Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 4, December 2024, 1200–1215 DOI:10.12150/jnma.2024.1200

# On Nonlocal Neutral Stochastic Integro Differential Equations with Impulsive Random

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**Abstract** In this work, we discuss the existence and continuous dependence on initial data of solutions for non-local random impulsive neutral stochastic integrodifferential delayed equations. First, we prove the existence of mild solutions to the equations by using Krasnoselskii's-Schaefer type fixed point theorem. Next, we prove the continuous dependence on initial data results under the Lipschitz condition on a bounded and closed interval. Finally, we propose an example to validate the obtained results.

**Keywords** Existence, continuous dependence, random impulsive, integro differential equations, Krasnoselskii's-Schaefer type fixed point theorem

**MSC(2010)** 34K20, 34K45, 45J05.

## 1. Introduction

The theory of neutral differential equations (NDES) in Banach spaces has been studied by several authors [7], [9], [10], [12]. A neutral functional differential equation is one that includes both the current state of the system and the implied derivatives of the past history or functionals of the past history. When dealing with problems involving electric networks with lossless transmission lines, NDEs are required. Such networks first appeared, for instance, in high-speed computers where switching circuits were connected by lossless transmission lines. The problem's importance stems from the fact that it differs from the traditional initial condition in that it is more general and has a finer influence. The presence of solutions for neutral functional integrodifferential equations (IDEs) in Banach spaces was investigated by the authors [11], [13], [30]. The authors [5], [14], [29] proved that several classes of IDEs in abstract spaces exist as well as controllability results.

The impulses are either deterministic or random in that they occur at predetermined times or at random periods. There are numerous articles that examine the qualitative characteristics of fixed-type impulses [3], [4], [8], [15], [21], [24], [25], [26], [31]. but few that examine random-type impulses. The first random impulsive ordinary differential equations (ODEs) were presented by Wu and Meng [16], who also investigated the boundedness of these models' solutions using Liapunov's direct function. Some qualitative characteristics of differential equations (DEs) with random impulses have been researched by Wu et al. [17], [18], [22]. Anguraj et al. [2] established the stability of random impulsive stochastic functional DEs

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driven by Poisson jumps with finite delays by using Banach fixed point theorem. Li et al. [19] investigated the existence and Hyers-Ulam (HU) stability of mild solutions for random impulsive stochastic functional ODEs using Krasnoselskii's fixed point theorem. In Baleanu et al. [6] the existence, uniqueness, and HU (Hyers-Ulam) stability of random impulsive stochastic IDEs with nonlocal conditions have been investigated. By using Banach fixed point theorem,

$$
\begin{aligned}
d(z(t)) &= \big[\mathfrak{A}z(t) + f(t, z_t) + \int_0^t k(t-s)z(s)ds\big]dt + g(t, z_t)dW(t),\ t \geq t_0,\ t \neq \sigma_q,\\
z(\sigma_q) &= \ b_q(\delta_q)z(\sigma_q^-),\ q = 1, 2, ...,\\
z_0 &= z_{t_0} + r(z).
\end{aligned}
$$

Motivated by the above works, this paper aims to fill this gap by investigating the existence and continuous dependence on initial data of solutions of non-local random impulsive neutral stochastic integrodifferential equations (NRINSIDEs) with finite delays. By using Krasnoselskii's-Schaefer type fixed point theorem.

We consider the following NRINSIDEs with finite delays of the type

$$d[z(t) + h(t, z_t)] = \big[f(t, z_t) + \int_0^t k(t, s, z_s)ds\big]dt + g(t, z_t)dW(t), \tag{1.1}$$

$$z(\sigma_q) = b_q(\delta_q)z(\sigma_q^-),\ q = 1, 2, ..., \tag{1.2}$$

$$z_{t_0} + r(z) = z_0 = \sigma = \{\sigma(\theta) : -\delta \leq \theta \leq 0\}, \tag{1.3}$$

where $\delta_q$ is random variable defined from $\Omega$ to $\mathcal{D}_q \overset{def}{=} (0, d_q)$ for $q = 1, 2, ...,\ 0 < d_q < \infty$. Moreover, suppose that $\delta_\imath$ and $\delta_\jmath$ are independent of each other as $\imath \neq \jmath$ for $\imath, \jmath = 1,2....$ Here $f : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $h : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $g : [t_0, \mathcal{T}] \times \mathfrak{C} \times \to \mathbb{R}^{d\times m}$, $k : [t_0, \mathcal{T}] \times [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, $r : \mathfrak{C} \to \mathfrak{C}$ and $b_q : \mathcal{D}_q \to \mathbb{R}^{d\times d}$ are Borel measurable functions, and $z_t$ is $\mathbb{R}^d$-valued stochastic process such that

$$z_t = \{z(t + \theta) : -\delta \leq \theta \leq 0\},\ z_t \in \mathbb{R}^d.$$

We assume that $\sigma_0 = t_0$ and $\sigma_q = \sigma_{q-1} + \tau_q$ for $q = 1, 2, ....$ Obviously, $\{\sigma_q\}$ is a process with independent increments. The impulsive moments $\sigma_q$ from a strictly increasing sequence, i.e $\sigma = \sigma_0 < \sigma_1 < \sigma_2 < ... < \lim_{q\to\infty} \sigma_q = \infty$, and $z(\sigma_q^-) = \lim_{t\to\sigma_q-0} z(t)$. Denote by $\{\mathbb{G}(t), t \geq 0\}$ the simple counting process generated by $\{\sigma_q\}$, let $\{\mathbb{K}(t), t \geq 0\}$ be a given m-dimensional Wiener process, and denote $\mathfrak{F}_t^{(1)}$ the $\sigma$-algebra generated by $\{\mathbb{G}(t), t \geq 0\}$. Denote $\mathfrak{F}_t^{(2)}$ the $\sigma$-algebra generated by $\{\mathbb{K}(s), s \leq t\}$.

For considering the main Eq. (1.1), we have

$$d(x(0)) = 0.$$

Here, extra conditions have to be imposed to guarantee the existence of a solution, so we refer to Lemmas 3.1, 3.2 and 4.1 in [27], and also, see Lemma 3.4 in [28].

**Highlights:**

1. This work extends the work of A. Vinodkumar. [6].
2. Time delay of NRINSIDEs is taken care of by the prescribed phase space $\mathcal{B}$.

The structure of this article is as follows. In section 2, we mention some concepts and principles. Section 3 discusses the existence of solutions for NRINSIDEs with finite delays. Section 4 studies continuous dependence on initial data of NRINSIDEs with finite delays. An example to illustrate the obtained results is given in section 5. Finally, section 6 gives the conclusion with acknowledgements of the study.

## 2. Preliminaries and notations

Suppose that $(\Omega, \mathfrak{F}_t, \mathcal{P})$ is a probability space with filtration $\{\mathfrak{F}_t\},\ t \geq 0$ fulfilling $\mathfrak{F}_t = \mathfrak{F}_t^{(1)} \cup \mathfrak{F}_t^{(2)}$. Let $\pounds^p = (\Omega, \mathbb{R}^d)$ be the collection of all strongly measurable, $p^{th}$ integrable, $\mathfrak{F}_t$ measurable, $\mathbb{R}^d$-random variables in z with the norm $\|z\|_{\pounds_p} = (\mathbb{E}\|z\|_t^p)^{1/p}$, where the expectation $\mathbb{E}$ is denoted by $\mathbb{E}z = \int_\Omega z d\mathcal{P}$. Suppose that $\delta > 0$ and denote the Banach space of all piecewise continuous $\mathbb{R}^d$-valued stochastic process $\{\sigma(t),\ t \in [-\delta, 0]\}$ by $\mathfrak{C}([-\delta, 0], \pounds(\Omega, \mathbb{R}^d))$ random variables equipped with the norm

$$\|\psi\|_{\mathfrak{C}} = \Big( \sup_{-\delta \leq \theta \leq 0} \mathbb{E}\|\psi(\theta)\|_t^p \Big)^{1/p}.$$

The initial data

$$z_{t_0} + r(z) = z_0 = \sigma = \{\sigma(\theta) : -\delta \leq \theta \leq 0\} \tag{2.1}$$

is an $\mathfrak{F}_{t_0}$ - measurable, $\mathbb{R}^d$-valued random variable such that $\mathbb{E}\|\sigma\|^p \leq \infty$.

**Definition 2.1.** For a given $\mathcal{T} \in (t_0, \infty)$, an $\mathbb{R}^d$-valued stochastic process $z(t)$ on $t_0 - \delta \leq t \leq \mathcal{T}$ is called the solution to equations (1.1)-(1.3) with the initial data (2.1), if for each $t_0 \leq t \leq \mathcal{T}$, $z_{t_0} = \sigma$, $\{z_{t_0}\}_{t_0 \leq t \leq \mathcal{T}}$ is $\mathfrak{F}_t$-adapted and

$$\begin{aligned}
z(t) = \sum_{q=0}^{\infty} \Bigg[ & \prod_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(z) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath(\delta_\imath) h(t, z_t) \\
& + \sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s, z_s) ds + \int_{\sigma_q}^{t} f(s, z_s) ds \\
& + \sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} \int_0^s k(s, \varsigma, z_\varsigma) d\varsigma ds + \int_{\sigma_q}^{t} \int_0^s k(s, \varsigma, z_\varsigma) d\varsigma ds \\
& + \sum_{\imath=1}^{q} \prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) \int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s, z_s) dW(s) + \int_{\sigma_q}^{t} g(s, z_s) dW(s) \Bigg] I_{[\sigma_q, \sigma_{q+1})}(t),
\end{aligned}$$

where $\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath) = b_q(\delta_q) b_{q-1}(\delta_{q-1}), \ldots b_\imath(\delta_\imath)$, and $I_{\mathcal{L}}(.)$ is the index function, i.e.,

$$I_{\mathcal{L}}(t) = \begin{cases} 1 & if \quad t \in \mathcal{L}, \\ 0 & if \quad t \notin \mathcal{L}. \end{cases}$$

**Lemma 2.1.** *[23] For any $p \geq 1$ and for any predictable process $z \in \pounds^p_{d \times m}[0, T]$, the inequality holds,*

$$\sup_{s \in [0,t]} \mathbb{E}\|z(t) dw(t)\|^p \leq (p/2(p-1))^{p/2} \Big( \int_0^t (\mathbb{E}\|z(s)\|^p)^{2/p} ds \Big)^{p/2},\ t \in [0, T].$$

**Lemma 2.2.** *[20] (Krasnoselskiis-Schaefer type Fixed Point Theorem): Let $X$ be a Banach space and Let $P$ and $Q$ be two operator satisfying:*
*(1) $P$ is a contraction mapping, and*
*(2) $Q$ is completely continuous.*
*Then, either*
*(a) the operator $Pz + Qz = z$ has a solution, or*
*(b) the set $\Omega = \{v \in X : \lambda p(\frac{v}{\lambda}) + \lambda Qv = v, 0 < \lambda < 1\}$ is unbounded.*

# 3. Main results

We may take into consideration the following hypotheses:

(A1): The functions $h : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$ and $f : [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$. There exist positive constants $L_h > 0$, $L_f > 0$ and $L_g > 0$ such that,

$$\mathbb{E}\|h(t,\psi_1) - h(t,\psi_2)\|^p \leq L_h \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},$$

$$\mathbb{E}\|h(t,\psi)\|^p \leq L_h \mathbb{E}\|\psi\|^p_{\mathfrak{C}}.$$

$$\mathbb{E}\|f(t,\psi_1) - f(t,\psi_2)\|^p \leq L_f \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},$$

$$\mathbb{E}\|f(t,\psi)\|^p \leq L_f \mathbb{E}\|\psi\|^p_{\mathfrak{C}}.$$

(A2): The function $g : [t_0, T] \times \mathcal{C} \to \mathbb{R}^{d\times m}$ fulfills:
(I) For each $t \in [t_0, T]$, the function $g(t, .) : \mathcal{C} \to \mathbb{R}^{d\times m}$ is continuous, and for each $\psi \in \mathcal{C}$, the function $g(., \psi) : [t_0, T] \to \mathbb{R}^{d\times m}$ is measurable.

(II)There exists a continuous function $m(t) : [t_0, T] \to [0, +\infty)$, and a $L^q$ integrable, continuous, increasing function $\Theta : [0, +\infty) \to [0, +\infty)$ such that

$$\mathbb{E}\|g(t,\psi)\|^p \leq m(t)\Theta(\|\psi\|^p_{\mathcal{C}}),$$

for arbitrary $(t, \psi) \in [t_0, T] \times \mathcal{C}, m^* = \sup_{t\in[t_0,T]} m(t)$, and the function $\Theta$ satisfies

$$\lim_{\tau\to\infty} \inf \frac{\Theta(\tau)}{\tau} = \gamma < \infty.$$

(A3): The function $k : [t_0, \mathcal{T}] \times [t_0, \mathcal{T}] \times \mathfrak{C} \to \mathbb{R}^d$, there exists a positive constant $L_k > 0$ such that,

$$\int_0^t \mathbb{E}\|k(t,s,\psi_1) - k(t,s,\psi_2)\|^p \leq L_k \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},$$

$$\int_0^t \mathbb{E}\|k(t,s,\psi)\|^p \leq L_k \mathbb{E}\|\psi\|^p_{\mathfrak{C}},$$

for all $t \in [t_0, \mathcal{T}]$ and $\psi_1, \psi_2$ *and* $\psi \in \mathfrak{C}$.

(A4): The condition $\max\limits_{\imath,q}\{\prod\limits_{\jmath=\imath}^{q} \|b_\jmath(\tau_\jmath)\|\} < \infty$. That is to say, there exists a constant $C > 0$ such that

$$\mathbb{E}\bigg(\max_{\imath,q}\{\prod_{\jmath=\imath}^{q} \|b_\imath(\tau_\jmath)\|\}\bigg)^p \leq C.$$

(A5): The function $r : \mathfrak{C} \to \mathfrak{C}$ is continuous and there exists some constant $L_r > 0$ such that,

$$\mathbb{E}\|r(t,\psi_1) - r(t,\psi_2)\|^p \leq L_r \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},$$

$$\mathbb{E}\|r(t,\psi)\|^p \leq L_r \mathbb{E}\|\psi\|^p_{\mathfrak{C}},$$

for all $t \in [t_0, \mathcal{T}]$ and $\psi_1, \psi_2$ *and* $\psi \in \mathfrak{C}$.

**Theorem 3.1.** *Assume that the assumptions (A1)–(A5) are satisfied. Then there exists at least one mild solution to systems* (1.1)-(1.3) *provided that*

$$4^{p-1}C(L_r + L_h) + 4^{p-1}\max\{1, C\}(t - t_0)^p(L_f + L_k) < 1.$$

**Proof.** Let $\mathcal{B}$ be the phase space $\mathcal{B} = \mathfrak{C}([t_0 - \delta, \mathcal{T}], \mathcal{L}^p(\Omega, \mathbb{R}^d))$ endowed with the norm

$$\|z\|^p_{\mathcal{B}} = \sup_{t\in[t_0,\mathcal{T}]} \|z_t\|^p_{\mathfrak{C}},$$

where $\|z_t\|_{\mathfrak{C}} = \sup_{-\delta\leq s\leq t} \mathbb{E}\|z_t\|^p$. Denote $B_m = \{z \in \mathcal{B}, \|z\|^p_{\mathcal{B}} \leq m\}$, which is the closed ball with Center $z$ and radius $m > 0$. For any initial value $(t_0, z_0,)$ with $t_0 \geq 0$ and $z_0 \in B_m$, we define the operator $S : \mathcal{B} \to \mathcal{B}$ by
$(Sz)(t) =$

$$\begin{cases} \sigma(t) - r(t), \qquad t \in (-\infty, t_0] \\ \sum\limits_{q=0}^{\infty} \Big[ \prod\limits_{\imath=1}^{q} b_\imath(\delta_\imath)\sigma(0) - r(t) + h(0,\sigma) \\ - \prod\limits_{\imath=1}^{q} b_\imath[(\delta_\imath)h(t,z_t) + \sum\limits_{\imath=1}^{q}\prod\limits_{\jmath=\imath}^{q} b_\jmath(\tau_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} f(s,z_s)ds \\ + \int_{\sigma_q}^{t} f(s,z_s)ds + \sum\limits_{\imath=1}^{q}\prod\limits_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma ds + \int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma \\ + \sum\limits_{\imath=1}^{q}\prod\limits_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath} g(s,z_s)dW(s) + \int_{\sigma_q}^{t} g(s,z_s)dW(s)\Big] I_{[\sigma_q,\sigma_{q+1})}(t), \;\; t \in [t_0,\mathcal{T}]. \end{cases}$$

Then the problem of finding mild solutions for problems (1.1)-(1.3) is reduced to finding the fixed point of S. Decomposing the operator S, we obtain

$$\begin{aligned}(Pz)(t) = \sum_{q=0}^{\infty}\Big[&\prod_{i=1}^{q} h_i(\tau_i)\sigma(0) - r(t) + h(0,\sigma) \\ &+ \int_{\sigma_q}^{t} f(s,z_s)ds - \prod_{\imath=1}^{q} b_\imath[(\delta_\imath)h(t,z_t) + \sum_{i=1}^{q}\prod_{j=\imath}^{q} h_j(\tau_j)\int_{\sigma_{i-1}}^{\sigma_i} f(s,z_s)ds \\ &+ \sum_{i=1}^{q}\prod_{j=i}^{q} h_j(\tau_j)\int_{\sigma_{i-1}}^{\sigma_i}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma ds + \int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma ds\Big] I_{[\sigma_q,\sigma_{q+1})}(t).\end{aligned}$$

$$(Qz)(t) = \sum_{q=0}^{\infty}\Big[\sum_{i=1}^{q}\prod_{j=i}^{q} h_j(\tau_j)\int_{\sigma_{i-1}}^{\sigma_i} g(s,z_s)dW(s) + \int_{\sigma_q}^{t} g(s,z_s)dW(s)\Big] I_{[\sigma_q,\sigma_{q+1})}(t).$$

We divide the proof into the following steps:
**Step 1.** We prove that P is a contraction mapping.

Let $z, w \in B_r$ and $t \in [t_0-\tau, T]$. Then

$$\mathbb{E}\|(Pz)(t)-(Pw)(t)\|^p$$

$$\leq 4^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\|r(z)-r(w)\|I_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p$$

$$+4^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\|h(t,z_t)-h(t,w_t)\|I_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p$$

$$+4^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\|f(s,z_s)-f(s,w_s)\|dsI_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p$$

$$+4^{p-1}\mathbb{E}\Big[\max_{\imath,q}\Big\{1,\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p\Big[\int_{t_0}^{t}\int_0^s\|k(s,\varsigma,z_\varsigma)-k(s,\varsigma,z_\varsigma)\|d\varsigma dsI_{[\sigma_q,\sigma_{q+1})}(t)\Big]^p$$

$$\leq 4^{p-1}CL_r\mathbb{E}\|z-w\|_{\mathfrak{C}}^p+4^{p-1}CL_h\mathbb{E}\|z-w\|_{\mathfrak{C}}^p$$
$$+2^{p-1}\max\{1,C\}(t-t_0)^pL_f\mathbb{E}\|z_s-w_s\|_{\mathfrak{C}}^pds+4^{p-1}\max\{1,C\}(t-t_0)^pL_k\mathbb{E}\|z_s-w_s\|_{\mathfrak{C}}^pds$$
$$\leq 4^{p-1}C(L_r+L_h)+4^{p-1}\max\{1,C\}(t-t_0)^p(L_f+L_k)\|z_t-w_t\|_{\mathfrak{C}}^p,$$

where

$$\|z_t-w_t\|_{\mathfrak{C}}^p=\sup_{t\in[-\tau,0]}\mathbb{E}\|z(t+\theta)-w(t+\theta)\|^p=\sup_{t\in[t_0-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p. \quad (3.1)$$

Taking the supremum over z, by (6), we obtain

$$\|(Pz)(t)+(Pz)(t)\|_{\mathcal{B}}^p\leq G\|z-w\|_{\mathcal{B}}^p,$$

where $G=4^{p-1}C(L_r+L_h)+4^{p-1}\max\{1,C\}(t-t_0)^p(L_f+L_k)$.
Since $0<G<1$. P is a contraction on $B_r$.

**Step 2.** Next, we prove that Q is completely continuous.

**Step 2.1.** We first prove that Q is continuous on $B_r$.

Let $\{z^n\}\subset B_r$ with $z^n\to z$ (as $n\to\infty$). For $t\in[t_0,T]$, and by the continuity of g in (A2)(I), we have

$$\mathbb{E}\|(Qz^n)(t)+(Qw)(t)\|^p\leq\mathbb{E}\Big\|\sum_{q=0}^{\infty}\Big[\sum_{i=1}^{q}\prod_{j=i}^{q}h_j(\tau_j)\int_{\sigma_{i-1}}^{\sigma_i}\big[g(s,z_s^n)-g(s,z_s)\big]dW(s)$$
$$+\int_{\sigma_q}^{t}\big[g(s,z_s^n)-g(s,z_s)\big]dW(s)\Big]I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p$$
$$\leq\max\{1,C\}(t-t_0)^pL_p\int_{t_0}^{t}\|g(s,z_s^n)-g(s,z_s)\|^pds$$
$$\longrightarrow 0 \ as\ n\to\infty.$$

Thus

$$\mathbb{E}\|(Qz^n)(t)+(Qz)(t)\|_{\mathcal{B}}^p\to 0 \quad (n\to\infty),$$

and Q is continuous on $B_r$.

**Step 2.2.** We prove that Q maps bounded sets $B_r$ into equicontinuous sets of $B_r$.
Since $B$ is a piecewise space, we assume that $\sigma_q < t_1 < t_2 < \sigma_{q+1},\ q = 1, 2, 3, ....$ and $z \in B_r$. Then for any fixed $z \in B_r$, by using hypotheses (A2)–(A4) and lemma 2.2, we have

$$\begin{aligned}\mathbb{E}\|(Qz)(t_1)+(Qw)(t_2)\|^p &= 2^{p-1}\mathbb{E}\Big\|\sum_{q=0}^{\infty}\int_{t_1}^{t_2}\big[g(s,z_s)dW(s)I_{[\sigma_q,\sigma_{q+1})}(t_2)\Big\|^p\\ &\leq 2^{p-1}(t_2-t_1)^{p/2-1}L_p\int_{t_1}^{t_2}m(t)\Theta(\|z_s\|)^p ds\\ &\longrightarrow 0\ as\ (t_2\to t_1).\end{aligned}$$

Thus, Q maps bounded sets $B_r$ into equicontinuous sets of $B_r$.

**Step 2.3.** We prove that Q$(B_r)$ is uniformly bounded. It follows from

$$\begin{aligned}\sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\|(Qz)(t)\|^p &= \sup_{t\in[t_0,\mathcal{T}]}\mathbb{E}\Big\|\sum_{q=0}^{\infty}\Big[\sum_{i=1}^{q}\prod_{j=i}^{q}h_j(\tau_j)\int_{\sigma_{i-1}}^{\sigma_i}\big[g(s,z_s)dW(s)\\ &+\int_{\sigma_q}^{t}\big[g(s,z_s)dW(s)I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p\\ &\leq \sup_{t\in[t_0,\mathcal{T}]}\max\{1,C\}\mathbb{E}\Big\|\int_{t_0}^{t}\big[g(s,z_s)dW(s)I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p\\ &\leq \max\{1,C\}(T-t_0)^{p/2}L_p\|m^*\|_{L_q}\Theta(r).\end{aligned}$$

Then $\{Q(B_r)\}$ is uniformly bounded.

**Step 2.4.** We show that $Q$ maps $B_r$ into a precompact set for every $t \in [t_0, T]$. Let $t_0 < t < T$ be fixed and let $\epsilon$ be a positive real number such that $0 < \epsilon < t - t_0$. For $z \in B_r$, we consider that

$$\begin{aligned}(Q^\epsilon z)(t) &= \sum_{q=0}^{\infty}\Big[\sum_{i=1}^{q}\prod_{j=i}^{q}h_j(\tau_j)\int_{\sigma_{i-1}}^{\sigma_i}\big[g(s,z_s)dW(s)\\ &+\int_{\sigma_q}^{t-\epsilon}\big[g(s,z_s)dW(s)\big]I_{[\sigma_q,\sigma_{q+1})}(t),\ t\in(t_0,t-\epsilon).\end{aligned}\tag{3.2}$$

The set $U_\epsilon(t) = (Q^\epsilon z)(t) : z \in B_r$ is relatively compact in $\mathcal{B}$ for every $\epsilon \in (0, t-t_0)$. We then have

$$\begin{aligned}\mathbb{E}\|(Qz)(t)+Q^\epsilon z(t)\|^p &\leq \mathbb{E}\Big\|\sum_{q=0}^{\infty}\int_{t-\epsilon}^{t}\big[g(s,z_s)dW(s)I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p\\ &\leq (\epsilon)^{p/2-1}L_p\int_{t-\epsilon}^{t}m^*\Theta(r)ds.\end{aligned}\tag{3.3}$$

As $\epsilon \to 0$, the right hand side of the inequality (8) tends to zero. Thus, there are precompact sets arbitrarily close to the set $U(t) = \{(Qz)(t) : z \in B_r\}$, and $U(t)$ is relatively compact in $\mathcal{B}$. Thus, by the Arzela-Ascoli theorem, we deuce that $Q$ is compact, and $Q$ is completely continuous.

Let

$$\begin{aligned}
\mathfrak{M}_1 &= 6^{p-1}C[\mathbb{E}\|\sigma(0)\|^p + L_h\mathbb{E}\|\sigma\|^p],\\
\mathfrak{M}_2 &= 6^{p-1}C(L_r + L_h),\\
\mathfrak{M}_3 &= 6^{p-1}\max\{1, C\}(t-t_0)^p.
\end{aligned}$$

To apply the Krasnoselskii-Schaefer theorem, it remains to show that the set

$$\Omega(S) = \{z(.) : \lambda P(\frac{z}{\lambda}) + \lambda Qz = z\}$$

is bounded for $\lambda \in (0,1)$. To this end, let $z(.) \in \Omega(S)$. Then $\lambda P(\frac{z}{\lambda}) + \lambda Qz = z$ for some $\lambda \in (0,1)$ and

$$\begin{aligned}
\mathbb{E}\|z(t)\|^p &\le |\lambda|\mathbb{E}\big\|\big(P(\frac{z}{\lambda})\big)(t) + (Qz)(t)\|^p\\
&\le \mathbb{E}\big\|\big(P(\frac{z}{\lambda})\big)(t) + (Qz)(t)\|^p.
\end{aligned}$$

For every $t \in [t_0, T]$, we have

$$\begin{aligned}
\mathbb{E}\|z(t)\|^p \le{}& 5^{p-1}\mathbb{E}\Big\|\sum_{q=0}^{\infty}\Big[\prod_{\nu=1}^{q}h_i(\tau_i)\sigma(0) - r(t) + h(0,\sigma)\\
&-\prod_{\imath=1}^{q}b_\imath[(\delta_\imath)h(t,\frac{z_t}{\lambda})\Big\|^p + 5^{p-1}\mathbb{E}\Big\|\sum_{q=0}^{\infty}\Big[\sum_{i=1}^{q}\prod_{j=i}^{q}h_j(\tau_j)\int_{\sigma_{i-1}}^{\sigma_i}f(s,\frac{z_s}{\lambda})ds\\
&+\int_{\sigma_q}^{t}f(s,\frac{z_s}{\lambda})ds\Big]I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p + 5^{p-1}\mathbb{E}\Big\|\sum_{q=0}^{\infty}\Big[\sum_{i=1}^{q}\prod_{j=i}^{q}h_j(\tau_j)\\
&\times\int_{\sigma_{i-1}}^{\sigma_i}\int_0^s k(s,\varsigma,\frac{z_\varsigma}{\lambda})d\varsigma ds + \int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,\frac{z_\varsigma}{\lambda})d\varsigma ds\Big]I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p\\
&+5^{p-1}\mathbb{E}\Big\|\sum_{q=0}^{\infty}\Big[\sum_{i=1}^{q}\prod_{j=i}^{q}h_j(\tau_j)\int_{\sigma_{i-1}}^{\sigma_i}g(s,z_s)dW(s)\\
&+\int_{\sigma_q}^{t}g(s,z_s)dW(s)\Big]I_{[\sigma_q,\sigma_{q+1})}(t).\Big\|^p\\
={}& 5^{p-1}\sum_{i=1}^{6}R_i,
\end{aligned}$$

where

$$\begin{aligned}
R_1 ={}& \mathbb{E}\Big[\max_{\imath,q}\Big\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Big\}\Big]^p[\|\sigma(0) - r(z) + h(0,\sigma)\|^p]\\
&\le C[\mathbb{E}\|\sigma(0)\|^p + L_r\mathbb{E}\|z\|^p] + CL_h\mathbb{E}\|\sigma\|^p,
\end{aligned}$$

$$R_2 = \mathbb{E}\bigg[\max_{\imath,q}\bigg\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\bigg\}\bigg]^p\|h(t,z_t)\|^p \leq CL_h\mathbb{E}\|z_t\|^p_{\mathfrak{C}},$$

$$R_3 = \mathbb{E}\bigg[\max_{i,q}\bigg\{1,\prod_{i=j}^{q}\|b_\jmath(\delta_\jmath)\|\bigg\}\bigg]^p\bigg[\int_{t_0}^{t}\mathbb{E}\|f(s,z_s)\|dsI_{[\sigma_q,\sigma_{q+1})}(t)\bigg]^p$$

$$\leq \max\{1,C\}(t-t_0)^pL_f\|z_s\|^p_{\mathfrak{C}},$$

$$R_4 = \mathbb{E}\bigg[\max_{i,q}\bigg\{1,\prod_{i=j}^{q}\|b_\jmath(\delta_\jmath)\|\bigg\}\bigg]^p\bigg[\int_{t_0}^{t}\int_0^s\mathbb{E}\|k(s,\varsigma,z_\varsigma)\|d\varsigma dsI_{[\sigma_q,\sigma_{q+1})}(t)\bigg]^p$$

$$\leq \max\{1,C\}(t-t_0)^pL_k\|z_s\|^p_{\mathfrak{C}},$$

$$R_5 = \mathbb{E}\bigg[\max_{i,q}\bigg\{1,\prod_{i=j}^{q}\|b_\jmath(\delta_\jmath)\|\bigg\}\bigg]^p\bigg[\int_{t_0}^{t}\mathbb{E}\|g(s,z_s)\|dW(s)I_{[\sigma_q,\sigma_{q+1})}(t)\bigg]^p$$

$$\leq \max\{1,C\}(t-t_0)^{p/2-1}L_p\int_{t_0}^{t}m(t)\Theta(\|z_s\|^p_{\mathfrak{C}})ds,$$

$$\mathbb{E}\|z(t)\|^p \leq 5^{p-1}C[\mathbb{E}\|\sigma(0)\|^p+L_r\mathbb{E}\|z\|^p]+5^{p-1}CL_h\mathbb{E}\|\sigma\|^p+5^{p-1}CL_h\mathbb{E}\|z_t\|^p_{\mathfrak{C}}$$
$$+5^{p-1}\max\{1,C\}(t-t_0)^pL_f\mathbb{E}\|z_s\|^p_{\mathfrak{C}}ds+5^{p-1}\max\{1,C\}(t-t_0)^p$$
$$\times L_k\mathbb{E}\|z_s\|^p_{\mathfrak{C}}+5^{p-1}\max\{1,C\}(t-t_0)^{p/2}L_p\int_{t_0}^{t}m(t)\Theta(\|z_s\|^p_{\mathfrak{C}})ds.$$

Thus

$$\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(t)\|^p \leq 5^{p-1}C[\mathbb{E}\|\sigma(0)\|^p+L_h\mathbb{E}\|\sigma\|^p]+5^{p-1}C(L_r+L_h)\mathbb{E}\|z_s\|^p_{\mathcal{B}}$$
$$+5^{p-1}\max\{1,C\}(t-t_0)^p\big(L_f+L_k\big)\sup_{s\in[t-\delta,t]}\mathbb{E}\|z_s\|^p_{\mathfrak{C}}$$
$$+5^{p-1}\max\{1,C\}(T-t_0)^{p/2-1}L_p\int_{t_0}^{t}\sup_{t\in[t_0,T]}m(t)\Theta(\|z_s\|^p_{\mathcal{C}})ds.$$

$$\|z(t)\|^p_{\mathcal{B}} \leq 5^{p-1}C[\mathbb{E}\|\sigma(0)\|^p+L_h\mathbb{E}\|\sigma\|^p]+6^{p-1}C(L_r+L_h)\mathbb{E}\|z_s\|^p_{\mathcal{B}}+5^{p-1}\max\{1,C\}$$
$$\times(t-t_0)^p\big[L_f+L_k+(t-t_0)^{p/2-2}L_p\int_{t_0}^{t}\frac{m^*}{\|z_s\|^p_{\mathfrak{C}}}\Theta(\|z_s\|^p_{\mathfrak{C}})ds\big]\mathbb{E}\|z_s\|^p_{\mathcal{B}},$$

where $L_p = (p(p-1)/2)^{p/2}$. Notice that $\sup_{t\in[t_0,T]}\mathbb{E}\|z_t\|^p_{\mathcal{C}} = \sup_{t\in[t_0-\tau,T]}\|z(t)\|^p \leq \|z(t)\|^p_{\mathcal{B}}$, and by (A4)(II), we get

$$\|z(t)\|^p_{\mathcal{B}} \leq \mathfrak{M}_1+\mathfrak{M}_2\mathbb{E}\|z_s\|^p_{\mathcal{B}}+\mathfrak{M}_3\big[L_f+L_k+(t-t_0)^{p/2-2}L_pm^*\gamma\big]\mathbb{E}\|z_s\|^p_{\mathcal{B}},$$

$$\|z\|^p_{\mathcal{B}} \leq \frac{\mathfrak{M}_1}{1-\big[\mathfrak{M}_2+\mathfrak{M}_3\big(L_f+L_k+(t-t_0)^{p/2-2}L_pm^*\gamma\big)\big]} = \mathfrak{A}.$$

This, implies that the set $\Omega(S) = \{z(.) : \lambda P(\frac{z}{\lambda})+\lambda Qz = z\}$ is bounded for $\lambda \in (0,1)$. Hence, by Krasnoselskii-Schaefer fixed point theorem, $S$ has a fixed point, which is the required mild solution of equations (1.1)-(1.3). □

Next, we use the Banach contraction principle to give another proof of existence for the solution of equations (1.1)-(1.3). We make the following assumption:

$(A'4)$ For the continuous $g \in \mathcal{L}^p([t_0, T] \times \mathfrak{C} \to \mathbb{R}^{d\times m})$ , there exists a positive constant $L_g > 0$ such that

$$\mathbb{E}\|g(t,\psi_1) - g(\zeta,\psi_2)\|^p \le L_g \mathbb{E}\|\psi_1 - \psi_2\|^p_{\mathfrak{C}},$$

for all $t \in [t_0, \mathcal{T}]$ and $\psi_1, \psi_2 \in \mathfrak{C}$. Under the assumptions (A1), (A3) and $(A'4)$. We have the following theorem.

**Theorem 3.2.** *If the hypotheses (A1), (A3) and (A′4) hold, then there exists a unique mild solution to equations* (1.1)-(1.3).

**Proof.** For any initial value $(t_0, z_0)$ $t_0 \le 0$ $z_0 \in B_r$, we define the operator $S : \mathcal{B} \to \mathcal{B}$ as follows: $(Sz)(t + t_0) = \sigma(\phi) \in \mathcal{L}^p(\Omega, \mathcal{C}),\ t \in [t_0, T]$.

$$\begin{aligned}(Sz)(t) &= \sum_{q=0}^{\infty}\Bigg[\prod_{i=1}^{q} h_i(\tau_i)\sigma(0) - r(t) + h(0,\sigma) - \prod_{\imath=1}^{q} b_\imath[(\delta_\imath)h(t,z_t)\\
&\quad + \sum_{i=1}^{q}\prod_{j=i}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{i-1}}^{\sigma_i} f(s,z_s)ds + \int_{\sigma_q}^{t} f(s,z_s)ds + \sum_{i=1}^{q}\prod_{j=i}^{q} b_\jmath(\delta_\jmath)\\
&\quad \times \int_{\sigma_{i-1}}^{\sigma_i}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma ds + \int_{\sigma_q}^{t}\int_0^s k(s,\varsigma,z_\varsigma)d\varsigma ds + \sum_{i=1}^{q}\prod_{j=i}^{q} b_\jmath(\delta_\jmath)\\
&\quad \times \int_{\sigma_{i-1}}^{\sigma_i} g(s,z_s)dW(s) + \int_{\sigma_q}^{t} g(s,z_s)dW(s)\Bigg] I_{[\sigma_q,\sigma_{q+1})}(t),\ t \in [t_0, T],\end{aligned}$$

$$\begin{aligned}&\mathbb{E}\|(Sz)(t) - (Sw)(t)\|^p\\
&\le 5^{p-1}\mathbb{E}\Bigg[\max_{\imath,q}\Bigg\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Bigg\}\Bigg]^p\Bigg[\|r(z){-}r(w)\|I_{[\sigma_q,\sigma_{q+1})}(t)\Bigg]^p\\
&+5^{p-1}\mathbb{E}\Bigg[\max_{\imath,q}\Bigg\{\prod_{\imath=\jmath}^{q}\|b_\jmath(\delta_\jmath)\|\Bigg\}\Bigg]^p\Bigg[\|h(t,z_t){-}h(t,w_t)\|I_{[\sigma_q,\sigma_{q+1})}(t)\Bigg]^p\\
&\le 5^{p-1}\mathbb{E}\Bigg[\max_{i,q}\Bigg\{1,\prod_{i=j}^{q}\|h_j(\tau_j)\|\Bigg\}\Bigg]^p\Bigg[\int_{t_0}^{t}\mathbb{E}\|f(s,z_s){-}f(s,w_s)\|ds\Bigg]^p\\
&+5^{p-1}\mathbb{E}\Bigg[\max_{i,q}\Bigg\{1,\prod_{i=j}^{q}\|h_j(\tau_j)\|\Bigg\}\Bigg]^p\Bigg[\int_{t_0}^{t}\int_0^s\mathbb{E}\|k(s,\varsigma,z_\varsigma){-}k(s,\varsigma,w_\varsigma)\|d\varsigma ds\Bigg]^p\\
&+5^{p-1}\mathbb{E}\Bigg[\max_{i,q}\Bigg\{1,\prod_{i=\jmath}^{q}\|h_j(\tau_j)\|\Bigg\}\Bigg]^p\Bigg[\int_{t_0}^{t}\mathbb{E}\|g(s,z_s){-}g(s,w_s)\|dW(s)\Bigg]^p\end{aligned}$$

$$\begin{aligned}&\le 5^{p-1}CL_r\mathbb{E}\|z-w\|^p_{\mathfrak{C}} + 5^{p-1}CL_h\mathbb{E}\|z-w\|^p_{\mathfrak{C}} + 5^{p-1}\max\{1,C\}\\
&\times (t-t_0)^p L_f\mathbb{E}\|z_s - w_s\|^p_{\mathfrak{C}}ds + 5^{p-1}\max\{1,C\}(t-t_0)^p L_k\mathbb{E}\|z_s\end{aligned}$$

$$
\begin{aligned}
&- w_s\|^p_{\mathfrak{C}} ds + 5^{p-1}\max\{1,C\}(t-t_0)^{p/2}L_pL_g\mathbb{E}\|z_s - w_s\|^p_{\mathfrak{C}} ds \\
&\leq \big\{5^{p-1}C(L_r+L_h) + 5^{p-1}\max\{1,C\}\big[\big(t-t_0)^pL_f + (t-t_0)^pL_k \\
&+ (t-t_0)^{p/2}L_pL_g\big]\big\} \sup_{\theta\in[-\delta,0]} \mathbb{E}\|z(t+\theta) - w(t+\theta)\|^p_{\mathfrak{C}} \\
&\leq \Bigg[5^{p-1}C(L_r+L_h) + 5^{p-1}\max\{1,C\}\big[\big(t-t_0)^pL_f + (t-t_0)^pL_k \\
&+ (t-t_0)^{p/2}L_pL_g\big]\Bigg] \sup_{s\in[t-\delta,t]} \mathbb{E}\|z(s) - w(s)\|^p_{\mathfrak{C}}.
\end{aligned}
$$

Taking the supremum over $t$, we get

$$\|(Sz)(t) - (Sw)(t)\|^p_{\mathcal{B}} \leq \mathfrak{A}(\mathcal{T})\mathbb{E}\|z(s) - w(s)\|^p_{\mathcal{B}},$$

with

$$\mathfrak{A}(\mathcal{T}) = 6^{p-1}C(L_r+L_h)+6^{p-1}\max\{1,C\}\big[\big(t-t_0)^pL_f+(t-t_0)^pL_k+(t-t_0)^{p/2}L_pL_g\big].$$

By taking a suitable $0 < \mathcal{T}_1 < \mathcal{T}$ sufficiently small where $\mathfrak{A}(\mathcal{T}) < 1$, hence $S$ is a contraction on $\mathcal{B}_{\mathcal{T}_1}$. $Sz = z$ is a unique solution of equations (1.1)-(1.3) by Banach fixed point theorem. □

# 4. Continuous dependence on initial data

**Theorem 4.1.** *If the assumptions of Theorem 3.2 are satisfied and*

$$2^{p-1}\max\{1,C\}\big[(L_1+L_2)(t-t_0)^p + L_pL_3(t-t_0)^{p/2}\big] < 1,$$

*then* $\forall\sigma,\bar{\sigma}\in\mathcal{B}$ *and for the corresponding mild solutions* $z,\bar{z}$ *of equations* (1.1)-(1.3) *the following inequality holds*

$$\|z-w\|^p_{\mathcal{B}} \leq \frac{\mathfrak{L}_1}{1-\big(\mathfrak{L}_2 + \mathfrak{M}\big[(T-t_0)^p(L_f+L_k) + (T-t_0)^{p/2}L_pL_g\big]\big)}\mathbb{E}\|\sigma_1-\sigma_2\|^p,$$

*where* $\mathfrak{L}_1 = 5^{p-1}C(1+L_h), \mathfrak{L}_2 = 5^{p-1}C(L_r+L_h)$ *and* $\mathfrak{M} = 5^{p-1}\max\{1,C\}$.

**Proof.** Let $\sigma,\bar{\sigma}\in\mathcal{B}$ be arbitrary functions and let $z,\bar{z}$ be the mild solutions of equations (1.1)-(1.3). Then we have

$$
\begin{aligned}
z(t) - w(t) = \sum_{q=0}^{\infty}\Bigg[&\prod_{\imath=1}^{q} b_\imath(\delta_\imath)[\sigma_1-\sigma_2] + [r(z)-r(w)] \\
&+ [h(0,\sigma_1) - h(0,\sigma_2)] + \prod_{\imath=1}^{q} b_\imath(\delta_\imath)[h(t,z_t) - h(t,w_t)] \\
&+ \Bigg[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}[f(s,z_s) - f(s,w_s)]ds + \int_{\sigma_q}^{t}[f(s,z_s)-f(s,w_s)]ds\Bigg] \\
&+ \Bigg[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}\int_0^s [k(s,\varsigma,z_\varsigma) - k(s,\varsigma,w_\varsigma)]d\varsigma ds
\end{aligned}
$$

$$+\int_{\sigma_q}^{t}\int_0^s [k(s,\varsigma,z_\varsigma)-k(s,\varsigma,w_\varsigma)]d\varsigma\Big] + \Big[\sum_{\imath=1}^{q}\prod_{\jmath=\imath}^{q} b_\jmath(\delta_\jmath)\int_{\sigma_{\imath-1}}^{\sigma_\imath}[g(s,z_s)$$
$$-g(s,w_s)]dW(s)+\int_{\sigma_q}^{t}[g(s,z_s)-g(s,w_s)]dW(s)\Big]\Big] I_{[\sigma_q,\sigma_{q+1})}(t)\Big\|^p.$$

Then,

$$\begin{aligned}\mathbb{E}\|z(t)-w(t)\|^p &\le 5^{p-1}C(1+L_h)\mathbb{E}\|\sigma_1-\sigma_2\|^p\\ &+5^{p-1}CL_r\mathbb{E}\|z-w\|^p+5^{p-1}C^pL_h\mathbb{E}\|z(t)-w(t)\|^p\\ &+5^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_f\int_{t_0}^{t}\mathbb{E}\|z(s)-w(s)\|^p ds\\ &+5^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_k\int_{t_0}^{t}\mathbb{E}\|z(s)-w(s)\|^p ds\\ &+5^{p-1}\max\{1,C\}(t-t_0)^{p/2-1}L_gL_p\int_{t_0}^{t}\mathbb{E}\|z(s)-w(s)\|^p ds.\end{aligned}$$

Furthermore,

$$\begin{aligned}&\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(t)-w(t)\|^p\\ &\le 5^{p-1}C(1+L_h)\mathbb{E}\|\sigma_1-\sigma_2\|^p+5^{p-1}C(L_r+L_h)\sup_{t\in[t-\tau,t]}\mathbb{E}\|z(t)-w(t)\|^p\\ &+5^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_f\int_{t_0}^{t}\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds\\ &+5^{p-1}\max\{1,C\}(t-t_0)^{p-1}L_k\int_{t_0}^{t}\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds\\ &+5^{p-1}\max\{1,C\}(t-t_0)^{p/2-1}L_gL_p\int_{t_0}^{t}\sup_{s\in[t-\tau,t]}\mathbb{E}\|z(s)-w(s)\|^p ds.\end{aligned}$$

Define the function $y:[-\tau,T]\to\mathbb{R}$ by

$$y(t)=\big\{\sup\mathbb{E}\|z_t-w_t\|^p : -\tau\le s\le t\big\},\ t\in[0,T].$$

Let $t^*\in[-\tau,t]$ be such that

$$y(t)=\mathbb{E}\|z_{t^*}-w_{t^*}\|^p.$$

If $t^*\in[0,t]$, then from the above, we have

$$\begin{aligned}y(t)&=\mathbb{E}\|z_{t^*}-w_{t^*}\|^p\le\mathfrak{L}\mathbb{E}\|\sigma_1-\sigma_2\|^p\\ &+5^{p-1}C(L_r+L_h)\mathbb{E}\|z-w\|^p+\mathfrak{M}(T-t_0)^{p-1}(L_f+L_k)\\ &\times\int_{t_0^*}^{t^*}\mathbb{E}\|z_s-w_s\|_{\mathfrak{C}}^p ds+\mathfrak{M}(T-t_0)^{p/2-1}L_pL_g\int_{t_0^*}^{t^*}\mathbb{E}\|z_s-w_s\|_{\mathfrak{C}}^p ds,\end{aligned}$$

$$y(t)\le\ \mathfrak{L}_1\mathbb{E}\|\sigma_1-\sigma_2\|^p+\mathfrak{L}_2y(t)+\mathfrak{M}(T-t_0)^p(L_f+L_k)y(t)+\mathfrak{M}(T-t_0)^{p/2}L_pL_gy(t),$$
$$y(t)\le\mathfrak{L}1\mathbb{E}\|\sigma_1-\sigma_2\|^p+\big[\mathfrak{L}_2+\mathfrak{M}(T-t_0)^p(L_f+L_k)+\mathfrak{M}(T-t_0)^{p/2}L_pL_g\big]\|z-w\|_{\mathcal{B}}^p.$$

Hence, we get,

$$\|z-w\|^p_{\mathcal{B}} \leq \frac{\mathfrak{L}_1}{1-\big(\mathfrak{L}_2+\mathfrak{M}\big[(T-t_0)^p(L_f+L_k+(T-t_0)^{p/2}L_pL_3\big]\big)}\mathbb{E}\|\sigma_1-\sigma_2\|^p.$$

□

## 5. An example

The considered NRINSIDEs with finite delays is of the form

$$\begin{aligned} d\Big[(z(\zeta)+\int_{-\alpha}^{0}u_1(\theta)z(\zeta+\theta)\Big] &= \Big[\int_{-\alpha}^{0}u_2(\theta)z(\zeta+\theta)+\int_{-\alpha}^{0}\int_{0}^{\zeta}u_3(\theta)z(\zeta+\theta)\Big]d\zeta \\ &+\Big[\int_{-\alpha}^{0}u_4(\theta)z(\zeta+\theta)\Big]dW(\zeta),\ t\geq t_0,\ t\neq \zeta_q, \\ z(\sigma_q) &= b_q(\delta_q)z(\sigma_q^-),\ q=1,2,..., \\ z(0)+\sum_{\imath}^{m}c_\imath(r_{\imath,z}) &= z_0,\ 0\leq r_1\leq r_1...\leq r_p\leq \mathcal{T}. \end{aligned} \tag{5.1}$$

Let $\alpha > 0$, z be R-valued stochastic process, and $\sigma \in \mathfrak{C}([-\delta,0],\pounds^2(\Omega,\mathbb{R}))$. $\delta_q$ is defined from $\Omega$ to $\mathcal{D}_q \stackrel{def}{=} (0,d_q)$ for $q=1,2,....$ Suppose that $\tau_q$ follows Erlang distribution and let $\delta_\imath$ and $\delta_\jmath$ be independent of each other as $\imath\neq\jmath$ for $\imath,\jmath=1$,2.... $\zeta_0=\sigma_0<\sigma_1<\sigma_2<...$ and $\sigma_q=\sigma_{q-1}+\tau_q$ for $q=1,2,....$ Let $W(t)\in\mathbb{R}$ be a one-dimensional Brownian motions, where b is a function of q. $u_1,u_2,u_3:[-\delta,0]\to\mathbb{R}$ are continuous functions. Define $h:[\zeta_0,\mathcal{T}]\times\mathfrak{C}\to\mathbb{R}^d$, $f:[\zeta_0,\mathcal{T}]\times\mathfrak{C}\to\mathbb{R}^d$, $g:[\zeta_0,\mathcal{T}]\times\mathfrak{C}\to\mathbb{R}^{d\times m}$, $r:\mathfrak{C}\to\mathfrak{C}$, $k:[\zeta_0,\mathcal{T}]\times[\zeta_0,\mathcal{T}]\times\mathfrak{C}\to\mathbb{R}^d$and $P:[\zeta_0,\mathcal{T}]\times\mathfrak{C}\times\mathfrak{U}\to\mathbb{R}^d$, and $b_q:\mathcal{D}_q\to\mathbb{R}^{d\times d}$ by

$$h(\zeta,z(\zeta))(.)=\int_{-\alpha}^{0}u_1(\theta)z(\zeta+\theta)d\theta(.),\quad f(\zeta,z(\zeta))(.)=\int_{-\alpha}^{0}u_2(\theta)z(\zeta+\theta)d\theta(.),$$

$$k(\zeta,z(\zeta))(.)=\int_{-\alpha}^{0}u_3(\theta)z(\zeta+\theta)d\theta(.),\quad g(\zeta,z(\zeta))(.)=\int_{-\alpha}^{0}u_4(\theta)z(\zeta+\theta)d\theta(.).$$

For $z(t+\theta)\in\mathfrak{C}$, we suppose that the following conditions hold:

(1) $\max\limits_{\imath,q}\{\prod\limits_{\jmath=\imath}^{q}\mathbb{E}\|a_\imath(\delta_\imath)\|^2\}<\infty$,

(2) $\int_{-\alpha}^{0}u_1(\theta)^2d\theta, \int_{-\alpha}^{0}u_2(\theta)^2d\theta, \int_{-\alpha}^{0}u_3(\theta)^2d\theta < \int_{-\alpha}^{0}u_4(\theta)^2d\theta < \int_{-\alpha}^{0}u_5(\theta)^2d\theta < \infty.$

Suppose that the states (1) and (2) are obtained from which we can prove that the assumptions (A1)-(A5) hold. As a result, the systems (1.1)-(1.3) have a unique mild solution z.

## 6. Conclusion

This article is devoted to discussing the existence and continuous dependence on initial data. First, we use Krasnoselskii's-Schaefer type fixed point theorem to demonstrate the existence of mild solutions to equations (1.1)-(1.3). Next, we examine the continuous dependence of solutions on the initial data. In addition, this result could be extended to investigate the controllability of random impulsive neutral stochastic differential equations with finite/infinite state-dependent delay in the future. The fractional order of NRINSDEs with Poisson jumps would be quite interesting. This will be the focus of future research.

# Acknowledgements

We are thankful to DST-FIST for providing infrastructural facilities at School of Mathematical Sciences, SRTM University, Nanded with the aid of which this research work has been carried out.

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