then $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-Bessel sequence for $\mathcal{M}$ if and only if there is a unique bounded operator $\mathcal{G} : \mathcal{M} \to \mathcal{M}$ such that $\mathcal{D}_l = \mathcal{E}_l\mathcal{G}^*$ for all $l \in \mathbb{L}$.*

The operator $\mathcal{G}$ in Lemma 2.3 is called the g-preframe operator associated with $\mathcal{D}$.

Lemma 2.4 ([13]). *Suppose that $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-ONB for $\mathcal{M}$, $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-Bessel sequence for $\mathcal{M}$, $\mathcal{G}$ and S are the g-preframe operator and g-frame operator associated with $\mathcal{D}$, respectively. Then $S = \mathcal{G}\mathcal{G}^*$.*

Lemma 2.5 ([18]). *Suppose that $K \in \mathcal{B}(\mathcal{M})$ and $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-ONB for $\mathcal{M}$. Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a g-Bessel sequence with g-preframe operator $\mathcal{G}$. Then $\mathcal{D}$ is a K-g-frame for $\mathcal{M}$ if and only if* ran$K \subseteq$ ran$\mathcal{G}$.

Lemma 2.6 ([18]). *Suppose that $K \in \mathcal{B}(\mathcal{M})$ and $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-ONB for $\mathcal{M}$. $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-frame for $\mathcal{M}$ with g-preframe operator $\mathcal{G}$. $\mathcal{F}$ is the g-preframe operator of g-Bessel sequence $\Gamma = \{\Gamma_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$. Then Γ is the dual K-g-Bessel sequence of $\mathcal{D}$ if and only if $K = \mathcal{G}\mathcal{F}^*$.*

Lemma 2.7 ([10]). *Suppose that $L, S \in \mathcal{B}(\mathcal{M})$. Then the following are equivalent:*
(1) *$S = LQ$ for some invertible operator Q on $\mathcal{M}$.*
(2) ran$L$ = ran$S$ *and* ker$L$ = ker$S$.

Lemma 2.8 ([7]). *Suppose that $L, S \in \mathcal{B}(\mathcal{M})$. Then the following are equivalent:*
(1) *$S^*S \le \lambda L^*L$ for some $\lambda > 0$.*
(2) *There exists a solution $Q \in \mathcal{B}(\mathcal{M})$ such that $S = QL$.*
(3) ran$S^* \subseteq$ ranL^*.

Recall from [3] that the linear mapping $L\mid_{(\ker L)^\perp}$ is a bijective linear transformation from $(\ker L)^\perp$ onto ran$L$. Suppose L' is the inverse of $L\mid_{(\ker L)^\perp}$. However, L' may not necessarily be continuous. Therefore, assuming $S^*S \le \lambda L^*L$, it follows that $Q_0 = SL'$ is a bounded linear mapping. Any continuous extension of Q_0 on $\mathcal{M}$ is a solution of $QL = S$. We consider the operator $\widetilde{Q}$ that coincides with Q_0 on ran$L$ and vanishes on $(\text{ran}L)^\perp$. Let us call the operator $\widetilde{Q}$ the *canonical solution* of $QL = S$.

3. Characterization of dual K-g-Bessel sequences by g-preframe operators

Let $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a g-ONB for $\mathcal{M}$ and $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-frame for $\mathcal{M}$. Then by Lemma 2.5 and Lemma 2.8, this is equivalent to $KK^* \le \lambda\mathcal{G}\mathcal{G}^*$ for some positive constant λ, where $\mathcal{G}$ is the g-preframe operator of $\mathcal{D}$. Thus $K = \mathcal{G}Q^*$ is solvable in $\mathcal{B}(\mathcal{M})$. Every solution Q induces the dual K-g-Bessel sequence $\{\mathcal{E}_lQ^*\}_{l\in\mathbb{L}}$ of $\mathcal{D}$. In fact, for any $\xi \in \mathcal{M}$,

$$\sum_{l\in\mathbb{L}} \mathcal{D}_l^*\mathcal{E}_lQ^*\xi = \sum_{l\in\mathbb{L}} \mathcal{G}\mathcal{E}_l^*\mathcal{E}_lQ^*\xi = K\xi.$$

Let $\mathcal{G}_{\widetilde{\mathcal{D}}}$ be the canonical solution of $K = \mathcal{G}Q^*$. Therefore,
(1) $\mathcal{G}_{\widetilde{\mathcal{D}}}$ maps $\mathcal{M}$ to $\overline{\text{ran}K^*}$.
(2) $\mathcal{G}_{\widetilde{\mathcal{D}}}$ coincides with $K^*(\mathcal{G}^*)'$ on $\text{ran}\mathcal{G}^*$.
(3) $\ker\mathcal{G}_{\widetilde{\mathcal{D}}} = \ker\mathcal{G}$.
For more information about $\mathcal{G}_{\widetilde{\mathcal{D}}}$ we can refer to [3].

It is worthwhile to investigate the existence of g-ONB for $\mathcal{M}$. However, in this section, we will not delve into the question of their existence and will assume that they exist for $\mathcal{M}$. In the following paragraphs, we will present some characterizations of the dual K-g-Bessel sequence along with g-preframe operations.

Theorem 3.1. *Suppose that $K \in \mathcal{B}(\mathcal{M})$ has closed range, $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-ONB for $\mathcal{M}$. Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-frame for $\mathcal{M}$ with g-preframe operator $\mathcal{G}$ and $\Gamma = \{\Gamma_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a g-Bessel sequence for $\mathcal{M}$ with g-preframe operator $\mathcal{F}$. Then Γ is the dual K-g-Bessel sequence of $\mathcal{D}$ for $\mathcal{M}$ if and only if there is a bounded linear operator $\varphi \in \mathcal{B}(\mathcal{M})$ such that $\Gamma_l = \mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*$ for any $l \in \mathbb{L}$ and $\mathcal{G}\varphi^* = 0$.*

Proof. Since $\mathcal{D}$ and Γ are both g-Bessel sequences for $\mathcal{M}$, then by Lemma 2.3,

$$\mathcal{D}_l = \mathcal{E}_l\mathcal{G}^*, \quad \Gamma_l = \mathcal{E}_l\mathcal{F}^*.$$

We first prove the necessary condition. It is obvious that $\{\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*\}_{l\in\mathbb{L}}$ is a g-Bessel sequence for $\mathcal{M}$. For any $\xi \in \mathcal{M}$,

$$\begin{aligned}\sum_{l\in\mathbb{L}} \mathcal{D}_l^*\Gamma_l\xi =& \sum_{l\in\mathbb{L}} \mathcal{D}_l^*\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*\xi \\ =& \sum_{l\in\mathbb{L}} \mathcal{D}_l^*\mathcal{E}_l\mathcal{G}_{\widetilde{\mathcal{D}}}^*\xi + \sum_{l\in\mathbb{L}} \mathcal{D}_l^*\mathcal{E}_l\varphi^*\xi \\ =& \sum_{l\in\mathbb{L}} \mathcal{G}\mathcal{E}_l^*\mathcal{E}_l\mathcal{G}_{\widetilde{\mathcal{D}}}^*\xi + \sum_{l\in\mathbb{L}} \mathcal{G}\mathcal{E}_l^*\mathcal{E}_l\varphi^*\xi \\ =& \mathcal{G}\mathcal{G}_{\widetilde{\mathcal{D}}}^*\xi + \mathcal{G}\varphi^*\xi = K\xi.\end{aligned}$$

Thus Γ is the dual K-g-Bessel sequence of $\mathcal{D}$ for $\mathcal{M}$.

Conversely, define the operator $\varphi = \mathcal{F} - \mathcal{G}_{\widetilde{\mathcal{D}}}$, where $\mathcal{F} \in \mathcal{B}(\mathcal{M})$ is the g-preframe operator associated with Γ. Then $\varphi \in \mathcal{B}(\mathcal{M})$. By Lemma 2.6, we know $\mathcal{G}\mathcal{F}^* = K$. Hence for $\xi \in \mathcal{M}$,

$$\mathcal{G}\varphi^*\xi = \mathcal{G}(\mathcal{F} - \mathcal{G}_{\widetilde{\mathcal{D}}})^*\xi = K\xi - \mathcal{G}\mathcal{G}_{\widetilde{\mathcal{D}}}^*\xi = 0.$$

Moreover,

$$\mathcal{E}_l\varphi^* = \mathcal{E}_l\mathcal{F}^* - \mathcal{E}_l\mathcal{G}_{\widetilde{\mathcal{D}}}^* = \Gamma_l - \mathcal{E}_l\mathcal{G}_{\widetilde{\mathcal{D}}}^*,$$

then

$$\Gamma_l = \mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*.$$

□

Corollary 3.1. *Suppose that $K \in \mathcal{B}(\mathcal{M})$ has closed range, $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-ONB for $\mathcal{M}$. Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-frame for $\mathcal{M}$ with g-preframe operator $\mathcal{G}$ and $\Gamma = \{\Gamma_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a g-Bessel sequence for $\mathcal{M}$ with g-preframe operator $\mathcal{F}$. Then Γ is the dual K-g-Bessel sequence of $\mathcal{D}$ for $\mathcal{M}$ if and only if there is a bounded linear operator $\varphi \in \mathcal{B}(\mathcal{M})$ such that $\mathcal{F} = \mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi$ and $\mathcal{G}\varphi^* = 0$.*

Let $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a g-ONB for $\mathcal{M}$ and $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-frame for $\mathcal{M}$. We denote $\mathcal{S}_\mathcal{D}$ by the set of all dual K-g-Bessel sequences of $\mathcal{D}$ with g-preframe operator $\mathcal{G}$. Thus, $\mathcal{S}_\mathcal{D}$ contains $\{\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}}+\varphi)^*\}_{l\in\mathbb{L}}$, where $\mathcal{G}\varphi^* = 0$.

Proposition 3.1. *Suppose that $K \in \mathcal{B}(\mathcal{M})$ has closed range and $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-ONB for $\mathcal{M}$. Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-frame for $\mathcal{M}$ with g-preframe operator $\mathcal{G}$. If dim $\ker K \leq$dim $\ker V$, then $\mathcal{S}_\mathcal{D}$ contains some g-frame for $\mathcal{M}$.*

Proof. By assumption we have that ranK^* is closed. Since dim $\ker K \leq$dim $\ker \mathcal{G}$, we can find a bounded surjective operator $\varphi : \mathcal{M} \to \ker K$ vanishing on $\ker \mathcal{G}^\perp$. Then $\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi$ is surjective. Thus, $(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*$ is a bounded below operator, i.e. there is an $a > 0$ such that for any $\xi \in \mathcal{M}$,

$$a\|\xi\|^2 \leq \|(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*\xi\|^2.$$

Then for $\xi \in \mathcal{M}$,

$$a\|\xi\|^2 \leq \sum_{l\in\mathbb{L}} \|\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*\xi\|^2 = \|(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*\xi\|^2 \leq \|\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi\|^2\|\xi\|^2,$$

which implies that $\{\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}}+\varphi)^*\}_{\ l\in\mathbb{L}}$ is a g-frame for $\mathcal{M}$. Moreover, $\mathcal{G}\varphi^* = 0$. Thus, $\{\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \varphi)^*\}_{l\in\mathbb{L}}$ is a dual K-g-Bessel sequence of $\mathcal{D}$. □

Proposition 3.2. *Suppose that $K \in \mathcal{B}(\mathcal{M})$ has closed range and $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a g-ONB for $\mathcal{M}$. Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-frame for $\mathcal{M}$ with g-preframe operator $\mathcal{G}$. If dim $\ker \mathcal{G} = \infty$, then $\mathcal{S}_\mathcal{D}$ contains a tight g-frame for $\mathcal{M}$.*

Proof. Let $\lambda I - \mathcal{G}_{\widetilde{\mathcal{D}}}\mathcal{G}^*_{\widetilde{\mathcal{D}}}$ be positive and $\mathcal{G}_1 = (\lambda I - \mathcal{G}_{\widetilde{\mathcal{D}}}\mathcal{G}^*_{\widetilde{\mathcal{D}}})^{\frac{1}{2}}$, where λ is a positive constant. By assumption dim $\ker \mathcal{G} = \infty$, we can find an isometry $\mathcal{G}_2$ on $\mathcal{M}$ whose range is $\ker \mathcal{G}$. Thus $\mathcal{G}_{\widetilde{\mathcal{D}}}\mathcal{G}_2 = 0$. For any $\xi \in \mathcal{M}$, we have

$$\begin{aligned}\sum_{l\in\mathbb{L}} \|\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \mathcal{G}_1^*\mathcal{G}_2^*)^*\xi\|^2 =& \sum_{l\in\mathbb{L}} \|\mathcal{E}_l\mathcal{G}^*_{\widetilde{\mathcal{D}}}\xi + \mathcal{E}_l\mathcal{G}_2\mathcal{G}_1\xi\|^2 \\ =&\langle \mathcal{G}_{\widetilde{\mathcal{D}}}\mathcal{G}^*_{\widetilde{\mathcal{D}}}\xi, \xi\rangle + \langle \mathcal{G}_1^*\mathcal{G}_1\xi, \xi\rangle = \lambda\|\xi\|^2,\end{aligned}$$

which implies that $\{\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \mathcal{G}_1^*\mathcal{G}_2^*)^*\}_{l\in\mathbb{L}}$ is a tight g-frame for $\mathcal{M}$. Moreover, $\mathcal{G}\mathcal{G}_2\mathcal{G}_1 = 0$. Thus, $\{\mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}} + \mathcal{G}_1^*\mathcal{G}_2^*)^*\}_{l\in\mathbb{L}}$ is a dual K-g-Bessel sequence of $\mathcal{D}$. □

Theorem 3.2. *Let $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a g-ONB for $\mathcal{M}$ and $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ and $\Gamma = \{\Gamma_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be K-g-frames for $\mathcal{M}$ with g-preframe operators $\mathcal{G}$ and $\mathcal{F}$, respectively. Suppose that Γ is a dual K-g-Bessel sequence of $\mathcal{D}$ for $\mathcal{M}$. If $\mathcal{F}\mathcal{F}^* + K + K^*$ is a positive operator, then $\mathcal{D} + \Gamma = \{\mathcal{D}_l + \Gamma_l\}_{l\in\mathbb{L}}$ is a K-g-frame for $\mathcal{M}$.*

Proof. Since Γ is a dual K-g-Bessel sequence of $\mathcal{D}$, we have $K = \mathcal{G}\mathcal{F}^*$ by Lemma 2.6. Since $\mathcal{D}$ is a K-g-frame for $\mathcal{M}$ with g-preframe operator $\mathcal{G}$, there is a $\lambda > 0$ such that $\lambda KK^* \leq \mathcal{G}\mathcal{G}^*$. It is easy to see that $\mathcal{D} + \Gamma$ is a g-Bessel sequence with g-preframe operator $\mathcal{G} + \mathcal{F}$. Then we have

$$\begin{aligned}(\mathcal{G} + \mathcal{F})(\mathcal{G} + \mathcal{F})^* &= \mathcal{G}\mathcal{G}^* + \mathcal{F}\mathcal{F}^* + \mathcal{G}\mathcal{F}^* + \mathcal{F}\mathcal{G}^* \\ &= \mathcal{G}\mathcal{G}^* + \mathcal{F}\mathcal{F}^* + (K + K^*)\end{aligned}$$

$$\geq \mathcal{G}\mathcal{G}^* \geq \lambda KK^*,$$

which implies that $\mathcal{D}+\Gamma$ is a K-g-frame for $\mathcal{M}$. □

Theorem 3.3. *Let $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M},\mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a g-ONB for $\mathcal{M}$ and $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M},\mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-frame for $\mathcal{M}$ with g-preframe operator $\mathcal{G}$. If dim $\mathrm{ran}K^* = \infty$ and $\mathrm{ran}K^* \subset \ker\mathcal{G}$, then there is a dual K-g-Bessel sequence $\Gamma = \{\Gamma_l \in \mathcal{B}(\mathcal{M},\mathcal{M}_l)\}_{l\in\mathbb{L}}$ of $\mathcal{D}$ such that $\mathcal{D}+\Gamma = \{\mathcal{D}_l+\Gamma_l\}_{l\in\mathbb{L}}$ is a K-g-frame for $\mathcal{M}$.*

Proof. Let $T = \mathcal{G}_{\widetilde{\mathcal{D}}}\mathcal{G}^*_{\widetilde{\mathcal{D}}} + (K+K^*)$. Then $-\|T\|I \leq T \leq \|T\|I$. Since dim $\mathrm{ran}K^* = \infty$, we can find an isometry $\psi : \mathcal{M} \to \overline{\mathrm{ran}K^*}$. Set $\Gamma_l = \mathcal{E}_l(\mathcal{G}_{\widetilde{\mathcal{D}}}+\varphi)^*$, where $\varphi = \sqrt{\|T\|}\psi^*$. Then Γ is a g-Bessel sequence for $\mathcal{M}$. We denote $\mathcal{F}$ the g-preframe operator associated with Γ. Then $\mathcal{F} = \mathcal{G}_{\widetilde{\mathcal{D}}}+\varphi$. It is directly checked that $\mathcal{D}+\Gamma$ is a g-Bessel sequence with g-preframe operator $\mathcal{G}+\mathcal{F}$. Since $\mathrm{ran}K^* \subset \ker\mathcal{G}$, $\mathcal{G}\varphi^* = 0$, we also have $\mathcal{G}_{\widetilde{\mathcal{D}}}\varphi^* = 0$. Thus

$$\begin{aligned}
(\mathcal{G}+\mathcal{F})(\mathcal{G}+\mathcal{F})^* =&(\mathcal{G}+\mathcal{G}_{\widetilde{\mathcal{D}}}+\varphi)(\mathcal{G}^*+\mathcal{G}^*_{\widetilde{\mathcal{D}}}+\varphi^*)\\
=&\mathcal{G}\mathcal{G}^*+\mathcal{G}\mathcal{G}^*_{\widetilde{\mathcal{D}}}+\mathcal{G}_{\widetilde{\mathcal{D}}}\mathcal{G}^*+\mathcal{G}_{\widetilde{\mathcal{D}}}\mathcal{G}^*_{\widetilde{\mathcal{D}}}+\varphi\varphi^*\\
=&\mathcal{G}\mathcal{G}^*+K+K^*+\mathcal{G}_{\widetilde{\mathcal{D}}}\mathcal{G}^*_{\widetilde{\mathcal{D}}}+\varphi\varphi\\
=&\mathcal{G}\mathcal{G}^*+T+\|T\|\psi^*\psi\\
=&\mathcal{G}\mathcal{G}^*+(T+\|T\|I) \geq \mathcal{G}\mathcal{G}^*.
\end{aligned}$$

Since $\mathcal{D}$ is a K-g-frame for $\mathcal{M}$, there is a $\lambda > 0$ such that

$$\lambda KK^* \leq \mathcal{G}\mathcal{G}^* \leq (\mathcal{G}+\mathcal{F})(\mathcal{G}+\mathcal{F})^*,$$

which implies that $\mathcal{D}+\Gamma$ is a K-g-frame for $\mathcal{M}$. □

4. Characterization of K-g-orthonormal bases

Motivated by the definition of g-ONB for $\mathcal{M}$, we give the following definition of K-g-orthonormal basis for $\mathcal{M}$.

Definition 4.1. We say that $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M},\mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-orthonormal basis (K-g-ONB) for $\mathcal{M}$, if it satisfies the following:
(1) $\mathcal{D}$ is an orthonormal system, i.e.,

$$\langle \mathcal{D}^*_l\eta_l, \mathcal{D}^*_j\eta_j\rangle = \delta_{lj}\langle \eta_l,\eta_j\rangle, l,j\in\mathbb{L}, \eta_l\in\mathcal{M}_l, \eta_j\in\mathcal{M}_j.$$

(2) For any $\xi \in \mathcal{M}$ one has

$$\sum_{l\in\mathbb{L}} \|\mathcal{D}_l\xi\|^2 = \|K^*\xi\|^2.$$

Obviously, if $K = I$, $\mathcal{D}$ is a g-ONB for $\mathcal{M}$. So the g-ONB is a special case of K-g-ONB.

Example 4.1. Let $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{M},\mathcal{M}_i)\}_{i\in\mathbb{N}}$ be a g-ONB for $\mathcal{M}$. Define the operator K on $\mathcal{M}$ by $\Lambda_iK^* = \Lambda_{i+1}$. Then $\{\Lambda_iK^*\}_{i\in\mathbb{N}}$ is a K-g-ONB for $\mathcal{M}$.

The question is whether a K-g-ONB is related to a g-ONB of the Hilbert space. Then we have the following theorem.

Theorem 4.1. *Let K be an isometry operator and $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a g-ONB for $\mathcal{M}$. Then $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-ONB for $\mathcal{M}$ if and only if there exists an isometry operator T such that $\mathcal{E}_l = \mathcal{D}_l T^*$ for any $l \in \mathbb{L}$ and* $\text{ran}T = \text{ran}K$.

Proof. Suppose that $\mathcal{D}$ is a g-ONB for $\mathcal{M}$ and T is an isometry operator such that $\text{ran}T = \text{ran}K$. Since K and T are isometry operators and $\text{ran}T = \text{ran}K$, $\ker T = \ker K = \{0\}$. We have

$$\langle T\mathcal{D}_l^*\eta_l, T\mathcal{D}_j^*\eta_j\rangle = \delta_{lj}\langle \eta_l, \eta_j\rangle, l, j \in \mathbb{L}, \eta_l \in \mathcal{M}_l, \eta_j \in \mathcal{M}_j.$$

Lemma 2.7 implies $T = KU$ for some invertible operator U and we have

$$\langle h, h\rangle = \langle Th, Th\rangle = \langle KUh, KUh\rangle = \langle Uh, Uh\rangle.$$

Then U is a unitary operator. Thus for all $\xi \in \mathcal{M}$,

$$\begin{aligned}\sum_{l\in\mathbb{L}} \|\mathcal{E}_l\xi\|^2 &= \sum_{l\in\mathbb{L}} \|\mathcal{D}_l T^*\xi\|^2 = \|T^*\xi\|^2 \\ &= \|U^*K^*\xi\|^2 = \|K^*\xi\|^2.\end{aligned}$$

Conversely assume that $\mathcal{D}$ is a g-ONB for $\mathcal{M}$ and $\mathcal{E}$ is a K-g-ONB for $\mathcal{M}$. Define T by

$$T\xi = \sum_{l\in\mathbb{L}} \mathcal{E}_l^*\mathcal{D}_l\xi, \quad \xi \in \mathcal{M}.$$

For any finite set $\mathbb{J} \subset \mathbb{L}$, we have

$$\left\|\sum_{l\in\mathbb{J}} \mathcal{E}_l^*\mathcal{D}_l\xi\right\| = \sup_{\|\eta\|=1} \left|\left\langle \sum_{l\in\mathbb{J}} \mathcal{E}_l^*\mathcal{D}_l\xi, \eta\right\rangle\right| \le \|K^*\|\|\xi\|.$$

Then T is well defined and bounded. Since $\mathcal{D}$ is a g-ONB,

$$\mathcal{D}_l\mathcal{D}_j^*\eta = \delta_{lj}\eta$$

and

$$T\mathcal{D}_j^*\eta = \sum_{l\in\mathbb{L}} \mathcal{E}_l^*\mathcal{D}_l\mathcal{D}_j^*\eta = \mathcal{E}_j^*\eta$$

for all $\eta \in \mathcal{M}_j, j \in \mathbb{L}$. Thus $\mathcal{E}_l = \mathcal{D}_l T^*, l \in \mathbb{L}$. For any $\xi \in \mathcal{M}$,

$$\begin{aligned}\|T\xi\|^2 =& \langle T\xi, T\xi\rangle = \left\langle \sum_{l\in\mathbb{L}} \mathcal{E}_l^*\mathcal{D}_l f, \sum_{j\in\mathbb{L}} \mathcal{E}_j^*\mathcal{D}_j\xi\right\rangle \\ =& \left\langle \sum_{l\in\mathbb{L}} \mathcal{D}_l\xi, \mathcal{D}_l\xi\right\rangle = \sum_{l\in\mathbb{L}} \|\mathcal{D}_l\xi\|^2 = \|\xi\|^2.\end{aligned}$$

We obtain that T is an isometry. However,

$$\begin{aligned}\|T^*\xi\|^2 &= \langle T^*\xi, T^*\xi\rangle = \left\langle \sum_{l\in\mathbb{L}} \mathcal{D}_l^*\mathcal{E}_l\xi, \sum_{j\in\mathbb{L}} \mathcal{D}_j^*\mathcal{E}_j\xi\right\rangle \\ &= \sum_{l\in\mathbb{L}} \langle \mathcal{E}_l\xi, \mathcal{E}_l\xi\rangle = \sum_{l\in\mathbb{L}} \|\mathcal{E}_l\xi\|^2 = \|K^*\xi\|^2.\end{aligned}$$

Then we have $\ker T^* = \ker K^*$ and thus $\text{ran}T = \text{ran}K$. □

Theorem 4.2. *Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-ONB for $\mathcal{M}$. Then $\{\mathcal{D}_l K\}_{l\in\mathbb{L}}$ is a dual K-g-Bessel sequence of $\mathcal{D}$.*

Proof. Since $\mathcal{D}$ is a K-g-ONB for $\mathcal{M}$. By Lemma 2.1, there is a g-Bessel sequence $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ such that

$$K\xi = \sum_{l\in\mathbb{L}} \mathcal{D}_l^* \mathcal{E}_l \xi, \ \forall \xi \in \mathcal{M}.$$

Since $\mathcal{D}$ is an orthonormal system, for any $\xi \in \mathcal{M}, \eta_j \in \mathcal{M}_j, \forall j \in \mathbb{L}$, we have

$$\begin{aligned}\langle K\xi, \mathcal{D}_j^* \eta_j\rangle &= \langle \xi, K^* \mathcal{D}_j^* \eta_j\rangle = \left\langle \sum_{l\in\mathbb{L}} \mathcal{D}_l^* \mathcal{E}_l \xi, \mathcal{D}_j^* \eta_j \right\rangle \\ &= \left\langle \mathcal{E}_l \xi, \sum_{l\in\mathbb{L}} \mathcal{D}_l \mathcal{D}_j^* \eta_j \right\rangle = \langle \mathcal{E}_j \xi, \eta_j\rangle = \langle \xi, \mathcal{E}_j^* \eta_j\rangle.\end{aligned}$$

Then $K^* \mathcal{D}_j^* = \mathcal{E}_j^*, \forall j \in \mathbb{L}$. Therefore, $\{\mathcal{D}_j K\}_{j\in\mathbb{L}} = \{\mathcal{E}_j\}_{j\in\mathbb{L}}$ is a dual K-g-Bessel sequence of $\mathcal{D}$. □

Corollary 4.1. *Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-ONB for $\mathcal{M}$. Then the dual K-g-Bessel sequence $\{\mathcal{D}_l K\}_{l\in\mathbb{L}}$ of $\mathcal{D}$ is a K^*-g-ONB for $\mathcal{M}$ if and only if K is co-isometry.*

Corollary 4.2. *Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-ONB for $\mathcal{M}$ and let $T : \mathcal{M} \to \mathcal{M}$ be a closed range operator such that $TK = KT$ and $\mathrm{ran}K^* \subset \mathrm{ran}T$. Then $\mathcal{D}_T = \{\mathcal{D}_l T^*\}_{l\in\mathbb{L}}$ is a K-g-frame for $\mathcal{M}$.*

Theorem 4.3. *Let $K \in \mathcal{B}(\mathcal{M})$ and $\{e_{lm}\}_{m\in\mathbb{M}_l}$ be an ONB for $\mathcal{M}_l$ for each $l \in \mathbb{L}$. Then $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-ONB for $\mathcal{M}$ if and only if there exists an isometry operator $h : \left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2} \to \mathcal{M}$ such that $h\widetilde{e}_{lm} = \mathcal{D}_l^* e_{lm}$ and for all $\zeta \in \mathcal{M}$, $\|K^*\zeta\| = \|h^*\zeta\|$, where $\{\widetilde{e}_{lm}\}_{l\in\mathbb{L}, m\in\mathbb{M}_l}$ is an ONB for $\left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2}$.*

Proof. Assume that $\mathcal{D}$ is a K-g-ONB for $\mathcal{M}$. Then

$$\sum_{l\in\mathbb{L}} \|\mathcal{D}_l \xi\|^2 = \|K^*\xi\|^2, \ \xi \in \mathcal{M}.$$

Define $L : \mathcal{M} \to \left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2}$ by $L\xi = \{\mathcal{D}_l \xi\}_{l\in\mathbb{L}}$. Then

$$\langle L^* \widetilde{e}_{lm}, \xi\rangle = \langle \widetilde{e}_{lm}, \{\mathcal{D}_l \xi\}_{l\in\mathbb{L}}\rangle = \langle e_{lm}, \mathcal{D}_l \xi\rangle = \langle \mathcal{D}_l^* e_{lm}, \xi\rangle.$$

Consider $h = L^*$. Then for any $\zeta \in \mathcal{M}$,

$$\|K^*\zeta\|^2 = \sum_{l\in\mathbb{L}} \|\mathcal{D}_l \zeta\|^2 = \|h^*\zeta\|^2.$$

For any $\{\eta_l\}_{l\in\mathbb{L}}, \{\zeta_l\}_{l\in\mathbb{L}} \in \left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2}$,

$$\begin{aligned}\langle h\{\eta_l\}_{l\in\mathbb{L}}, h\{\zeta_l\}_{l\in\mathbb{L}}\rangle &= \left\langle \sum_{l\in\mathbb{L}} \mathcal{D}_l^* \eta_l, \sum_{j\in\mathbb{L}} \mathcal{D}_j^* \zeta_j \right\rangle \\ &= \sum_{l\in\mathbb{L}} \langle \mathcal{D}_l^* \eta_l, \mathcal{D}_l^* \zeta_l\rangle = \sum_{l\in\mathbb{L}} \langle \eta_l, \zeta_l\rangle\end{aligned}$$

$$= \langle \{\eta_l\}_{l\in\mathbb{L}}, \{\zeta_l\}_{l\in\mathbb{L}} \rangle.$$

We obtain that h is an isometry.

Conversely, let $h : \left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2} \to \mathcal{M}$ such that $h\widetilde{e}_{lm} = \mathcal{D}_l^* e_{lm}$ is an isometry and for all $\xi \in \mathcal{M}$, $\|K^*\xi\| = \|h^*\xi\|$. Then

$$\begin{aligned}\sum_{l\in\mathbb{L}} \|\mathcal{D}_l\xi\|^2 &= \sum_{l\in\mathbb{L}} \sum_{m\in\mathbb{M}_l} |\ \langle \mathcal{D}_l\xi, e_{lm}\rangle\ |^2 = \sum_{l\in\mathbb{L}} \sum_{m\in\mathbb{M}_l} |\ \langle \xi, \mathcal{D}_l^* e_{lm}\rangle\ |^2 \\ &= \sum_{l\in\mathbb{L}} \sum_{m\in\mathbb{M}_l} |\ \langle \xi, h\widetilde{e}_{lm}\rangle\ |^2 = \sum_{l\in\mathbb{L}} \sum_{m\in\mathbb{M}_l} |\ \langle h^*\xi, \widetilde{e}_{lm}\rangle\ |^2 \\ &= \|h^*\xi\|^2 = \|K^*\xi\|^2.\end{aligned}$$

On the other hand, we need to show that $\mathcal{D}$ is an orthonormal system. For any $l_1 \neq l_2, l_1, l_2 \in \mathbb{L}, g_{l_1} \in \mathcal{M}_{l_1}, g_{l_2} \in \mathcal{M}_{l_2}$,

$$\begin{aligned}\langle \mathcal{D}_{l_1}^* g_{l_1}, \mathcal{D}_{l_2}^* g_{l_2}\rangle &= \left\langle \sum_{m_1\in\mathbb{M}_{l_1}} \langle g_{l_1}, e_{l_1 m_1}\rangle \mathcal{D}_{l_1}^* e_{l_1 m_1}, \sum_{m_2\in\mathbb{M}_{l_2}} \langle g_{l_2}, e_{l_2 m_2}\rangle \mathcal{D}_{l_2}^* e_{l_2 m_2} \right\rangle \\ &= \left\langle \sum_{m_1\in\mathbb{M}_{l_1}} \langle g_{l_1}, e_{l_1 m_1}\rangle h\widetilde{e}_{l_1 m_1}, \sum_{m_2\in\mathbb{M}_{l_2}} \langle g_{l_2}, e_{l_2 m_2}\rangle h\widetilde{e}_{l_2 m_2} \right\rangle \\ &= \left\langle \sum_{m_1\in\mathbb{M}_{l_1}} \langle g_{l_1}, e_{l_1 m_1}\rangle \widetilde{e}_{l_1 m_1}, \sum_{m_2\in\mathbb{M}_{l_2}} \langle g_{l_2}, e_{l_2 m_2}\rangle \widetilde{e}_{l_2 m_2} \right\rangle = 0,\end{aligned}$$

and for $g_1, g_2 \in \mathcal{M}_l$,

$$\begin{aligned}\langle \mathcal{D}_l^* g_1, \mathcal{D}_l^* g_2\rangle &= \left\langle \sum_{m_1\in\mathbb{M}_l} \langle g_1, e_{lm_1}\rangle \mathcal{D}_l^* e_{lm_1}, \sum_{m_2\in\mathbb{M}_l} \langle g_2, e_{lm_2}\rangle \mathcal{D}_l^* e_{lm_2} \right\rangle \\ &= \left\langle \sum_{m_1\in\mathbb{M}_l} \langle g_1, e_{lm_1}\rangle \widetilde{e}_{lm_1}, \sum_{m_2\in\mathbb{M}_l} \langle g_2, e_{lm_2}\rangle \widetilde{e}_{lm_2} \right\rangle = \langle g_1, g_2\rangle.\end{aligned}$$

Then $\mathcal{D}$ is a K-g-ONB for $\mathcal{M}$. □

The following theorem presents an intriguing relationship between K-g-frames and K-g-ONBs for $\mathcal{M}$.

Theorem 4.4. *Let $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-ONB for $\mathcal{M}$. If $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-frame for $\mathcal{M}$, then there is an operator $T : \mathcal{M} \to \mathcal{M}$ such that* $\mathrm{ran}K \subset \mathrm{ran}T$ *and $\mathcal{E}_l = \mathcal{D}_l T^*, l \in \mathbb{L}$.*

Proof. Suppose that $\mathcal{E}$ is a K-g-frame with analysis operator $T_{\mathcal{E}}$. Then $\mathrm{ran}K \subset \mathrm{ran}T_{\mathcal{E}}^*$ by Lemma 2.8, and $T_{\mathcal{E}}^*\widetilde{e}_{lm} = \mathcal{E}_l^* e_{lm}$, where $\{\widetilde{e}_{lm}\}_{l\in\mathbb{L},m\in\mathbb{M}_l}$ is an ONB for $\left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2}$ and $\{e_{lm}\}_{m\in\mathbb{M}_l}$ is an ONB for $\mathcal{M}_l$ for each $l \in \mathbb{L}$. By Theorem 4.3, there exists an isometry operator $h : \left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2} \to \mathcal{M}$ such that $h\widetilde{e}_{lm} = \mathcal{D}_l^* e_{lm}$ and for all $\xi \in \mathcal{M}$, $\|K^*\xi\| = \|h^*\xi\|$. For any $\xi \in \mathcal{M}$, we have

$$\begin{aligned}\langle T_{\mathcal{E}}^* h^* \mathcal{D}_l^* e_{lm}, \xi\rangle &= \langle \mathcal{D}_l^* e_{lm}, hT_{\mathcal{E}}\xi\rangle = \langle h\widetilde{e}_{lm}, hT_{\mathcal{E}}\xi\rangle \\ &= \langle \widetilde{e}_{lm}, T_{\mathcal{E}}\xi\rangle = \langle T_{\mathcal{E}}^*\widetilde{e}_{lm}, \xi\rangle = \langle \mathcal{E}_l^* e_{lm}, \xi\rangle.\end{aligned}$$

So $\mathcal{E}_l^* e_{lm} = T_{\mathcal{E}}^* h^* \mathcal{D}_l^* e_{lm}$. Since $\{e_{lm}\}_{m\in\mathbb{M}_l}$ is an ONB for $\mathcal{M}_l$ for each $l \in \mathbb{L}$, we have $\mathcal{E}_l^* = T_{\mathcal{E}}^* h^* \mathcal{D}_l^*$ for each $l \in \mathbb{L}$. Then $\mathcal{E}_l = \mathcal{D}_l h T_{\mathcal{E}}$ for each $l \in \mathbb{L}$. Let $T = T_{\mathcal{E}}^* h^*$. Since h is an isometry, h^* is surjective. Then we have ran$K \subset$ ran$T$. □

In the following, we present new findings about K-g-Riesz bases using K-g-ONBs in Hilbert spaces. The following theorem establishes that if K is an operator in $\mathcal{B}(\mathcal{M})$ with a closed range, then $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-ONB for $\mathcal{M}$, and $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-Riesz basis for $\mathcal{M}$ if and only if there exists a bounded linear operator T on $\mathcal{M}$ such that $\mathcal{E}_l = \mathcal{D}_l T^*$ for all $l \in \mathbb{L}$, and $T : \text{ran}K \to \text{ran}K$ is a bijective operator.

Theorem 4.5. *Suppose that $K \in \mathcal{B}(\mathcal{M})$ has the closed range, and $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-ONB for $\mathcal{M}$. Then $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-Riesz basis for $\mathcal{M}$ if and only if there is an operator $T \in \mathcal{B}(\mathcal{M})$ such that $\mathcal{E}_l = \mathcal{D}_l T^*$ for each $l \in \mathbb{L}$, where T is a bounded operator on $\mathcal{M}$ which satisfies that $T : \text{ran}K \to \text{ran}K$ is a bijective operator.*

Proof. Suppose that $\mathcal{E}_l = \mathcal{D}_l T^*$ for each $l \in \mathbb{L}$, where $T \in \mathcal{B}(\mathcal{M})$ such that $T : \text{ran}K \to \text{ran}K$ is a bijective operator and $\mathcal{D}$ is a K-g-ONB. If $\mathcal{E}_l\xi = \mathcal{D}_l T^*\xi = 0$, then

$$\|K^*T^*\xi\|^2 = \sum_{l\in\mathbb{L}} \|\mathcal{D}_l T^*\xi\|^2 = 0.$$

Then $K^*T^*\xi = 0$, which implies $\xi \in \ker K^*T^* = (\text{ran}TK)^\perp$. Since $T : \text{ran}K \to \text{ran}K$ is a bounded bijective operator, $\text{ran}TK = \text{ran}K$. Therefore, $\xi \in (\text{ranK})^\perp$, i.e., $\mathcal{E}$ is ran$K$-g-complete. For any finite set $\mathbb{J} \subset \mathbb{L}, \eta_l \in \mathcal{M}_l, l \in \mathbb{J}$,

$$\left\|\sum_{l\in\mathbb{J}} \mathcal{E}_l^*\eta_l\right\|^2 = \left\|T\left(\sum_{l\in\mathbb{J}} \mathcal{D}_l^*\eta_l\right)\right\|^2 \le \|T\|^2 \left\|\sum_{l\in\mathbb{J}} \mathcal{D}_l^*\eta_l\right\|^2 = \|T\|^2 \sum_{l\in\mathbb{J}} \|\eta_l\|^2.$$

Since $\mathcal{D}$ is a K-g-ONB, ranK $=$ ran$T_{\mathcal{D}}^*$ by Lemma 2.8. For any finite set $\mathbb{J} \subset \mathbb{L}, \eta_l \in \mathcal{M}_l, l \in \mathbb{J}$, $\sum_{l\in\mathbb{J}} \mathcal{D}_l^*\eta_l \in \text{ran}T_{\mathcal{D}}^* = \text{ranK}$. Let $T_1 = T\mid_{\text{ran}K}$. Then there is a bounded invertible operator T_1^{-1} on ranK such that

$$\left\|\sum_{l\in\mathbb{J}} \mathcal{D}_l^*\eta_l\right\|^2 = \left\|T_1^{-1}T_1\left(\sum_{l\in\mathbb{J}} \mathcal{D}_l^*\eta_l\right)\right\|^2 \le \|T_1^{-1}\|^2 \left\|\sum_{l\in\mathbb{J}} \mathcal{E}_l^*\eta_l\right\|^2.$$

From this we deduce that

$$\|T_1^{-1}\|^{-2} \sum_{l\in\mathbb{J}} \|\eta_l\|^2 \le \left\|\sum_{l\in\mathbb{J}} \mathcal{E}_l^*\eta_l\right\|^2 \le \|T\|^2 \sum_{l\in\mathbb{J}} \|\eta_l\|^2.$$

Then $\mathcal{E}$ is a K-g-Riesz basis for $\mathcal{M}$.

Conversely, suppose that $\mathcal{E}$ is a K-g-Riesz basis for $\mathcal{M}$. The right-hand inequality in (2.5) implies that $\mathcal{E}$ is a g-Bessel sequence with bound b. Let $T_{\mathcal{E}}$ be the analysis operator of $\mathcal{E}$. Then by (2.5), $T_{\mathcal{E}}^*$ satisfies

$$a\|\eta\|^2 \le \|T_{\mathcal{E}}^*\eta\| \le b\|\eta\|^2, \quad \eta \in \left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}.$$

Then $\ker T_{\mathcal{E}}^* = \{0\}$. Let $\{\eta_l\}_{l\in\mathbb{L}} \subset \operatorname{ran}T_{\mathcal{E}}^*$ such that $\lim_{l\to\infty}\eta_l = \eta \in \mathcal{M}$. Then $\{\eta_l\}_{l\in\mathbb{L}}$ is a Cauchy sequence in $\mathcal{M}$ and there is $\{\xi_l\}_{l\in\mathbb{L}} \in \left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}$ such that $\eta_l = T_{\mathcal{E}}^*\xi_l$. For any $l, j \in \mathbb{L}$,

$$\|\xi_{l+j} - \xi_l\|^2 \leq a^{-1}\|T_{\mathcal{E}}^*(\xi_{l+j} - \xi_l)\|^2 = a^{-1}\|\eta_{l+j} - \eta_l\|^2.$$

Then $\{\xi_l\}_{l\in\mathbb{L}}$ is a Cauchy sequence in $\left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}$. Therefore there is a $\xi \in \left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}$ such that $\xi_l \to \xi$ and

$$\eta = lim_{l\to\infty}\eta_l = lim_{l\to\infty}T_{\mathcal{E}}^*\xi_l = T_{\mathcal{E}}^*\xi.$$

Hence $T_{\mathcal{E}}^*$ has closed range. Define an operator $T : \mathcal{M} \to \mathcal{M}$ by

$$T\xi = \sum_{l\in\mathbb{L}} \mathcal{E}_l^*\mathcal{D}_l\xi, \quad \xi \in \mathcal{M}.$$

For any finite set $\mathbb{J} \subset \mathbb{L}$, we have

$$\left\|\sum_{l\in\mathbb{J}} \mathcal{E}_l^*\mathcal{D}_l\xi\right\| = \sup_{\|\eta\|=1} \mid \langle\sum_{l\in\mathbb{J}} \mathcal{E}_l^*\mathcal{D}_l\xi, \eta\rangle \mid \leq \|K^*\|\|\xi\|.$$

Then T is well defined and bounded. Since $\mathcal{D}$ is a K-g-ONB,

$$\mathcal{D}_l\mathcal{D}_j^*\eta = \delta_{lj}\eta$$

and

$$T\mathcal{D}_j^*\eta = \sum_{l\in\mathbb{L}} \mathcal{E}_l^*\mathcal{D}_l\mathcal{D}_j^*\eta = \mathcal{E}_j^*\eta$$

for all $\eta \in \mathcal{M}_j, j \in \mathbb{L}$. Thus $\mathcal{E}_l = \mathcal{D}_lT^*, l \in \mathbb{L}$. Let $\xi \in (\operatorname{ran}T_{\mathcal{E}}^*)^\perp$. For any $\{\eta_l\}_{l\in\mathbb{L}} \in \left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}$,

$$0 = \langle\xi, T_{\mathcal{E}}^*(\{\eta_l\}_{l\in\mathbb{L}})\rangle = \langle T_{\mathcal{E}}\xi, \{\eta_l\}_{l\in\mathbb{L}}\rangle.$$

Then $T_{\mathcal{E}}\xi = 0$, i.e., $\|T_{\mathcal{E}}\xi\|^2 = \sum_{l\in\mathbb{L}} \|\mathcal{E}_l\xi\|^2 = 0$, which implies $\mathcal{E}_l\xi = 0, \forall l \in \mathbb{L}$. Since $\mathcal{E}$ is $\operatorname{ran}K$-g-complete, $\xi \in \operatorname{ran}K^\perp$, then $\operatorname{ran}K \subset \operatorname{ran}T_{\mathcal{E}}^*$. Let $\xi \in \operatorname{ran}K \subset \operatorname{ran}T_{\mathcal{E}}^*$. There is $\{\eta_l\}_{l\in\mathbb{L}} \in \left(\sum_{l\in\mathbb{L}} \oplus\mathcal{M}_l\right)_{l^2}$ such that $T_{\mathcal{E}}^*(\{\eta_l\}_{l\in\mathbb{L}}) = \xi$. Since $\mathcal{D}$ is a K-g-ONB, ranK $= \operatorname{ran}T_{\mathcal{D}}^*$. Let $\eta = T_{\mathcal{D}}^*(\{\eta_l\}_{l\in\mathbb{L}}) \in$ ranK. Then

$$T\eta = T\sum_{l\in\mathbb{L}} \mathcal{D}_l^*\eta_l = \sum_{l\in\mathbb{L}} \mathcal{E}_l^*\eta_l = \xi.$$

Therefore, $T : \operatorname{ran}K \to \operatorname{ran}K$ is a surjective operator. Assume

$$0 = T_{\mathcal{E}}^*(\{\eta_l\}_{l\in\mathbb{L}}) = \sum_{l\in\mathbb{L}} \mathcal{E}_l^*\eta_l = T\left(\sum_{l\in\mathbb{L}} \mathcal{D}_l^*\eta_l\right).$$

Since $\ker T_{\mathcal{E}}^* = \{0\}$, $\eta_l = 0$. We obtain that T is injective on $\operatorname{ran}K$. □

As explained in [12], a sequence $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is said to be *g-linearly independent with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$ if for any $\sum_{l\in\mathbb{L}} \mathcal{D}_l^*\eta_l = 0$, we have $\eta_l = 0$ for all $l \in \mathbb{L}$, where $\eta_l \in \mathcal{M}_l$. Moreover, the following theorem provides an equivalent characterization of K-g-Riesz bases using K-g-ONBs in Hilbert spaces.*

Theorem 4.6. *Suppose that $\mathcal{D} = \{\mathcal{D}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ is a K-g-ONB for $\mathcal{M}$, where $K \in \mathcal{B}(\mathcal{M})$ has closed range. Let $\mathcal{E} = \{\mathcal{E}_l \in \mathcal{B}(\mathcal{M}, \mathcal{M}_l)\}_{l\in\mathbb{L}}$ be a K-g-frame for $\mathcal{M}$. Then the following statements are equivalent:*
(1) *$\mathcal{E}$ is g-linearly independent with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$.*
(2) *$\mathcal{E}$ is a K-g-Riesz basis for $\mathcal{M}$.*

Proof. Suppose that $\mathcal{E}$ is a K-g-Riesz basis for $\mathcal{M}$. By Theorem 4.5, let $\mathcal{E}_l = \mathcal{D}_l T^*, l \in \mathbb{L}$, where $T \in \mathcal{B}(\mathcal{M})$ such that $T : \text{ran}K \to \text{ran}K$ is a bijective operator and $\mathcal{D}$ is a K-g-ONB for $\mathcal{M}$. Suppose that $0 = \sum_{l\in\mathbb{L}} \mathcal{E}_l^* \eta_l$, where $\eta_l \in \mathcal{M}_l, l \in \mathbb{L}$. Then

$$0 = \sum_{l\in\mathbb{L}} \mathcal{E}_l^* \eta_l = \sum_{l\in\mathbb{L}} T\mathcal{D}_l^* \eta_l = T\left(\sum_{l\in\mathbb{L}} \mathcal{D}_l^* \eta_l\right).$$

Since T is injective on $\text{ran}K$, $\text{ran}K = \text{ran}T_{\mathcal{D}}^*$. We have $\sum_{l\in\mathbb{L}} \mathcal{D}_l^* \eta_l = 0$. However, $\mathcal{D}$ is a K-g-ONB for $\mathcal{M}$. For any $j \in \mathbb{L}$,

$$\mathcal{D}_j \sum_{l\in\mathbb{L}} \mathcal{D}_l^* \eta_l = \sum_{l\in\mathbb{L}} \mathcal{D}_j \mathcal{D}_l^* \eta_l = \eta_j = 0.$$

Then $\mathcal{E}$ is g-linearly independent with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$.

Conversely, let $\mathcal{E}$ be g-linearly independent with respect to $\{\mathcal{M}_l\}_{l\in\mathbb{L}}$. Suppose that $\mathcal{E}$ is a K-g-frame with analysis operator $T_{\mathcal{E}}$. Then $\text{ran}K \subset \text{ran}T_{\mathcal{E}}^*$ and $T_{\mathcal{E}}^* \widetilde{e}_{lm} = \mathcal{E}_l^* e_{lm}$, where $\{\widetilde{e}_{lm}\}_{l\in\mathbb{L}, m\in\mathbb{M}_l}$ is an ONB for $\left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2}$ and $\{e_{lm}\}_{m\in\mathbb{M}_l}$ is an ONB for $\mathcal{M}_l$ for each $l \in \mathbb{L}$. By Theorem 4.3, there exists an isometry operator $h : \left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2} \to \mathcal{M}$ such that $h\widetilde{e}_{lm} = \mathcal{D}_l^* e_{lm}$ and for all $\xi \in \mathcal{M}$, $\|K^*\xi\| = \|h^*\xi\|$. So $\mathcal{E}_l^* e_{lm} = T_{\mathcal{E}}^* h^* \mathcal{D}_l^* e_{lm}$. Since $\{e_{lm}\}_{m\in\mathbb{M}_l}$ is an ONB for $\mathcal{M}_l$ for each $l \in \mathbb{L}$, we have $\mathcal{E}_l^* = T_{\mathcal{E}}^* h^* \mathcal{D}_l^*$ for each $l \in \mathbb{L}$. Then $\mathcal{E}_l = \mathcal{D}_l h T_{\mathcal{E}}$ for each $l \in \mathbb{L}$. Let $T = T_{\mathcal{E}}^* h^*$. We have $\text{ran}K \subset \text{ran}T$. Suppose

$$0 = \sum_{l\in\mathbb{L}} \mathcal{E}_l^* \eta_l = \sum_{l\in\mathbb{L}} T\mathcal{D}_l^* \eta_l = T(\sum_{l\in\mathbb{L}} \mathcal{D}_l^* \eta_l).$$

We can obtain $\eta_l = 0$ since $\mathcal{E}$ is g-linearly independent. This implies T is a bounded injective operator on $\text{ran}K$. Let $\xi \in \text{ran}K \subset \text{ran}T_{\mathcal{E}}^*$. There is $\{\eta_l\}_{l\in\mathbb{L}} \in \left(\sum_{l\in\mathbb{L}} \oplus \mathcal{M}_l\right)_{l^2}$ such that $T_{\mathcal{E}}^*(\{\eta_l\}_{l\in\mathbb{L}}) = \xi$. Since $\mathcal{D}$ is a K-g-ONB for $\mathcal{M}$, ranK = $\text{ran}T_{\mathcal{D}}^*$. Let $\eta = T_{\mathcal{D}}^*(\{\eta_l\}_{l\in\mathbb{L}}) \in$ ranK. Then

$$T\eta = T\sum_{l\in\mathbb{L}} \mathcal{D}_l^* \eta_l = \sum_{l\in\mathbb{L}} \mathcal{E}_l^* \eta_l = \xi.$$

Therefore, $T : \text{ran}K \to \text{ran}K$ is a surjective operator. By Theorem 4.5, $\mathcal{E}$ is a K-g-Riesz basis for $\mathcal{M}$. □

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