Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 4, December 2024, 873–889 DOI:10.12150/jnma.2024.873

The First Eigenvalue of (p, q)-Laplacian System on C-Totally Real Submanifold in Sasakian Manifolds

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Abstract Consider (M, g) as an n-dimensional compact Riemannian manifold. Our main aim in this paper is to study the first eigenvalue of (p, q)-Laplacian system on C-totally real submanifold in Sasakian space of form $\bar{M}^{2m+1}(\kappa)$. Also in the case of $p, q > n$ we show that for $\lambda_{1,p,q}$ arbitrary large there exists a Riemannian metric of volume one conformal to the standard metric of S^n.

Keywords Eigenvalue, (p, q)-Laplacian system, geometric estimate, Sasakian manifolds

MSC(2010) 65N25, 53C21, 58C40.

1. Introduction

Studying the bounds of the eigenvalue of the Laplacian on a given manifold is a key aspect in Riemannian geometry. A major objective of this purpose is to study eigenvalue that appears as a solution of Dirichlet or Neumann boundary value problems for curvatures functions. By reason of the theory of self-adjoint operators, the spectral properties of linear Laplacian were studied extensively. As an example, mathematicians are generally attracted to the spectrum of the Laplacian on compact manifolds with or without boundary or on noncompact complete manifolds in these two cases the linear Laplacian can be uniquely extended to self-adjoint operators (see [7,8]).

Consider (M^n, g) as an n-dimensional compact Riemannian manifold. Let $u : M \longrightarrow \mathbb{R}$ be a smooth function on M or $u \in W^{1,p}(M)$ where $W^{1,p}(M)$ is the Sobolev space. The p-Laplacian of u for $1 < p < \infty$ is defined as

$$\begin{aligned}\Delta_p u &= div\left(|\nabla u|^{p-2}\nabla u\right)\\ &= |\nabla u|^{p-2}\Delta u + (p-2)\,|\nabla u|^{p-4}\,(Hess\, u)\,(\nabla u, \nabla u)\,,\end{aligned}$$

where

$$\begin{aligned}(Hess\, u)\,(X, Y) &= \nabla\,(\nabla u)\,(X, Y)\\ &= X\,(Yu) - (\nabla_X Y)\,u \qquad X, Y \in \chi\,(M)\,.\end{aligned}$$

The first eigenvalues of p-Laplace operator in both Dirichlet and Neumann cases have been studied in many papers (see for example [14]).

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In this paper we are going to study the first Dirichlet eigenvalue of the system

$$\begin{cases} \Delta_p u = -\lambda |u|^{\alpha} |v|^{\beta} v, & \text{in M}, \\ \Delta_q v = -\lambda |u|^{\alpha} |v|^{\beta} u, & \text{in M}, \\ u = v = 0, & \text{on } \partial\text{M}, \end{cases} \tag{1.1}$$

where $p, q > 1$ and α, β are real numbers such that

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1.$$

Let (M, g) be an n-dimensional compact Riemannian manifold. The first Dirichlet eigenvalue of system (1.1) is defined as

$$\lambda_{1,p,q}(M) = \inf_{u,v \neq 0} \left\{ \frac{1}{\int_M |u|^{\alpha+1} |v|^{\beta+1} dv} \left[\frac{\alpha+1}{p} \int_M |\nabla u|^p dv + \frac{\beta+1}{q} \int_M |\nabla v|^q dv \right] \right\},$$

where

$$(u, v) \in W_0^{1,p}(M) \times W_0^{1,q}(M) \setminus \{0\}.$$

As an example, the second author studied the first eigenvalue of the system (1.1) in [3].

The (p, q)-Laplacian system (1.1) was studied before in many papers. As an example the authors of this paper studied the first eigenvalue of the general case of the system (1.1) under the integral curvature condition in [9]. These types of systems have been found in different cases in physics. For example, they are relevant to the study of transport of electron temperature in a confined plasma and also to the study of electromagnetic phenomena in nonhomogeneous super conductors (see [5,6]).

The study of submanifolds, especially Legendrian submanifolds in contact manifolds from the Riemannian geometric perspective was initiated in the 1970s. The main problem in this area is to establish the classes which include such submanifolds. As an example, nonharmonic biharmonic submanifolds in Sasakian space forms of low dimension were studied before in [11,17]. The importance of studying eigenvalues of Laplacian was clearly obtained by Reilly in [15]. As a quick remark, we recall that the n-manifold M^n is called a minimal submanifold, if the mean curvature vector vanishes on M^n everywhere. In this case Reilly showed that the first eigenvalue of the Laplacian for a compact n-manifold isometrically immersed in Euclidean space is bounded above by n times the average value of the square of the norm of the mean curvature vector. Moreover, if the eigenvalue achieves this bound, then the submanifold is a minimal submanifold of some hypersphere in the Euclidean space. Ali et al. studied the first non-zero eigenvalue of p-Laplacian operator in [1].

Proposition 1.1. *Let Σ^n be an n-dimensional closed oriented C-totally real submanifold in a Sasakian space form $\bar{M}^{2m+1}(\kappa)$. The first non-zero eigenvalue $\lambda_{1,p}$ of the p-Laplacian satisfies the following conditions.*

- *If $1 < p \leq 2$ then*

$$\lambda_{1,p} \leq \frac{2^{\left(1-\frac{p}{2}\right)} (m+1)^{\left(1-\frac{p}{2}\right)} n^{\frac{p}{2}}}{(Vol(\Sigma))^{\frac{p}{2}}} \left(\int_{\Sigma^n} \left(\left(\frac{\kappa+3}{4} \right) + |H|^2 \right) dv \right)^{\frac{p}{2}}.$$

- *If* $2 < p \leq \frac{n}{2} + 1$

$$\lambda_{1,p} \leq \frac{2^{\left(1-\frac{p}{2}\right)} (m+1)^{\left(1-\frac{p}{2}\right)} n^{\frac{p}{2}}}{(Vol\,(\Sigma))^{\frac{p}{2}}} \int_{\Sigma^n} \left(\left(\frac{\kappa+3}{4}\right) + |H|^2\right)^{\frac{p}{2}} dv,$$

where H *denotes the mean curvature of the immersion of* Σ^n *into* $\mathbb{R}^m$.

As a quick review, in this paper, by improving this result for the first Dirichlet eigenvalue of system (1.1) in the Sasakian space form, also we are going to prove the following theorems.

Theorem 1.1. *Let* M^n *be an n-dimensional compact Riemannian manifold and consider* Σ^n *as an n-dimensional closed oriented C-totally real submanifold in a Sasakian space form* $\bar{M}^{2m+1}(\kappa)$ *with a constant sectional curvature* κ. *If* $\lambda_{1,p,q}$ *denotes the first eigenvalue of system (1.1), then for arbitrary natural N:*

- *For* $p \geq q \geq 2$ *we get*

$$\lambda_{1,p,q} \leq \sqrt{2}^{p^2} (m+1)^{\frac{1}{2}p^2} n^{\frac{p}{2}} \int_{\Sigma^n} \left(\left(\frac{\kappa+3}{4}\right) + |H|^2\right)^{\frac{p}{2}} dv.$$

- *For* $1 < q \leq p < 2$ *we get*

$$\lambda_{1,p,q} \leq \sqrt{2}^{-q(q+1)} (m+1)^{-\frac{1}{2}q(q+1)} n^{\frac{p}{2}} \int_{\Sigma^n} \left(\left(\frac{\kappa+3}{4}\right) + |H|^2\right)^{\frac{q}{2}} dv.$$

- *For* $p \geq q > 2$ *we get*

$$\lambda_{1,p,q} \leq \sqrt{2}^{\left(p^2-q\right)} (m+1)^{\frac{1}{2}\left(p^2-q\right)} n^{\frac{p}{2}} \int_{\Sigma^n} \left(\left(\frac{\kappa+3}{4}\right) + |H|^2\right)^{\frac{p}{2}} dv,$$

where H *is the same as the previous one.*

Theorem 1.2. *Consider* M *as an n-dimensional compact manifold. If* $\lambda_{1,p,q}$ *denotes the first eigenvalue of the* (p,q)*-Laplacian system (1.1) and* $p \geq q > n$ *then for* $\epsilon > 0$

$$\limsup_{\epsilon \to 0} \lambda_{1,p,q}(\epsilon)\, \epsilon^{\frac{p}{n}} = \infty.$$

2. Preliminaries in Sasakian spaces

In this section, we are going to introduce the Sasakian space forms and some other useful notations. Consider $\left(\bar{M}, g\right)$ as an odd dimensional smooth manifold with an almost contact structure (φ, ξ, η), to be an almost contact metric manifold and also for every $U, V \in \Gamma\left(T\bar{M}\right)$ the following relations hold.

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0,$$
$$g(\varphi U, \varphi V) = g(U,V) - \eta(U)\,\eta(V), \quad \eta(U) = g(U,\xi),$$

where φ, ξ and η are called a $(1,1)$-type tensor field, a structure vector field and a dual 1-form respectively. In this case an almost contact metric manifold is called Sasakian manifold with Riemannian connection

$$\left(\bar{\nabla}_U \varphi\right) V = g(U,V)\,\xi - \eta(V)\,U,$$

where

$$\bar{\nabla}_U \xi = -\varphi U,$$

U, V are vector fields on $\bar{M}^{2m+1}$ and $\bar{\nabla}$ denotes the Riemannian connection with respect to g (for more details see [2]). We recall that for tangent vectors X, Y, Z, W on Σ^n, the Gauss equation is given by

$$\bar{R}(X,Y,Z,W) = R(X,Y,Z,W) + g(h(X,Y), h(Y,W)) - g(h(X,W), h(Y,Z)),$$

where $\bar{R}$, R are the curvature tensors on $\bar{M}^{2m+1}$ and Σ^n respectively. If $\bar{M}^{2m+1}(\kappa)$ denotes the Sasakian space form of φ-sectional curvature κ, then the curvature tensor $\bar{R}$ of $\bar{M}^{2m+1}(\kappa)$ can be explained as

$$\begin{aligned}\bar{R}(X,Y,Z,W) = &\left(\frac{\kappa+3}{4}\right)\Big\{g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\Big\} \qquad (2.1)\\ &+ \left(\frac{\kappa-1}{4}\right)\Big\{\eta(X)\,\eta(Z)\,g(Y,W) + \eta(W)\,\eta(Y)\,g(X,Z)\\ &- \eta(Y)\,\eta(Z)\,g(X,W) - \eta(X)\,g(Y,Z)\,\eta(W)\\ &+ g(\varphi Y,Z)\,g(\varphi X,W) - g(\varphi X,Z)\,g(\varphi Y,W)\\ &+ 2g(X,\varphi Y)\,g(\varphi Z,W)\Big\},\end{aligned}$$

where $X, Y, Z \in \Gamma\left(T\bar{M}\right)$.

Let $\bar{M}^{2m+1}(\kappa)$ be a Sasakian space form of constant φ-sectional curvature κ. The n-dimensional Riemannian submanifold Σ^n of $\bar{M}^{2m+1}(\kappa)$ is called C-totally real submanifold in the case that ξ is a normal vector field on Σ^n. That is, the standard almost contact structure φ of $\bar{M}^{2m+1}(\kappa)$ maps any tangent space of Σ^n into the corresponding normal space (see [16]).

By the quantity (2.1) the curvature tensor $\bar{R}$ for a C-totally real submanifold in a Sasakian space form $\bar{M}^{2m+1}(\kappa)$ is given as

$$\bar{R}(X,Y,Z,W) = \left(\frac{\kappa+3}{4}\right)\Big\{g(Y,Z)\,g(X,W) - g(X,Z)\,g(Y,W)\Big\}.$$

In the case where χ is a C-totally real immersion from Σ^n to an $(2m+1)$-dimensional Riemannian manifold $\left(\bar{M}, g\right)$, Σ^n has an induced metric $g_\Sigma = \chi^* g$. If $\{e_i\}_{i=1}^{2m+1}$ is an orthogonal frame for $\bar{M}^{2m+1}(\kappa)$ where $\{e_j\}_{j=1}^{n}$ are tangent to Σ^n and $\{e_k\}_{k=n+1}^{2m+1}$ are normal to Σ^n as well, then A. Ali et al. have already proved in [1] that

$$R = \left(\frac{\kappa+3}{4}\right) n(n-1) + n^2|H|^2 - S,$$

where R is the scalar curvature of Σ^n, $S = \sum_{i,j,k}\left(\pi_{ij}^k\right)^2$ is the norm square of the second fundamental form. $H = \sum_k H^k e_k = \frac{1}{n}\sum_k\left(\sum_i \pi_{ij}^k\right) e_k$ is the mean curvature vector of Σ^n and

$$1 \le i, j \le n, \qquad n+1 \le k \le 2m+1.$$

We recall that if $\bar{M}^{2m+1}$ is equipped with a new metric $\bar{g} = e^{2\mu} g$ which is conformal to g, then $\{\bar{e}_i\}_{i=1}^{2m+1} = e^{-\mu}\{e_i\}_{i=1}^{2m+1}$ is an orthogonal frame for $\left(\bar{M}, \bar{g}\right)$ where $\mu \in C^\infty\left(\bar{M}\right)$. In this case it was proved in [1] that

$$e^{2\mu}\left(\bar{S} - n|\bar{H}|^2\right) = S - n|H|^2.$$

Also, there are some other valuable works in which Sasakian space forms are examined (see [10, 12, 18]). We will use these relations to provide proof for our main results.

3. Main results

In this section, we are going to give appropriate proofs for our main results. At the first step, we must mention some lemmas which make our proofs more possible. The conformal bounds for the first eigenvalue of the p-Laplacian operator were studied by Matei in [13] extensively. The first eigenvalue of the (p,q)-Laplacian system (**1.1**) can be viewed as a functional as

$$g \longmapsto \lambda_{1,p,q}(M,g),$$

on the space of Riemannian metrics on M. Since $\lambda_{1,p,q}$ is not invariant under dilatations i.e. $\lambda_{1,p,q}(M,Kg) \neq \lambda_{1,p,q}(M,g)$ for a real constant K, we need a normalisation when we study the uniform boundedness for the mentioned functional. Consider $\mathcal{M}(M)$ as a set of Riemannian metrics of volume one. We consider the unit sphere S^n and we construct metrics in $\mathcal{M}(M)$ conformal to the standard metric which is denoted by *can*. For a Riemannian metric on M as g, we denote the conformal class of g by

$$[g] = \Big\{ fg | f \in C^{\infty}(M), \quad f > 0 \Big\},$$

and also

$$G(N) = \Big\{ \gamma \in Diff\left(S^N\right) | \gamma^* can \in [can] \Big\},$$

for a group of conformal diffeomorphisms of $\left(S^N, can\right)$ where N is an arbitrary natural number. We will divide our main issue into two different parts, first of all we consider the case where $p,q \leq n$ and then the case of $p,q > n$ will be assumed. In the first case we are going to study the upper bound for the first eigenvalue of the (p,q)-Laplacian system (**1.1**) with respect to the Sasakian space form. For this purpose, we have to prove some useful lemmas. For the first lemma which is called Chebyshev inequality, a suitable proof can be found in many papers such as [4].

Lemma 3.1 (Chebyshev inequality). *Consider $\{a_i\}_{i=1}^n$ and $\{b_i\}_{i=1}^n$ as two decreasing real sequences. Then*

$$\frac{1}{n}\sum_{i=1}^{n} a_i b_i \geq \left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} b_i\right).$$

Lemma 3.2. *Consider $\phi : (M,g) \longrightarrow \left(S^N, can\right)$ as a smooth map whose level sets are of measure zero in (M,g). Then for $\alpha, \beta > 0$ there exist $\gamma, \delta \in G(N)$ such that:*

- *If $p,q \geq 2$, then*

$$\lambda_{1,p,q}(M) \leq (N+1)^{\frac{1}{2}p^2}\left(\frac{\alpha+1}{p}\int_M |d\psi|^p dv + \frac{\beta+1}{q}\int_M |d\zeta|^q dv\right).$$

- *If* $1 < p, q < 2$, *then*

$$\lambda_{1,p,q}(M) \leq (N+1)^{-\frac{1}{2}q(q+1)} \left(\frac{\alpha+1}{p} \int_M |d\psi|^p dv + \frac{\beta+1}{q} \int_M |d\zeta|^q dv \right).$$

- *If* $1 < q < 2 \leq p$, *then*

$$\lambda_{1,p,q}(M) \leq (N+1)^{\frac{1}{2}(p^2-q)} \left(\frac{\alpha+1}{p} \int_M |d\psi|^q dv + \frac{\beta+1}{q} \int_M |d\zeta|^q dv \right),$$

where for arbitrary natural N *and* $1 \leq i \leq N+1$

$$\psi_i = (\gamma \circ \phi)_i, \qquad \zeta_i = (\delta \circ \phi)_i.$$

Proof. Consider $\psi_i = (\gamma \circ \phi)_i$ and $\zeta_i = (\delta \circ \phi)_i$. For $1 \leq i \leq N+1$ we get

$$\lambda_{1,p,q}(M) \leq \frac{1}{\int_M |\check{\psi}_i|^{\alpha+1} |\check{\zeta}_i|^{\beta+1} dv} \Big[\frac{\alpha+1}{p} \int_M |d\check{\psi}_i|^p dv + \frac{\beta+1}{q} \int_M |d\check{\zeta}_i|^q \Big],$$

where $\{\check{\psi}\}_i$ and $\{\check{\zeta}\}_i$ denote the decreasing rearrangement of $\{\psi\}_i$ and $\{\zeta\}_i$ respectively. Thus

$$\lambda_{1,p,q}(M) \leq \frac{1}{\int_M \sum_{i=1}^{N+1} |\check{\psi}_i|^{\alpha+1} |\check{\zeta}_i|^{\beta+1} dv} \Big[\frac{\alpha+1}{p} \int_M \sum_{i=1}^{N+1} |d\check{\psi}_i|^p dv + \frac{\beta+1}{q} \int_M \sum_{i=1}^{N+1} |d\check{\zeta}_i|^q \Big],$$

where the last inequality comes from

$$\lambda \int_M |\check{\psi}_i|^{\alpha+1} |\check{\zeta}_i|^{\beta+1} dv \leq \frac{\alpha+1}{p} \int_M |d\check{\psi}_i|^p + \frac{\beta+1}{q} \int_M |d\check{\zeta}_i|^q dv,$$

and summing both sides from 1 to $n+1$,

$$\sum_{i=1}^{n+1} \left(\lambda \int_M |\check{\psi}_i|^{\alpha+1} |\check{\zeta}_i|^{\beta+1} dv \right) \leq \sum_{i=1}^{n+1} \left(\frac{\alpha+1}{p} \int_M |d\check{\psi}_i|^p + \frac{\beta+1}{q} \int_M |d\check{\zeta}_i|^q \right).$$

First consider $p, q \geq 2$. Then

$$\sum_{i=1}^{N+1} |d\check{\psi}_i|^p = \sum_{i=1}^{N+1} \left(|d\check{\psi}_i|^2 \right)^{\frac{p}{2}} \leq \left(\sum_{i=1}^{N+1} |d\check{\psi}_i|^2 \right)^{\frac{p}{2}} = |d\check{\psi}|^p,$$

and

$$\sum_{i=1}^{N+1} |d\check{\zeta}_i|^q = \sum_{i=1}^{N+1} \left(|d\check{\zeta}_i|^2 \right)^{\frac{q}{2}} \leq \left(\sum_{i=1}^{N+1} |d\check{\zeta}_i|^2 \right)^{\frac{q}{2}} = |d\check{\zeta}|^q.$$

By Chebyshev inequality, we have

$$\sum_{i=1}^{N+1} |\check{\psi}_i|^{\alpha+1} |\check{\zeta}_i|^{\beta+1} \geq \frac{1}{N+1} \sum_{i=1}^{N+1} |\check{\psi}_i|^{\alpha+1} |\check{\zeta}_i|^{\beta+1}$$

$$
\begin{aligned}
&\geq \left(\frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\psi}_i|^{\alpha+1}\right)\left(\frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\zeta}_i|^{\beta+1}\right)\\
&\geq \left(\frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\psi}_i|^{p(\alpha+1)}\right)\left(\frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\zeta}_i|^{q(\beta+1)}\right)\\
&= \left(\sum_{i=1}^{N+1}\frac{1}{N+1}\left(|\check{\psi}_i|^2\right)^{\frac{p}{2}(\alpha+1)}\right)\left(\sum_{i=1}^{N+1}\frac{1}{N+1}\left(|\check{\zeta}_i|^2\right)^{\frac{q}{2}(\beta+1)}\right).
\end{aligned}
\tag{3.1}
$$

Since $x\mapsto x^{\frac{R}{2}}$ for $\frac{R}{2}\geq 1$ is convex, and given that $\sum_{i=1}^{N+1}|\psi_i|^2=1$ and $\sum_{i=1}^{N+1}|\zeta_i|^2=1$, by applying Jensen's inequality to formula (**3.1**) we get

$$
\begin{aligned}
&\sum_{i=1}^{N+1}|\check{\psi}_i|^{\alpha+1}|\check{\zeta}_i|^{\beta+1}\\
&\geq \left(\sum_{i=1}^{N+1}\frac{1}{N+1}|\check{\psi}_i|^2\right)^{\frac{p}{2}(\alpha+1)}\left(\sum_{i=1}^{N+1}\frac{1}{N+1}|\check{\zeta}_i|^2\right)^{\frac{q}{2}(\beta+1)}\\
&= \left(\frac{1}{N+1}\right)^{\frac{p}{2}(\alpha+1)}\left(\sum_{i=1}^{N+1}|\check{\psi}_i|^2\right)^{\frac{p}{2}(\alpha+1)}\left(\frac{1}{N+1}\right)^{\frac{q}{2}(\beta+1)}\left(\sum_{i=1}^{N+1}|\check{\zeta}_i|^2\right)^{\frac{q}{2}(\beta+1)}\\
&= (N+1)^{-\frac{1}{2}(p(\alpha+1)+q(\beta+1))}.
\end{aligned}
$$

Now under consideration $p\geq q$, we have

$$
\lambda_{1,p,q}(M)\leq (N+1)^{\frac{1}{2}p^2}\left(\frac{\alpha+1}{p}\int_M|d\psi|^p dv+\frac{\beta+1}{q}\int_M|d\zeta|^q dv\right).
$$

For $1<p,q<2$, since

$$
|\psi_i|\leq 1,\quad |\zeta_i|\leq 1,
$$

also $x\mapsto x^{\frac{p+1}{2}}$ and $x\mapsto x^{\frac{q+1}{2}}$ are convex, by the similar process we see

$$
\begin{aligned}
\sum_{i=1}^{N+1}|\check{\psi}_i|^{\alpha+1}|\check{\zeta}_i|^{\beta+1} &\geq \frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\psi}_i|^{\alpha+1}|\check{\zeta}_i|^{\beta+1}\\
&\geq \left(\frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\psi}_i|^{\alpha+1}\right)\left(\frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\zeta}_i|^{\beta+1}\right)\\
&\geq \left(\frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\psi}_i|^{(p+1)(\alpha+1)}\right)\left(\frac{1}{N+1}\sum_{i=1}^{N+1}|\check{\zeta}_i|^{(q+1)(\beta+1)}\right)\\
&= \left(\sum_{i=1}^{N+1}\frac{1}{N+1}\left(|\check{\psi}_i|^2\right)^{\frac{p+1}{2}(\alpha+1)}\right)\left(\sum_{i=1}^{N+1}\frac{1}{N+1}\left(|\check{\zeta}_i|^2\right)^{\frac{q+1}{2}(\beta+1)}\right).
\end{aligned}
$$

Continuing from the last line of the previous formula by Jensen's inequality, we get

$$
\sum_{i=1}^{N+1}|\check{\psi}_i|^{\alpha+1}|\check{\zeta}_i|^{\beta+1}
$$

$$\geq \left(\sum_{i=1}^{N+1} \frac{1}{N+1}|\check{\psi}_i|^2\right)^{\frac{p+1}{2}(\alpha+1)} \left(\sum_{i=1}^{N+1} \frac{1}{N+1}|\check{\zeta}_i|^2\right)^{\frac{q+1}{2}(\beta+1)}$$
$$= \left(\frac{1}{N+1}\right)^{\frac{p+1}{2}(\alpha+1)} \left(\sum_{i=1}^{N+1} |\check{\psi}_i|^2\right)^{\frac{p+1}{2}(\alpha+1)} \left(\frac{1}{N+1}\right)^{\frac{q+1}{2}(\beta+1)} \left(\sum_{i=1}^{N+1} |\check{\zeta}_i|^2\right)^{\frac{q+1}{2}(\beta+1)}$$
$$= (N+1)^{-\frac{1}{2}((p+1)(\alpha+1)+(q+1)(\beta+1))}.$$

Since $x \mapsto x^{\frac{p}{2}}$ and $x \mapsto x^{\frac{q}{2}}$ are concave, we conclude that

$$\sum_{i=1}^{N+1} |d\psi_i|^p = \sum_{i=1}^{N+1} \left(|d\psi_i|^2\right)^{\frac{p}{2}} \leq (N+1)^{1-\frac{p}{2}} \left(\sum_{i=1}^{N+1} |d\psi_i|^2\right)^{\frac{p}{2}}$$
$$= (n+1)^{1-\frac{p}{2}} |d\psi|^p,$$

and

$$\sum_{i=1}^{N+1} |d\zeta_i|^q = \sum_{i=1}^{N+1} \left(|d\zeta_i|^2\right)^{\frac{q}{2}} \leq (N+1)^{1-\frac{q}{2}} \left(\sum_{i=1}^{N+1} |d\zeta_i|^2\right)^{\frac{q}{2}}$$
$$= (N+1)^{1-\frac{q}{2}} |d\zeta|^q.$$

Now similar to the previous case, under consideration $p \geq q$, we have

$$\lambda_{1,p,q}(M) \leq (N+1)^{-\frac{1}{2}q(q+1)} \left(\frac{\alpha+1}{p} \int_M |d\psi|^p dv + \frac{\beta+1}{q} \int_M |d\zeta|^q dv\right).$$

In the third case for $1 < q < 2 \leq p$, since $x \mapsto x^{\frac{p}{2}}$ is convex, by the similar process we get

$$\sum_{i=1}^{N+1} |\check{\psi}_i|^{\alpha+1}|\check{\zeta}_i|^{\beta+1} \geq \frac{1}{N+1} \sum_{i=1}^{N+1} |\check{\psi}_i|^{\alpha+1}|\check{\zeta}_i|^{\beta+1}$$
$$\geq \left(\frac{1}{N+1} \sum_{i=1}^{N+1} |\check{\psi}_i|^{\alpha+1}\right) \left(\frac{1}{N+1} \sum_{i=1}^{N+1} |\check{\zeta}_i|^{\beta+1}\right)$$
$$\geq \left(\frac{1}{N+1} \sum_{i=1}^{N+1} |\check{\psi}_i|^{p(\alpha+1)}\right) \left(\frac{1}{N+1} \sum_{i=1}^{N+1} |\check{\zeta}_i|^{p(\beta+1)}\right)$$
$$= \left(\sum_{i=1}^{N+1} \frac{1}{N+1} \left(|\check{\psi}_i|^2\right)^{\frac{p}{2}(\alpha+1)}\right) \left(\sum_{i=1}^{N+1} \frac{1}{N+1} \left(|\check{\zeta}_i|^2\right)^{\frac{p}{2}(\beta+1)}\right),$$

and by Jensen's inequality

$$\sum_{i=1}^{N+1} |\psi_i|^{\alpha+1}|\zeta_i|^{\beta+1} \geq (N+1)^{-\frac{p}{2}(\alpha+\beta+2)}.$$

Furthermore, since $x \mapsto x^{\frac{q}{2}}$ is convex,

$$\sum_{i=1}^{N+1} |d\psi_i|^p \leq \sum_{i=1}^{N+1} |d\psi_i|^q = \sum_{i=1}^{N+1} \left(|d\psi_i|^2\right)^{\frac{q}{2}} \leq \left(\sum_{i=1}^{N+1} |d\psi_i|^2\right)^{\frac{q}{2}} = (N+1)^{1-\frac{q}{2}} |d\psi|^q,$$

and

$$\sum_{i=1}^{N+1} |d\zeta_i|^q = \sum_{i=1}^{N+1} \left(|d\zeta_i|^2\right)^{\frac{q}{2}} \leq \left(\sum_{i=1}^{N+1} |d\zeta_i|^2\right)^{\frac{q}{2}} = (N+1)^{1-\frac{q}{2}} |d\zeta|^q,$$

which conclude that

$$\lambda_{1,p,q}(M) \leq (N+1)^{\frac{1}{2}(p^2-q)} \left(\frac{\alpha+1}{p} \int_M |d\psi|^q dv + \frac{\beta+1}{q} \int_M |d\zeta|^q dv\right).$$

□

Proof. [**Proof of Theorem 1.1**] Taking account of Lemma 3.2, there exists a function ϕ and $\gamma, \delta \in G(2m+2)$ such that

$$\lambda_{1,p,q} \int_{\Sigma^n} \sum_i |\psi_i|^{\alpha+1} |\zeta_i|^{\beta+1} dv \leq \frac{\alpha+1}{p} \int_{\Sigma^n} \sum_i |d\psi_i|^p dv + \frac{\beta+1}{q} \int_{\Sigma^n} \sum_i |d\zeta_i|^q dv,$$

where $\psi_i = (\gamma \circ \phi)_i$, $\zeta_i = (\delta \circ \phi)_i$ and

$$1 \leq i \leq 2(m+1).$$

It should be mentioned that if

$$\sum_{i=1}^{2m+2} |\psi_i|^2 = 1,$$

then $|\psi_i| \leq 1$. Now with respect to the preliminaries section, consider $\{e_i\}_{i=1}^{2m+1}$ as an orthogonal frame for $\bar{M}^{2m+1}(\kappa)$ where $\{e_i\}_{i=1}^{n}$ are tangent to Σ^n, we have

$$\sum_{i=1}^{2m+2} |d\psi_i|^2 = \sum_{i=1}^{n} |\nabla_{e_i} \psi_i|^2 = ne^{2\mu},$$

and

$$\sum_{i=1}^{2m+2} |d\zeta_i|^2 = \sum_{i=1}^{n} |\nabla_{e_i} \zeta_i|^2 = ne^{2\mu},$$

where $\bar{g} = e^{2\mu} g$ and $\mu \in C^\infty(M)$. Now by applying Lemma 3.2 we have seen before, as an example for $p, q \geq 2$ and setting N as $2(m+1)$, we get

$$\lambda_{1,p,q} \leq \sqrt{2}^{p^2} (m+1)^{\frac{1}{2}p^2} \left[\frac{\alpha+1}{p} \int_{\Sigma^n} \sum_{i=1}^{2m+2} |d\psi_i|^p dv + \frac{\beta+1}{q} \int_{\Sigma^n} \sum_{i=1}^{2m+2} |d\zeta_i|^q dv\right].$$

By Hölder's inequality and under consideration of $p \geq q$ we easily get

$$\begin{aligned} &\lambda_{1,p,q} \\ \leq &\sqrt{2}^{p^2} (m+1)^{\frac{1}{2}p^2} \left[\frac{\alpha+1}{p} \int_{\Sigma^n} \left(\sum_{i=1}^{2m+2} |d\psi_i|^2\right)^{\frac{p}{2}} dv + \frac{\beta+1}{q} \int_{\Sigma^n} \left(\sum_{i=1}^{2m+2} |d\zeta_i|^2\right)^{\frac{q}{2}} dv\right] \\ \leq &\sqrt{2}^{p^2} (m+1)^{\frac{1}{2}p^2} \left(\int_{\Sigma^n} \left(ne^{2\mu}\right)^{\frac{p}{2}} dv\right), \end{aligned}$$

where we used the fact for system (1.1) that

$$\frac{\alpha+1}{p}+\frac{\beta+1}{q}=1.$$

Since by some conformal relationships and also by some calculations it was proved before in [1] that

$$\int_{\Sigma^n} e^{p\mu}dv \leq \int_{\Sigma^n}\left(|\frac{\kappa+3}{4}+|H|^2|\right)^{\frac{p}{2}}dv,$$

we conclude that

$$\lambda_{1,p,q}\leq \sqrt{2}^{p^2}\,(m+1)^{\frac{1}{2}p^2}\,n^{\frac{p}{2}}\int_{\Sigma^n}\left(\left(\frac{\kappa+3}{4}\right)+|H|^2\right)^{\frac{p}{2}}dv.$$

By the same process for other cases, we get

- If $1<p,q<2$, then

$$\lambda_{1,p,q}\leq \sqrt{2}^{-q(q+1)}\,(m+1)^{-\frac{1}{2}q(q+1)}\,n^{\frac{p}{2}}\int_{\Sigma^n}\left(\left(\frac{\kappa+3}{4}\right)+|H|^2\right)^{\frac{p}{2}}dv.$$

- If $1<q<2\leq p$, then

$$\lambda_{1,p,q}\leq \sqrt{2}^{\left(p^2-q\right)}\,(m+1)^{\frac{1}{2}\left(p^2-q\right)}\,n^{\frac{p}{2}}\int_{\Sigma^n}\left(\left(\frac{\kappa+3}{4}\right)+|H|^2\right)^{\frac{p}{2}}dv.$$

□

Consider $r\in[0,\pi]$ as a geodesic distance and $\epsilon>0$. In this case the radial function $f_\epsilon: S^n\to\mathbb{R}$ is defined as

$$f_\epsilon(r)=\epsilon^{\frac{4p}{n(p-n)}}.\chi_{[0,\frac{\pi}{2}-\epsilon]\cup[\frac{\pi}{2}+\epsilon,\pi]}(r)+\chi_{\left(\frac{\pi}{2}-\epsilon,\frac{\pi}{2}+\epsilon\right)}(r),$$

thus the parametrisation of the first eigenvalue of the (p,q)-Laplacian (**1.1**) is given by

$$\lambda_{1,p,q}(\epsilon)=\inf_{u,v\neq 0}\left\{R_\epsilon(u,v)\,|\,(u,v)\in W_0^{1,p}\times W_0^{1,q}\setminus\{0\}\right\},$$

where

$$R_\epsilon(u,v):=\frac{1}{\int_{S^{n-1}} f_\epsilon^{\frac{n}{2}}|u|^{\alpha+1}|v|^{\beta+1}dv_{can}}\Big[\frac{\alpha+1}{p}\int_{S^{n-1}} f_\epsilon^{\frac{n-p}{2}}|du|^p dv_{can} + \frac{\beta+1}{q}\int_{S^{n-1}} f_\epsilon^{\frac{n-p}{2}}|dv|^q dv_{can}\Big].$$

Proof. [**Proof of Theorem 1.2**] The radial functions $\bar{u}_\epsilon,\bar{v}_\epsilon: S^n\to\mathbb{R}$ are defined as

$$\bar{u}_\epsilon^p(r)=\frac{1}{V}\int_{S^{n-1}}|u_\epsilon(r,.)|^p dv_{can},$$
$$\bar{v}_\epsilon^q(r)=\frac{1}{V}\int_{S^{n-1}}|v_\epsilon(r,.)|^q dv_{can},$$

where $V = \text{vol}\left(S^{n-1}, can\right)$. We just give calculations in the case of p and u and also the same process holds for q and v. By taking derivatives with respect to r we get

$$p\bar{u}_\epsilon^{p-1}\bar{u}_\epsilon' = \frac{p}{V}\int_{S^{n-1}} |u_\epsilon|^{p-2}u_\epsilon \frac{\partial u_\epsilon}{\partial r} dv_{can}.$$

By Hölder's inequality we see

$$\begin{aligned}
\bar{u}_\epsilon^{p-1}|\bar{u}_\epsilon'| &\leq \frac{1}{V}\int_{S^{n-1}} |u_\epsilon|^{p-1}|\frac{\partial u_\epsilon}{\partial r}| dv_{can} \\
&\leq \frac{1}{V}\left(\int_{S^{n-1}} |u_\epsilon|^p dv_{can}\right)^{\frac{p-1}{p}} \left(\int_{S^{n-1}} |\frac{\partial u_\epsilon}{\partial r}|^p dv_{can}\right)^{\frac{1}{p}},
\end{aligned}$$

which concludes that

$$|\bar{u}_\epsilon'|^p \leq \frac{1}{V}\int_{S^{n-1}} |\frac{\partial u_\epsilon}{\partial r}|^p dv_{can} \leq \frac{1}{V}\int_{S^{n-1}} |du_\epsilon|^p dv_{can}. \tag{3.2}$$

Since $\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1$, by Hölder's inequality we get

$$\int_{S^n} |u_\epsilon|^{\alpha+1}|v_\epsilon|^{\beta+1} dv_{can} \leq \left(\int_{S^n} |u_\epsilon|^p dv_{can}\right)^{\frac{\alpha+1}{p}} \left(\int_{S^n} |v_\epsilon|^q dv_{can}\right)^{\frac{\beta+1}{q}}.$$

Now by some calculations we obtain

$$\begin{aligned}
&\int_{S^n} f_\epsilon^{\frac{n}{2}} |\bar{u}_\epsilon|^{\alpha+1}|\bar{v}_\epsilon|^{\beta+1} dv_{can} \qquad (3.3)\\
=&V.\int_0^\pi f_\epsilon^{\frac{n}{2}} |\bar{u}_\epsilon|^{\alpha+1}|\bar{v}_\epsilon|^{\beta+1} \sin r^{n-1} dr \\
=&V.\int_0^\pi f_\epsilon^{\frac{n}{2}} \left(\frac{1}{V}\int_{S^{n-1}} |u_\epsilon|^p dv_{can}\right)^{\frac{\alpha+1}{p}} \left(\frac{1}{V}\int_{S^{n-1}} |v_\epsilon|^q dv_{can}\right)^{\frac{\beta+1}{q}} \sin r^{n-1} dr \\
=&\int_0^\pi f_\epsilon^{\frac{n}{2}} \left(\int_{S^{n-1}} |u_\epsilon|^p dv_{can}\right)^{\frac{\alpha+1}{p}} \left(\int_{S^{n-1}} |v_\epsilon|^q dv_{can}\right)^{\frac{\beta+1}{q}} \sin r^{n-1} dr \\
\geq&\int_0^\pi f_\epsilon^{\frac{n}{2}} \left(\int_{S^{n-1}} |u_\epsilon|^{\alpha+1}|v_\epsilon|^{\beta+1} dv_{can}\right) \sin r^{n-1} dr \\
\geq&\int_{S^n} f_\epsilon^{\frac{n}{2}} |u_\epsilon|^{\alpha+1}|v_\epsilon|^{\beta+1} dv_{can}.
\end{aligned}$$

By formula (3.2) we conclude that

$$\begin{aligned}
\int_{S^n} f_\epsilon^{\frac{n-p}{2}} |\bar{u}_\epsilon'|^p dv_{can} &= V.\int_0^\pi f_\epsilon^{\frac{n-p}{2}} |\bar{u}_\epsilon'|^p \sin r^{n-1} dr \\
&\leq \int_0^\pi \Big[\int_{S^{n-1}} |du_\epsilon|^p dv_{can}\Big] f_\epsilon^{\frac{n-p}{2}} \sin r^{n-1} dr \\
&= \int_{S^n} f_\epsilon^{\frac{n-p}{2}} |du_\epsilon|^p dv_{can},
\end{aligned}$$

also by the similar process

$$\int_{S^n} f_\epsilon^{\frac{n-p}{2}} |\bar{v}'_\epsilon|^q dv_{can} \leq \int_{S^n} f_\epsilon^{\frac{n-p}{2}} |dv_\epsilon|^q dv_{can}.$$

If S^n_+ and S^n_- denote the upper and the lower hemispheres centered at x_0 and $-x_0$ respectively, then

$$\begin{aligned}&\lambda_{1,p,q}(\epsilon)\\ \geq&\frac{1}{\vartheta_!}\Big[\frac{\alpha+1}{p}\int_{S^n} f_\epsilon^{\frac{n-p}{2}}|\bar{u}'_\epsilon|^p dv_{can}+\frac{\beta+1}{q}\int_{S^n} f_\epsilon^{\frac{n-p}{2}}|\bar{v}'_\epsilon|^q dv_{can}\Big]\\ \geq&\min\{\lambda^+_{1,p,q},\lambda^-_{1,p,q}\},\end{aligned}$$

where $\lambda^+_{1,p,q}$ and $\lambda^-_{1,p,q}$ are determined by the case of taking integrals with respect to upper or lower hemispheres respectively, and $\vartheta_1 = \int_{S^n} f_\epsilon^{\frac{n}{2}} |\bar{u}_\epsilon|^{\alpha+1} |\bar{v}_\epsilon|^{\beta+1} dv_{can}$. Without loss of generality, we assume

$$\lambda_{1,p,q}(\epsilon) \geq \lambda^+_{1,p,q}(\epsilon),$$

which means that

$$\begin{aligned}&\lambda_{1,p,q}(\epsilon)\\ \geq&\frac{1}{\vartheta_2}\Big[\frac{\alpha+1}{p}\int_{S^n_+} f_\epsilon^{\frac{n-p}{2}}|\bar{u}'_\epsilon|^p dv_{can}+\frac{\beta+1}{q}\int_{S^n_+} f_\epsilon^{\frac{n-p}{2}}|\bar{v}'_\epsilon|^q dv_{can}\Big],\end{aligned}$$

where $\vartheta_2 = \int_{S^n_+} f_\epsilon^{\frac{n}{2}} |\bar{u}_\epsilon|^{\alpha+1} |\bar{v}_\epsilon|^{\beta+1} dv_{can}$. Functions $a_\epsilon \in W^{1,p}(M)$ and $c_\epsilon \in W^{1,q}(M)$ are defined as

$$a_\epsilon = \begin{cases} \bar{u}_\epsilon & [0, \frac{\pi}{2} - \epsilon], \\ \bar{u}_\epsilon\left(\frac{\pi}{2} - \epsilon\right) & \left(\frac{\pi}{2} - \epsilon, \frac{\pi}{2}\right], \end{cases}$$

and $b_\epsilon = \bar{u}_\epsilon - a_\epsilon$ as well as

$$c_\epsilon = \begin{cases} \bar{v}_\epsilon & [0, \frac{\pi}{2} - \epsilon], \\ \bar{v}_\epsilon\left(\frac{\pi}{2} - \epsilon\right) & \left(\frac{\pi}{2} - \epsilon, \frac{\pi}{2}\right], \end{cases}$$

where similarly $d_\epsilon = \bar{v}_\epsilon - c_\epsilon$. It is obvious that on $\left[0, \frac{\pi}{2} - \epsilon\right]$ we have

$$b_\epsilon = d_\epsilon = 0,$$

and on $\left(\frac{\pi}{2} - \epsilon, \frac{\pi}{2}\right]$ also

$$a'_\epsilon = c'_\epsilon = 0,$$

and by the above definition we conclude that

$$\begin{aligned}|\bar{u}'_\epsilon|^p &= |a'_\epsilon|^p + |b'_\epsilon|^p,\\ |\bar{v}'_\epsilon|^q &= |c'_\epsilon|^q + |d'_\epsilon|^q,\\ |\bar{u}_\epsilon|^{\alpha+1} &\leq 2^\alpha\left(|a_\epsilon|^{\alpha+1} + |b_\epsilon|^{\alpha+1}\right),\\ |\bar{v}_\epsilon|^{\beta+1} &\leq 2^\beta\left(|c_\epsilon|^{\beta+1} + |d_\epsilon|^{\beta+1}\right).\end{aligned}$$

By the definition of $f_\epsilon(r)$ and also substituting in the formula of $\lambda_{1,p,q}$ we get

$$\lambda_{1,p,q}(\epsilon) \geq \frac{2^{-(\alpha+\beta)}}{\mathcal{A}}\Big[\epsilon^{-\frac{2p}{n}}\left(\frac{\alpha+1}{p}\int_{S_+^n}|a_\epsilon'|^p dv_{can} + \frac{\beta+1}{q}\int_{S_+^n}|c_\epsilon'|^q dv_{can}\right) + \frac{\alpha+1}{p}\int_{S_+^n}|b_\epsilon'|^p dv_{can} + \frac{\beta+1}{q}\int_{S_+^n}|d_\epsilon'|^q dv_{can}\Big],$$

where

$$\mathcal{A} = \int_{S_+^n} f_\epsilon^{\frac{n}{2}}|a_\epsilon|^{\alpha+1}|c_\epsilon|^{\beta+1}dv_{can} + \int_{S_+^n}|a_\epsilon|^{\alpha+1}|d_\epsilon|^{\beta+1}dv_{can} + \int_{S_+^n}|b_\epsilon|^{\alpha+1}|c_\epsilon|^{\beta+1}dv_{can} + \int_{S_+^n}|b_\epsilon|^{\alpha+1}|d_\epsilon|^{beta+1}dv_{can}.$$

Let

$$\mathcal{A} = 1.$$

It is clear that

$$\lambda_{1,p,q}(\epsilon) \geq 2^{-(\alpha+\beta)}\Big[\epsilon^{-\frac{2p}{n}}\left(\frac{\alpha+1}{p}\int_{S_+^n}|a_\epsilon'|^p dv_{can} + \frac{\beta+1}{q}\int_{S_+^n}|c_\epsilon'|^q dv_{can}\right) + \left(\frac{\alpha+1}{p}\int_{S_n^+}|b_\epsilon'|^p dv_{can} + \frac{\beta+1}{q}\int_{S_n^+}|d_\epsilon'|^q dv_{can}\right)\Big]. \tag{3.4}$$

There are two different cases. First of all,

$$\limsup_{\epsilon\to 0}\Big[\frac{\alpha+1}{p}\int_{S_+^n}|a_\epsilon'|^p dv_{can} + \frac{\beta+1}{q}\int_{S_+^n}|c_\epsilon'|^q dv_{can}\Big] > 0.$$

In this case

$$\lambda_{1,p,q}(\epsilon).\epsilon^{\frac{p}{n}} \geq 2^{-(\alpha+\beta)}\epsilon^{-\frac{p}{n}}\left(\frac{\alpha+1}{p}\int_{S_+^n}|a_\epsilon'|^p dv_{can} + \frac{\beta+1}{q}\int_{S_+^n}|c_\epsilon'|^q dv_{can}\right),$$

which concludes that

$$\limsup_{\epsilon\to 0}\lambda_{1,p,q}(\epsilon).\epsilon^{\frac{p}{n}} = \infty.$$

In the other case

$$\lim_{\epsilon\to 0}\Big[\frac{\alpha+1}{p}\int_{S_+^m}|a_\epsilon'|^p dv_{can} + \frac{\beta+1}{q}\int_{S_+^n}|c_\epsilon'|^q dv_{can}\Big] = 0.$$

Now consider the sequence $\epsilon_N \to 0$ as $a_{\epsilon_N} + c_{\epsilon_n} \to a + c$ where a and c are some constants. Since

$$\lim_{N\to\infty}\int_{S_+^n} f_{\epsilon_N}^{\frac{n}{2}}|a_{\epsilon_N}|^{\alpha+1}|c_{\epsilon_N}|^{\beta+1}dv_{can} = \lim_{N\to\infty}\int_{S_+^n} f_{\epsilon_N}^{\frac{n}{2}}\left(|a_{\epsilon_N}|^{\alpha+1}|c_{\epsilon_N}|^{\beta+1} - |a|^{\alpha+1}|c|^{\beta+1}\right)dv_{can}$$

$$+\left(|a|^{\alpha+1}|c|^{\beta+1}\right)\lim_{N\to\infty}\int_{S^n_+} f_{\epsilon_N}^{\frac{n}{2}}dv_{can}=0,$$

for $p, q > n$ and arbitrary N, $\{f_{\epsilon_N}\}$ is uniformly bounded. Thus we have

$$\lim_{N\to 0}\int_{S^n_+} f_{\epsilon_N}^{\frac{n}{2}}dv_{can}=0.$$

By substituting all those in (3.4), we get

$$\begin{aligned}
\lambda_{1,p,q}(\epsilon) &\geq \frac{2^{-(\alpha+\beta)}}{\mathcal{B}}\Big[\frac{\alpha+1}{p}\int_{S^n_+}|b'_\epsilon|^p dv_{can}+\frac{\beta+1}{q}\int_{S^n_+}|d'_\epsilon|^q dv_{can}\Big]\\
&=\frac{2^{-(\alpha+\beta)}}{\int_{\frac{\pi}{2}-\epsilon_N}^{\frac{\pi}{2}}\mathcal{D}\sin r^{n-1}dr}\int_{\frac{\pi}{2}-\epsilon_N}^{\frac{\pi}{2}}\left(\frac{\alpha+1}{p}|b'_\epsilon|^p+\frac{\beta+1}{q}|d'_\epsilon|^q\right)\sin r^{n-1}dr\\
&\geq 2^{-(\alpha+\beta)}\left(\sin\left(\frac{\pi}{2}-\epsilon_N\right)\right)^{n-1}\frac{\frac{\alpha+1}{p}\int_{\frac{\pi}{2}-\epsilon_N}^{\frac{\pi}{2}}|b'_\epsilon|^p dr+\frac{\beta+1}{q}\int_{\frac{\pi}{2}-\epsilon_N}^{\frac{\pi}{2}}|d'_\epsilon|^q dr}{\int_{\frac{\pi}{2}-\epsilon_N}^{\frac{\pi}{2}}\mathcal{D}dr},
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{B}=&\int_{S^n_+}|a_\epsilon|^{\alpha+1}|d_\epsilon|^{\beta+1}dv_{can}+\int_{S^n_+}|b_\epsilon|^{\alpha+1}|c_\epsilon|^{\beta+1}dv_{can}\\
&+\int_{S^n_+}|b_\epsilon|^{\alpha+1}|d_\epsilon|^{\beta+1}dv_{can},
\end{aligned}$$

and

$$\mathcal{D}=|a_\epsilon|^{\alpha+1}|d_\epsilon|^{\beta+1}+|b_\epsilon|^{\alpha+1}|c_\epsilon|^{\beta+1}+|b_\epsilon|^{\alpha+1}|d_\epsilon|^{\beta+1}.$$

Similar to the previous one, we define $\bar{a}_{\epsilon_N}\in W_0^{1,p}(-\epsilon_N,\epsilon_N)$ as

$$\bar{a}_{\epsilon_N}(x)=a_{\epsilon_N}\left(x+\frac{\pi}{2}-\epsilon_N\right).$$

Also, the similar definition works for $\bar{b}_{\epsilon_N}$ from $W_0^{1,p}(-\epsilon_N,\epsilon_N)$ and $\bar{c}_{\epsilon_N}$, $\bar{d}_{\epsilon_N}\in W_0^{1,q}(-\epsilon_N,\epsilon_N)$. Thus

$$\begin{aligned}
\frac{\frac{\alpha+1}{p}\int_{\frac{\pi}{2}-\epsilon_N}^{\frac{\pi}{2}}|b'_\epsilon|^p dr+\frac{\beta+1}{q}\int_{\frac{\pi}{2}-\epsilon_N}^{\frac{\pi}{2}}|d'_\epsilon|^q dr}{\int_{\frac{\pi}{2}-\epsilon_N}^{\frac{\pi}{2}}\mathcal{D}dr} &=\frac{\frac{\alpha+1}{p}\int_0^{\epsilon_n}|b'_\epsilon|^p dr+\frac{\beta+1}{q}\int_0^{\epsilon_N}|d'_\epsilon|^q dr}{\int_0^{\epsilon_N}\mathcal{D}dr}\\
&=\frac{\frac{\alpha+1}{p}\int_{-\epsilon_N}^{\epsilon_N}|b'_\epsilon|^p dr+\frac{\beta+1}{q}\int_{-\epsilon_N}^{\epsilon_N}|d'_\epsilon|^q dr}{\int_{-\epsilon_N}^{\epsilon_n}\mathcal{D}dr}\\
&\geq\lambda_{1,p,q}(-\epsilon_N,\epsilon_N)\\
&=\epsilon_N^{-p}\lambda_{1,p,q}(-1,1),
\end{aligned}$$

and

$$\lambda_{1,p,q}(\epsilon)\geq 2^{-(\alpha+\beta)}.\epsilon_N^{-p}\left(\sin\left(\frac{\pi}{2}-\epsilon_N\right)\right)^{n-1}\lambda_{1,p,q}(-1,1),$$

which finally concludes

$$\limsup_{\epsilon \to 0} \lambda_{1,p,q} (\epsilon) . \epsilon^{\frac{p}{n}} = \infty.$$

□

For $\epsilon > 0$, $\bar{f} \in C^{\infty} (S^n)$ is a radial function such that $\bar{f}_\epsilon \leq f_\epsilon$. Also, on $\left[\frac{\pi}{2} - \frac{\epsilon}{2}, \frac{\pi}{2} + \frac{\epsilon}{2}\right]$ we have

$$\bar{f}_\epsilon (r) = f_\epsilon (r) = 1,$$

and

$$\bar{f}_\epsilon (\pi - r) = \bar{f} (r) .$$

Thus

$$\begin{aligned}
\operatorname{vol}\left(S^n, \bar{f}_\epsilon can\right) = \int_{S^n} \bar{f}_\epsilon^{\frac{n}{2}} dv_{can} &= \int_{S^{n-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{f}_\epsilon^{\frac{n}{2}} \sin r^{n-1} dr dv_{can} \\
&> V. \int_{\frac{\pi}{2}-\frac{\epsilon}{2}}^{\frac{\pi}{2}+\frac{\epsilon}{2}} \sin r^{n-1} dr \\
&> \epsilon V \Big[\sin \Big(\frac{\pi}{2} - \epsilon \Big) \Big]^{n-1},
\end{aligned}$$

where $V = \operatorname{vol}\left(S^{n-1}, can\right)$.

Corollary 3.1. *Let $p \geq q > n$ and $\lambda_{1,p,q}$ denote the first eigenvalue of the (p, q)-Laplacian system (1.1). Then for arbitrary large $\lambda_{1,p,q}$ there exists a Riemannian metric of volume one conformal to standard metric can.*

Proof. By improving a method from [13], let $\bar{u}_\epsilon$ and $\bar{v}_\epsilon$ denote the eigenfunctions for $\lambda_{1,p,q}\left(S^n, \bar{f}_\epsilon can\right)$. Also $\bar{u}_\epsilon^+$, $\bar{u}_\epsilon^-$, $\bar{v}_\epsilon^+$ and $\bar{v}_\epsilon^-$ are assumed as the positive and negative values of $\bar{u}_\epsilon$ and $\bar{v}_\epsilon$ respectively. For the system (1.1) we conclude that

$$\begin{aligned}
\lambda_{1,p,q}\left(S^n, \bar{f}_\epsilon can\right) &= \frac{1}{\int_{S^n} |\bar{u}_\epsilon^+|^{\alpha+1} |\bar{v}_\epsilon^+|^{\beta+1} \bar{f}_\epsilon^{\frac{n}{2}} dv_{can}} \Big[\frac{\alpha+1}{p} \int_{S^n} |d\bar{u}_\epsilon^+|^p \bar{f}_\epsilon^{\frac{n-p}{2}} dv_{can} \\
&+ \frac{\beta+1}{q} \int_{S^n} |d\bar{v}_\epsilon^+|^q \bar{f}_\epsilon^{\frac{n-p}{2}} dv_{can} \Big] \\
&= \frac{1}{\int_{S^n} |\bar{u}_\epsilon^-|^{\alpha+1} |\bar{v}_\epsilon^-|^{\beta+1} \bar{f}_\epsilon^{\frac{n}{2}} dv_{can}} \Big[\frac{\alpha+1}{p} \int_{S^n} |d\bar{u}_\epsilon^-|^p \bar{f}_\epsilon^{\frac{n-p}{2}} dv_{can} \\
&+ \frac{\beta+1}{q} \int_{S^n} |d\bar{v}_\epsilon^-|^q \bar{f}_\epsilon^{\frac{n-p}{2}} dv_{can} \Big].
\end{aligned}$$

Set $t \in \mathbb{R}$ such that

$$\bar{u}_{\epsilon,t} = t\bar{u}_\epsilon^+ + \bar{u}_\epsilon^- .$$

So we get

$$\lambda_{1,p,q}\left(S^n, \bar{f}_\epsilon can\right) = \frac{1}{\int_{S^n} |\bar{u}_\epsilon|^{\alpha+1} |\bar{v}_\epsilon|^{\beta+1} \bar{f}_\epsilon^{\frac{n}{2}} dv_{can}} \Big[\frac{\alpha+1}{p} \int_{S^n} |d\bar{u}_\epsilon|^p \bar{f}_\epsilon^{\frac{n-p}{2}} dv_{can}$$

$$
\begin{aligned}
&+ \frac{\beta+1}{q}\int_{S^n} |d\bar{v}_\epsilon|^q \bar{f}_\epsilon^{\frac{n-p}{2}} dv_{can}\Big] \\
&\geq \frac{1}{\int_{S^n} |\bar{u}_\epsilon|^{\alpha+1}|\bar{v}_\epsilon|^{\beta+1} f_\epsilon^{\frac{n}{2}} dv_{can}}\Big[\frac{\alpha+1}{p}\int_{S^n} |d\bar{u}_\epsilon|^p f_\epsilon^{\frac{n-p}{2}} dv_{can} \\
&+ \frac{\beta+1}{q}\int_{S^n} |d\bar{v}_\epsilon|^q f_\epsilon^{\frac{n-p}{2}} dv_{can}\Big] \\
&\geq \lambda_{1,p,q}(\epsilon).
\end{aligned}
$$

Under the consideration that $p \geq q$ and by Theorem 1.2, we finally see

$$
\limsup_{\epsilon\to 0} \lambda_{1,p,q}\left(S^n, \bar{f}_\epsilon can\right) \operatorname{vol}\left(S^n, \bar{f}_\epsilon can\right)^{\frac{p}{n}} \geq V^{\frac{p}{n}}. \limsup_{\epsilon\to 0} \lambda_{1,p,q}(\epsilon).\epsilon^{\frac{p}{n}} = \infty.
$$

Just setting

$$
h_\epsilon = \operatorname{vol}\left(S^n, \bar{f}_\epsilon can\right)^{-\frac{2}{n}} \bar{f}_\epsilon,
$$

we have

$$
\operatorname{vol}(S^n, h_\epsilon can) = 1,
$$

and

$$
\limsup_{\epsilon\to 0} \lambda_{1,p,q}(S^n, h_\epsilon can) = \infty.
$$

□

Remark 3.1. Someone may consider the situation $q < n < p$. In this case by the same process as in Theorem 1.2, consider the radial function $f_\epsilon : S^n \to \mathbb{R}$ as

$$
f_\epsilon(r) = \epsilon^{\frac{4q}{n(n-q)}}.\chi_{[0,\frac{\pi}{2}-\epsilon]\cup[\frac{\pi}{2}+\epsilon,\pi]}(r) + \chi_{\left(\frac{\pi}{2}-\epsilon,\frac{\pi}{2}+\epsilon\right)}(r),
$$

and we have

$$
R_\epsilon(u,v) := \frac{1}{\vartheta_3}\Big[\frac{\alpha+1}{p}\int_{S^{n-1}} f_\epsilon^{\frac{q-n}{2}} |du|^p dv_{can} + \frac{\beta+1}{q}\int_{S^{n-1}} f_\epsilon^{\frac{q-n}{2}} |dv|^q dv_{can}\Big],
$$

where $\vartheta_3 = \int_{S^{n-1}} f_\epsilon^{\frac{n}{2}} |u|^{\alpha+1}|v|^{\beta+1} dv_{can}$ and

$$
\lambda_{1,p,q}(\epsilon) = \inf_{u,v\neq 0}\Big\{R_\epsilon(u,v) \,|\, (u,v) \in W_0^{1,p} \times W_0^{1,q} \setminus \{0\}\Big\}.
$$

By the same process as in Theorem 1.2 we finally get

$$
\limsup_{\epsilon\to 0} \lambda_{1,p,q}(\epsilon).\epsilon^{\frac{q}{n}} = \infty,
$$

which means that the same statement as Corollary 3.1 holds for the first eigenvalue of system (1.1) when $q < n < p$.

Acknowledgments

The authors hereby declare that they have not recieved any grant in the process of conducting this research.

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