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Solution of Some Non-homogeneous Fractional Integral Equations by Aboodh Transform

Wagdi F. S. Ahmed[1,2,†] and D. D. Pawar[1]

Abstract The present paper introduces the Aboodh transform technique as a method for obtaining solutions to a class of non-homogeneous fractional integral equations. The emphasis is placed on equations characterized by expressions involving Riemann-Liouville fractional integrals of orders $1, \frac{1}{2}$, and $\frac{1}{3}$. The paper includes illustrative examples that demonstrate the application of the Aboodh transform technique. These examples elucidate how the technique can effectively yield solutions for specific instances of the mentioned equations. The obtained solutions are presented in the form of Mellin-Ross functions.

Keywords Fractional differential equations, Aboodh transform, fractional integral equations, Mellin-Ross function, Riemann-Liouville fractional integral

MSC(2010) MSC(2010): 26A33; 46F10; 34A08; 46A12.

1. Introduction

Fractional calculus is indeed a mathematical theory that deals with derivatives and integrals of arbitrary complex or real order. Its origins can be traced back to the early 18^{th} century when mathematicians like Leibniz and L'Hopital started to investigate the meaning of fractional derivatives.

In particular, in 1695, L'Hopital posed the problem of finding the meaning of the derivative of order $n = 1/2, i.e., d^n y/dx^n$, and asked Leibniz for a solution. Leibniz himself was intrigued by the problem and tried to find a way to define fractional derivatives and integrals. However, it was not until the 19^{th} century that the concept of fractional calculus was fully developed by mathematicians like Liouville, Riemann, and Grunwald. They introduced the concept of fractional derivatives and integrals as a natural extension of the classical calculus. Since then, fractional calculus has found numerous applications in various fields, including engineering, economics, physics, and biology [3, 5, 6, 15, 19, 20, 22, 23, 29, 37, 38]. It has proven to be a powerful tool for modeling and analyzing complex systems with non-local and non-linear behavior, such as fractional differential equations.

Fractional derivatives, a fundamental component of fractional calculus, have garnered significant attention in recent years. They play a crucial role in modeling phenomena across various branches of engineering and science when dealing with

[†]the corresponding author.
Email address: wagdialakel@gmail.com (Wagdi F. S. Ahmed), dypawar@yahoo.com (D. D. Pawar)
[1]School of Mathematical Sciences, Swami Ramanand Teerth Marathwada University, Nanded-431606, India.
[2]Department of Mathematics, Aden University, Aden, Yemen.

real-world problems. Fractional calculus has facilitated the development of mathematical models for practical issues encountered in diverse fields such as dielectric polarization, viscoelasticity, electromagnetic waves, and electrode-electrolyte polarization [7–9, 11, 12, 16–18, 21, 31–35]. These applications highlight the practical relevance and broad impact of fractional calculus in addressing complex phenomena in engineering and scientific domains.

The Aboodh transform is a mathematical tool used to solve fractional differential equations. It is based on the fractional derivative of the generalized Mittag-Leffler function, which is a special function that arises in the study of fractional calculus [1,2,4,10,29,30]. The application of the Aboodh transform to non-homogeneous fractional integral equations is a relatively new area of research, and there is still much to be discovered and understood about its potential applications and limitations. However, preliminary findings suggest that it has the potential to be a powerful tool for solving a wide range of problems in fractional calculus and related fields.

In 2005, T. Morita [26] conducted a study on the initial value problem of fractional differential equations, employing the Laplace transform. In his work, he derived solutions for fractional differential equations involving the Riemann-Liouville fractional derivative as well as the Caputo fractional derivative or its modified form. Morita's research focused on obtaining solutions to these equations by utilizing the Laplace transform technique.

In 2010, T. Morita and K. Sato [27] conducted a study on the initial value problem of fractional differential equations with constant coefficients. The equations they considered were of the following forms:

$$
{}_0D_t^{\zeta}u(t) + l \cdot u(t) = f(t),
$$

$$
{}_0D_t^{\zeta}u(t) + k \cdot {}_0D_t^{\xi}u(t) + l \cdot u(t) = f(t),
$$

and

$$
{}_0D_t^{\gamma_n}u(t) + \sum_{r=0}^{n-1} l_r \cdot {}_0D_t^{\gamma_r}u(t) = f(t).
$$

In these equations, ${}_0D_t^{\gamma_n}$ represents the (RL) fractional derivative, l_r are constants for $r = 0, 1, 2, 3, ..., n-1$, and $t \in \mathbb{R}^+$. Morita and Sato obtained solutions to these equations using techniques involving Green's function and distribution theory. Furthermore, they also studied the solution of a fractional differential equation of the form:

$$
(j_2t + k_2)_0D_t^{2\gamma}u(t) + (j_1t + k_1)_0D_t^{\gamma}u(t) + (j_0t + k_0)u(t) = f(t),
$$

where $\gamma = \frac{1}{2}, \gamma = 1, t \in \mathbb{R}^+$ and j_r, k_r are constants for r=0,1,2,3. For more details, refer to [28].

In 2018, C. Li [24] conducted a study on Abel's integral equation of the second kind. The equation is given by:

$$
y(t) + \frac{\lambda}{\Gamma(\gamma)} \int_0^t (t-\phi)^{\gamma-1} y(\phi) d\phi = f(t), \quad t > 0. \tag{1.1}
$$

Here, Γ is the gamma function,$\gamma \in \mathbb{R}$, and λ is a constant . Equation (1.1) can be written in the form

$$
(1 + \lambda I_{0+}^{\gamma})y(t) = f(t).
$$

Here, I_{0+}^{γ} denotes the (RL) fractional integral. Li and Clarkson employed Babenko's method and the fractional integral technique to solve the aforementioned equation. The linear fractional order integral equations with constant coefficients have the form:

$$l_1 I_{a^+}^{\gamma_1} y(t) + l_2 I_{a^+}^{\gamma_2} y(t) + ... + l_n I_{a^+}^{\gamma_n} y(t) = f(t), \tag{1.2}$$

where $a \in \mathbb{R}, \gamma_r \in \mathbb{R}, \gamma_1 > \gamma_2 > ... > \gamma_n \geq 0, l_r \in \mathbb{C}$ for $r \in \{1, 2, 3, .., n\}$, and f is a real valued function of real variable defined on an interval (a, b). The general solution of equation (1.2) can be found in [36], considering γ_r as real numbers. The solution lies in the space $S_+^{'}$ of tempered distributions with support in $[0, \infty)$.

In 2021, K. Kaewnimit et al. [13,14] conducted a study on the solutions of nonhomogeneous fractional integral equations employing the Laplace transform technique. This paper presents the solution to nonhomogeneous fractional integral equations of the following forms:

$$I_{0+}^{2\gamma} y(t) + j \cdot I_{0+}^{\gamma} y(t) + k \cdot y(t) = t^n,$$

and

$$I_{0+}^{3\gamma} y(t) + j \cdot I_{0+}^{2\gamma} y(t) + k \cdot I_{0+}^{\gamma} y(t) + l \cdot y(t) = t^n.$$

In these equations I_{0+}^{γ} denotes the (RL) fractional integral of order $\gamma = \frac{1}{2}, \gamma = 1$, and $\gamma = \frac{1}{3}, \gamma = 1$, respectively, $n \in \mathbb{N} \cup \{0\}$, and j, k, l are constants. The Aboodh transform technique is utilized to obtain the solutions.

Section 2 provides the definitions of the (RL) fractional integral and the Aboodh transform, which are essential for deriving the main results. In Section 3, the main results are established along with illustrative examples. Finally, the conclusions of the study are presented in Section 4.

2. Preliminaries

In order to proceed to the main results, we need to establish the following lemmas, definitions, examples, and concepts.

Definition 2.1. [25] The Riemann-Liouville (RL) fractional integral of a function $f(t)$ of order(γ) is defined as follows:

$$I_{0+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t - \phi)^{\gamma - 1} f(\phi) d\phi, \qquad (t > 0, \Re(\gamma) > 0),$$

where $\Gamma(\gamma)$ denotes the gamma function, which is defined as:

$$\Gamma(\gamma) = \int_0^t t^{\gamma - 1} e^{-t} dt.$$

Definition 2.2. [25] The Riemann-Liouville (RL) fractional derivative of a function $f(t)$ denoted by ${}_0D_t^{\gamma} f(t)$ is defined as follows:

$${}_0D_t^{\gamma} f(t) = \frac{1}{\Gamma(n - \gamma)} \frac{d^n}{dt^n} \int_0^t (t - \phi)^{n - \gamma - 1} f(\phi) d\phi, \qquad (n - \gamma > 0, \Re(\gamma) > 0).$$

Here, n is an integer that satisfies $n - 1 \leq \gamma < n$ and $\Gamma(.)$ represents the gamma function.

Definition 2.3. [1] The Aboodh transform is defined for functions of exponential order. We consider functions in the set A given by:

$$A = \{f(t) : \exists \quad \Im, k_1, k_2 > 0, \quad |f(t)| < \Im e^{-pt}\}.$$

For a given function in the set A, the constant$\Im$ must be a finite number, k_1, k_2 may be finite or infinite.

The Aboodh transform denoted by the operator $\mathcal{A}(.)$ *is* defined by the integral equation:

$$\mathcal{A}[f(t)](p) = K(p) = \frac{1}{p}\int_0^\infty f(t)e^{-pt}dt, \qquad t \geq 0, k_1 \leq p \leq k_2.$$

Definition 2.4. [1] The inverse Aboodh transform of a function $f(t)$ is defined as

$$f(t) = \mathcal{A}^{-1}[K(p)].$$

Definition 2.5. [25] The Mellin-Ross function $E_t(\beta, \gamma)$ is defined as:

$$E_t(\beta, \gamma) = t^\beta e^{\gamma t}\Gamma^*(\beta, \gamma t),$$

where β is a real number ,γ is a constant, $t \in \mathbb{R}^+$, and Γ^* is the incomplete gamma function:

$$\Gamma^*(\beta, t) = e^{-t}\sum_{r=0}^{\infty}\frac{t^r}{\Gamma(\beta + r + 1)}.$$

Alternatively, we can express the Mellin-Ross function as:

$$E_t(\beta, \gamma) = t^\beta \sum_{r=0}^{\infty}\frac{(\gamma t)^r}{\Gamma(\beta + r + 1)}.$$

Example 2.1. Let n be a real number, γ a constant, t and p positive real numbers. Then the following Aboodh transforms hold:
(i) $\mathcal{A}\{1\} = \frac{1}{p^2}$;
(ii) $\mathcal{A}\{t\} = \frac{1}{p^3}$;
(iii) $\mathcal{A}\{t^2\} = \frac{2!}{p^4}$;
(iv) $\mathcal{A}\{t^n\} = \frac{n!}{p^{n+2}} \qquad , n \geqslant 0$;
(v) $\mathcal{A}\{e^{\gamma t}\} = \frac{1}{(p^2 - \gamma p)}$.

Lemma 2.1 (Lemma 1, [10])**.** *The Aboodh transformation of (RL) fractional integral operator of order $\gamma > 0$ can be written in the form:*

$$\mathcal{A}[I^\gamma f(t)] = p^{-\gamma}\mathcal{A}[f(t)].$$

Example 2.2. Let n be a real number, γ a constant, t and p positive real numbers. Then the following Aboodh transforms hold:
(i) $\mathcal{A}^{-1}\{\frac{1}{p^2}\} = 1$;
(ii) $\mathcal{A}^{-1}\{\frac{1}{p^3}\} = t$;
(iii) $\mathcal{A}^{-1}\{\frac{1}{p^4}\} = \frac{t^2}{2!}$;
(iv) $\mathcal{A}^{-1}\{\frac{1}{p^{\gamma+2}}\} = \frac{t^\gamma}{\Gamma(\gamma+1)}$;
(v) $\mathcal{A}^{-1}\{\frac{1}{p^2 - \gamma p}\} = e^{\gamma t}$;

(vi) $\mathcal{A}^{-1}\Big\{\frac{1}{p+\gamma}\Big\} = E_t\Big(-1,-\gamma\Big)$;

(vii) $\mathcal{A}^{-1}\Big\{\frac{1}{p^{\frac{1}{2}}+\gamma}\Big\} = E_t\Big(-\frac{3}{2},\gamma^2\Big) - \gamma E_t\Big(-1,\gamma^2\Big)$;

(viii) $\mathcal{A}^{-1}\Big\{\frac{1}{p^{\frac{1}{3}}+\gamma}\Big\} = E_t\Big(-\frac{5}{3},-\gamma^3\Big) - \gamma E_t\Big(-\frac{4}{3},-\gamma^3\Big) + \gamma^2 E_t\Big(-1,-\gamma^3\Big)$.

3. Main results

In this section, we present our main results along with their corresponding proofs.

- (**1**). The Solution of non-homogeneous fractional integral equations is given by:

$$I_{0^+}^{2\gamma} y(t) + j \cdot I_{0^+}^{\gamma} y(t) + k \cdot y(t) = f(t), \qquad \gamma = \frac{1}{2}, \gamma = 1, and \;\; f(t) = t^n.$$

Theorem 3.1. *Consider the non-homogeneous fractional integral equation of the form:*

$$I_{0^+}^{2\gamma} y(t) + j \cdot I_{0^+}^{\gamma} y(t) + k \cdot y(t) = t^n. \tag{3.1}$$

Here, $I_{0^+}^{\gamma}$ represents the (RL) fractional integral of order $\gamma = \frac{1}{2}, \gamma = 1, t \in \mathbb{R}^+, j, k$ are constants and $n \in \mathbb{N} \cup \{0\}$. Then the solutions of equation (3.1) are as follows:

(**i**) *If $\gamma = \frac{1}{2}$, and $z_1, z_2 \in \mathbb{R} \setminus \{0\}$ with $z_1 \neq z_2$ such that $j = z_1 + z_2$ and $k = z_1 z_2$, then the solution of equation (3.1) is of the form:*

$$\begin{aligned} y(t) =& \frac{n!}{z_1 - z_2} \sum_{r=0}^{2n+2} (-1)^{r+1} \left[\frac{z_1^{2n+3} - z_2^{2n+3}}{(z_1 z_2)^{(r+1)}}\right] \frac{(t^{(r-1)})}{\Gamma(r)} + \frac{n!\; z_1^{2n+2}}{z_1 - z_2} \left[E_t\Big(-\frac{3}{2}, \frac{1}{z_1^2}\Big)\right] \\ & - \frac{n!\; z_1^{2n+2}}{z_1 - z_2} \left[\frac{1}{z_1} E_t\Big(-1, \frac{1}{z_1^2}\Big)\right] - \frac{n!\; z_2^{2n+2}}{z_1 - z_2} \left[E_t\Big(-\frac{3}{2}, \frac{1}{z_2^2}\Big) - \frac{1}{z_2} E_t\Big(-1, \frac{1}{z_2^2}\Big)\right]. \end{aligned} \tag{3.2}$$

(**ii**) *If $\gamma = 1$, and $z_1, z_2 \in \mathbb{R} \setminus \{0\}$ with $z_1 \neq z_2$ such that $j = z_1 + z_2$ and $k = z_1 z_2$, then the solution of equation (3.1) is of the form:*

$$\begin{aligned} y(t) =& \frac{n!}{z_1 - z_2} \sum_{r=1}^{n} (-1)^{n+r} \left[\frac{z_1^{n+1-r} - z_2^{n+1-r}}{\Gamma(r-1)}\right] t^{(r-2)} \\ & + \frac{(-1)^n n!}{z_1 - z_2} \left[z_1^n E_t\Big(-1, -\frac{1}{z_1}\Big)\right] - \frac{(-1)^n n!}{z_1 - z_2} \left[z_2^n E_t\Big(-1, -\frac{1}{z_2}\Big)\right]. \end{aligned} \tag{3.3}$$

Proof. Applying the Aboodh transform to both sides of equation (3.1), we have

$$\mathcal{A}\big\{I_{0^+}^{2\gamma} y(t)\big\} + j\mathcal{A}\big\{I_{0^+}^{\gamma} y(t)\big\} + k\mathcal{A}\big\{y(t)\big\} = \mathcal{A}\big\{t^n\big\}. \tag{3.4}$$

Using Example 2.1 (iv), Lemma 2.1, and denoting the Aboodh transform $\mathcal{A}\big\{y(t)\big\} = Y(p)$ to equation (3.4), we obtain

$$Y(p) = \frac{n!}{p^{n+2}(p^{-2\gamma} + j\; p^{-\gamma} + k)}, \tag{3.5}$$

and turn it

$$Y(p) = \frac{n! p^{2\gamma}}{p^{n+2}(k\; p^{2\gamma} + j\; p^{\gamma} + 1)}. \tag{3.6}$$

(**i**). For $\gamma = \frac{1}{2}$, equation (3.6) becomes

$$Y(p) = \frac{n!}{p^{n+1}(k\ p + j\ p^{\frac{1}{2}} + 1)}. \tag{3.7}$$

Substituting of ($p^{\frac{1}{2}} = u$) we get

$$Y(p) = \frac{n!}{u^{2n+2}(k\ u^2 + j\ u + 1)}. \tag{3.8}$$

Using partial fractions with explicit values of j, k, we can rewrite

$$\begin{aligned} Y(p) =& \frac{n!}{z_1 - z_2} \sum_{r=0}^{2n+2} (-1)^{r+1} \left[\frac{z_1^{2n+3} - z_2^{2n+3}}{(z_1 z_2)^{(r+1)}} \right] \frac{1}{(u^2)^{r+1}} + \frac{n!\ z_1^{2n+2}}{z_1 - z_2} \left(\frac{1}{u + \frac{1}{z_1}} \right) \\ & - \frac{n!\ z_1^{2n+2}}{z_1 - z_2} \left(\frac{1}{u + \frac{1}{z_1}} \right). \end{aligned} \tag{3.9}$$

Now, resubstituting ($p^{\frac{1}{2}} = u$) and taking the inverse Aboodh transform to equation (3.9) with the help of Example 2.2 (iv),(vii) we obtain a solution of equation (3.1) as the form

$$\begin{aligned} y(t) =& \frac{n!}{z_1 - z_2} \sum_{r=0}^{2n+2} (-1)^{r+1} \left[\frac{z_1^{2n+3} - z_2^{2n+3}}{(z_1 z_2)^{(r+1)}} \right] \frac{(t^{(r-1)})}{\Gamma(r)} + \frac{n!\ z_1^{2n+2}}{z_1 - z_2} \left[E_t\Big(-\frac{3}{2}, \frac{1}{z_1^2} \Big) \right] \\ & - \frac{n!\ z_1^{2n+2}}{z_1 - z_2} \left[\frac{1}{z_1} E_t\Big(-1, \frac{1}{z_1^2} \Big) \right] - \frac{n!\ z_2^{2n+2}}{z_1 - z_2} \left[E_t\Big(-\frac{3}{2}, \frac{1}{z_2^2} \Big) - \frac{1}{z_2} E_t\Big(-1, \frac{1}{z_2^2} \Big) \right]. \end{aligned} \tag{3.10}$$

(**ii**). For $\gamma = 1$, equation (3.6) becomes

$$Y(p) = \frac{n!}{p^n(k\ p^2 + j\ p + 1)}. \tag{3.11}$$

Using partial fractions with explicit values of $j; k$, we can rewrite the above equation as

$$\begin{aligned} Y(p) =& \frac{n!}{z_1 - z_2} \sum_{r=1}^{n} (-1)^{n+r} \left[z_1^{n+1-r} - z_2^{n+1-r} \right] \frac{1}{p^r} + \frac{(-1)^n\ n!\ z_1^n}{z_1 - z_2} \left(\frac{1}{p + \frac{1}{z_1}} \right) \\ & - \frac{(-1)^n\ n!\ z_2^n}{z_1 - z_2} \left(\frac{1}{p + \frac{1}{z_2}} \right). \end{aligned} \tag{3.12}$$

Now, taking the inverse Aboodh transform to equation (3.12) with the help of Example 2.2 (iv),(vi) we obtain a solution of equation (3.1) as the form:

$$\begin{aligned} y(t) =& \frac{n!}{z_1 - z_2} \sum_{r=1}^{n} (-1)^{n+r} \left[\frac{z_1^{n+1-r} - z_2^{n+1-r}}{\Gamma(r-1)} \right] t^{(r-2)} \\ & + \frac{(-1)^n n!}{z_1 - z_2} \left[z_1^n E_t\Big(-1, -\frac{1}{z_1} \Big) \right] - \frac{(-1)^n n!}{z_1 - z_2} \left[z_2^n E_t\Big(-1, -\frac{1}{z_2} \Big) \right]. \end{aligned} \tag{3.13}$$

□

Example 3.1. For $j = \frac{5}{6}, k = \frac{1}{6}$, and $\gamma = \frac{1}{2}$, equation (3.1) changes to

$$I_{0^+} y(t) + \frac{5}{6} \cdot I_{0^+}^{\frac{1}{2}} y(t) + \frac{1}{6} \cdot y(t) = t^n. \tag{3.14}$$

From Theorem 3.1 (i), equation (3.14) has a solution

$$\begin{aligned} y(t) =& n! \sum_{r=0}^{2n+2} (-1)^{r+1} \left[\frac{3\left(\frac{1}{2}\right)^{2n+2} - 2\left(\frac{1}{3}\right)^{2n+2}}{\left(\frac{1}{6}\right)^{r+1}} \right] \frac{t^{r-1}}{\Gamma(r)} + \frac{3n!}{2^{2n+1}} \left[E_t\Big(-\frac{3}{2}, 4\Big) \right] \\ & - \frac{3n!}{2^{2n+1}} \left[2E_t\Big(-1, 4\Big) \right] - \frac{2n!}{3^{2n+1}} \left[E_t\Big(-\frac{3}{2}, 9\Big) - 3E_t\Big(-1, 9\Big) \right]. \end{aligned} \tag{3.15}$$

Example 3.2. For $j = \frac{5}{2}, k = 1$, and $\gamma = \frac{1}{2}$, equation (3.1) changes to

$$I_{0^+} y(t) + \frac{5}{2} \cdot I_{0^+}^{\frac{1}{2}} y(t) + y(t) = t^n. \tag{3.16}$$

From Theorem 3.1 (i), equation (3.16) has a solution

$$\begin{aligned} y(t) =& \frac{n!}{3} \sum_{r=0}^{2n+2} (-1)^{r+1} \left[\frac{2^{2n+4} - (\frac{1}{2})^{2n+2}}{\Gamma(r)} \right] (t^{(r-1)}) + \frac{n!\ 2^{2n+3}}{3} \left[E_t\Big(-\frac{3}{2}, \frac{1}{4}\Big) \right] \\ & - \frac{n!\ 2^{2n+3}}{3} \left[\frac{1}{2} E_t\Big(-1, \frac{1}{4}\Big) \right] - \frac{n!\ (\frac{1}{2})^{2n+1}}{3} \left[E_t\Big(-\frac{3}{2}, 4\Big) - 2E_t\Big(-1, 4\Big) \right]. \end{aligned} \tag{3.17}$$

Example 3.3. For $j = \frac{5}{2}, k = 1$, and $\gamma = 1$, equation (3.1) changes to

$$I_{0^+}^2 y(t) + \frac{5}{2} \cdot I_{0^+} y(t) + y(t) = t^n. \tag{3.18}$$

From Theorem 3.1 (ii), equation (3.18) has a solution

$$\begin{aligned} y(t) =& \frac{n!}{3} \sum_{r=1}^{n} (-1)^{n+r} \left[\frac{2^{n+2-r} - (\frac{1}{2})^{n-r}}{\Gamma(r-1)} \right] t^{(r-2)} + \frac{(-1)^n n!}{3} \left[2^{n+1} E_t\Big(-1, -\frac{1}{2}\Big) \right] \\ & - \frac{(-1)^n n!}{3} \left[\Big(\frac{1}{2}\Big)^{n-1} E_t\Big(-1, -2\Big) \right]. \end{aligned} \tag{3.19}$$

Example 3.4. For $j = \frac{5}{6}, k = \frac{1}{6}$, and $\gamma = 1$, equation (3.1) changes to

$$I_{0^+}^2 y(t) + \frac{5}{6} \cdot I_{0^+} y(t) + \frac{1}{6} y(t) = t^n. \tag{3.20}$$

From Theorem 3.1 (ii), equation (3.20) has a solution

$$\begin{aligned} y(t) =& n! \sum_{r=1}^{n} (-1)^{n+r} \left[\frac{3\left(\frac{1}{2}\right)^{n-r} - 2\left(\frac{1}{3}\right)^{n-r}}{\Gamma(r-1)} \right] t^{(r-2)} + \frac{3(-1)^n n!}{(2)^{n-1}} \left[E_t\Big(-1, -2\Big) \right] \\ & - \frac{2(-1)^n n!}{(3)^{n-1}} \left[E_t\Big(-1, -3\Big) \right]. \end{aligned} \tag{3.21}$$

Example 3.5. For $j = \frac{10}{3}, k = 1$, and $\gamma = \frac{1}{2}$, equation (3.1) changes to

$$I_{0^+}y(t) + \frac{10}{3} \cdot I_{0^+}^{\frac{1}{2}}y(t) + y(t) = t^n. \tag{3.22}$$

From Theorem 3.1 (i), equation (3.22) has a solution

$$\begin{aligned} y(t) =& \frac{n!}{8}\sum_{r=0}^{2n+2}(-1)^{r+1}\left[\frac{3^{2n+4}-(\frac{1}{3})^{2n+2}}{\Gamma(r)}\right](t^{(r-1)}) + \frac{n!\ 3^{2n+3}}{8}\left[E_t\Big(-\frac{3}{2},\frac{1}{9}\Big)\right] \\ &- \frac{n!\ 3^{2n+3}}{8}\left[\frac{1}{3}E_t\Big(-1,\frac{1}{9}\Big)\right] - \frac{n!\ (\frac{1}{3})^{2n+1}}{8}\left[E_t\Big(-\frac{3}{2},9\Big) - 3E_t\Big(-1,9\Big)\right]. \end{aligned} \tag{3.23}$$

Example 3.6. For $j = \frac{10}{3}, k = 1$, and $\gamma = 1$, equation (3.1) changes to

$$I_{0^+}^2y(t) + \frac{10}{3} \cdot I_{0^+}y(t) + y(t) = t^n. \tag{3.24}$$

From Theorem 3.1 (ii), equation (3.24) has a solution

$$\begin{aligned} y(t) =& \frac{n!}{8}\sum_{r=1}^{n}(-1)^{n+r}\left[\frac{3^{n+2-r}-(\frac{1}{3})^{n-r}}{\Gamma(r-1)}\right]t^{(r-2)} + \frac{(-1)^n n!}{8}\left[3^{n+1}E_t\Big(-1,-\frac{1}{3}\Big)\right] \\ &- \frac{(-1)^n n!}{8}\left[\Big(\frac{1}{3}\Big)^{n-1}E_t\Big(-1,-3\Big)\right]. \end{aligned} \tag{3.25}$$

- (**2**). The Solution of non-homogeneous fractional integral equations is given by:

$$I_{0^+}^{3\gamma}y(t) + j \cdot I_{0^+}^{2\gamma}y(t) + k \cdot I_{0^+}^{\gamma}y(t) + l \cdot y(t) = f(t),$$

$$\gamma = \frac{1}{3}, \gamma = 1, and \ \ f(t) = t^n.$$

Theorem 3.2. *Consider the non-homogeneous fractional integral equation of the form:*

$$I_{0^+}^{3\gamma}y(t) + j \cdot I_{0^+}^{2\gamma}y(t) + k \cdot I_{0^+}^{\gamma}y(t) + l \cdot y(t) = t^n. \tag{3.26}$$

Here, $I_{0^+}^{\gamma}$ *represents the (RL) fractional integral of order* $\gamma = \frac{1}{3}, \gamma = 1, t \in \mathbb{R}^+, j, k, l$ *are constants and,* $n \in \mathbb{N} \cup \{0\}$*. Then the solutions of Equation* (3.26) *are as the following:*
(**i**) *If* $\gamma = \frac{1}{3}$*, and* $z_1, z_2, z_3 \in \mathbb{R} \setminus \{0\}$ *with* z_1, z_2, z_3 *being different such that* $j = z_1 + z_2 + z_3, k = z_1z_2, +z_1z_3 + z_2z_3$ *and* $l = z_1z_2z_3$*, then*

$$
\begin{aligned}
&y(t)\\
&=\sum_{r=0}^{3n+3} \frac{n!(-1)^{3n+4+r}}{(z_1z_2z_3)^{(r+1)}}\\
&\quad\times \frac{t^{(r-1)}}{\Gamma(r)}\left[\frac{z_1^{3n+5}}{(z_1-z_2)(z_1-z_3)}-\frac{z_2^{3n+5}}{(z_1-z_2)(z_2-z_3)}+\frac{z_3^{3n+5}}{(z_1-z_3)(z_2-z_3)}\right]\\
&\quad+\frac{n!\ (-1)^{3n+3}\ z_1^{3n+4}}{(z_1-z_2)(z_1-z_3)}\left[E_t\Big(-\frac{5}{3},-\frac{1}{z_1^3}\Big)-\frac{1}{z_1}E_t\Big(-\frac{4}{3},-\frac{1}{z_1^3}\Big)+\frac{1}{z_1^2}E_t\Big(-1,-\frac{1}{z_1^3}\Big)\right]\\
&\quad+\frac{n!\ (-1)^{3n+4}\ z_2^{3n+4}}{(z_1-z_2)(z_2-z_3)}\left[E_t\Big(-\frac{5}{3},-\frac{1}{z_2^3}\Big)-\frac{1}{z_2}E_t\Big(-\frac{4}{3},-\frac{1}{z_2^3}\Big)+\frac{1}{z_2^2}E_t\Big(-1,-\frac{1}{z_2^3}\Big)\right]\\
&\quad+\frac{n!\ (-1)^{3n+5}\ z_3^{3n+4}}{(z_1-z_3)(z_2-z_3)}\left[E_t\Big(-\frac{5}{3},-\frac{1}{z_3^3}\Big)-\frac{1}{z_3}E_t\Big(-\frac{4}{3},-\frac{1}{z_3^3}\Big)+\frac{1}{z_3^2}E_t\Big(-1,-\frac{1}{z_3^3}\Big)\right]
\end{aligned}
\tag{3.27}
$$

as the solution to (3.26).

(**ii**) *If* $\gamma = 1$, *and* $z_1, z_2, z_3 \in \mathbb{R}\setminus\{0\}$ *with* z_1, z_2, z_3 *being different such that* $j = z_1+z_2+z_3, k = z_1z_2, +z_1z_3+z_2z_3$ *and* $l = z_1z_2z_3$, *then*

$$
\begin{aligned}
y(t) = &\sum_{r=1}^{n} n!(-1)^{(n+1)-r}\\
&\times\left[\frac{z_1^{(n+1)-r}(z_2-z_3)-z_2^{(n+1)-r}(z_1-z_3)+z_3^{(n+1)-r}(z_1-z_2)}{(z_1-z_2)(z_1-z_3)(z_2-z_3)}\right]\\
&\times\frac{t^{(r-2)}}{\Gamma(r-1)}+\frac{n!(-1)^{n-1}z_1^n}{(z_1-z_2)(z_1-z_3)}\left[E_t\Big(-1,-\frac{1}{z_1}\Big)\right]+\frac{n!(-1)^n z_2^n}{(z_1-z_2)(z_2-z_3)}\\
&\times\left[E_t\Big(-1,-\frac{1}{z_2}\Big)\right]+\frac{n!\ (-1)^{n+1}\ z_3^n}{(z_1-z_3)(z_2-z_3)}\left[E_t\Big(-1,-\frac{1}{z_3}\Big)\right]
\end{aligned}
\tag{3.28}
$$

as the solution to (3.26).

Proof. Applying the Aboodh transform to both sides of Equation (3.26), we have

$$\mathcal{A}\big\{I_{0^+}^{3\gamma}y(t)\big\}+j\cdot\mathcal{A}\big\{I_{0^+}^{2\gamma}y(t)\big\}+k\cdot\mathcal{A}\big\{I_{0^+}^{\gamma}y(t)\big\}+l\cdot\mathcal{A}\big\{y(t)\big\}=\mathcal{A}\big\{t^n\big\}. \tag{3.29}$$

Using Example 2.1 (iv), Lemma 2.1, and denoting the Aboodh transform $\mathcal{A}\big\{y(t)\big\} = Y(p)$ to equation (3.29), we obtain

$$Y(p)=\frac{n!}{p^{n+2}(p^{-3\gamma}+j\ p^{-2\gamma}+k\ p^{-\gamma}+l)}, \tag{3.30}$$

and turn it

$$Y(p)=\frac{n!\ p^{3\gamma}}{p^{n+2}(l\ p^{3\gamma}+k\ p^{2\gamma}+j\ p^{\gamma}+1)}. \tag{3.31}$$

(**i**). For $\gamma=\frac{1}{3}$, equation (3.31) becomes

$$Y(p)=\frac{n!}{p^{n+1}(l\ p+k\ p^{\frac{2}{3}}+j\ p^{\frac{1}{3}}+1)}. \tag{3.32}$$

Substituting ($p^{\frac{1}{3}} = u$), we get

$$Y(p) = \frac{n!}{u^{3n+3}(l\ u^3 + k\ u^2 + j\ u + 1)}. \tag{3.33}$$

Using partial fractions with explicit values of j, k, l; we can rewrite it as:

$$\begin{aligned} y(t) = &\sum_{r=0}^{3n+3} \frac{n!(-1)^{3n+4+r}}{(z_1 z_2 z_3)^{(r+1)}} \\ &\times \frac{1}{(u^3)^{(r+1)}} \left[\frac{z_1^{3n+5}}{(z_1-z_2)(z_1-z_3)} - \frac{z_2^{3n+5}}{(z_1-z_2)(z_2-z_3)} + \frac{z_3^{3n+5}}{(z_1-z_3)(z_2-z_3)} \right] \\ &+ \frac{n!\ (-1)^{3n+3} z_1^{3n+4}}{(z_1-z_2)(z_1-z_3)} \left(\frac{1}{(u+\frac{1}{z_1})} \right) + \frac{n!(-1)^{3n+4} z_2^{3n+4}}{(z_1-z_2)(z_2-z_3)} \left(\frac{1}{(u+\frac{1}{z_2})} \right) \\ &+ \frac{n!\ (-1)^{3n+5}\ z_3^{3n+4}}{(z_1-z_3)(z_2-z_3)} \left(\frac{1}{(u+\frac{1}{z_3})} \right). \end{aligned} \tag{3.34}$$

Now, resubstituting ($p^{\frac{1}{3}} = u$) and taking the inverse Aboodh transform to equation (3.34) with the help of Example 2.2 (iv),(viii) we obtain equation (3.27) as the solution to equation (3.26).

(**ii**). For $\gamma = 1$, equation (3.31) becomes

$$Y(p) = \frac{n!}{p^{n-1}(l\ p^3 + k\ p^2 + j\ p^{\gamma} + 1)}. \tag{3.35}$$

Using partial fractions with explicit values of j, k, l, we can rewrite equation (3.35) as:

$$\begin{aligned} y(t) = &\sum_{r=1}^{n} n!(-1)^{(n+1)-r} \\ &\times \frac{1}{p^{(r)}} \left[\frac{z_1^{(n+1)-r}(z_2-z_3) - z_2^{(n+1)-r}(z_1-z_3) + z_3^{(n+1)-r}(z_1-z_2)}{(z_1-z_2)(z_1-z_3)(z_2-z_3)} \right] \\ &+ \frac{n!(-1)^{n-1} z_1^n}{(z_1-z_2)(z_1-z_3)} \left(\frac{1}{(p+\frac{1}{z_1})} \right) + \frac{n!(-1)^n z_2^n}{(z_1-z_2)(z_2-z_3)} \left(\frac{1}{(p+\frac{1}{z_2})} \right) \\ &+ \frac{n!\ (-1)^{n+1}\ z_3^n}{(z_1-z_3)(z_2-z_3)} \left(\frac{1}{(p+\frac{1}{z_3})} \right). \end{aligned} \tag{3.36}$$

Applying the inverse Aboodh transform to equation (3.36) and using Example 2.2 (iv) and (vi), we obtain equation (3.28) as the solution to equation (3.26). □

Example 3.7. For $j = \frac{1}{2}, k = -1, l = -\frac{1}{2}$, and $\gamma = \frac{1}{3}$, equation (3.26) changes to

$$I_{0^+} y(t) + \frac{1}{2} \cdot I_{0^+}^{\frac{2}{3}} y(t) - I_{0^+}^{\frac{1}{3}} y(t) - \frac{1}{2} \cdot y(t) = t^n. \tag{3.37}$$

From Theorem 3.2 (i), equation (3.37) has a solution

$$
\begin{aligned}
y(t) = &\sum_{r=0}^{3n+3} \frac{n!(-1)^{3n+5+r}}{(-\frac{1}{2})^{(r+1)}} \left[\frac{(\frac{1}{2})^{3n+3} - 3 + (-1)^{3n+4}}{3} \right] \frac{t^{(r-1)}}{\Gamma(r)} \\
&+ \frac{n!\ (-1)^{3n+4}\ (\frac{1}{2})^{3n+2}}{3} \left[E_t\Big(-\frac{5}{3}, -8\Big) - 2E_t\Big(-\frac{4}{3}, -8\Big) + 4E_t\Big(-1, -8\Big) \right] \\
&+ n!\ (-1)^{3n+5} \left[E_t\Big(-\frac{5}{3}, -1\Big) - E_t\Big(-\frac{4}{3}, -1\Big) + E_t\Big(-1, -1\Big) \right] \\
&+ \frac{n!\ (-1)^{6n+9}}{3} \left[E_t\Big(-\frac{5}{3}, 1\Big) + E_t\Big(-\frac{4}{3}, 1\Big) + E_t\Big(-1, 1\Big) \right].
\end{aligned} \tag{3.38}
$$

Example 3.8. For $j = \frac{1}{2}, k = -1, l = -\frac{1}{2}$, and $\gamma = 1$, equation (3.26) changes to

$$
I_{0+}^{3} y(t) + \frac{1}{2} \cdot I_{0+}^{2} y(t) - I_{0+} y(t) - \frac{1}{2} \cdot y(t) = t^n. \tag{3.39}
$$

From Theorem 3.2 (ii), equation (3.39) has a solution

$$
\begin{aligned}
y(t) = &\sum_{r=1}^{n} (-1)^{(n+2)-r} n! \left[\frac{(\frac{1}{2})^{(n-1)-r} - 3 + (-1)^{(n+2)-r}}{3} \right] \frac{t^{(r-2)}}{\Gamma(r-1)} + \frac{n!(-1)^n (\frac{1}{2})^{n-2}}{3} \\
&\times \left[E_t\Big(-1, -2\Big) \right] + n!(-1)^{n+1} \left[E_t\Big(-1, -1\Big) \right] + \frac{n!(-1)^{2n+1}}{3} \left[E_t\Big(-1, 1\Big) \right].
\end{aligned} \tag{3.40}
$$

Example 3.9. For $j = \frac{11}{6}, k = 1, l = -\frac{1}{6}$, and $\gamma = \frac{1}{3}$, equation (3.26) changes to

$$
I_{0+} y(t) + \frac{11}{6} \cdot I_{0+}^{\frac{2}{3}} y(t) + I_{0+}^{\frac{1}{3}} y(t) + \frac{1}{6} \cdot y(t) = t^n. \tag{3.41}
$$

From Theorem 3.2 (i), equation (3.41) has a solution

$$
\begin{aligned}
y(t) = &\sum_{r=0}^{3n+3} \frac{n!(-1)^{3n+4+r}}{(\frac{1}{6})^{(r+1)}} \left[\frac{3 + (\frac{1}{3})^{(3n+3)} - 3(\frac{1}{2})^{(3n+3)}}{\Gamma(r)} \right] t^{(r-1)} \\
&+ n!(3)(-1)^{3n+3} \left[E_t\Big(-\frac{5}{3}, -1\Big) - E_t\Big(-\frac{4}{3}, -1\Big) + E_t\Big(-1, -1\Big) \right] \\
&+ n!(-1)^{3n+5} \Big(\frac{1}{3}\Big)^{3n+2} \left[E_t\Big(-\frac{5}{3}, -27\Big) - 3E_t\Big(-\frac{4}{3}, -27\Big) + 9E_t\Big(-1, -27\Big) \right] \\
&+ n!(3)(-1)^{3n+6} \Big(\frac{1}{2}\Big)^{3n+2} \left[E_t\Big(-\frac{5}{3}, -8\Big) - 2E_t\Big(-\frac{4}{3}, -8\Big) + 4E_t\Big(-1, 8\Big) \right].
\end{aligned} \tag{3.42}
$$

Example 3.10. For $j = \frac{1}{2}, k = -1, l = -\frac{1}{2}$, and $\gamma = 1$, equation (3.26) changes to

$$
I_{0+}^{3} y(t) + \frac{11}{6} \cdot I_{0+}^{2} y(t) + I_{0+} y(t) + \frac{1}{6} \cdot y(t) = t^n. \tag{3.43}
$$

From Theorem 3.2 (ii), equation (3.43) has a solution

$$\begin{aligned} y(t) = &\sum_{r=1}^{n} n!(-1)^{(n+1)-r}\left[\frac{3+(\frac{1}{3})^{(n-1)-r}-3(\frac{1}{2})^{(n-1)-r}}{\Gamma(r-1)}\right]t^{(r-2)} \\ &+ n!3(-1)^{n-1}\left[E_t\Big(-1,-1\Big)\right] + n!(-1)^{n+1}\Big(\frac{1}{3}\Big)^{(n-2)}\left[E_t\Big(-1,-3\Big)\right] \\ &+ n!(3)(-1)^{n+2}\Big(\frac{1}{2}\Big)^{(n-2)}\left[E_t\Big(-1,-2\Big)\right]. \end{aligned} \tag{3.44}$$

4. Conclusions

We applied the Aboodh transform technique to solve non-homogeneous fractional integral equations of the following forms:

$$I_{0+}^{2\gamma}y(t) + j\cdot I_{0+}^{\gamma}y(t) + k\cdot y(t) = t^n,$$

and

$$I_{0+}^{3\gamma}y(t) + j\cdot I_{0+}^{2\gamma}y(t) + k\cdot I_{0+}^{\gamma}y(t) + l\cdot y(t) = t^n.$$

Here, I_{0+}^{γ} represents the (RL) fractional integral of order $\gamma = \frac{1}{2}, \gamma = 1$, and $\gamma = \frac{1}{3}, \gamma = 1$, respectively, $n \in \mathbb{N}\cup\{0\}$, and j, k, l are constants. To illustrate the effectiveness of our results, we provide examples. The solutions obtained using the Aboodh transform technique are demonstrated. We anticipate that these findings will stimulate further research in this specific field.

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