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On Some Relations of *R*-Projective Curvature Tensor in Recurrent Finsler Space

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Abstract In this paper, we present a novel class of relations and investigate the connection between the R-projective curvature tensor and other tensors of Finsler space F_n. This space is characterized by the property for Cartan's the third curvature tensor R^i_{jkh} which satisfies the certain relationship with given covariant vectors field, as follows:

$$\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} = a_{lmn}R^i_{jkh} + b_{lmn}(\delta^i_h g_{jk} - \delta^i_k\ g_{jh}) - 2[c_{lm}\mathcal{B}_r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})y^r$$

$$+ d_{ln}\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})y^r + \mu_l\ \mathcal{B}_n\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})y^r],$$

where $R^i_{jkh} \neq 0$ and $\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l$ is the Berwald's third order covariant derivative with respect to x^l, x^m and x^n respectively. The quantities $a_{lmn} = \mathcal{B}_n u_{lm} + u_{lm}\ \lambda_n$, $b_{lmn} = \mathcal{B}_n v_{lm} +\ u_{lm}\ \mu_n$, $c_{lm}\ = v_{lm}$, and $d_{ln} = \mathcal{B}_n \mu_l$ are non-zero covariant vector fields. We define this space a generalized $\mathcal{B}R$-$3rd$ recurrent space and denote it briefly by$G\mathcal{B}R$-$3RF_n$. This paper aims to derive the third-order Berwald covariant derivatives of the torsion tensor H^i_{kh} and the deviation tensor H^i_h. Additionally, it demonstrates that the curvature vector K_j, the curvature vector H_k , and the curvature scalar H are all non-vanishing within the considered space. We have some relations between Cartan's third curvature tensor R^i_{jkh} and some tensors that exhibit self-similarity under specific conditions. Furthermore, we have established the necessary and sufficient conditions for certain tensors in this space to have equal third-order Berwald covariant derivatives with their lower-order counterparts.

Keywords n-dimensional Finsler space F_n, generalized $\mathcal{B}R$-$3rd$ recurrent spaces, employing Berwald's third order covariant derivative, R^i_{jkh} Cartan's third curvature tensor

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1. Introduction

The study of recurrent Finsler spaces began in 1973 with the work of Sinha and Singh [24], who explored the properties of recurrent tensors in these spaces. The differential geometry of Finsler spaces subsequent research on recurrent Finsler spaces

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was conducted by Rund [20] in 1959 and 1981. While Abdallah [3] and Baleedi [15] in 2017, investigated the recurrence of Berwald's curvature tensors R^i_{jkh} and K^i_{jkh}. Building upon these foundational works, Ahsan and Ali [4] in 2014, studied the properties of W-curvature tensor. Opondo [18] and Abu-Donia et al. [10] introduced and analyzed the recurrence conditions of the curvature tensor W^i_{jkh} using Berwald's approach.

From 2019 to 2023, Ali et al. [11–13] and Shaikh et al. [21, 22] presented some properties of the tensors W and M. They delved into the semi-conformal symmetry a new symmetry of the spacetime manifold of the general relativity. Qasem and Abdallah [19] furthered this research by defining the generalized $\mathcal{B}$R-recurrent Finsler space and establishing the necessary and sufficient conditions for both the Berwald curvature tensor and Cartan's fourth curvature tensor to exhibit generalized recurrence. Subsequently, Al-Qashbari and Qasem [5] investigated generalized $\mathcal{B}$R-trirecurrent Finsler spaces. Then in 2020, Al-Qashbari [6–8] derived various identities for generalized curvature tensors in $\mathcal{B}$-recurrent Finsler spaces and other tensors.

The most recent contribution to this field is the work of Al-Qashbari and Al-Maisary [9], who studied generalized BW-fourth recurrent Finsler spaces in 2023. Chen, Decu et al. [16, 17] in 2021, introduced the concept of classification of Roter type spacetimes and recent developments in Wintgen inequality and Wintgen ideal submanifolds. In 2021 and 2022, Atashafrouz et al. [1] and Saleem et al. [23] studied the notions of D-recurrent Finsler metrics and the U-recurrent Finsler space respectively. Recently, Abdallah [2] studied the relationships between two curvature tensors in Finsler space. Embarking on an exploration of the inherent attributes of an n-dimensional Finsler space F_n, we presuppose that its metric function F adheres to the well-defined stipulations outlined in [18].

1. Positively homogeneous: $F(x, ky) = k\ F(x, y)$, $k > 0$.
2. Positively: $F\,(x, y) > 0 \quad , \quad y \neq 0$.
3. $\{\ \dot{\partial}_i\ \dot{\partial}_j F^2(x, y)\ \}\ \xi^i\ \xi^j,\ \ \dot{\partial}_i = \frac{\partial}{\partial y^i}$ is the positive definite for all variables ξ^i.

The corresponding metric tenser denoted by g_{ij}, the connection coefficients of Cartan represented by Γ^{*i}_{jk} and the connection coefficients of Berwald designated by G^i_{jk}, are all related to the metric function F.

$$(a)\ \ g_{ij}\ y^i\ y^j = F^2,\ (b)\ g_{ij}\ y^j = y_i,\ (c)\ g_{ij} = \frac{1}{2}\ \dot{\partial}_i y_j,\ (d)\ y_i\ y^{\ i} = F^2,$$

$$(e)\ g_{ij}\ g^{ik} = \delta^k_j = \begin{cases} 1 & if\ \ j = k \\ 0 & if\ \ j \neq k \end{cases}\ ,\ (f)\quad \delta^i_h\ g_{ik} = g_{hk}, \tag{1.1}$$

$$(g)\qquad \delta^i_k\ y^k = y^i,\ and\ (h)\ \ \delta^i_i = n.$$

The torsion tensor C_{ijk} is defined by [20]

$$C_{ijk} = \frac{1}{2}\ \dot{\partial}_i\ g_{jk} = \frac{1}{4}\ \dot{\partial}_i\ \dot{\partial}_j\ \dot{\partial}_k\ F^2, \tag{1.2}$$

and its associate is the torsion tensor C^i_{jk} which is defined by:

$$(a)\ C^h_{ik} = g^{hj}\ C_{ijk}\ ,\ (b)\ \ C^i_{jk}\ \ y^k = C^i_{kj}\ \ y^k = 0. \tag{1.3}$$

These tensors satisfy the following conditions.

$$(a)\ C_{ijk}\ y^k = C_{kij}\ y^k = C_{jki}\ y^k = 0,\ (b)\ \ G^i_{jkh}\ y^j = G^i_{hjk}\ y^j = G^i_{khj}\ y^j = 0,$$
$$(c)\ \delta^i_k\ C_{jin} = C_{jkn},\ (d)\ C_{jkr}\ g^{jk} = C_r,$$
$$(e)\ \Gamma^{*i}_{jkh}\ y^h = \ G^i_{jkh}\ y^h = 0,\ where\ G^i_{jkh} = \dot{\partial}_j\ G^i_{kh}\ \ and\ \dot{\partial}_j = \frac{\partial}{\partial y^j}. \tag{1.4}$$

The Berwald covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is defined as:

$$\mathcal{B}_k\ T^i_j\ = \partial_k\ T^i_j\ - \left(\dot{\partial}_r\ T^i_j\ \right)\ G^r_k + T^r_j\ \ G^i_{rk} - T^i_r\ \ G^r_{jk}. \tag{1.5}$$

The Berwald covariant derivatives of the metric function F, the vectors y^i , y_i and the unit vector l^i are all identically zero [3]. In other words,

$$(a)\ \mathcal{B}_k\ F = 0\ ,\ (b)\ \mathcal{B}_k\ y^i = 0\ ,\ (c)\ \mathcal{B}_k\ y_i = 0,\ and\ (d)\ \mathcal{B}_k l^i = 0. \tag{1.6}$$

However, Berwald's covariant derivative of the metric tensor g_{ij} is not identically zero, that is $\mathcal{B}_k\ g_{ij} \neq 0$. It is expressed as:

$$\mathcal{B}_k g_{ij} = -2\ y^h\ \mathcal{B}_h\ C_{ijk} = -2\ C_{ijk|h}\ y^h. \tag{1.7}$$

The covariant differential operator of Berwald with respect to x^h and the partial differential operator with respect to y^k commute, as defined by:

$$(\dot{\partial}_k\ \mathcal{B}_h - \mathcal{B}_h \dot{\partial}_k t)\ T^i_j = T^r_j\ G^i_{khr} - T^i_r\ \ G^r_{khj}\ ,where\ T^i_j\ is\ any\ arbitrary\ tensor. \tag{1.8}$$

The second Berwald covariant derivative of the vector field X^i, with respect to x^k and x^h is given by:

$$\mathcal{B}_k \mathcal{B}_h\ X^i = \dot{\partial}_k \mathcal{B}_h X^i - (\ \dot{\partial}_s \mathcal{B}_h X^i)\ G^s_k + (\mathcal{B}_h X^r) G^i_{rk} - (\mathcal{B}_r X^i) G^r_{hk}. \tag{1.9}$$

The tensors R^i_{jkh} and K^i_{jkh} are defined by:

$$(a)\ R^i_{jkh} = \partial_h \Gamma^{*i}_{jk} + (\dot{\partial}_r \Gamma^{*i}_{jk})\ \Gamma^{*r}_{sh} y^s + C^i_{jm}(\partial_k \Gamma^{*m}_{sh} y^s - \Gamma^{*m}_{kr} \Gamma^{*r}_{sh}\ y^s) + \Gamma^{*i}_{mk} \Gamma^{*m}_{jh}$$
$$- \partial_k \Gamma^{*i}_{jh} - (\dot{\partial}_r \Gamma^{*i}_{jh})\ \Gamma^{*r}_{sk}\ y^s - C^i_{jm}(\partial_h\ \Gamma^{*m}_{sk}\ y^s - \Gamma^{*m}_{hr}\ \Gamma^{*r}_{sk}\ y^s) - \Gamma^{*i}_{mh}\ \Gamma^{*m}_{jk},$$
$$(b)\ K^i_{jkh} = \partial_k \Gamma^{*i}_{hj} + (\dot{\partial}_s \Gamma^{*i}_{jk}) \Gamma^{*s}_{th}\ y^t + \Gamma^{*i}_{mk} \Gamma^{*m}_{hj} - \partial_h \Gamma^{*i}_{kj} - (\dot{\partial}_s \Gamma^{*i}_{jh}) \Gamma^{*s}_{tk}\ y^t - \Gamma^{*i}_{mh} \Gamma^{*m}_{kj}. \tag{1.10}$$

The aforementioned tensors, namely Cartan's third curvature tensor and Cartan's fourth curvature tensor, respectively, display skew-symmetry regarding their last two lower indices and maintain positive homogeneity of degree zero in their directional arguments. These tensors are governed by the following relations:

$$(a)\ R^i_{jkh}\ y^j = K^i_{jkh}\ y^j = H^i_{kh}\ ,\ (b)\ K^i_{jkh} = H^i_{jkh} -\ y^m\ (\dot{\partial}_j\ K^i_{mkh}),$$
$$(c)\ R^i_{jkh} = K^i_{jkh} + C^i_{js} H^s_{kh}\ ,\ (d)\ K^i_{jkh} = H^i_{jkh} - P^i_{jkh} - P^r_{jk} P^i_{rh} + P^i_{jhk} + P^r_{jh} P^i_{rk},$$
$$(e)\ R_{ijhk} = K_{ijhk} + C_{ijm}\ K^s_{rhk}\ y^m,\ and\ (f)\ R_{ijkh} = g_{rj} R^r_{ikh}. \tag{1.11}$$

Ricci tensor R_{jk}, the deviation tensor R^i_h and curvature scalar R derived from the curvature tensor R^i_{jkh} are defined as:

$$(a)\ \ R^r_{jkr} = R_{jk}\ ,\ (b)\ R^i_{jkh}\ g^{jk} = R^i_h\ ,\ and\ (c)\ R^i_i = R. \tag{1.12}$$

The curvature tensor of Berwald H^i_{jkh}, torsion tensor H^i_{kh}, Ricci tensor H_{jk}, deviation tensor H^i_h and curvature scalar H are defined as:

$$(a)\ H^i_{jkh}\ y^j = H^i_{kh}\ ,\ (b)\ \ H^i_{kh}\ y^k = H^i_h,\ and\ (c)\ H_j\ y^j = (n-1)H. \tag{1.13}$$

The Concircular curvature tensor M^i_{jkh}, the torsion tensor M^i_{jk} , the Ricci tensor M_{jk}, the curvature vector M_k and the scalar curvature M satisfy the following conditions:

$$\begin{aligned}&(a)\ M^i_{jkh}\ y^j = M^i_{kh}\ \ ,\ (b)\ M^i_{kh}\ y^k = M^i_h\ ,\ (c)\ M^i_{jki} = M_{jk},\\ &(d)\ M^i_{ki} = M_k\ \ and\ (e)\ M^i_i = M.\end{aligned} \tag{1.14}$$

The conformal curvature tensor Z^i_{jkh}, the torsion tensor Z^i_{jk}, the Ricci tensor Z_{jk}, the curvature vector Z_k and the scalar curvature Z are satisfying the following conditions:

$$(a)\ Z^i_{jkh}\ y^j = Z^i_{kh}\ \ ,\ (b)\ Z^i_{kh}\ y^k = Z^i_h\ ,\ and\ (c)\ Z^i_{ki} = Z_k. \tag{1.15}$$

Notations. R^i_{jkh}: Cartan's third Curvature Tensor, Z^i_{jkh}: Conformal curvature tensor, H^i_{jkh} : Berwald Curvature Tensor, R_{jk}: Ricci Tensor, R^i_h: the deviation tensor, R: Scalar Curvature.

2. On generalized $\mathcal{B}$R-3rd recurrent Finsler space

Let us explore in $G\mathcal{B}K\text{-}RF_n$ for whose Cartan's third curvature tensor R^i_{jkh} is defined as [9]:

$$\mathcal{B}_m R^i_{jkh} = a_m R^i_{jkh} + b_m\left(\delta^i_h g_{jk} - \delta^i_k\ g_{jh}\right), R^i_{jkh} \neq 0.$$

This space is designated as a generalized $\mathcal{B}R$-recurrent space, where $\mathcal{B}_m$ represents the first-order covariant derivative (Berwald's covariant differential operator) with respect to x^m. By taking the third-order covariant derivative of curvature tensor R^i_{jkh} in the Berwald sense with respect to x^l, x^m and x^n, we obtain:

$$\begin{aligned}\mathcal{B}_n\mathcal{B}_m\mathcal{B}_l R^i_{jkh} =& a_{lmn}R^i_{jkh} + b_{lmn}\ (\delta^i_h\ g_{jk} - \delta^i_k\ g_{jh})\\ &-2\left[\ c_{lm}\ \mathcal{B}_r\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right) y^r + d_{ln}\ \mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right) y^r\right.\\ &\left.+\mu_l \mathcal{B}_n\mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\ y^r\right].\end{aligned} \tag{2.1}$$

Multiplying (2.1) by y^j, using (1.6b), (1.11a), (1.4a) and (1.1b), we obtain

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_{kh} = a_{lnm}H^i_{kh} + b_{lnm}(\delta^i_h y_k - \delta^i_k y_h). \tag{2.2}$$

Multiplying (2.2) by y^k, using (1.6b), (1.14b), (1.1d) and (1.1g), we obtain

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H^i_h = a_{lnm}H^i_h + b_{lnm}(\delta^i_h F^2 - y^i y_h). \tag{2.3}$$

In conclusion, we find the following theorem.

Theorem 2.1. *In the $G\mathcal{B}R$-$3RF_n$, Berwald's covariant derivatives of the third order for the torsion tensor H^i_{kh} and the deviation tensor H^i_h are given by the conditions* (2.2) *and* (2.3)*, respectively.*

Summing over the indices i and h in condition (2.1), using (1.12a), (1.4c), (1.1f), (1.1h) and setting $n = 4$, we obtain,

$$\begin{aligned}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_{jk} =& a_{lnm}R_{jk} + 3[b_{lnm}g_{jk} - c_{lnm}C_{jkn} \\ & - d_{lnm}C_{jkl} - e_{lnm}C_{jkq} - 2b_{nm}y^r\mathcal{B}_rC_{jks}.]\end{aligned} \tag{2.4}$$

Multiplying (2.4) by y^k, using (1.6b), (1.12b), (1.4a), (1.1b) and setting $n = 4$, we obtain

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R_j = a_{lnm}R_j + 3\ b_{lnm}y_j. \tag{2.5}$$

Multiplying (2.5) by y^j, using (1.6b), (1.14i) and (1.1d), we obtain

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H = a_{lnm}H + b_{lnm}F^2. \tag{2.6}$$

Multiplying (2.4) by y^j, using (1.6b), (1.14h), (1.4a), (1.1b) and setting $n = 4$, we obtain

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m H_k = a_{lnm}H_k + 3\ b_{lnm}\ y_k. \tag{2.7}$$

In conclusion, we find the following theorem.

Theorem 2.2. *In the $G\mathcal{B}R$-$3RF_n$, the curvature vector R_j, the curvature vector R_k and the curvature scalar H are all nonzero.*

3. Relations between curvature tensor R^i_{jkh} and other curvature tensors

In this section we presented the relationship between Cartan's third curvature tensor R^i_{jkh} and some curvature tensors in $G\mathcal{B}R$-$3RF_n$.

The relation between Cartan's third curvature tensor R^i_{jkh} and the Concircular curvature tensor M^i_{jkh} for a V_4 is defined as:

$$M^i_{jkh} = R^i_{jkh} - \frac{R}{12}(g_{jk}\delta^i_h - g_{jh}\delta^i_k\ ). \tag{3.1}$$

Taking the covariant derivative of the third order for (3.1) in the sense of Berwald, we obtain,

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M^i_{jkh} = \mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} - \frac{R}{12}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(\ g_{jk}\ \delta^i_h\ - g_{jh}\ \delta^i_k). \tag{3.2}$$

Using the conditions (2.1) in (3.2), we obtain,

$$\begin{aligned}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M^i_{jkh} =& a_{lmn}R^i_{jkh} + b_{lmn}\ (\delta^i_h\ g_{jk} - \delta^i_k\ g_{jh}) \\ & - 2[c_{lm}\ \mathcal{B}_r(\delta^i_hC_{jkn} - \delta^i_kC_{jhn})y^r + d_{ln}\ \mathcal{B}_r(\delta^i_hC_{jkm} - \delta^i_kC_{jhm})y^r \\ & + \mu_l\mathcal{B}_n\mathcal{B}_r(\delta^i_hC_{jkm} - \delta^i_kC_{jhm})y^r] - \mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\frac{R}{12}(g_{jk}\delta^i_h\ - g_{jh}\ \delta^i_k).\end{aligned} \tag{3.3}$$

In view of condition (3.2), condition (3.3) devolves to

$$\begin{aligned}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M^i_{jkh} =& a_{lmn}M^i_{jkh} + a_{lmn}\frac{R}{12}(g_{jk}\delta^i_h - g_{jh}\delta^i_k) + b_{lmn}(\delta^i_h g_{jk} - \delta^i_k g_{jh}) \\ &- 2[c_{lm}\mathcal{B}_r(4C_{jkn} - C_{jkn})y^r + d_{ln}\ \mathcal{B}_r(4C_{jkm} - C_{jkm})y^r \\ &+ \mu_l\mathcal{B}_n\mathcal{B}_r(4C_{jkm} - C_{jkm})y^r] - \frac{R}{12}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{jk}\delta^i_h - g_{jh}\delta^i_k.\end{aligned} \tag{3.4}$$

We can express the above equation in a different way as

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M^i_{jkh} = a_{lmn}M^i_{jkh} + \ b_{lmn}\left(\delta^i_h\ g_{jk} - \delta^i_k\ g_{jh}\right)$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R\left(\ g_{jk}\ \delta^i_h\ \ - g_{jh}\ \delta^i_k\right) = a_{lmn}R\left(\ g_{jk}\ \delta^i_h\ \ - g_{jh}\ \delta^i_k\ \right)$$

and

$$c_{lm}\mathcal{B}_r(4C_{jkn} - C_{jkn})y^r + d_{ln}\mathcal{B}_r(4C_{jkm} - C_{jkm})y^r + \mu_l\mathcal{B}_n\mathcal{B}_r\left(4C_{jkm} - C_{jkm}\right)y^r = 0. \tag{3.5}$$

In conclusion, we find the following theorem.

Theorem 3.1. *In the $G\mathcal{B}R$-$3RF_n$ (in the sense of Berwald space), the covariant derivative of Berwald on the third order for the Concircular curvature tensor M^i_{jkh} is $G\mathcal{B}M$-$3RF_n$ if and only if the tensor $R(g_{jk}\delta^i_h - g_{jh}\delta^i_k)$ is Trirecurrent in Finsler space and condition* (3.5) *is hold.*

Multiplying (3.4) by y^j, using (1.6b), (1.14a), (1.4a) and (1.1b), we obtain

$$\begin{aligned}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M^i_{kh} =& a_{lmn}M^i_{kh} + a_{lmn}\frac{R}{12}(\ y_k\ \delta^i_h\ \ - y_h\ \delta^i_k) \\ &+\ b_{lmn}(y_k\delta^i_h - y_h\delta^i_k) - \mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\frac{R}{12}(y_k\ \delta^i_h\ \ - y_h\ \delta^i_k).\end{aligned} \tag{3.6}$$

We can express the above equation in a different way as:

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\ M^i_{kh} = a_{lmn}M^i_{kh} + \ b_{lmn}\left(\ y_k\ \delta^i_h\ \ - y_h\ \delta^i_k\ \right)$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R\left(\ y_k\ \delta^i_h\ \ - y_h\ \delta^i_k\ \right) = a_{lmn}R\left(\ y_k\ \delta^i_h\ \ - y_h\ \delta^i_k\ \right).$$

In conclusion, we find the following theorem.

Theorem 3.2. *In the $G\mathcal{B}R$-$3RF_n$ (in the sense of Berwald space), the covariant derivative of Berwald on the third order for the curvature tensor M^i_{kh} is $G\mathcal{B}M$-$3RF_n$, if and only if the tensor $R(g_{jk}\delta^i_h - g_{jh}\delta^i_k)$ is Trirecurrent in Finsler space.*

Multiplying (3.6) by y^k, using (1.6b), (1.14b), (1.1d) and (1.1g), we obtain,

$$\begin{aligned}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M^i_h =& a_{lmn}M^i_h + a_{lmn}\frac{R}{12}(F^2\delta^i_h - y_h y^i) \\ &+ b_{lmn}(F^2\delta^i_h - y_h y^i) - \mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\frac{R}{12}(F^2\delta^i_h - y_h y^i).\end{aligned} \tag{3.7}$$

We can express the above equation in a different way as:

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\ M_h^i = a_{lmn}M_h^i +\ b_{lmn}(F^2\delta_h^i - y_h y^i)$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(F^2\delta_h^i - y_h y^i) = a_{lmn}(F^2\delta_h^i - y_h y^i).$$

In conclusion, we find the following theorem.

Theorem 3.3. *In the $G\mathcal{B}R$-$3RF_n$(in the sense of Berwald space), the covariant derivative of Berwald on the third order for the curvature tensor M_h^i is $G\mathcal{B}M$-$3RF_n$ if and only if the tensor $F^2\delta_h^i - y_h y^i$ is Trirecurrent in Finsler space.*

Summing over the indices i and h in condition (3.4), using (1.14c), (1.4c), (1.1f), (1.1h) and setting $n = 4$, we obtain,

$$\begin{aligned}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{jk} =& a_{lmn}M_{jk} + a_{lmn}\frac{R}{4}g_{jk} + 3b_{lmn}\ g_{jk} - \mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\frac{R}{4}g_{jk} \\ & - 6[\ c_{lm}\mathcal{B}_r C_{jkn}y^r + d_{ln}\ \mathcal{B}_r C_{jkm}y^r + \mu_l\mathcal{B}_n\mathcal{B}_r C_{jkm}y^r].\end{aligned} \tag{3.8}$$

We can express the above equation in a different way as:

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_{jk} = a_{lmn}M_{jk} + 3b_{lmn}\ g_{jk}$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R g_{jk} = a_{lmn}R g_{jk}$$

$$c_{lm}\ \mathcal{B}_r C_{jkn}y^r + d_{ln}\ \mathcal{B}_r C_{jkm}y^r + \mu_l\mathcal{B}_n\mathcal{B}_r C_{jkm}y^r = 0. \tag{3.9}$$

In conclusion, we find the following theorem.

Theorem 3.4. *In the $G\mathcal{B}R$-$3RF_n$(in the sense of Berwald space), the covariant derivative of Berwald on the third order for the Ricci tensor M_{jk} is $G\mathcal{B}M$-$3RF_n$, if and only if condition (3.9) is hold.*

Further, summing over the indices i and h in conditions (3.6) and (3.7), using (1.14d), (1.14e), (1.4c), (1.1f), (1.1h) and setting $n = 4$, we obtain,

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_k = a_{lmn}M_k + a_{lmn}\frac{R}{4}y_k + 3\ b_{lmn}y_k - \mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\frac{R}{4}y_k, \tag{3.10}$$

and

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M = a_{lmn}M + a_{lmn}\frac{R}{3}\left(F^2 - 1\right) + 3b_{lmn}\left(F^2 - 1\right) - \mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\frac{R}{4}\left(F^2 - 1\right). \tag{3.11}$$

We can express the above equation in a different way as:

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M_k = a_{lmn}M_k + 3b_{lmn}\ y_k$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R\ y_k\ = a_{lmn}R\ y_k;$$

and

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m M = a_{lmn}M + 3b_{lmn}\ y_k$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R\left(F^2 - 1\right) = a_{lmn}R\left(F^2 - 1\right).$$

In conclusion, we find the following theorem.

Theorem 3.5. *In the $G\mathcal{B}R$-$3RF_n$ (in the sense of Berwald space), the covariant derivative of Berwald on the third order for the vector tensor M_k and scalar tensor M are $G\mathcal{B}M$-$3RF_n$, if and only if the tensors (Ry_k) and $R(F^2-1)$ are Trirecurrent in Finsler space.*

For a Riemannian space V4, the Conharmonic curvature tensor Z^i_{jkh} is defined as [19]:

$$Z^i_{jkh} = R^i_{jkh} + \frac{1}{2}(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k\ R_{jh} - g_{jh}\ R^i_k). \tag{3.12}$$

Taking the covariant derivative of the third order for (3.1) in the sense of Berwald, we obtain

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m Z^i_{jkh} = \mathcal{B}_l\mathcal{B}_n\mathcal{B}_m R^i_{jkh} - \frac{1}{2}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k). \tag{3.13}$$

Using the condition (2.1) in (3.13), we obtain,

$$\begin{aligned}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m Z^i_{jkh} =& a_{lmn}R^i_{jkh} + b_{lmn}(\delta^i_h\ g_{jk} - \delta^i_k\ g_{jh}) - 2[c_{lm}\ \mathcal{B}_r(\delta^i_h C_{jkn} - \delta^i_k C_{jhn})y^r \\ &+ d_{ln}\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})y^r + \mu_l\mathcal{B}_n\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})y^r] \\ &- \frac{1}{2}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k).\end{aligned} \tag{3.14}$$

In view of condition (3.12), condition (3.14) devolves to,

$$\begin{aligned}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m Z^i_{jkh} =& a_{lmn}Z^i_{jkh} - \frac{1}{2}a_{lmn}(\ g_{jk}\ R^i_h\ + \delta^i_h\ R_{jk} - \delta^i_k\ R_{jh} - g_{jh}\ R^i_k) \\ &+\ b_{lmn}\ \left(\delta^i_h\ g_{jk} - \delta^i_k\ g_{jh}\right) - 2[c_{lm}\ \mathcal{B}_r\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)y^r \\ &+ d_{ln}\ \mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})y^r + \mu_l\mathcal{B}_n\mathcal{B}_r(\delta^i_h C_{jkm} - \delta^i_k C_{jhm})y^r] \\ &+ \frac{1}{2}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k).\end{aligned} \tag{3.15}$$

We can express the above equation in a different way as:

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m Z^i_{jkh} = a_{lmn}Z^i_{jkh} +\ b_{lmn}(\delta^i_h\ g_{jk} - \delta^i_k\ g_{jh})$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k) = a_{lmn}(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k)$$

and

$$\begin{aligned}c_{lm}\mathcal{B}_r\left(\delta^i_h C_{jkn} - \delta^i_k C_{jhn}\right)y^r &+ d_{ln}\ \mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\ y^r \\ &+ \mu_l\mathcal{B}_n\mathcal{B}_r\left(\delta^i_h C_{jkm} - \delta^i_k C_{jhm}\right)\ y^r = 0.\end{aligned} \tag{3.16}$$

In conclusion, we find the following theorem.

Theorem 3.6. *In the $G\mathcal{B}R$-$3RF_n$ (in the sense of Berwald space), the covariant derivative of Berwald on the third order for the Conharmonic curvature tensor Z^i_{jkh} is $G\mathcal{B}M$-$3RF_n$, if and only if the tensor $(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k)$ is Trirecurrent in Finsler space and condition (3.16) is hold.*

Multiplying (3.15) by y^j, using (1.6b), (1.15a), (1.4a), (1.11a) and (1.1b), we obtain

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m Z^i_{kh} = a_{lmn} Z^i_{kh} + \frac{1}{2} a_{lmn}(y_k\ R^i_h\ + \delta^i_h\ H_k - \delta^i_k\ H_h - y_h R^i_k\ ) + b_{lmn}(y_k \delta^i_h - y_h \delta^i_k) - \frac{1}{2}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(y_k R^i_h + \delta^i_h\ H_k - \delta^i_k\ H_h - y_h\ R^i_k). \tag{3.17}$$

We can express the above equation in a different way as

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\ Z^i_{kh} = a_{lmn} Z^i_{kh} + + b_{lmn}\left(\ y_k\ \delta^i_h\ - y_h\ \delta^i_k\ \right)$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(y_k R^i_h + \delta^i_h H_k - \delta^i_k H_h - y_h R^i_k) = a_{lmn}(y_k R^i_h + \delta^i_h H_k - \delta^i_k H_h - y_h R^i_k).$$

In conclusion, we find the following theorem.

Theorem 3.7. *In the $G\mathcal{B}R$-$3RF_n$ (in the sense of Berwald space), the covariant derivative of Berwald on the third order for the curvature tensor Z^i_{kh} is $G\mathcal{B}M$-$3RF_n$, if and only if the tensor $(y_k R^i_h\ + \delta^i_h\ H_k - \delta^i_k\ H_h - y_h\ R^i_k)$ is Trirecurrent in Finsler space.*

Multiplying (3.17) by y^k, using (1.6b), (1.15b), (1.1d), (1.14c) and (1.1g), we obtain

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m Z^i_h = a_{lmn} Z^i_h + \frac{1}{2} a_{lmn}(F^2 R^i_h\ + 3\ \delta^i_h\ H - H_h y^i - y_h\ R^i_k y^k) + b_{lmn}\left(F^2 - 1\right) \delta^i_h - \frac{1}{2}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(F^2 R^i_h\ + 3\delta^i_h\ H - H_h y^i - y_h\ R^i_k\ y^k). \tag{3.18}$$

We can express the above equation in a different way as:

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\ Z^i_h = a_{lmn} Z^i_h + + b_{lmn}(F^2 \delta^i_h - y_h y^i)$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(F^2 R^i_h + 3\ \delta^i_h\ H - H_h y^i - y_h\ R^i_k\ y^k\right) = a_{lmn}\left(F^2 R^i_h + 3\ \delta^i_h\ H - H_h y^i - y_h\ R^i_k\ y^k\right).$$

In conclusion, we find the following theorem.

Theorem 3.8. *In the $G\mathcal{B}R$-$3RF_n$ (in the sense of Berwald space), the covariant derivative of Berwald on the third order for the curvature tensor Z^i_h is $G\mathcal{B}M$-$3RF_n$, if and only if the tensor $(F^2 R^i_h\ + 3\ \delta^i_h H - H_h y^i - y_h R^i_k y^k)$ is Trirecurrent in Finsler space.*

Summing over the indices i and h in condition (3.15), using (1.15c), (1.12c), (1.4c), (1.1f), (1.1h) and setting $n = 4$, we obtain,

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m Z_{jk} = a_{lmn} Z_{jk} - \frac{1}{2} a_{lmn}(g_{jk} R + 4R_{jk} - R_{jk} - R_{kj}) + 3b_{lmn} g_{jk} - 6[c_{lm}\mathcal{B}_r C_{jkn}\ y^r + d_{ln}\mathcal{B}_r C_{jkm} y^r + \mu_l \mathcal{B}_n\mathcal{B}_r C_{jkm} y^r] + \frac{1}{2}\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m(g_{jk} R + 4R_{jk} - R_{jk} - R_{kj}). \tag{3.19}$$

We can express the above equation in a different way as

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m Z_{jk} = a_{lmn} Z_{jk} + 3b_{lmn}\ g_{jk}$$

if and only if

$$\mathcal{B}_l\mathcal{B}_n\mathcal{B}_m\left(g_{jk}\ R + 4R_{jk} - R_{jk} - R_{kj}\right) = a_{lmn}\left(g_{jk}\ R + 4R_{jk} - R_{jk} - R_{kj}\right)$$

$$c_{lm}\mathcal{B}_r C_{jkn} y^r + d_{ln}\mathcal{B}_r C_{jkm}\ y^r + \mu_l\mathcal{B}_n\mathcal{B}_r C_{jkm}\ y^r = 0. \tag{3.20}$$

In conclusion, we find the following theorem.

Theorem 3.9. *In the $G\mathcal{B}R$-$3RF_n$(in the sense of Berwald space), the covariant derivative of Berwald on the third order for the Ricci tensor Z_{jk} is $G\mathcal{B}M$-$3RF_n$ if and only if condition* (3.20) *is hold.*

4. Conclusions

The generalized BR-third recurrent space meets the criteria outlined in condition (2.1). In the context of Berwald spaces, $G\mathcal{B}R$-$3RF_n$ (in the sense of Berwald space) exhibits the B-derivative of the third order for the Ricci tensor R_{jk} and the curvature vector R_j , as defined in equations (2.4) and (2.5), respectively. Within $G\mathcal{B}R$-$3RF_n$, the curvature tensor M^i_{jkh} satisfies the conditions for a generalized third recurrent Finsler space if and only if the condition $R(g_{jk}\delta^i_h - g_{jh}\delta^i_k)$is trirecurrent in Finsler space and condition (3.5) holds. Furthermore, the curvature tensor M^i_{kh} qualifies as $G\mathcal{B}R$-$3RF_n$ if and only if the tensor $R(g_{jk}\delta^i_h - g_{jh}\delta^i_k)$ is trirecurrent in Finsler space.

The Conharmonic curvature tensor Z^i_{jkh} in $G\mathcal{B}R$-$3RF_n$ is categorized as $G\mathcal{B}R$-$3RF_n$ if and only if the tensor $(g_{jk}R^i_h + \delta^i_h R_{jk} - \delta^i_k R_{jh} - g_{jh}R^i_k)$ is trirecurrent in Finsler space and condition (3.16) holds. The curvature tensor Z^i_{kh} falls under the $G\mathcal{B}R$-$3RF_n$ classification if and only if the tensor $\left(y_k\ R^i_h + \delta^i_h H_k - \delta^i_k\ H_h - y_h\ R^i_k\ \right)$ is trirecurrent in Finsler space. Additionally, the third-order covariant derivative of Berwald for the Ricci tensor Z_{jk} belongs to the $G\mathcal{B}R$-$3RF_n$category if and only if condition (3.20) is satisfied.

The authors advocate for further investigation and exploration of generalized $\mathcal{B}$K-higher recurrent Finsler spaces, emphasizing the potential connections to specialized Finsler spaces.

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