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On the Cauchy Problem for a Viscous Cahn-Hilliard-Oono System with Chemotaxis

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Abstract In this paper, we are concerned with the well-posedness and large time behavior of Cauchy problem for viscous Cahn-Hilliard-Oono system with chemotaxis in 3D whole space. By using the pure energy method, standard continuity arguments together with negative Sobolev norm estimates, one proves the global well-posedness and time decay estimates.

Keywords Viscous Cahn-Hilliard-Oono system, global existence, decay

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1. Introduction

In this paper, we consider the following system of partial differential equations [11] in 3D whole space

$$\partial_t\varphi = \Delta\left(\Psi'(\varphi) - \Delta\varphi - \chi\sigma + \partial_t\varphi\right) - \alpha\varphi, \quad \text{in } \boldsymbol{R}^3 \times (0, +\infty), \tag{1.1}$$

$$\partial_t\sigma = \Delta(\sigma - \chi\varphi), \quad \text{in } \boldsymbol{R}^3 \times (0, +\infty), \tag{1.2}$$

together with the initial conditions

$$\varphi|_{t=0} = \varphi_0, \, \sigma|_{t=0} = \sigma_0, \quad \text{in } \boldsymbol{R}^3. \tag{1.3}$$

Systems (1.1)-(1.2) can be seen as a simplified, fluid-free version of the general thermodynamically consistent diffuse interface model derived in [12]. This model is suitable for a two-phase incompressible fluid mixture with chemical species subject, which are influenced by important mechanisms such as diffusion, chemotactic interactions, and active transport. The order parameter (phase function) φ is the difference in volume fractions between the two components, while the variable σ is the standard for nutrient concentration. $\mu = \Psi'(\varphi) - \Delta\varphi - \chi\sigma + \partial_t\varphi$ is regarded as the chemical potential associated with (φ, σ), in which the function $\Psi'(\varphi)$ is the derivative of a potential Ψ with double-well structure. A physically significant example of Ψ is given by the so-called Flory-Huggins logarithmic potential [8,10,16]

$$\Psi(r) = \frac{\theta}{2}[(1-r)\ln(1-r) + (1+r)\ln(1+r)] + \frac{\theta_0}{2}\left(1-r^2\right), \forall r \in (-1, 1) \tag{1.4}$$

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with $0 < \theta < \theta_0$. It is referred to as a singular potential since its derivative $\Psi'(\varphi)$ blows up at the pure phases ± 1. In the literature, the singular potential Ψ is often approximated by a fourth order polynomial [17,18]

$$\Psi(r) = \frac{1}{4}\left(1 - s^2\right)^2, \quad \forall s \in R. \tag{1.5}$$

The nontrivial coupling between Cahn-Hilliard equations (1.1) and the diffusion equation (1.2) for the nutrient is characterized by the constant χ, which models some specific mechanisms such as chemotaxis/active transport in the context of tumor growth modeling (see, e.g., Garcke and Lam [6] and Garcke et al [5]). Cahn-Hilliard equation (1.1) also involves some nonlocal interaction that is given by Oono's type $-\alpha\varphi$ for the sake of simplicity(cf., e.g., Giorgini et al. and Miranville [7,13]), where $\alpha \geqslant 0$. Recently, He [11] considered the properties of solutions for the initial-boundar value problem of equations (1.1)-(1.2) with singular potentials including the physically relevant logarithmic potential in the 3D bounded domain. The authors proved the existence and uniqueness of a global weak solution, obtained some regularity properties of the weak solution when $t > 0$, and studied the longtime behavior of the system. We remark that there is no paper related to the Cauchy problem of equations (1.1)-(1.2). This is just the main purpose of this paper. In this paper, we consider the global existence and long time behavior of global strong solutions for equations (1.1)-(1.2) in 3D whole space.

Remark 1.1. If $\sigma = 0$ in equations (1.1)-(1.2), we obtain the well-known classical Cahn-Hilliard equation, which has been employed as an efficient mathematical tool for the study on dynamics of binary mixtures, particularly, recently for the tumor growth modeling [3, 9, 15]. Concerning the mathematical analysis of the Cahn-Hilliard equation and its variants, we refer to several studies [1, 2, 4, 13]. And the references cited therein (see also the recent book [14]).

Our main result is stated as follows:

Theorem 1.1. *Assume that $\varphi_0, \sigma_0 \in R^3$ for an integer $N \geq 2$ and $2\sqrt{\alpha} - 1 - 2\chi^2 > 0$. Then there exists a constant δ_0 such that if*

$$\|\varphi_0\|_{H^2} + \|\nabla\varphi_0\|_{H^2} + \|\sigma_0\|_{H^2} < \delta_0, \tag{1.6}$$

then problem (1.1) admits a unique global solution (φ, σ) satisfying that for all $t \geq 0$,

$$\begin{aligned}
&\|\varphi(t)\|_{H^N}^2 + \|\nabla\varphi(t)\|_{H^N}^2 + \|\sigma(t)\|_{H^N}^2 \\
&+ \int_0^t \left(\|\nabla\varphi(\tau)\|_{H^N}^2 + \left\|\nabla^2\varphi(\tau)\right\|_{H^N}^2 + \|\nabla\sigma(\tau)\|_{H^N}^2\right) d\tau \\
\leq & C\left(\|\varphi_0\|_{H^N}^2 + \|\nabla\varphi_0\|_{H^N}^2 + \|\sigma_0\|_{H^N}^2\right).
\end{aligned} \tag{1.7}$$

If further, $\varphi_0, \sigma_0 \in \dot{H}^{-s}$ for some $s \in \left[0, \frac{3}{2}\right)$, then for all $t \geq 0$,

$$\|\varphi(t)\|_{\dot{H}^{-s}}^2 + \|\nabla\varphi(t)\|_{\dot{H}^{-s}}^2 + \|\sigma(t)\|_{\dot{H}^{-s}}^2 \leqslant C_0, \tag{1.8}$$

and the following decay result holds for $l = 0, \cdots, N-1$:

$$\left\|\nabla^l\varphi(t)\right\|_{H^{N-l}} + \left\|\nabla^{l+1}\varphi(t)\right\|_{H^{N-l}} + \left\|\nabla^l\sigma(t)\right\|_{H^{N-l}} \leqslant C_0(1+t)^{-\frac{l+s}{2}}.$$

Note that the Hardy-Littlewood-Sobolev theorem implies that for p $\in (1,2]$, $L^p \subset \dot{H}^{-s}$ with $S = 3\left(\frac{1}{p}-\frac{1}{2}\right) \in \left[0,\frac{3}{2}\right)$. Then by Theorem 1.1, we have the following corollary of the usual $L^p - L^2$ type of the optimal decay results:

Corollary 1.1. *Under the assumptions of Theorem 1.1 except that we replace the $\dot{H}^{-S}$ assumption with the condition that $\varphi_0, \sigma_0 \in R^3$ for some* p $\in (1,2]$, *then the following decay results hold for $l = 0, \cdots, N-1$:*

$$\left\|\nabla^l \varphi(t)\right\|_{H^{N-l}} + \left\|\nabla^l \nabla\varphi(t)\right\|_{H^{N-l}} + \left\|\nabla^l \sigma(t)\right\|_{H^{N+1-l}} \leqslant C_0(1+t)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{2}\right)+\frac{l}{2}}.$$

The rest of this paper is organized as follows. In the next section, we establish the energy estimates for problems (1.1)-(1.2); Section 3 is devoted to proving some negative Sobolev norm estimates which are useful for the study of the longtime behavior. The proof of Theorem 1 is shown in Section 4.

2. Energy estimates

In this section, we derive the a priori energy estimates for problems (1.1)-(1.2). Suppose that there exists a small positive constant $\delta > 0$ such that

$$\sqrt{\mathcal{E}_0^2(t)} = \|\varphi(t)\|_{H^2} + \|\nabla\varphi\|_{H^2} + \|\sigma\|_{H^2} < \delta. \tag{2.1}$$

Lemma 2.1. *If $\sqrt{\mathcal{E}_0^2(t)} < \delta$, N_2, and $2\sqrt{\alpha} - 1 - 2\chi^2 > 0$, then for k=0,...,N, then we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla^k\varphi\right\|_{L^2}^2 + \left\|\nabla^{k+1}\varphi\right\|_{L^2}^2 + \left\|\nabla^k\sigma\right\|_{L^2}^2\right)\\ &+\left(2\sqrt{\alpha}-1-2\chi^2\right)\left\|\nabla^{k+1}\varphi\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla^{k+1}\sigma\right\|_{L^2}^2\\ \leq& C\|\varphi\|_{H^2}^2\left\|\nabla^{k+1}\varphi\right\|_{L^2}^2.\end{aligned} \tag{2.2}$$

Proof. Applying ∇^k to (1.1), multiplying the resulting identity by $\nabla^k\varphi$, and then integrating over R^3 by parts, we arrive at

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla^k\varphi\right\|_{L^2}^2 + \left\|\nabla^{k+1}\varphi\right\|_{L^2}^2\right) + \left\|\nabla^{k+2}\varphi\right\|_{L^2}^2 + \alpha\left\|\nabla^k\varphi\right\|_{L^2}^2\\ = \int_{R^3}\nabla^k\varphi\nabla^{k+2}\left(\varphi^3-\varphi-\chi\sigma\right)dx.\end{aligned} \tag{2.3}$$

Applying ∇^k to (1.2), multiplying the resulting identity by $\nabla^k\sigma$, and integrating over R^3 by parts, we arrive at

$$\frac{1}{2}\frac{d}{dt}\left\|\nabla^k\sigma\right\|_{L^2}^2 + \left\|\nabla^{k+1}\sigma\right\|_{L^2}^2 = -\chi\int_{R^3}\nabla^k\sigma\nabla^{k+2}\varphi dx. \tag{2.4}$$

Adding (2.3)-(2.4) together gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla^k\varphi\right\|_{L^2}^2 + \left\|\nabla^{k+1}\varphi\right\|_{L^2}^2 + \left\|\nabla^k\sigma\right\|_{L^2}^2\right)\\ &+\left\|\nabla^{k+2}\varphi\right\|_{L^2}^2 + \alpha\left\|\nabla^k\varphi\right\|_{L^2}^2 + \left\|\nabla^{k+1}\sigma\right\|_{L^2}^2\\ =&\int_{R^3}\nabla^k\varphi\nabla^{k+2}\left(\varphi^3-\varphi-\chi\sigma\right)dx - \chi\int_{R^3}\nabla^k\sigma\nabla^{k+2}\varphi dx =: J_1 + J_2.\end{aligned} \tag{2.5}$$

Using Holder's inequality and Sobolev's embedding theorem, we have

$$\begin{aligned} J_1 &= \int_{R^3} \nabla^k \varphi \nabla^{k+2} \left(\varphi^3 - \varphi\right) dx \leqslant \left\|\nabla^{k+1}\varphi\right\|_{L^2} \left\|\nabla^{k+1}\left(\varphi^3 - \varphi\right)\right\|_{L^2} \\ &\leqslant C \left\|\nabla^{k+1}\varphi\right\|_{L^2} \left(\left\|\nabla^{k+1}\varphi^3\right\|_{L^2} + \left\|\nabla^{k+1}\varphi\right\|_{L^2}\right) \\ &\leqslant C \left\|\nabla^{k+1}\varphi\right\|_{L^2} \left(\|\varphi\|_{L^\infty}^2 \left\|\nabla^{k+1}\varphi\right\|_{L^2} + \left\|\nabla^{k+1}\varphi\right\|_{L^2}\right) \\ &\leqslant C\|\varphi\|_{H^2}^2 \|\nabla^{k+1}\varphi\|_{L^2}^2 + \left\|\nabla^{k+1}\varphi\right\|_{L^2}^2, \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} J_2 &= -2\chi \int_{R^3} \nabla^{k+2}\sigma \nabla^k \varphi dx \\ &\leqslant 2\chi \int_{R^3} \nabla^{k+1}\sigma \nabla^{k+1}\varphi dx \\ &\leqslant \frac{1}{2} \left\|\nabla^{k+1}\sigma\right\|_{L^2}^2 + 2\chi^2 \left\|\nabla^{k+1}\varphi\right\|_{L^2}^2. \end{aligned} \tag{2.7}$$

Substituting (2.6),(2.7) into (2.5), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla^k\varphi\right\|_{L^2}^2 + \left\|\nabla^{k+1}\varphi\right\|_{L^2}^2 + \left\|\nabla^k\sigma\right\|_{L^2}^2\right) \\ &+ \left\|\nabla^{k+2}\varphi\right\|_{L^2}^2 + \alpha\left\|\nabla^k\varphi\right\|_{L^2}^2 + \left\|\nabla^{k+1}\sigma\right\|_{L^2}^2 \\ \leqslant& C\|\varphi\|_{H^2}^2 \left\|\nabla^{k+1}\varphi\right\|_{L^2}^2 + \left\|\nabla^{k+1}\varphi\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla^{k+1}\sigma\right\|_{L^2}^2 + 2\chi^2\left\|\nabla^{k+1}\varphi\right\|_{L^2}^2. \end{aligned} \tag{2.8}$$

Note that

$$\left\|\nabla^{k+2}\varphi\right\|_{L^2}^2 + \alpha\left\|\nabla^k\varphi\right\|_{L^2}^2 \geqslant 2\sqrt{\alpha}\left\|\nabla^{k+1}\varphi\right\|_{L^2}^2. \tag{2.9}$$

Summing up, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\nabla^k\varphi\right\|_{L^2}^2 + \left\|\nabla^{k+1}\sigma\right\|_{L^2}^2 + \left\|\nabla^k\sigma\right\|_{L^2}^2\right) + \left\|\nabla^{k+1}\sigma\right\|_{L^2}^2 + 2\sqrt{\alpha}\left\|\nabla^{k+1}\varphi\right\|_{L^2}^2 \\ \leqslant& C\|\varphi\|_{H^2}^2\left\|\nabla^{k+1}\varphi\right\|_{L^2}^2 + \frac{1}{2}\left\|\nabla^{k+1}\sigma\right\|_{L^2}^2 + \left(1+2\chi^2\right)\left\|\nabla^{k+1}\varphi\right\|_{L^2}^2, \end{aligned} \tag{2.10}$$

then (2.2) is obtained directly. □

3. Negative Sobolev estimate

Lemma 3.1. *If* $\sqrt{\mathcal{E}_0^2(t)} < \delta$, $N \geq 2$, *and* $2\sqrt{\alpha} - 1 - 2\chi^2 > 0$, *then for* $s \in \left(0, \frac{1}{2}\right]$, *we have*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^{-s}\varphi\right\|_{L^2}^2 + \left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}^2 + \left\|\Lambda^{-s}\sigma\right\|_{L^2}^2\right) + \left(2\sqrt{\alpha} - 1 - 2\chi^2\right)\left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}^2 \\ &+ \frac{1}{2}\left\|\Lambda^{-s}\nabla\sigma\right\|_{L^2}^2 \leqslant C\left(\left\|\nabla^2\varphi\right\|_{L^2}^2 + \|\nabla\varphi\|_{H^1}^2\right)\left\|\Lambda^{-s}\varphi\right\|_{L^2}. \end{aligned} \tag{3.1}$$

Moreover, if $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$, *the following inequality holds:*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^{-s}\varphi\right\|_{L^2}^2 + \left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}^2 + \left\|\Lambda^{-2}\sigma\right\|_{L^2}^2\right) + \left(2\sqrt{\alpha} - 1 - 2\chi^2\right)\left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}^2 \\ &+ \frac{1}{2}\left\|\Lambda^{-s}\nabla\sigma\right\|_{L^2}^2 \leqslant C\left\|\Lambda^{-s}\varphi\right\|_{L^2}\|\Delta\varphi\|_{L^2}\|\varphi\|_{L^2}^{s-\frac{1}{2}}\|\nabla\varphi\|_{L^2}^{\frac{3}{2}-5}. \end{aligned} \tag{3.2}$$

Proof. Taking Λ^{-s} to (1.1) and (1.2), multiplying by $\Lambda^{-s}\varphi$ and $\Lambda^{-s}\sigma$ respectively and integrating over R^3, we deduce that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^{-s}\varphi\right\|_{L^2}^2+\left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}^2\right)+\left\|\Lambda^{-s}\Delta\varphi\right\|_{L^2}^2+\alpha\left\|\Lambda^{-s}\varphi\right\|_{L^2}^2\\ =\int_{R^3}\Lambda^{-s}\varphi\Lambda^{-s}\Delta\left(\varphi^3-\varphi-\chi\sigma\right)dx,\end{aligned}\tag{3.3}$$

and

$$\frac{1}{2}\frac{d}{dt}\left\|\Lambda^{-s}\sigma\right\|_{L^2}^2+\left\|\Lambda^{-s}\nabla\sigma\right\|_{L^2}^2=-\chi\int_{R^3}\Lambda^{-s}\sigma\Lambda^{-s}\Delta\varphi dx.\tag{3.4}$$

Adding (3.3) and (3.4) together gives

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|\Lambda^{-s}\varphi\right\|_{L^2}^2+\left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}^2+\left\|\Lambda^{-s}\sigma\right\|_{L^2}^2\right)+\left\|\Lambda^{-s}\Delta\varphi\right\|_{L^2}^2+\left\|\Lambda^{-s}\nabla\sigma\right\|_{L^2}^2\\ &+\alpha\left\|\Lambda^{-s}\varphi\right\|_{L^2}^2=\int_{R^3}\Lambda^{-s}\Delta\left(\varphi^3-\varphi\right)\Lambda^{-s}\varphi dx-2\chi\int_{R^3}\Lambda^{-s}\Delta\sigma\Lambda^{-s}\varphi dx\\ &=W_1+W_2,\end{aligned}\tag{3.5}$$

where

$$\begin{aligned}W_1&=\int_{R^3}\Lambda^{-s}\Delta\varphi^3\Lambda^{-s}\varphi dx+\left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}^2=W_{11}+W_{12},\\ W_2&=-2\chi\int_{R^3}\Lambda^{-s}\Delta\sigma\Lambda^{-s}\varphi dx.\end{aligned}\tag{3.6}$$

If s $\in\left(0,\frac{1}{2}\right]$, we easily obtain $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{s}{3}\geq 6$. Then, applying the Kato-Ponce inequality and Sobolev's embedding theorem together with Hölder's and Young's inequalities, it yields that

$$\begin{aligned}W_{11}\leqslant&\|\Lambda^{-s}\varphi\|_{L^2}\|\Lambda^{-s}\Delta\varphi^3\|_{L^2}\leqslant C\|\Lambda^{-s}\varphi\|_{L^2}\|\Delta\varphi^3\|\\ \leqslant&C\left\|\Lambda^{-3}\varphi\right\|_{L^2}\|\varphi\|_{L^2}\|\Delta\varphi\|_{L^2}\|\varphi\|_{L^{3/s}}\\ \leqslant&C\left\|\Lambda^{-s}\varphi\right\|_{L^2}\|\Delta\varphi\|_{L^2}\|\varphi\|_{L^{3/s}}\\ \leqslant&C\left\|\Lambda^{-s}\varphi\right\|_{L^2}\left\|\nabla^2\varphi\right\|_{L^2}\|\nabla\varphi\|_{L^2}^{1/2+s}\left\|\nabla^2\varphi\right\|_{L^2}^{1/2-s}\\ \leqslant&C\left(\left\|\nabla^2\varphi\right\|_{L^2}^2+\|\nabla\varphi\|_{H^2}^2\right)\left\|\Lambda^{-s}\varphi\right\|_{L^2},\end{aligned}\tag{3.7}$$

and

$$W_2\leqslant 2\chi\left\|\Lambda^{-s}\nabla\sigma\right\|_{L^2}\left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}\leqslant\frac{1}{2}\left\|\Lambda^{-s}\nabla\sigma\right\|_{L^2}^2+2\chi^2\left\|\Lambda^{-s}\nabla\varphi\right\|_{L^2}^2.\tag{3.8}$$

Summing up, we obtain (3.1) directly. On the other hand, if s $\in\left(\frac{1}{2},\frac{3}{2}\right)$, we will estimate the right-hand sides of (3.5) and obtain the negative Sobolev norm estimates in a different way. Note that s $\in\left(\frac{1}{2},\frac{3}{2}\right)$, it is easy to see that $\frac{1}{2}+\frac{s}{3}<1$ and $\frac{s}{3}\in(2,6)$. Therefore, by using Kato-Ponce inequality and Sobolev's embedding theorem, we derive that

$$W_{11}\leqslant C\left\|\Lambda^{-s}\varphi\right\|_{L^2}\|\Delta\varphi\|_{L^2}\|\varphi\|_{L^2}^{S-1/2}\|\nabla\varphi\|_{L^2}^{3/2-s}.\tag{3.9}$$

Substituting (3.7)-(3.9) into (3.5), we obtain (3.2) directly. Then, the proof is complete. □

4. Global well-posedness and decay

In this section, we shall combine all the energy estimates that we have derived in the previous two sections and the Sobolev interpolation to prove the main result.

Let $N \geq 1$ and $0 \leq l \leq m$ with $1 \leq m \leq N$. Summing up the estimates (2.2) from $k = l$ to m , since $\sqrt{\mathcal{E}_0^2(t)} < \delta$ is small, we obtain

$$\begin{aligned} &\frac{d}{dt} \sum_{l \leqslant k \leqslant m} \left(\left\| \nabla^k \varphi \right\|_{L^2}^2 + \left\| \nabla^{k+1} \varphi \right\|_{L^2}^2 + \left\| \nabla^k \sigma \right\|_{L^2}^2 \right) \\ &+ C_0 \sum_{l+1 \leqslant k \leqslant m+1} \left(\left\| \nabla^k \varphi \right\|_{L^2}^2 + \left\| \nabla^k \sigma \right\|_{L^2}^2 \right) \leqslant \delta^2 \sum_{l \leqslant k \leqslant m} \left\| \nabla^{k+1} \varphi \right\|_{L^2}^2 . \end{aligned} \tag{4.1}$$

It is easy to see that there exists a constant $C_0 > 0$ such that for $0 \leq l \leq m - 1$,

$$\begin{aligned} &\frac{d}{dt} \sum_{l \leqslant k \leqslant m} \sum_{l \leqslant k \leqslant m} \left(\left\| \nabla^k \varphi \right\|_{L^2}^2 + \left\| \nabla^{k+1} \varphi \right\|_{L^2}^2 + \left\| \nabla^k \sigma \right\|_{L^2}^2 \right) \\ &+ C_0 \sum_{l \leqslant k \leqslant m} \left(\left\| \nabla^{k+1} \varphi \right\|_{L^2}^2 + \left\| \nabla^k \sigma \right\|_{L^2}^2 \right) \leqslant 0. \end{aligned} \tag{4.2}$$

Define $\mathcal{E}_l^m(t)$ to be $1/C_0$ times the expression under the time derivative in (4.2). Hence, we may write (4.2) as that for $0 \leq l \leq m - 1$,

$$\frac{d}{dt} \mathcal{E}_l^m(t) + \left(\left\| \nabla^{l+1} \varphi \right\|_{H^{m-l}}^2 + \left\| \nabla^l \sigma \right\|_{H^{m-l}}^2 \right) \leqslant 0. \tag{4.3}$$

Taking $l = 0$ and $m = 2$ in (4.3) and integrating directly in time, we deduce that

$$\| \nabla \varphi(t) \|_{H^2}^2 + \| \sigma(t) \|_{H^2}^2 \leqslant C \mathcal{E}_0^2(0) \leqslant C \left(\| \nabla \varphi_0 \|_{H^2}^2 + \| \sigma_0 \|_{H^2}^2 \right). \tag{4.4}$$

By a standard continuity argument, if at the initial time is sufficiently small. This in turn allows us to take $l = 0$ and m $= N$ in (4.4) and obtain

$$\frac{d}{dt} \mathcal{E}_0^N(t) + \| \nabla \varphi \|_{H^N}^2 + \| \sigma \|_{H^N}^2 \leqslant 0, \tag{4.5}$$

which implies that

$$\mathcal{E}_0^N(t) + \int_0^t \left(\| \nabla \varphi \|_{H^N}^2 + \| \sigma \|_{H^N}^2 \right) ds \leqslant \mathcal{E}_0^N(0). \tag{4.6}$$

By a standard continuity argument, this establishes the a priori estimates (1.1) if at the initial time we assume that $_0$ is sufficiently small.

In the following, we consider the decay estimates for problems (1.1)-(1.2). It is worth pointing out that we are not able to prove them for all $s \in \left[0, \frac{3}{2}\right)$ at this moment. We shall first prove them for $s \in \left[0, \frac{1}{2}\right]$.

Define

$$\mathcal{E}_l(t) = \left\| \nabla^l \varphi \right\|_{L^2}^2 + \left\| \nabla^{l+1} \varphi \right\|_{L^2}^2 + \left\| \nabla^l \sigma \right\|_{L^2}^2 ,$$

and

$$\mathcal{E}_{-s}(t) = \left\| \Lambda^{-s} \varphi \right\|_{L^2}^2 + \left\| \Lambda^{-s} \nabla \varphi \right\|_{L^2}^2 + \left\| \Lambda^{-s} \sigma \right\|_{L^2}^2 .$$

Integrating in time, we obtain

$$\begin{aligned}\mathcal{E}_{-s}(t) \leqslant & \mathcal{E}_{-s}(0) + C\int_0^t \left(\left\|\nabla^2\varphi\right\|_{L^2}^2 + \|\nabla\varphi\|_{H^1}^2\right)\sqrt{\mathcal{E}_{-s}(\tau)}d_\tau \\ \leqslant & C_0\left(1 + \sup_{0\leqslant\tau\leqslant t}\sqrt{\mathcal{E}_{-s}(\tau)}d\tau\right),\end{aligned} \tag{4.7}$$

which implies that for s $\in \left[0, \frac{1}{2}\right]$,

$$\left\|\Lambda^{-s}\varphi(t)\right\|_{L^2} + \left\|\Lambda^{-s}\nabla\varphi(t)\right\|_{L^2} + \left\|\Lambda^{-s}\sigma(t)\right\|_{L^2}^2 \leqslant C_0. \tag{4.8}$$

Moreover, if $l = 0, 1, 2, \cdots, N$,

$$\left\|\nabla^{l+1}f\right\|_{L^2} \geqslant C\left\|\Lambda^{-s}f\right\|_{L^2}^{-\frac{1}{l+s}} + \left\|\nabla^l f\right\|_{L^2}^{1+\frac{1}{l+s}}. \tag{4.9}$$

Then, by this facts, we deduce that

$$\left\|\nabla^{l+1}(\varphi, \nabla\varphi, \sigma)\right\|_{L^2}^2 \geqslant \left(C_0\left\|\nabla^l(\varphi, \nabla\varphi, \sigma)\right\|_{L^2}^2\right)^{1+\frac{1}{l+s}}. \tag{4.10}$$

Thus, we deduce from (4.3) the following inequality:

$$\frac{d}{dt}\mathcal{E}_l + C_0\left(\mathcal{E}_l\right)^{1+\frac{1}{l+s}} \leqslant 0. \text{ for } l = 0, 1, 2, \cdots N, \tag{4.11}$$

which implies

$$\mathcal{E}_l(t) \leqslant C_0(1+t)^{-l-s}, \text{ for } l = 0, 1, 2, \cdots N. \tag{4.12}$$

On the other hand, the arguments for $s \in \left[0, \frac{1}{2}\right]$ cannot be applied to s $\in \left(\frac{1}{2}, \frac{3}{2}\right)$. However, observing that $\varphi_0, \sigma_0 \in \dot{H}^{-1/2}$ hold since $\dot{H}^{-s} \cap L^2 \subset \dot{H}^{-s'}$ for any $s' \in [0, s]$, we can deduce from what we have proved for (1.4) with $s = \frac{1}{2}$ that the following estimate holds:

$$\left\|\nabla^l\varphi\right\|_{L^2}^2 + \left\|\nabla^l\nabla\varphi\right\|_{L^2}^2 + \left\|\nabla^l\sigma\right\|_{L^2}^2 \leqslant C_0(1+t)^{-1/2-l}. \text{ for } l = 0.1\cdots N. \tag{4.13}$$

Therefore, for s $\in \left(\frac{1}{2}, \frac{3}{2}\right)$,

$$\begin{aligned}\mathcal{E}_{-s}(t) \leqslant & \mathcal{E}_{-s}(0) + C\int_0^t \|\varphi\|_{L^2}^{s-1/2}\|\nabla\varphi\|_{L^2}^{3/2-s}\sqrt{\mathcal{E}_{-s}(\tau)}d\tau \\ \leqslant & C + C\int_0^t (1+\tau)^{-(7/4-s/2)}d\tau \sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)} + C\sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)} \\ \leqslant & C + C\sup_{\tau\in[0,t]}\sqrt{\mathcal{E}_{-s}(\tau)},\end{aligned} \tag{4.14}$$

that is

$$\left\|\Lambda^{-s}\varphi(t)\right\|_{L^2} + \left\|\Lambda^{-s}\nabla\varphi(t)\right\|_{L^2} + \left\|\Lambda^{-s}\sigma(t)\right\|_{L^2} \leqslant C_0. \tag{4.15}$$

We may repeat the arguments for s $\in \left[0, \frac{1}{2}\right]$ to prove that they also hold for $s \in \left(\frac{1}{2}, \frac{3}{2}\right)$. Hence, the proof of the main result is complete.

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