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Analytical and Numerical Investigation of Fractional Delay Differential Equations under Relaxed Lipschitz Assumptions

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Abstract Fractional delay differential equations constitute a powerful mathematical framework for modeling complex dynamical phenomena exhibiting memory and delay effects. In this study, we investigate a class of fractional delay differential equations incorporating Caputo and Riemann-Liouville fractional derivatives with a delay term. Unlike previous approaches, we establish the existence and uniqueness of the analytical solution under relaxed Lipschitz conditions on the nonlinear terms, without requiring contraction assumptions. Utilizing Picard iteration techniques, we demonstrate convergence of the numerical method under these Lipschitz conditions, thereby broadening the applicability of our model to a wider range of real-world scenarios. Additionally, numerical tests are conducted to validate the effectiveness and accuracy of the proposed method, further highlighting its utility in practical applications. Our findings offer new insights into the modeling and analysis of complex dynamical systems, with implications for various scientific and engineering disciplines.

Keywords Fractional differential equation, delay term, Picard method, numerical integration

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1. Introduction

Fractional differential equations (FDEs) provide a powerful mathematical framework for modeling complex dynamical phenomena across various scientific domains, ranging from physics to biology to engineering. The introduction of fractional derivatives generalizes classical derivatives, allowing for a more precise description of non-local and non-stationary phenomena.

Incorporating both Riemann-Liouville and Caputo fractional derivatives [9] into differential equations with a delay term constitutes a particularly fascinating and relevant area of research. This class of problems offers several scientific and practical advantages: Modeling Non-Local Temporal Dynamics [10], Flexibility in Modeling Real-World Phenomena [12,13], In-Depth Mathematical Analysis and Practical Applications in Technology and Engineering [11].

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In this study, we focus on a specific class of FDEs that incorporate both Caputo and Riemann-Liouville fractional derivatives along with a delay term introduced by Hallaci et al [1]. The equation under investigation is given by:

$$\begin{cases} \forall t \in [0,T], \quad {}^{RL}D^{\alpha}\left({}^{C}D^{\beta}u(t) - g(t,u(t-\tau))\right) = f(t,u(t)), \\ \forall t \in [-\tau,0], u(t) = \phi(t), \\ \lim\limits_{t \longrightarrow 0} t^{1-\alpha}\, {}^{C}D^{\beta}u(t) = 0, \\ u'(0) = 0. \end{cases}$$

This equation captures the intricate interplay between fractional calculus, delay dynamics, and nonlinearity, making it a subject of considerable interest in both theoretical and applied mathematics. The Caputo and Riemann-Liouville derivatives offer a comprehensive framework for describing the temporal behavior of systems with memory and non-local effects, while the delay term accounts for delayed interactions in the system.

In this paper, we aim to explore the analytical properties and numerical solutions of this fractional delay differential equation. We investigate the existence and uniqueness of solutions using Picard's sequence under some lipschitz conditions. For the numerical analysis, we employ numerical integration techniques to approximate the solutions.

By delving into this class of equations, we contribute to the broader understanding of fractional calculus and its applications in dynamical systems theory, paving the way for advancements in modeling and analysis of complex phenomena.

In our study, we utilized Picard iteration techniques to establish the existence and uniqueness of the analytical solution, under relaxed Lipschitz conditions on f and g, without the need for contraction assumptions as in previous works [1, 10, 11]. Moreover, these Lipschitz conditions were found to be sufficient for ensuring convergence of the numerical method. This approach allowed us to broaden the scope of applicable scenarios and enhance the robustness of our model in handling various real-world complexities.

2. Problem position

In this section, we introduce the fractional differential problem to be studied in this paper and his integral equivalent form. Let $f, g : [0,T] \times \mathbb{R} \to \mathbb{R}$ be two continuous functions, where $T > 0$. Our goal is to solve numerically the following fractional differential equation (FDE) with a delay $\tau > 0$

$$\begin{cases} \forall t \in [0,T], \quad {}^{RL}D^{\alpha}\left({}^{C}D^{\beta}u(t) - g(t,u(t-\tau))\right) = f(t,u(t)), \\ \forall t \in [-\tau,0], u(t) = \phi(t), \\ \lim\limits_{t \longrightarrow 0} t^{1-\alpha}\, {}^{C}D^{\beta}u(t) = 0, \\ u'(0) = 0, \end{cases}$$

where $0 < \alpha < 1$, $1 < \beta < 2$, ϕ is a given continuous function and u is the unknown to be found in $C^0[0,T]$, ${}^{RL}D^{\alpha}$ the Riemann-Liouville fractional derivative is given, for $\alpha \in \,]n-1, n[$, $n \in \mathbb{N}$, by

$$\forall t \geq 0, \quad {}^{RL}D^{\alpha}\psi(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}\psi(s)\,ds,$$

and ${}^{C}D^{\alpha}$ is the Caputo fractional derivative, which is given for $\alpha \in]n-1, n[$, $n \in \mathbb{N}$ by

$$\forall t \geq 0, \quad {}^{C}D^{\alpha}\psi(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}\psi^{(n)}(s)\,ds,$$

where, Γ is the Euler gamma function defined for α by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-\eta}\eta^{\alpha-1}d\eta.$$

Hallaci et al. [1] showed that (FDE) is equivalent to the following fractional integral equation (FIE)

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s, u(s-\tau))ds \\ &+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s, u(s-\tau))ds + \phi(0), \end{aligned}$$

with,

$$\forall t \in [-\tau, 0], u(t) = \phi(t).$$

In the next section, we are going to deal directly with the integral version (FIE) of our problem. Firstly, we show that (FIE) has a unique solution $u \in C^0[0, T]$ supposed to be equipped with its usual norm,

$$\forall \psi \in C^0[0, T], \|\psi\| = \max_{0 \leq t \leq T} |\psi(t)|.$$

Secondly, we use numerical integration to build an approximation of u.

3. Analytical approach

Our goal in this section is to build acceptable hypotheses in order to obtain a unique solution u of the following (FIE)

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}f(s, u(s-\tau))ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}g(s, u(s-\tau))ds + \phi(0). \end{aligned} \tag{3.1}$$

Changing s by $s+\tau$, we get

$$\begin{aligned} u(t) = &\frac{1}{\Gamma(\alpha+\beta)}\int_{-\tau}^{t-\tau} (t-\tau-s)^{\alpha+\beta-1}f(s+\tau, u(s))ds \\ &+ \frac{1}{\Gamma(\beta)}\int_{-\tau}^{t-\tau} (t-\tau-s)^{\beta-1}g(s+\tau, u(s))ds + \phi(0). \end{aligned} \tag{3.2}$$

But, for all $t \in [-\tau, 0]$, $u(t) = \phi(t)$, then

$$\begin{aligned} u(t) =& \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t-\tau} (t-\tau-s)^{\alpha+\beta-1} f(s+\tau, u(s)) ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^{t-\tau} (t-\tau-s)^{\beta-1} g(s+\tau, u(s)) ds + h(t), \end{aligned} \tag{3.3}$$

where $\forall t \in [0, \tau]$, and

$$\begin{aligned} h(t) =& \frac{1}{\Gamma(\alpha+\beta)} \int_{-\tau}^{0} (t-\tau-s)^{\alpha+\beta-1} f(s+\tau, \phi(s)) ds \\ &+ \frac{1}{\Gamma(\beta)} \int_{-\tau}^{0} (t-\tau-s)^{\beta-1} g(s+\tau, \phi(s)) ds + \phi(0). \end{aligned} \tag{3.4}$$

Theorem 3.1. *Assume that f and g verify the following Lipshitz proprieties*

$$\begin{aligned} &(C_1) : \exists L_1 > 0,\ \forall t \in [0, T],\ \forall x, \overline{x} \in \mathbb{R}\ |f(t, x) - f(t, \overline{x})| \leq L_1 |x - \overline{x}|, \\ &(C_2) : \exists L_2 > 0,\ \forall t \in [0, T],\ \forall x, \overline{x} \in \mathbb{R}\ |g(t, x) - g(t, \overline{x})| \leq L_2 |x - \overline{x}|. \end{aligned}$$

Then, equation (3.3) admits a unique solution $u \in C^0[0, T]$.

Proof. We introduce a Picard sequence $\{u_n\}_{n \in \mathbb{N}}$ as

$$\begin{cases} u_0 = h, \\ \forall t \in [0, T],\ \forall n \geq 0,\ u_{n+1}(t) = \dfrac{1}{\Gamma(\alpha+\beta)} \displaystyle\int_0^{t-\tau} (t-\tau-s)^{\alpha+\beta-1} f(s+\tau, u_n(s)) ds \\ \qquad + \dfrac{1}{\Gamma(\beta)} \displaystyle\int_0^{t-\tau} (t-\tau-s)^{\beta-1} g(s+\tau, u_n(s)) ds + h(t). \end{cases}$$

We obtain, $\forall t \in [0, T]$,

$$\begin{aligned} &|u_{n+1}(t) - u_n(t)| \\ &= \Big| \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t-\tau} (t-\tau-s)^{\alpha+\beta-1} \big[f(s+\tau, u_n(s)) - f(s+\tau, u_{n-1}(s)) \big] ds \\ &+ \frac{1}{\Gamma(\beta)} \int_0^{t-\tau} (t-\tau-s)^{\beta-1} \big[g(s+\tau, u_n(s)) - g(s+\tau, u_{n-1}(s)) \big] ds \Big| \\ &\leq \frac{L_1}{\Gamma(\alpha+\beta)} \int_0^{t-\tau} (t-\tau-s)^{\alpha+\beta-1} \big| u_n(s) - u_{n-1}(s) \big| ds \\ &+ \frac{L_2}{\Gamma(\beta)} \int_0^{t-\tau} (t-\tau-s)^{\beta-1} \big| u_n(s) - u_{n-1}(s) \big| ds. \end{aligned}$$

Therefore, $\forall\, n \geq 1$, $\forall\, t \in [0, T]$,

$$|u_{n+1}(t) - u_n(t)| \leq \left(\frac{L_1 (T-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{L_2 (T-\tau)^{\beta-1}}{\Gamma(\beta)} \right) \int_0^t |u_n(s) - u_{n-1}(s)| ds.$$

Repeating the last inequality $n-$times, we obtain: $\forall\, t \in [0,T]$,

$$|u_{n+1}(t) - u_n(t)| \leq \left(\frac{L_1(T-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{L_2(T-\tau)^{\beta-1}}{\Gamma(\beta)}\right)\frac{T^n}{n!}H,$$

where,

$$H = \frac{L_1(T-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^{T-\tau}|f(s+\tau,h(0))|ds + \frac{L_2(T-\tau)^{\beta-1}}{\Gamma(\beta)}\int_0^{T-\tau}|g(s+\tau,h(0))|ds,$$

proving that $\sum_{n\geq 1}(u_n - u_{n-1})$ is uniformly convergent in $C^0[0,T]$ to a function u, i.e

$$\forall t \in [0,T] : \sum_{n=1}^{+\infty}(u_n(t) - u_{n-1}(t)) = \lim_{n\longrightarrow+\infty} u_n(t) = u(t).$$

We use the continuity of $f$and g to prove that u is solution of (3.3). Now, to show the uniqueness, we set $\tilde{u} \in C^0[0,T]$ as a new solution of (3.3).

Then, $\forall t \in [0,T]$

$$\begin{aligned}|u(t) - \tilde{u}(t)| &= \left(\frac{L_1\,(T-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{L_2\,(T-\tau)^{\beta-1}}{\Gamma(\beta)}\right)\int_0^t |u(s) - \tilde{u}(s)|\,ds \\ &\leq \left(\frac{L_1\,(T-\tau)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} + \frac{L_2\,(T-\tau)^{\beta-1}}{\Gamma(\beta)}\right)\int_0^t |u(s) - \tilde{u}(s)|\,ds.\end{aligned}$$

We use Lemma 3 from [2] to obtain $\forall t \in [0,T], u(t) - \tilde{u}(t) = 0$, proving the uniqueness. □

4. Numerical approach

From the previous section, we understand that if $(C1)$ and $(C2)$ are verified, the system

$$u(t) = \phi(t), \quad t \in [-\tau, 0], \tag{4.1}$$

$$\begin{aligned}u(t) =& \frac{1}{\Gamma(\alpha+\beta)}\int_0^{t-\tau}(t-\tau-s)^{\alpha+\beta-1} f(s+\tau,u(s))ds \\ &+ \frac{1}{\Gamma(\beta)}\int_0^{t-\tau}(t-\tau-s)^{\beta-1} g(s+\tau,u(s))ds + \phi(0)\end{aligned} \tag{4.2}$$

admits a unique solution $u \in C^0[-\tau,T]$.

The objective of this section is to build a numerical approximation of the exact solution u. For that, we use Nyström method because of its efficiency to deal with this type of Volterra equation as show in many previous works [2–6]. Let us introduce a subdivision of the interval $[-\tau,T]$ in the following sense.

For an integer $N \geq 2$, we define $h = \frac{\tau}{N}$ and $M \in \mathbb{N}$ such that $-\tau + (M-1)h < T \leq -\tau + Mh$, let now $\{t_i\}_{i=1}^{M}$ be a subdivision of $[-\tau,T]$ such that

$$\begin{aligned}t_i &= -\tau + ih, \quad 0 \leq i \leq M-1, \\ t_M &= T.\end{aligned}$$

With this subdivision, we use trapezoidal weights $\{W_i\}_{i=0}^{M}$ given by:

$$W_0 = \frac{h}{2}, \quad W_M = \frac{T+\tau-(M-1)h}{2},$$
$$W_i = h, \quad 1 \leq i \leq M-1,$$
$$W_M = \frac{T+\tau-(M-1)h}{2}$$

as a numerical integration method. This is the most adequate method with our equation because it verifies (see [7]):

$$\forall x \in C^0[-\tau, T], \quad \lim_{h \longrightarrow 0} \Big| \sum_{i=0}^{M} W_i x(t_i) - \int_{-\tau}^{T} x(t)dt \Big| = 0. \tag{4.3}$$

Applying Nyström method with trapezoidal rule to the systems (4.1)-(4.2) gives us the following discrete system

$$u_i = \phi(t_i), \quad 0 \leq i \leq N, \tag{4.4}$$

$$\begin{aligned} u_i =& \frac{1}{\Gamma(\alpha+\beta)} \sum_{j=0}^{i-N} W_j (t_i - \tau - t_j)^{\alpha+\beta-1} f(t_j+\tau, u_j) \\ &+ \frac{1}{\Gamma(\beta)} \sum_{j=0}^{i-N} W_j (t_i - \tau - t_j)^{\beta-1} g(t_j+\tau, u_j) + h(t_i) \quad N+1 \leq i \leq M, \end{aligned} \tag{4.5}$$

where, $U_i \approx u(t_i)$ for $0 \leq i \leq M$.

Unlike system obtained with Nyström method in [2,8] the delay term gives us an explicit numerical systems (4.4)-(4.5) which facilitates the convergence study and programming.

To study the convergence of our numerical method, we introduce the discrete error $\{\varepsilon_i\}_{i=0}^{M}$ as

$$\varepsilon_i = U_i - u(t_i), \quad 0 \leq i \leq M.$$

Now, we introduce the consistence errors for :
$N+1 \leq i \leq 2N$,

$$\begin{aligned} \hat{\delta}_i^1 =& \int_0^{t_i-\tau} (t_i - \tau - s)^{\alpha+\beta-1} f(s+\tau, \phi(s)) ds \\ &- \sum_{j=i-N}^{N} W_j (t_i - \tau - t_j)^{\alpha+\beta-1} f(t_j+\tau, \phi(t_j)), \\ \hat{\delta}_i^2 =& \int_0^{t_i-\tau} (t_i - \tau - s)^{\beta-1} g(s+\tau, \phi(s)) ds \\ &- \sum_{j=i-N}^{N} W_j (t_i - \tau - t_j)^{\beta-1} g(t_j+\tau, \phi(t_j)). \end{aligned}$$

But, for $2N+1 \leq i \leq M$, ($T >> \tau$ and $\hbar$ small enough to obtain $M \geq 2N+2$), the consistence errors are given by

$$\begin{aligned}
\delta_i^1 &= \int_0^{t_i-\tau} (t_i-\tau-s)^{\alpha+\beta-1} f(s+\tau, u(s))ds \\
&- \sum_{j=N}^{i-N} W_j (t_i-\tau-t_j)^{\alpha+\beta-1} f(t_j+\tau, u(t_j)), \\
\delta_i^2 &= \int_0^{t_i-\tau} (t_i-\tau-s)^{\beta-1} g(s+\tau, u(s))ds \\
&- \sum_{j=N}^{i-N} W_j (t_i-\tau-t_j)^{\beta-1} g(t_j+\tau, u(t_j)).
\end{aligned}$$

Using (4.3), it is obvious to obtain that

$$\begin{aligned}
\lim_{h\longrightarrow 0} \max_{N+1\leq i\leq 2N} |\hat{\delta}_i^1| &= \lim_{h\longrightarrow 0} \max_{N+1\leq i\leq 2N} |\hat{\delta}_i^2| = \lim_{h\longrightarrow 0} \max_{2N+1\leq i\leq M} |\delta_i^1| \\
&= \lim_{h\longrightarrow 0} \max_{2N+1\leq i\leq M} |\delta_i^2| = 0.
\end{aligned}$$

Also, it is clear that

$$\varepsilon_i = 0, \quad 0 \leq i \leq N.$$

Theorem 4.1. *Supposing that* (C_1), (C_2) *hold, then*

$$\lim_{h\longrightarrow 0} \max_{N\leq i\leq M} |\varepsilon_i| = 0.$$

Proof. For all $2N+1 \leq i \leq M$, we have:

$$\begin{aligned}
|\varepsilon_i| &= |U_i - u(t_i)| \\
&= | \frac{1}{\Gamma(\alpha+\beta)} \sum_{j=N}^{i-N} W_j (t_i-\tau-t_j)^{\alpha+\beta-1} f(t_j+\tau, u_j) \\
&+ \frac{1}{\Gamma(\beta)} \sum_{j=N}^{i-N} W_j (t_i-\tau-t_j)^{\beta-1} g(t_j+\tau, u_j) \\
&- \frac{1}{\Gamma(\alpha+\beta)} \int_0^{t_i-\tau} (t_i-\tau-s)^{\alpha+\beta-1} f(s+\tau, u(s))ds \\
&- \frac{1}{\Gamma(\beta)} \int_0^{t_i-\tau} (t_i-\tau-s)^{\beta-1} g(s+\tau, u(s))ds \,| \\
&\leq \frac{h}{\Gamma(\alpha+\beta)} \sum_{j=N}^{i-N} (t_i-\tau-t_j)^{\alpha+\beta-1} \mid f(t_j+\tau, u_j) - f(t_j+\tau, u(t_j)) \mid \\
&+ \frac{h}{\Gamma(\beta)} \sum_{j=N}^{i-N} (t_i-\tau-t_j)^{\beta-1} \mid g(t_j+\tau, u_j) - g(t_j+\tau, u(t_j)) \mid
\end{aligned}$$

$$+ \max_{2N+1\le i\le M} |\delta_i^1| + \max_{2N+1\le i\le M} |\delta_i^2|$$
$$\le \frac{2h\,(T-\tau)^{\alpha+\beta-1}}{\Gamma(\beta)} \max\{L_1, L_2\} \sum_{j=N}^{i-N} |\varepsilon_j| + \max_{2N+1\le i\le M} |\delta_i^1| + \max_{2N+1\le i\le M} |\delta_i^2|,$$

which means, for all $2N+1 \le i \le M$,

$$|\varepsilon_i| \le \frac{2h\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}}{\Gamma(\beta) - 2h\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}} \sum_{j=N}^{i-N-1} |\varepsilon_j|$$
$$+ \frac{\Gamma(\beta)}{\Gamma(\beta) - 2h\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}} \left(\max_{2N+1\le i\le M} \left(|\delta_i^1| + |\delta_i^2|\right)\right).$$

Applying Theorem 7.1 page 101 from [7], we obtain for all $2N+1 \le i \le M$,

$$|\varepsilon_i| \le A_{N,M} \cdot \frac{\Gamma(\beta)}{\Gamma(\beta) - 2h\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}} \left(\max_{2N+1\le i\le M} |\delta_i^1| + \max_{2N+1\le i\le M} |\delta_i^2| \right),$$

where

$$A_{N,M} = \left(1 + \frac{2h\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}}{\Gamma(\beta) - 2h\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}}\right)^{M-N}.$$

But,

$$\frac{1}{A_{N,M}} = \left(1 - \frac{2\tau\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}}{N\ \Gamma(\beta)}\right)^{M-N}.$$

Then,

$$A_{N,M} \sim \exp\left(\frac{2\tau\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}}{\Gamma(\beta)}\right).$$

Now, for $N+1 \le i \le 2N$, we use the same previous steps to obtain,

$$|\varepsilon_i| \le \frac{2h\,(T-\tau)^{\alpha+\beta-1}\max\{L_1,L_2\}}{\Gamma(\beta)} \sum_{j=i-N}^{N} |\varepsilon_j|$$
$$+ \max_{N+1\le i\le 2N} |\hat{\delta}_i^1| + \max_{N+1\le i\le 2N} |\hat{\delta}_i^2|.$$

We use $|\varepsilon_j| = 0$, $i-N \le j \le N$, to achieve the proof. □

5. Numerical example

We take the same analytical example studied in [1], i.e

$$\alpha = 0.5, \beta = 1.5, T = 1, f(t,x) = \frac{\sin(x)}{1+t^2},\ g(t,x) = \cos(t)\big(t^2 e^{-e^{-2}} - \frac{t^2}{2} + \frac{\sin(tx)}{e}\big).$$

If we take $\phi(t) = t\exp(t)$, $\tau = 0.2$, the numerical results are presented in figure 1. But, figure 2 represents the case where $\phi(t) = -0.1\sin(10\pi t)$, $\tau = 0.3$.

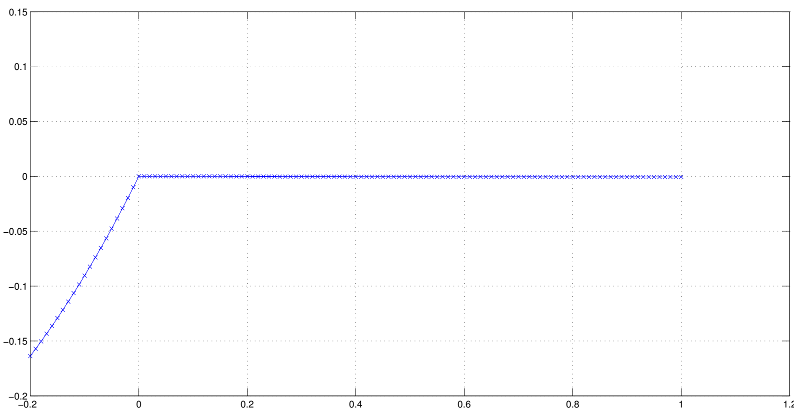

Figure 1. $\phi(t) = t\exp(t)$, $\tau = 0.2$

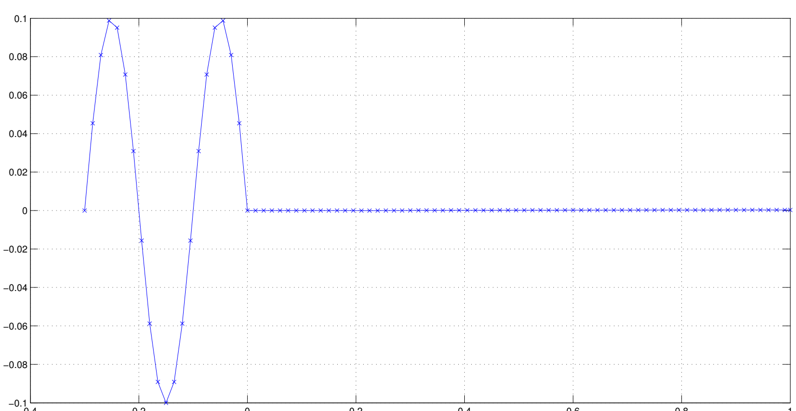

Figure 2. $\phi(t) = -0.1\sin(10\pi t)$, $\tau = 0.3$

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