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Exact Solutions and Optimal System of Hyperbolic Monge-Ampère Equation

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Abstract Based on Lie symmetry theory, the exact solutions of the hyperbolic Monge-Ampère equation are studied. Firstly, the invariance of the Lie symmetry group is applied to obtain the six-dimensional Lie algebras, then the commutator table and the adjoint representation of the equation are obtained, based on which the optimal system is found. Finally, the exact solutions are obtained by symmetry reduction which transforms the PDEs into easily solvable ODEs.

Keywords Optimal system, exact solution, Lie symmetry, hyperbolic Monge-Ampère equation

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1. Introduction

Many important scientific and engineering problems can be summarised by the study of nonlinear partial differential equations. Mathematical models in many areas of real life can be described by NLPDEs, and many important fundamental equations in physics, mechanics, fiber optic communication [1], biologic [2] and other disciplines [3] are NLPDEs. Progress has been made in the study of integrable models. For example, Ma [4–6] has proposed many new integrable equations, including coupled modified Korteweg-de Vries four-component equation, coupled nonlinear Schrodinger equation and AKNS type equation, and has achieved breakthrough research results. As research progresses, methods for solving PDEs become more sophisticated. Such as, linear superposition method [7], Bäcklund transformation method [8–11], Darboux transformation method [12–14], Hirota bilinear method [15–21], Inverse scattering method [22–24], Jacobi elliptic function expansion method [25–28], Lie symmetry analysis [29–35], maximum likelihood estimation [36], etc [37–39].

The Monge-Ampère equation was originally formulated by mathematicians Gaspard Monge and André-Marie Ampère in the late eighteenth century and first introduced as a concept in differential geometry. The Monge-Ampère equation is one of the fundamental equations widely used in the fields of elementary unitary calculus and geometric growth. It has demonstrated its importance in this field, not only for solving complex non-linear equations but also for calculating a large number of parameters in fluid mechanics and detecting characteristic parameters

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of electromagnetic and acoustic fields. These properties have made the Monge-Ampère equation very popular in the physical and mathematical community, and its theoretical study and applications have been widely explored [40, 41]. The hyperbolic Monge-Ampère equation is used to describe a class of mechanical systems with singular terminal constraints, and its greatest advantage lies in the fact that its definition can be expressed exactly. Because of the practicality of the hyperbolic Monge-Ampère equation, scholars have done a lot of research. In [42], Kong et al. found that the initial value problem of a one-dimensional hyperbolic mean curvature flow F of a closed plane curve which can be simplified to the initial value problem of a PDE satisfied by the support function of the curve, which is hyperbolic Monge-Ampère equation

$$S_{\tau\tau} = \frac{S_{\theta\tau}^2 - 1}{S + S_{\theta\theta}}.$$

In [43], Michal et al. solved the equivalence problem based on the construction of the hyperbolic Monge-Ampère equation, which had a fundamental differential invariants of the exposure transformation. In [44], Chen proved degenerate hyperbolic Monge-Ampère equation in the existence of smooth solution near zero. Moreover, the linearized equation was transformed into a simpler form by a transformation of variables, leading to its a priori estimate. Finally the existence of local solutions was proved by an iterative method. It was verified that the zero set of small perturbations has a simple structure. What has been studied by the above scholars, as well as the exact solutions in our paper helps us to better understand the structure and properties of the equation.

Nowadays, it is widely used in differential geometry, variational methods, optimization problems, and transmission problems. Its mathematical expression is as follows: the unknown function $z = z(x,t)$ defined in $(x,t) \in R^2$ corresponds to Monge-Ampère equation

$$F + Gz_{xx} + Hz_{xt} + Iz_{tt} + J(z_{xx}z_{tt} - z_{xt}^2) = 0, \tag{1.1}$$

where F, G, H, I, J are first-order variables and x, t, z_x, z_t are the only non-independent variables. F, G, H, I, J depend on x, t, z, z_t, z_x. If conditions

$$\Delta^2(x,t,z,z_t,z_x) \triangleq H^2 - 4GI + 4FJ > 0,$$

and

$$z_{tt} + G(x,t,z,z_t,z_x) \neq 0$$

are satisfied, then (1.1) is hyperbolic.

According to [45], we have

$$\det(D^2 z) = k(x,t)(1 + z_x^2 + z_t^2)^2, \tag{1.2}$$

where $G, H, I = 0$, $F = (1 + z_t^2 + z_x^2)^2$ and $k(x,t)$ stands for the Gaussian curvature of the surface. Hyperbolic Monge-Ampère equations are closely related to geometric applications. A surface with negative Gaussian curvature at each point is a solution to Equation (1.2). We call that a minimal surface, such as a Costa surface. Minimal surfaces cover a wide domain and have negative Gaussian curvature at every point, so the existence of minimal surfaces is closely related to the solvability of the hyperbolic Monge-Ampère equation. Therefore, it is extremely important to find the solutions of the hyperbolic Monge-Ampère equations.

In this article, we mainly apply the classic Lie symmetry method to study exact solutions of hyperbolic Monge-Ampère Equation (1.3)

$$z_{tt}z_{xx} - z_{tx}^2 = -(1 + z_t^2 + z_x^2)^2, \tag{1.3}$$

which is the special case of $k(x,t) = -1$ in Equation (1.2). Clearly, because of $\Delta > 0$, it follows that Equation (1.3) is hyperbolic.

The article is divided into five main parts. Section 2 focuses on the Lie symmetry analysis of the equation. Section 3 is devoted to finding the optimal system of the equation. Section 4 has the main aim of finding the exact solutions of the equation, and finally gives a small summary.

2. Lie symmetry analysis of Equation (1.3)

We will use the knowledge of group invariant solutions to obtain the Lie algebra of the equation, and the vector is

$$X = \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} + \phi\frac{\partial}{\partial z}, \tag{2.1}$$

where ξ, τ, ϕ are functions on x, t, z.

$$Pr^{(2)}X(\Delta)|_{\Delta=0} = 0, \tag{2.2}$$

in which $\Delta = z_{tt} - z_{xx} - z_{tx}^2 = -(1 + z_t^2 + z_x^2)^2$. $\mathrm{Pr}^{(2)}X$ is a second-order prolongation of X.

$$Pr^{(2)}X(\Delta) = X + \phi^x\frac{\partial}{\partial z_x} + \phi^t\frac{\partial}{\partial z_t} + \phi^{xx}\frac{\partial}{\partial z_{xx}} + \phi^{tt}\frac{\partial}{\partial z_{tt}} + \phi^{tx}\frac{\partial}{\partial z_{tx}}, \tag{2.3}$$

in which

$$\begin{aligned}
\phi^x &= D_x(\phi) - z_xD_x(\xi) - z_tD_x(\tau),\\
\phi^t &= D_t(\phi) - z_xD_t(\xi) - z_tD_t(\tau),\\
\phi^{xx} &= D_x(\phi^x) - z_{xx}D_x(\xi) - z_{tx}D_x(\tau),\\
\phi^{tt} &= D_t(\phi^t) - z_{tt}D_t(\xi) - z_{tx}D_t(\tau),\\
\phi^{tx} &= D_x(\phi^t) - z_{tt}D_x(\xi) - z_{tx}D_x(\tau),
\end{aligned} \tag{2.4}$$

where D_x, D_t are whole derivative with respect to x, t.

$$D_x = \frac{\partial}{\partial x} + z_x\frac{\partial}{\partial z} + z_{xt}\frac{\partial}{\partial z_t} + z_{xx}\frac{\partial}{\partial z_x} + ... \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3) yields the following determining equations:

$$\begin{aligned}
&\phi_t = -\tau_z, \phi_x = -\xi_z, \phi_z = 0, \tau_t = 0, \tau_{xx} = 0, \tau_{zz} = 0,\\
&\xi_t = -\tau_x, \xi_x = 0, \xi_{zz} = 0, \tau_{zx} = 0.
\end{aligned} \tag{2.6}$$

By solving (2.6), we can get :

$$\begin{aligned}
\xi &= -c_1t + c_4z + c_5,\\
\tau &= c_1x + c_2z + c_3,\\
\phi &= -c_2t - c_4x + c_6,
\end{aligned} \tag{2.7}$$

where $c_j (j = 1...6)$ are arbitrary constants. Substituting (2.7) into (2.1), we can get:

$$X = (-c_1 t + c_4 z + c_5)\frac{\partial}{\partial x} + (c_1 x + c_2 z + c_3)\frac{\partial}{\partial t} - c_2 t - c_4 x + c_6. \quad (2.8)$$

Then, we get the following subalgebra:

$$\begin{aligned} &X_1 = \frac{\partial}{\partial x}, X_2 = \frac{\partial}{\partial t}, X_3 = \frac{\partial}{\partial z}, X_4 = z\frac{\partial}{\partial x} - x, \\ &X_5 = z\frac{\partial}{\partial t} - t, X_6 = -t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}. \end{aligned} \quad (2.9)$$

3. The optimal system of the equation of Equation (1.3)

The commutator table for the equation is obtained from the Lie bracket, through which the adjoint representation for the equation is acquired. The Lie bracket is

$$[X_j, X_i] = X_j X_i - X_i X_j.$$

Table 1. Commutator table

$[X_i, X_j]$	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	0	0	$-X_3$	0	X_2
X_2	0	0	0	0	$-X_3$	$-X_1$
X_3	0	0	0	X_1	X_2	0
X_4	X_3	0	$-X_1$	0	$-X_6$	X_5
X_5	0	X_3	$-X_2$	X_6	0	$-X_4$
X_6	$-X_2$	X_1	0	$-X_5$	X_4	0

Table 2. Adjoint representation

Ad	X_1	X_2	X_3	X_4	X_5	X_6
X_1	X_1	X_2	X_3	$X_4 + \varepsilon X_3$	X_5	$X_6 - \varepsilon X_2$
X_2	X_1	X_2	X_3	X_4	$X_5 + \varepsilon X_3$	$X_6 + \varepsilon X_1$
X_3	X_1	X_2	X_3	$X_4 - \varepsilon X_1$	$X_5 - \varepsilon X_2$	X_6
X_4	$X_1 \cos(\varepsilon) - X_3 \sin(\varepsilon)$	X_2	$X_3 \cos(\varepsilon) + X_1 \sin(\varepsilon)$	X_4	$X_5 \cos(\varepsilon) + X_6 \sin(\varepsilon)$	$X_6 \cos(\varepsilon) - X_5 \sin(\varepsilon)$
X_5	X_1	$X_2 \cos(\varepsilon) - X_3 \sin(\varepsilon)$	$X_3 \cos(\varepsilon) + X_2 \sin(\varepsilon)$	$X_4 \cos(\varepsilon) - X_6 \sin(\varepsilon)$	X_5	$X_6 \cos(\varepsilon) + X_4 \sin(\varepsilon)$
X_6	$X_1 \cos(\varepsilon) + X_2 \sin(\varepsilon)$	$X_2 \cos(\varepsilon) - X_1 \sin(\varepsilon)$	X_3	$X_4 \cos(\varepsilon) + X_5 \sin(\varepsilon)$	$X_5 \cos(\varepsilon) - X_4 \sin(\varepsilon)$	X_6

We consider the vector X,

$$X = b_1 X_1 + b_2 X_2 + b_3 X_3 + b_4 X_4 + b_5 X_5 + b_6 X_6. \quad (3.1)$$

We find the optimal system of equation in the following way.
(A) $b_4 \neq 0, b_4 = 1$.

Letting X act on $Ad(\exp(-b_3X_1))$, $Ad(\exp(b_1X_3))$ successively, eliminating the coefficients of X_1 and X_3 successively, we get

$$X = (b_2 + b_3b_6 - b_1b_5)X_2 + X_4 + b_5X_5 + b_6X_6. \tag{3.2}$$

(a) If $b_6 = 0$, then (3.2) becomes

$$X = (b_2 - b_1b_5)X_2 + X_4 + b_5X_5. \tag{3.3}$$

(b) When $b_6 \neq 0$, let X act on $Ad(\exp(\arctan(\frac{b_5}{b_6})X_4))$

$$X = (b_2 + b_3b_6 - b_1b_5)X_2 + X_4 + hX_6, \tag{3.4}$$

where h is related to b_5 and b_6.

(B) $b_4 = 0$, assume $b_5 \neq 0$, $b_5 = 1$.

We obtain $X = b_1X_1 + b_2X_2 + b_3X_3 + X_5 + b_6X_6$, then let $Ad(\exp(-b_3X_2))$, $Ad(\exp(b_2X_3))$ work on X respectively. The vector is equivalent to

$$X = (b_1 - b_3b_6)X_1 + X_5 + b_6X_6. \tag{3.5}$$

(C) $b_4 = b_5 = 0$, assume $b_6 \neq 0$, $b_6 = 1$,

$$Ad(\exp(-b_1X_2)) \circ Ad(\exp(b_2X_1))X = b_3X_3 + X_6. \tag{3.6}$$

(D) $b_4 = b_5 = b_6 = 0$, assume $b_2 \neq 0$, $b_2 = 1$,

$$X = b_1X_1 + X_2 + b_3X_3. \tag{3.7}$$

(a) $b_1 = 0$,

$$X = X_2 + b_3X_3, \tag{3.8}$$

$$X^{(1)} = Ad(\exp(\arctan b_3X_5))X = pX_2, \tag{3.9}$$

where p depends on b_3.

(b) $b_1 \neq 0$,

$$X^{(1)} = Ad(\exp(\arctan \frac{b_3}{b_1}X_4))X = mX_1 + X_2, \tag{3.10}$$

$$X^{(2)} = Ad(\exp(\arctan(-\frac{1}{m})X_6))X^{(1)} = cX_1. \tag{3.11}$$

(E) $b_4 = b_5 = b_6 = b_2 = 0$, assume $b_1 \neq 0$,$b_1 = 1$,

$$X = X_1 + b_3X_3, \tag{3.12}$$

$$X^{(1)} = Ad(\exp(\arctan(b_3)X_4))X = X_1. \tag{3.13}$$

(F) $b_4 = b_5 = b_6 = b_2 = b_1 = 0$, and the vector X is tantamount to X_3.

In summary, the optimal system for Equation (1.3) is

$$\begin{aligned}
&X_2 \pm X_4 \pm X_5, b_4 \neq 0, b_6 = 0,\\
&X_2 \pm X_4 \pm X_6, b_4 = b_6 \neq 0,\\
&X_1 \pm X_5 \pm X_6, b_4 = 0, b_5 \neq 0,\\
&X_3 \pm X_6, b_4 = b_5 = 0, b_6 \neq 0,\\
&X_2 \pm X_3, b_4 = b_5 = b_6 = b_1 = 0, b_2 \neq 0,\\
&X_1, b_4 = b_5 = b_6 = 0, b_1 = b_2 \neq 0,\\
&X_3, b_1 = b_2 = b_4 = b_5 = b_6 = 0.
\end{aligned} \tag{3.14}$$

4. Exact solutions

Case 1. For generator $X_2+X_4+X_5 = (1+z)\frac{\partial}{\partial t}+z\frac{\partial}{\partial x}+(-t-x)\frac{\partial}{\partial z}$, its characteristic equation is the following form

$$\frac{dx}{z} = \frac{dt}{(1+z)} = \frac{dz}{(-t-x)}. \tag{4.1}$$

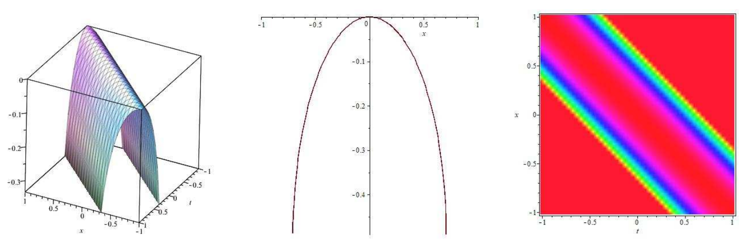
Based on its characteristic equation, we can obtain an expression for the implicit function on z,

$$z^2 + z + \frac{(x+t)^2}{2} = c, \tag{4.2}$$

then we get

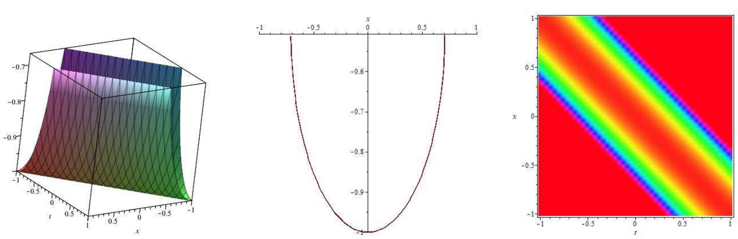
$$z_{1,2} = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-2t^2 - 4tx - 2x^2 - 4c + 1}. \tag{4.3}$$

When $c = 1$, (4.3) is a solution to Equation (1.3).



(a) (b) (c)

Figure 1. The dynamical structures of z_1: (a) 3D plot; (b) contour plot; (c) density plot.



(a) (b) (c)

Figure 2. The dynamical structures of z_2: (a) 3D plot; (b) contour plot; (c) density plot.

Case 2. For generator $X_2 + X_4 + X_6 = (1+x)\frac{\partial}{\partial t} + (z-t)\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}$, its corresponding characteristic equation can be written as

$$\frac{dx}{z-t} = \frac{dt}{1+x} = \frac{dz}{-x}. \tag{4.4}$$

Depending on (4.4), we get

$$(z-t)^2 - 2(-x^2 - x) = c, \tag{4.5}$$

$$z_{1,2} = \pm\sqrt{2(-x^2 - x) + c} + t, \tag{4.6}$$

where $c = -\dfrac{1}{2}$, (4.6) is the solution.

Case 3. For generator $X_1 + X_2 = \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial t}$, its corresponding invariant is $x - t$, and the corresponding form of the solution is

$$z = f(x - t). \tag{4.7}$$

Substituting (4.7) into (1.3), we have

$$(1 + 2(f')^2)^2 = 0, \tag{4.8}$$

so

$$f_{1,2} = \pm\frac{1}{2}\sqrt{2}\xi i + c_1. \tag{4.9}$$

$$z_{1,2} = \pm\frac{1}{2}\sqrt{2}(x - t)i + c_1, \tag{4.10}$$

where c_1 is an arbitrary constant. We get complex solutions.

Case 4. For generator $X_3 + X_6 = \dfrac{\partial}{\partial z} - t\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial t}$, we find its characteristic equation as

$$\frac{dx}{-t} = \frac{dt}{x} = \frac{dz}{1}, \tag{4.11}$$

and the corresponding group invariant solution is

$$z = f(\xi) + \arcsin\left(\frac{t}{\sqrt{\xi}}\right), \tag{4.12}$$

where $\xi = x^2 + t^2$. Substituting (4.12) into (1.3), we obtain an ODE that is not suitable for solving.

Case 5. For generator $X_1 + X_5 + X_6 = (1 - t)\dfrac{\partial}{\partial x} + (z + x)\dfrac{\partial}{\partial t} - t\dfrac{\partial}{\partial z}$, we get

$$\frac{dx}{1 - t} = \frac{dt}{z + x} = \frac{dz}{-t}. \tag{4.13}$$

Using (4.13), we can obtain the implicit equation of z,

$$(z + x)^2 - 2(t - t^2) = c, \tag{4.14}$$

and we have

$$z_{1,2} = \pm\sqrt{2(t - t^2) + c} - x, \tag{4.15}$$

where $c = -\dfrac{1}{2}$, (4.15) is the solution.

Case 6. For generators X_2 and X_3, we obtain the invariants as x, t, and the group invariant solutions are $z = f(x)$, $z = f(t)$,respectively. The solutions are $z = c_1 \pm ix$ and $z = c_1 \pm it$, which are the complex solutions.

5. Conclusion

In this paper, we aim to construct invariant solutions via Lie symmetry method. Firstly, we obtain the six-dimensional Lie algebras of the equation using the invariance of the Lie symmetry group solutions. Then, according to the Lie algebra, we obtain the commutator table and the adjoint representation of the equation respectively. Under these, we gain the optimal system of the equation. Finally the PDEs are symmetrically reduced to the ODEs to obtain the exact solutions of the equation. The solutions obtained have not appeared in the previous literature, which enriches the solutions of the equation. In addition, the solutions have the property that every point is a saddle point. The result proves that the Lie symmetry method could be applied to hyperbolic Monge-Ampère to derive exact solutions, and the method is effective.

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