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Exact Solution to the Compressible Euler System in 1-D*

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Abstract In this paper, the exact solution of one-dimensional isentropic Euler equations is studied. When the exponent of the state equation satisfies $\gamma = 2$, we get an exact solution which is linear with respect to the spatial variable x. For this end, we solve some ordinary differential equations with time dependent variable coefficients.

Keywords Euler equations, compressible, exact solutions, ordinary differential equation

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1. Introduction

As the most basic equations in the field of fluid mechanics, the compressible Euler system can be used to describe and simulate many physical phenomena in real fluids. The long time behavior of the solution to the equations has been widely studied. Although there is no complete explanation so far, the research in this aspect is going on all the time. There have been a lot of results. The isentropic compressible Euler equations in n-d are as follows

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^{\mathbf{n}}, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = 0, \quad (t,x) \in \mathbf{R}_+ \times \mathbf{R}^{\mathbf{n}}, \end{cases} \tag{1.1}$$

where $\rho = \rho(t,x)$, $u = u(t,x) = (u_1(t,x), u_2(t,x), \cdots, u_n(t,x))$ and $p = p(t,x)$ as the real value of the unknown functions, represent the density, velocity and pressure of the fluid, respectively.

System (1.1) is a nonlinear system of partial differential equations. General speaking, it is very difficult to obtain the exact solution. The exact solution results are mainly for one-dimensional Euler equations or high dimensional radial symmetry cases so far. Li and Wang considered radially symmetric solutions in any dimension and constructed an exact solution of the form $u = c(t)r$ (where $r = |x|$ and $c(t)$ satisfies a second-order ordinary differential equation) and the blow up of the solution of the compressible Euler equations was further analyzed [1]. Liang constructed an exact solution to the non-isentropic one-dimensional Euler equations in the form of $u = c(t)x$ [2]. Furthermore, the blow up result is discussed by setting

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appropriate initial values $c(0)$ and $c'(0)$. Using similar methods, Yuen obtained the following analytic solution in one-dimensional case:

$$\begin{cases} \rho^{\gamma-1}(x,t) = \max\left\{\rho^{\gamma-1}(0,t) - \frac{\gamma-1}{K\gamma}\left[\dot{b}(t) + b(t)\frac{\dot{a}(t)}{a(t)}\right]x \right. \\ \left. -\frac{(\gamma-1)\xi}{2K\gamma a^{\gamma+1}(t)}x^2, 0\right\}, \\ u(x,t) = \frac{\dot{a}(t)}{a(t)}x + b(t), \end{cases} \tag{1.2}$$

where $a(t)$, $b(t)$, and $c(t)$ satisfy a system of differential equations [3]. Dong and Li obtained radially symmetric and self-similar analytic solutions to compressible Euler equations with time dependent damping and free boundary in three dimensional space, and proved the global existence of such solutions [4]. Jia considered one-dimensional isothermal Euler equations ($p = a\rho$) with time dependent damping, and an exact solution of the form $\rho(x,t) = e^{c(t)x+d(t)}$ was obtained [5]. For compressible Navier-Stokes equations, there has also been a lot of work on solving the analytical solutions and further studying the large time behavior of the solutions, see [6–8] and the references therein.

The existing results show that the exact analytical solution is possible to obtain in the case of one-dimensional or high-dimensional radial symmetry. In other cases, although it is not easy to obtain the exact solution directly, it is very important to study the large time behavior of the solution. Since the compressible Euler equations can be written in the form of the symmetric hyperbolic system, which admits a common phenomenon of singularity formation. We refer to [9–20] and references therein for such kinds of results.

In this paper, the compressible Euler equations in one-dimension are mainly considered. If the state equation is $p = \frac{1}{2}\rho^2$, by solving an ordinary differential equation, we obtain an exact solution, in which the density $\rho(t,x)$ and the velocity $u(t,x)$ are linear with respect to the space variable x.

Specifically, we consider the following theorem.

$$\begin{cases} \rho_t + (\rho u)_x = 0, (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ (\rho u)_t + \left(\rho u^2\right)_x + \left(\frac{1}{2}\rho^2\right)_x = 0, (x,t) \in \mathbb{R} \times \mathbb{R}^+. \end{cases} \tag{1.3}$$

Then we have

Theorem 1.1. *For system* (1.3), *there exists an exact solution of the following form*

$$\begin{aligned} \rho(x,t) &= c(t)x + d(t), \\ u(x,t) &= -\frac{c'(t)}{2c(t)}x + \frac{c'(t)d(t) - 2c(t)d'(t)}{2c^2(t)}, \end{aligned} \tag{1.4}$$

where

$$\begin{aligned} c(t) &= \frac{c_2}{(t+2c_1)^2}, \\ d(t) &= \frac{c_3}{t+2c_1} + \frac{c_4 - c_2^2 - c_2^2\log(t+2c_1)}{(t+2c_1)^2}, \end{aligned} \tag{1.5}$$

and c_1, c_2, c_3, c_4 *are constants.*

2. The proof of Theorem 1.1

First, we verify that the solution given by (1.4) satisfies the first formula in system (1.3). According to the expression in (1.4), it can be calculated directly that

$$\begin{cases} \rho_x = c(t), \\ \rho_t = c'(t)x + d'(t), \\ u_x = -\frac{c'(t)}{2c(t)}, \\ u_t = \frac{c'^2(t)-c(t)c''(t)}{2c^2(t)}x + \frac{c(t)c''(t)d(t)+3c(t)c'(t)d'(t)-2c^2(t)d''(t)-2c'^2(t)d(t)}{2c^3(t)}. \end{cases} \tag{2.1}$$

From the first formula of equation (1.3), we substitute the above formula, and get

$$\begin{aligned} &\rho_t + (\rho u)_x \\ =&\rho_t + \rho_x u + \rho u_x \\ =&c'x + d' + cu + \rho \cdot \left(-\frac{c'}{2c}\right) \\ =&c'x + d' - \frac{1}{2}c'x + \frac{c'd - 2cd'}{2c} - \frac{1}{2}c'x - \frac{c'}{2c}d \\ =&0. \end{aligned} \tag{2.2}$$

Therefore, we can get that the solution (1.4) satisfies the above mass equation. Next we consider the second formula of equation (1.3). From the second formula of equation (1.3), it can be obtained

$$\rho_t u + \rho u_t + \rho_x u^2 + 2\rho u u_x + \rho\rho_x = 0, \tag{2.3}$$

then the first formula of the side end (1.3), that is

$$-\rho_x u^2 - \rho u u_x + \rho u_t + \rho_x u^2 + 2\rho u u_x + \rho\rho_x = 0, \tag{2.4}$$

which can be simplified to

$$u_x u + u_t + \rho_x = 0. \tag{2.5}$$

The formula (2.1) is substituted into

$$\begin{aligned} &u_x u + u_t + \rho_x \\ =& -\frac{c'}{2c}\left(-\frac{c'}{2c}\right)x - \frac{c'^2 d - 2cc'd'}{4c^3} + \frac{c'^2 - cc''}{2c^2}x + \frac{cc''d + 3cc'd' - 2c^2 d'' - 2c'^2 d}{2c^3} + c \\ =& \left(\frac{c^2}{4c^2} + \frac{c'^2 - cc''}{2c^2}\right)x - \frac{c'^2 d - 2cc'd'}{4c^3} + \frac{cc''d + 3cc'd' - 2c^2 d'' - 2c'^2 d}{2c^3} + c \\ =& \frac{3c'^2 - 2cc''}{4c^2}x - \frac{c'^2 d - 2cc'd'}{4c^3} + \frac{cc''d + 3cc'd' - 2c^2 d'' - 2c'^2 d}{2c^3} + c = 0. \end{aligned} \tag{2.6}$$

By comparing the coefficients of the primary term of x and the constant term, a system of equations about c and d is obtained as follows

$$\begin{cases} \frac{3c'^2 - 2cc''}{4c^2} = 0 \\ -\frac{c'^2 d - 2cc'd'}{4c^3} + \frac{cc''d + 3cc'd' - 2c^2 d'' - 2c'^2 d}{2c^3} + c = 0. \end{cases} \tag{2.7}$$

$$\begin{cases} 3c'^2(t) - 2c(t)c''(t) = 0 \\ d''(t) - \frac{2c'(t)}{c(t)}d'(t) + \frac{5c'^2(t)-2cc''(t)}{4c^2(t)}d(t) = c^2(t). \end{cases} \tag{2.8}$$

For the first equation in the above equation, we have

$$2\left(c'^2 - cc''\right) + c'^2 = 0, \tag{2.9}$$

that is

$$2\left(\frac{c'^2 - cc''}{c'^2}\right) + 1 = 0. \tag{2.10}$$

So, it can be converted to

$$\left(\frac{c}{c'}\right)' = -\frac{1}{2}, \tag{2.11}$$

and integrating both sides, we get

$$\frac{c(t)}{c'(t)} = -\frac{1}{2}t + c_1, \tag{2.12}$$

where c_1 is an arbitrary constant. Separating the variables, we get

$$\frac{\mathrm{d}c}{c} = \frac{1}{-\frac{1}{2}t + c_1}\mathrm{d}t, \tag{2.13}$$

so we integrate both sides again to get

$$\log|c| = -2\log\left|-\frac{1}{2}t + c_1\right| = \log\frac{1}{\left(-\frac{1}{2}t + c_1\right)^2} + c, \tag{2.14}$$

and hence,we have

$$c(t) = \frac{c_2}{\left(-\frac{1}{2}t + c_1\right)^2} = \frac{c_2}{\left(t + 2c_1\right)^2}. \tag{2.15}$$

Next we solve for $d(t)$. From (2.15), we obtain

$$\begin{cases} c'(t) = -2c_2\left(t + 2c_1\right)^{-3} = -2c\left(t + 2c_1\right)^{-1}, \\ c''(t) = 6c_2\left(t + 2c_1\right)^{-4} = 6c\left(t + 2c_1\right)^{-2}. \end{cases} \tag{2.16}$$

Therefore, the second equation in the above (2.8) can be reduced to

$$d''(t) + \frac{4}{t + 2c_1}d'(t) + \frac{2}{\left(t + 2c_1\right)^2}d(t) = \frac{c_2^2}{\left(t + 2c_1\right)^4}. \tag{2.17}$$

This is a second-order nonhomogeneous ordinary differential equation with variable coefficients. First,we try to obtain the general solution of the corresponding homogeneous equation, that is

$$(t + 2c_1)^2 d''(t) + 4(t + 2c_1)d'(t) + 2d(t) = 0. \tag{2.18}$$

This equation is a typical Euler equation, and it is considered to be transformed into an ordinary differential equation with constant coefficients.

Let $\tau = t + 2c_1$, then the above expression is converted to

$$\tau^2 d''(\tau - 2c_1) + 4\tau d'(\tau - 2c_1) + 2d(\tau + 2c_1) = 0. \tag{2.19}$$

Let $m = \log \tau$, then $\tau = e^m$. Thus it follows that

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}\tau} = \tfrac{\mathrm{d}}{\mathrm{d}m} \cdot \tfrac{\mathrm{d}m}{\mathrm{d}\tau} = \tfrac{1}{\tau}\tfrac{\mathrm{d}}{\mathrm{d}m}, \\ &\tfrac{\mathrm{d}^2}{\mathrm{d}\tau^2} = \tfrac{\mathrm{d}}{\mathrm{d}\tau}\tfrac{1}{\tau}\tfrac{\mathrm{d}}{\mathrm{d}m} = \tfrac{1}{\tau^2}\left(\tfrac{\mathrm{d}^2}{\mathrm{d}m^2} - \tfrac{\mathrm{d}}{\mathrm{d}m}\right) = \tfrac{1}{\tau^2}\tfrac{\mathrm{d}}{\mathrm{d}m}\left(\tfrac{\mathrm{d}}{\mathrm{d}m} - 1\right). \end{aligned} \tag{2.20}$$

Therefore, equation (2.18) can be reduced to

$$d''(m) - d'(m) + 4d'(m) + 2d(m) = d''(m) + 3d'(m) + 2d(m) = 0. \tag{2.21}$$

The second order ordinary differential equations with constant coefficients can be solved by eigenroot method. The characteristic equation of this equation is

$$r^2 + 3r + 2 = 0, \tag{2.22}$$

and two eigenroots can be solved as $r_1 = -1$, $r_2 = -2$. So the general solution is

$$d(m) = c_3 e^{-m} + c_4 e^{-2m}, \tag{2.23}$$

that is

$$d(t) = \frac{c_3}{t + 2c_1} + \frac{c_4}{(t + 2c_1)^2}. \tag{2.24}$$

We get the general solution of equation (2.17) corresponding to the homogeneous equation. Now let's think about the particular solution. Let's assume the general solution is

$$d(t) = c_3 y_1(t) + c_4 y_2(t), \tag{2.25}$$

where $y_1(t) = \frac{1}{t+2c_1}$ and $y_2(t) = \frac{1}{(t+2c_1)^2}$. It's easy to verify that both $y_1(t) = \frac{1}{t+2c_1}$ and $y_2(t) = \frac{1}{(t+2c_1)^2}$ satisfy the homogeneous equation (2.18), that is, $y_1(t) = \frac{1}{t+2c_1}$, $y_2(t) = \frac{1}{(t+2c_1)^2}$ are the corresponding solution of homogeneous equation for equation (2.17).

The following uses the method of constant variation to find the special solution of the above nonhomogeneous equation. We set

$$y^* = y_1(x)v_1(x) + y_2(x)v_2(x). \tag{2.26}$$

In order to make the above nonhomogeneous equation (2.17), we first need to determine the unknown functions $v_1(x)$ and $v_2(x)$. Taking the derivative of (2.26), we have

$$y^{*\prime} = y_1 v_1' + y_2 v_2' + y_1' v_1 + y_2' v_2. \tag{2.27}$$

By $v_1(x)$ and $v_2(x)$, as defined by the solution of y^* for the nonhomogeneous equation (2.17), it is better to make them meet another relationship. According to y^* expression, in order to make the y'' not contain the v_1'' and v_2'', we can set

$$y_1 v_1' + y_2 v_2' = 0. \tag{2.28}$$

Hence, we have

$$y^{*\prime} = y_1' v_1 + y_2' v_2. \tag{2.29}$$

Furthermore we take the second derivative to get

$$y^{*\prime\prime} = y_1' v_1' + y_2' v_2' + y_1'' v_1 + y_2'' v_2. \tag{2.30}$$

We substitute y^*,$y^{*\prime}$,$y^{*\prime\prime}$ into equation(2.17), which is simplified to

$$\begin{aligned} &y_1' v_1' + y_2' v_2' + \left(y_1'' + \tfrac{4}{t+2c_1} y_1' + \tfrac{2}{(t+2c_1)^2} y_1 \right) v_1 \\ &+ \left(y_2'' + \tfrac{4}{t+2c_1} y_2' + \tfrac{2}{(t+2c_1)^2} y_2 \right) v_2 = \tfrac{c_2^2}{(t+2c_1)^4}. \end{aligned} \tag{2.31}$$

Since y_1 and y_2 are the solutions of equation (2.17) corresponding to the homogeneous equation, the above formula can be simplified to

$$y_1' v_1' + y_2' v_2' = \frac{c_2^2}{(t+2c_1)^4} = f(t). \tag{2.32}$$

Simultaneously equation (2.28) and equation (2.32) imply that it can be calculated when the coefficient matrix determinant

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \frac{1}{t+2c_1} & \frac{1}{(t+2c_1)^2} \\ \frac{-1}{(t+2c_1)^2} & \frac{-2}{(t+2c_1)^3} \end{vmatrix} = -\frac{1}{(t+2c_1)^4} \neq 0. \tag{2.33}$$

We have

$$\begin{cases} v_1'(t) = -\frac{y_2 f}{W} = \frac{c_2^2}{(t+2c_1)^2} \\ v_2'(t) = \frac{y_1 f}{W} = -\frac{c_2^2}{t+2c_1}. \end{cases} \tag{2.34}$$

What we get when we integrate it is as follows

$$v_1(t) = C_1 - \frac{c_2^2}{t+2c_1}, v_2(t) = C_2 - c_2^2 \log(t + 2C_1). \tag{2.35}$$

Therefore, the special solution is

$$\begin{aligned} y^\star(t) &= y_1(t) v_1(t) + y_2(t) v_2(t) \\ &= \frac{-c_2^2}{(t+2c_1)^2} - \frac{c_2^2 \log(t+2c_1)}{(t+2c_1)^2}. \end{aligned} \tag{2.36}$$

According to the relevant theory of the linear differential equation solution, the general solution of the above non-homogeneous equation (2.17) is

$$d(t) = D(t) + y^*(t) = \frac{c_3}{t+2c_1} + \frac{c_4 - c_2^2 - c_2^2 \log(t+2c_1)}{(t+2c_1)^2}, \tag{2.37}$$

then Theorem 1.1 is proved.

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