Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 4, December 2024, 930–948 DOI:10.12150/jnma.2024.930

Some Euler-Maclaurin-Type Inequalities by Means of Tempered Fractional Integrals

Fatih Hezenci[1] and Hüseyin Budak[1,†]

Abstract This paper introduces an equality that is valid for tempered fractional integrals. By using this equality, we establish some Euler-Maclaurin-type inequalities for the case of differentiable convex functions involving tempered fractional integrals. Furthermore, our results are provided by using special cases of obtained theorems.

Keywords Euler-Maclaurin-type inequality, quadrature formulae, tempered fractional integrals, convex functions

MSC(2010) 26D07, 26D10, 26D15, 65D32.

1. Introduction and preliminaries

The theory of inequalities is well-known and is still an active research field that has numerous practical applications in many mathematical areas. It is also well-known that inequality theory is closely related to the study of convex functions. Many famous inequalities, such as the Hermite–Hadamard-type inequalities Simpson, Newton, and Euler-Maclaurin-type inequalities, are established for convex functions. In addition, convexity is an significant tool in the study of inequalities, as it allows for the use of various analytical and geometric methods to prove inequalities. Finally, the study of convex functions and the theory of inequalities are deeply interconnected and complement one another.

Fractional calculus is related to inequality theory in that it provides a generalization of traditional calculus, allowing for the consideration of non-integer orders of differentiation and integration. These types of inequalities can be useful in the study of various mathematical and physical phenomena, such as the behavior of complex systems or the properties of certain functions. Moreover, fractional calculus has gained important attention from mathematicians in various fields of mathematics due to its essential characteristics and practical use in real-world problems. Due to the significance of fractional calculus, various mathematical researchers have proved numerous inequalities including fractional integrals. One can obtain the bounds of the new inequalities not only by using Hermite–Hadamard-type inequalities but also by using Simpson, Newton, and Euler–Maclaurin-type inequalities.

Definition 1.1 (See [12, 18])**.** The *Riemann–Liouville integrals* $J_{a+}^{\alpha}f$ and $J_{b-}^{\alpha}f$ of

†the corresponding author.
Email address: fatihezenci@gmail.com(F. Hezenci),
hsyn.budak@gmail.com(H. Budak)
[1]Department of Mathematics, Faculty of Science and Arts, Duzce University, Duzce 81620, Türkiye.

order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{1.1}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b, \tag{1.2}$$

respectively. Here, $f \in L_1[a,b]$ and $\Gamma(\alpha)$ denotes the Gamma function defined as

$$\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du.$$

The fractional integral reduces to the classical integral for the case of $\alpha = 1$.

One can state the Simpson's rules in the following way:

i. The formula for Simpson's quadrature (also known as Simpson's 1/3 rule) is expressed as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]. \tag{1.3}$$

ii. The formula for Simpson's second formula or Newton-Cotes quadrature formula (also known as Simpson's 3/8 rule (cf. [4])) is stated as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[f(a) + 3f\left(\frac{2a+b}{3}\right) + 3f\left(\frac{a+2b}{3}\right) + f(b)\right]. \tag{1.4}$$

iii. The corresponding dual Simpson's 3/8 formula - the Maclaurin rule based on the Maclaurin formula (cf. [4]) is created as follows:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right]. \tag{1.5}$$

Formulae (1.3), (1.4), and (1.5) hold for every function f with continuous 4^{th} derivative on $[a,b]$.

The most popular Newton-Cotes quadrature containing three-point Simpson-type inequality is as follows.

Theorem 1.1. *Let $f : [a,b] \to \mathbb{R}$ be a four times differentiable and continuous function on (a,b), and let $\left\|f^{(4)}\right\|_{\infty} = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty$. Then, one has the inequality*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^4.$$

Dragomir established an approximation of the remainder for Simpson's quadrature formula for bounded variation functions and the applications in theory of special means in paper [7]. Budak et al. proved some variants of Simpson-type inequalities for the case of differentiable convex functions by generalized fractional integrals in paper [1]. The reader is referred to [2, 13, 28, 30, 31] and the references therein for further information about fractional integrals.

The Classical closed-type quadrature rule is the Simpson 3/8 rule derived from the Simpson 3/8 inequality as follows:

Theorem 1.2 (See [4]). *Suppose that $f:[a,b]\to\mathbb{R}$ is a four times differentiable and continuous function on (a,b). Also suppose that $\left\Vert f^{(4)}\right\Vert_\infty=\sup\limits_{x\in(a,b)}\left\vert f^{(4)}(x)\right\vert<\infty$. Then, the following inequality holds:*

$$\left\vert\frac{1}{8}\left[f(a)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f(b)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right\vert$$
$$\leq\frac{1}{6480}\left\Vert f^{(4)}\right\Vert_\infty(b-a)^4.$$

The evaluations for the three-step quadratic kernel in Simpson's second rule are sometimes referred to as Newton-type results because this rule is based on the three-point Newton-Cotes quadrature. Many researchers have conducted on Newton-type inequalities extensively. For instance, in paper [11], Gao and Shi investigated new Newton type inequalities based on convexity and some applications for special cases of real functions are also established. Erden et al. proved some new integral inequalities of Newton type for the case of functions whose first derivative in absolute value at certain power are arithmetically-harmonically convex in paper [8]. Moreover, several Newton type inequalities for the case of differentiable convex functions were proved and some fractional Newton type inequalities for the case of bounded variation functions were presented in [15]. For more information and unexpected subjects about Newton type of inequalities including convex differentiable functions,one can refer to [16, 17, 23, 26] and the references therein.

The Maclaurin rule, which is the dual of the Simpson's 3/8 formula, is based on the Maclaurin inequality and can be expressed as follows:

Theorem 1.3 (See [4]). *Let us consider that $f:[a,b]\to\mathbb{R}$ is a four times differentiable and continuous function on (a,b). If $\left\Vert f^{(4)}\right\Vert_\infty=\sup\limits_{x\in(a,b)}\left\vert f^{(4)}(x)\right\vert<\infty$, then the following inequality*

$$\left\vert\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right\vert$$
$$\leq\frac{7}{51840}\left\Vert f^{(4)}\right\Vert_\infty(b-a)^4$$

is valid.

Some Euler–Maclaurin-type inequalities are established for the case of differentiable convex functions by using Riemann-Liouville fractional integrals in paper [14]. By using graphs, example is given in order to indicate that our main result is correct. In paper [5], Dedić et al. is proved a set of inequalities by using the Euler-Maclaurin formulae and the results are applied to obtain some error estimates in the case of the Maclaurin quadrature rules. Moreover, a set of inequalities is considered by using the Euler-Simpson 3/8 formulae. The results are implemented to obtain several error estimates for the case of the Simpson 3/8 quadrature rules in [6]. For more details about these types of inequalities, the reader can refer to the sources such as [4, 9, 10, 24] and the references therein.

Recall that the incomplete gamma function and λ-incomplete gamma function are defined by

$$\Upsilon(\alpha, x) := \int_0^x t^{\alpha-1} e^{-t} dt$$

and

$$\Upsilon_\lambda(\alpha, x) := \int_0^x t^{\alpha-1} e^{-\lambda t} dt,$$

respectively. Here, $0 < \alpha < \infty$ and $\lambda \geq 0$.

Remark 1.1. [22] For the real numbers $\alpha > 0$; $\lambda \geq 0$ and $a < b$, we readily hold:

i. $\Upsilon_{\lambda(b-a)}(\alpha, 1) = \int_0^1 t^{\alpha-1} e^{-\lambda(b-a)t} dt = \frac{1}{(b-a)^\alpha} \Upsilon_\lambda(\alpha, b-a)$,

ii. $\int_0^1 \Upsilon_{\lambda(b-a)}(\alpha, x)\, dx = \frac{\Upsilon_\lambda(\alpha, b-a)}{(b-a)^\alpha} - \frac{\Upsilon_\lambda(\alpha+1, b-a)}{(b-a)^{\alpha+1}}$.

Next, we will introduce the essential concepts and novel symbols of tempered fractional operators.

Definition 1.2 (See [19, 21])**.** The fractional tempered integral operators $\mathcal{J}_{a+}^{(\alpha,\lambda)} f$ and $\mathcal{J}_{b-}^{(a,\lambda)} f$ of order $\alpha > 0$ and $\lambda \geq 0$ are presented by

$$\mathcal{J}_{a+}^{(\alpha,\lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{-\lambda(x-t)} f(t) dt, \quad x \in [a, b] \tag{1.6}$$

and

$$\mathcal{J}_{b-}^{(\alpha,\lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-\lambda(t-x)} f(t) dt, \quad x \in [a, b], \tag{1.7}$$

respectively for $f \in L_1[a, b]$.

If we choose $\lambda = 0$, then the fractional integrals in (1.6) and (1.7) coincide with the Riemann-Liouville fractional integrals in (1.1) and (1.2), respectively.

Tempered fractional calculus can be defined as an generalization of fractional calculus. The definitions of fractional integration with weak singular and exponential kernels were firstly reported in Buschman's earlier work [3]. For more information associated with the different definitions of the tempered fractional integration, see the references [20, 25, 27, 29] and references therein. Mohammed et al. [22] proved various Hermite–Hadamard-type inequalities involving the tempered fractional integrals for the case of convex functions, which cover the previously published results for Riemann integrals and Riemann-Liouville fractional integrals.

2. Main results

Lemma 2.1. *If $f : [a, b] \to \mathbb{R}$ is an absolutely continuous function (a, b) so that $f' \in L_1[a, b]$, then the equality*

$$\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right] \tag{2.1}$$

$$
\begin{aligned}
&-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\\
&=\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\sum_{i=1}^{4}I_i
\end{aligned}
$$

is valid. Here,

$$
\begin{cases}
I_1=\int\limits_0^{\frac{1}{6}}\Upsilon_{\lambda(b-a)}(\alpha,t)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\
I_2=\int\limits_{\frac{1}{6}}^{\frac{1}{2}}\left\{\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\\
\qquad\times\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\
I_3=\int\limits_{\frac{1}{2}}^{\frac{5}{6}}\left\{\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\\
\qquad\times\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt,\\
I_4=\int\limits_{\frac{5}{6}}^{1}\left\{\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\\
\qquad\times\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt.
\end{cases}
$$

Proof. By using the integration by parts, it yields

$$
\begin{aligned}
I_1&=\int\limits_0^{\frac{1}{6}}\Upsilon_{\lambda(b-a)}(\alpha,t)\left[f'(tb+(1-t)a)-f'(ta+(1-t)b)\right]dt \qquad (2.2)\\
&=\frac{1}{b-a}\Upsilon_{\lambda(b-a)}(\alpha,t)\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]\Big|_0^{\frac{1}{6}}\\
&\quad-\frac{1}{b-a}\int\limits_0^{\frac{1}{6}}t^{\alpha-1}e^{-\lambda(b-a)t}\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]dt\\
&=\frac{1}{b-a}\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{6}\right)\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right]\\
&\quad-\frac{1}{b-a}\int\limits_0^{\frac{1}{6}}t^{\alpha-1}e^{-\lambda(b-a)t}\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]dt.
\end{aligned}
$$

Then, similar to the previous process, we readily have

$$
\begin{aligned}
I_2&=\frac{2}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}f\left(\frac{a+b}{2}\right) \qquad (2.3)\\
&\quad-\frac{1}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{6}\right)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right]\\
&\quad-\frac{1}{b-a}\int\limits_{\frac{1}{6}}^{\frac{1}{2}}t^{\alpha-1}e^{-\lambda(b-a)t}\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]dt,
\end{aligned}
$$

$$I_3 = \frac{1}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{5}{6}\right) - \frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right] \tag{2.4}$$
$$-\frac{2}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right) - \frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right\} f\left(\frac{a+b}{2}\right)$$
$$-\frac{1}{b-a}\int_{\frac{1}{2}}^{\frac{5}{6}} t^{\alpha-1}e^{-\lambda(b-a)t}\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]dt,$$

and

$$I_4 = -\frac{1}{b-a}\left\{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{5}{6}\right) - \Upsilon_{\lambda(b-a)}(\alpha,1)\right\}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+5b}{6}\right)\right] \tag{2.5}$$
$$-\frac{1}{b-a}\int_{\frac{5}{6}}^{1} t^{\alpha-1}e^{-\lambda(b-a)t}\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]dt.$$

Let us add the equalities from (2.2) to (2.5). Then, we have

$$\sum_{i=1}^{4} I_i = \frac{\Upsilon_\lambda(\alpha,b-a)}{4(b-a)^{\alpha+1}}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{5a+b}{6}\right)\right] \tag{2.6}$$
$$-\frac{1}{b-a}\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}\left[f(tb+(1-t)a)+f(ta+(1-t)b)\right]dt.$$

With the aid of the change of the variable $x = tb+(1-t)a$ and $x = ta+(1-t)b$ for $t\in[0,1]$ respectively, the equality (2.6) can be rewritten as follows

$$\sum_{i=1}^{4} I_i = \frac{\Upsilon_\lambda(\alpha,b-a)}{4(b-a)^{\alpha+1}}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{5a+b}{6}\right)\right] \tag{2.7}$$
$$-\frac{\Gamma(\alpha)}{(b-a)^{\alpha+1}}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right].$$

If we multiply both sides of (2.7) by $\frac{(b-a)^{\alpha+1}}{2\Upsilon_\lambda(\alpha,b-a)}$, then equality (2.1) is obtained simultaneously. □

Theorem 2.1. *Suppose that the assumptions of Lemma 2.1 hold. Also suppose that the function $|f'|$ is convex on $[a,b]$. Then, we have the following Euler–Maclaurin-type inequality*

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right] \right. \tag{2.8}$$
$$\left. -\frac{\Gamma(\alpha)}{2\,\Upsilon_\lambda(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|$$

$$\leq \frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left(\Omega_1(\alpha,\lambda)+\Omega_2(\alpha,\lambda)+\Omega_3(\alpha,\lambda)+\Omega_4(\alpha,\lambda)\right)\left[|f'(a)|+|f'(b)|\right],$$

where

$$\begin{cases} \Omega_1(\alpha,\lambda)=\int\limits_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|dt,\\ \Omega_2(\alpha,\lambda)=\int\limits_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt,\\ \Omega_3(\alpha,\lambda)=\int\limits_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt,\\ \Omega_4(\alpha,\lambda)=\int\limits_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt. \end{cases}$$

Proof. If we take the modulus in Lemma 2.1, then we actually get

$$\begin{aligned} &\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right. \\ &\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right| \\ \leq&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\int\limits_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\right. \\ &\times\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt \\ &+\int\limits_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right| \\ &\times\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt \\ &+\int\limits_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right| \\ &\times\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt \\ &+\int\limits_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right| \\ &\left.\times\left|f'(tb+(1-t)a)-f'(ta+(1-t)b)\right|dt\right]. \end{aligned} \tag{2.9}$$

Since $|f'|$ is convex, it yields

$$\begin{aligned} &\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right. \\ &\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\right]\right| \end{aligned}$$

$$\leq \frac{(b-a)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,b-a)}\left[\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\right.$$
$$\times\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt$$
$$+\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|$$
$$\times\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt$$
$$+\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|$$
$$\times\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt$$
$$+\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|$$
$$\left.\times\left[t\left|f'(b)\right|+(1-t)\left|f'(a)\right|+t\left|f'(a)\right|+(1-t)\left|f'(b)\right|\right]dt\right]$$
$$=\frac{(b-a)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,b-a)}\left(\Omega_1(\alpha)+\Omega_2(\alpha)+\Omega_3(\alpha)+\Omega_4(\alpha)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right].$$

This completes the proof of Theorem 2.1. □

Remark 2.1. If we choose $\lambda = 0$ in Theorem 2.1, then the following Euler–Maclaurin-type inequality holds:

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)\right]\right|$$
$$\leq\frac{\alpha(b-a)}{2}\left(\Omega_1(\alpha,0)+\Omega_2(\alpha,0)+\Omega_3(\alpha,0)+\Omega_4(\alpha,0)\right)\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

which is established by Hezenci and Budak in paper [14, Theorem 4].

Remark 2.2. Let us consider $\lambda = 0$ and $\alpha = 1$ in Theorem 2.1. Then, we obtain the following Euler–Maclaurin-type inequality

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|$$
$$\leq\frac{25(b-a)}{576}\left[\left|f'(a)\right|+\left|f'(b)\right|\right],$$

which is presented in paper [14, Corollary 1].

Theorem 2.2. *Assume that the assumptions of Lemma 2.1 are valid and the function $|f'|^q$, $q > 1$ is convex on $[a,b]$. Then, the following Euler-Maclaurin-type* *inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\
\leq&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left\{\left(\varphi_1^p(\alpha,\lambda)+\varphi_4^p(\alpha,\lambda)\right)\right.\\
&\times\left[\left(\frac{11\left|f'(a)\right|^q+\left|f'(b)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+11\left|f'(b)\right|^q}{72}\right)^{\frac{1}{q}}\right]\\
&+\left(\varphi_2^p(\alpha,\lambda)+\varphi_3^p(\alpha,\lambda)\right)\\
&\left.\times\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{9}\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}$$

Here, $\frac{1}{p}+\frac{1}{q}=1$ *and*

$$\begin{cases}
\varphi_1^p(\alpha,\lambda)=\left(\int_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}},\\
\varphi_2^p(\alpha,\lambda)=\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}},\\
\varphi_3^p(\alpha,\lambda)=\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}},\\
\varphi_4^p(\alpha,\lambda)=\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}.
\end{cases}$$

Proof. If we apply Hölder inequality in (2.9), then we can conclude that

$$\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\
\leq&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left\{\left(\int_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}}\left[\left(\int_0^{\frac{1}{6}}\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\right.\right.\\
&\left.+\left(\int_0^{\frac{1}{6}}\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\right]\\
&+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}
\end{aligned}$$

$$\times\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)-\frac{5}{8}\,\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)-\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\left.\times\left[\left(\int_{\frac{5}{6}}^{1}\left|f'\left(tb+(1-t)a\right)\right|^{q}dt\right)^{\frac{1}{q}}+\left(\int_{\frac{5}{6}}^{1}\left|f'\left(ta+(1-t)b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]\right\}.$$

With the help of the convexity of $|f'|^{q}$, we readily have

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.$$

$$\left.-\frac{\Gamma\left(\alpha\right)}{2\,\Upsilon_{\lambda}\left(\alpha,b-a\right)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f\left(a\right)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f\left(b\right)\right]\right|$$

$$\leq\frac{\left(b-a\right)^{\alpha+1}}{2\,\Upsilon_{\lambda}\left(\alpha,b-a\right)}\left\{\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)\right|^{p}dt\right)^{\frac{1}{p}}\right.$$

$$\times\left[\left(\int_{0}^{\frac{1}{6}}t\left|f'\left(b\right)\right|^{q}+(1-t)\left|f'\left(a\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{0}^{\frac{1}{6}}t\left|f'\left(a\right)\right|^{q}+(1-t)\left|f'\left(b\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}\left(\alpha,t\right)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}\left(\alpha,1\right)\right|^{p}dt\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}t\left|f'\left(b\right)\right|^{q}+(1-t)\left|f'\left(a\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}} t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\Bigg[\left(\int_{\frac{1}{2}}^{\frac{5}{6}} t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}} t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\Bigg]$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}$$

$$\times\Bigg[\left(\int_{\frac{5}{6}}^{1} t\left|f'(b)\right|^q+(1-t)\left|f'(a)\right|^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{5}{6}}^{1} t\left|f'(a)\right|^q+(1-t)\left|f'(b)\right|^q dt\right)^{\frac{1}{q}}\Bigg]\Bigg\}$$

$$=\frac{(b-a)^{\alpha+1}}{2\Upsilon_{\lambda}(\alpha,b-a)}\Bigg\{\Bigg[\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|^p dt\right)^{\frac{1}{p}}$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}\Bigg]$$

$$\times\left[\left(\frac{11\left|f'(a)\right|^q+\left|f'(b)\right|^q}{72}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+11\left|f'(b)\right|^q}{72}\right)^{\frac{1}{q}}\right]$$

$$+\Bigg[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^p dt\right)^{\frac{1}{p}}$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|^{p}dt\right)^{\frac{1}{p}}\Bigg]$$
$$\times\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}\right]\Bigg\}.$$

□

Remark 2.3. Let us consider $\lambda = 0$ in Theorem 2.2. Then, we have the Euler-Maclaurin-type inequality

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\&\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)\right]\right|\\\leq&\frac{\alpha(b-a)}{2}\left\{\left(\varphi_{1}^{p}(\alpha,0)+\varphi_{4}^{p}(\alpha,0)\right)\right.\\&\times\left[\left(\frac{11|f'(a)|^{q}+|f'(b)|^{q}}{72}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+11|f'(b)|^{q}}{72}\right)^{\frac{1}{q}}\right]\\&+\left(\varphi_{2}^{p}(\alpha,0)+\varphi_{3}^{p}(\alpha,0)\right)\\&\left.\times\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}\right]\right\}.\end{aligned}$$

This coincides with [14, Theorem 5].

Remark 2.4. If we assign $\lambda = 0$ and $\alpha = 1$ in Theorem 2.2, then the following Euler-Maclaurin-type inequality holds:

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(t)\,dt\right|\\\leq&(b-a)\left[\left(\frac{1}{(p+1)6^{p+1}}\right)^{\frac{1}{p}}\right.\\&\times\left[\left(\frac{11|f'(a)|^{q}+|f'(b)|^{q}}{72}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+11|f'(b)|^{q}}{72}\right)^{\frac{1}{q}}\right]\\&+\left(\frac{1}{(p+1)}\left(\left(\frac{5}{24}\right)^{p+1}+\left(\frac{1}{8}\right)^{p+1}\right)\right)^{\frac{1}{p}}\\&\left.\times\left[\left(\frac{2|f'(a)|^{q}+|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^{q}+2|f'(b)|^{q}}{9}\right)^{\frac{1}{q}}\right]\right].\end{aligned}$$

This is proved by Hezenci and Budak in [14, Corollary 2].

Theorem 2.3. *Let us consider that the assumptions of Lemma 2.1 hold and the function $|f'|^q$, $q \geq 1$ is convex on $[a, b]$. Then, the following Euler-Maclaurin-type* *inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\
\leq&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left\{(\Omega_1(\alpha,\lambda))^{1-\frac{1}{q}}\right.\\
&\times\left[\left(\Omega_5(\alpha,\lambda)\left|f'(b)\right|^q+(\Omega_1(\alpha,\lambda)-\Omega_5(\alpha,\lambda))\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\Omega_5(\alpha,\lambda)\left|f'(a)\right|^q+(\Omega_1(\alpha,\lambda)-\Omega_5(\alpha,\lambda))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\\
&+(\Omega_2(\alpha,\lambda))^{1-\frac{1}{q}}\left[\left(\Omega_6(\alpha,\lambda)\left|f'(b)\right|^q+(\Omega_2(\alpha,\lambda)-\Omega_6(\alpha,\lambda))\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\Omega_6(\alpha,\lambda)\left|f'(a)\right|^q+(\Omega_2(\alpha,\lambda)-\Omega_6(\alpha,\lambda))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\\
&+(\Omega_3(\alpha,\lambda))^{1-\frac{1}{q}}\left[\left(\Omega_7(\alpha,\lambda)\left|f'(b)\right|^q+(\Omega_3(\alpha,\lambda)-\Omega_7(\alpha,\lambda))\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\Omega_7(\alpha,\lambda)\left|f'(a)\right|^q+(\Omega_3(\alpha,\lambda)-\Omega_7(\alpha,\lambda))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\\
&+(\Omega_4(\alpha,\lambda))^{1-\frac{1}{q}}\left[\left(\Omega_8(\alpha,\lambda)\left|f'(b)\right|^q+(\Omega_4(\alpha,\lambda)-\Omega_8(\alpha,\lambda))\left|f'(a)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\Omega_8(\alpha,\lambda)\left|f'(a)\right|^q+(\Omega_4(\alpha,\lambda)-\Omega_8(\alpha,\lambda))\left|f'(b)\right|^q\right)^{\frac{1}{q}}\right]\right\}.
\end{aligned}
$$

Here, $\Omega_1(\alpha,\lambda)$, $\Omega_2(\alpha,\lambda)$, $\Omega_3(\alpha,\lambda)$ *and* $\Omega_4(\alpha,\lambda)$ *are defined in Theorem 2.1 and*

$$
\begin{cases}
\Omega_5(\alpha,\lambda)=\int\limits_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|tdt,\\
\Omega_6(\alpha,\lambda)=\int\limits_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|tdt,\\
\Omega_7(\alpha,\lambda)=\int\limits_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|tdt,\\
\Omega_8(\alpha,\lambda)=\int\limits_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|tdt.
\end{cases}
$$

Proof. Firstly, let us apply power-mean inequality in (2.9). Then, we obtain

$$
\begin{aligned}
&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.\\
&\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|\\
\leq&\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left\{\left(\int\limits_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|dt\right)^{1-\frac{1}{q}}\right.
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\int_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right| dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right| dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right| dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right| dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)\right|^q dt\right)^{\frac{1}{q}}\\
&+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right| dt\right)^{1-\frac{1}{q}}\\
&\times\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(ta+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}
$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(tb+(1-t)a)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left|f'(ta+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}\right\}.$$

By using the fact that $|f'|^q$ is convex, we readily obtain

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left[\mathcal{J}_{b-}^{(\alpha,\lambda)}f(a)+\mathcal{J}_{a+}^{(\alpha,\lambda)}f(b)\right]\right|$$
$$\leq\frac{(b-a)^{\alpha+1}}{2\,\Upsilon_{\lambda}(\alpha,b-a)}\left\{\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|dt\right)^{1-\frac{1}{q}}\right.$$
$$\times\left[\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{0}^{\frac{1}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$
$$+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$
$$\times\left[\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{\frac{1}{6}}^{\frac{1}{2}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{3}{8}\,\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{\frac{5}{6}}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\frac{5}{8}\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]$$

$$+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(b)\right|^{q}+(1-t)\left|f'(a)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.$$

$$\left.\left.+\left(\int_{\frac{5}{6}}^{1}\left|\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right|\left[t\left|f'(a)\right|^{q}+(1-t)\left|f'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right]\right\}.$$

This ends the proof of Theorem 2.3. □

Remark 2.5. If we select $\lambda=0$ in Theorem 2.3, then the following Euler-Maclaurin-type inequality holds:

$$\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]\right.$$
$$\left.-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{b-}^{\alpha}f(a)+J_{a+}^{\alpha}f(b)\right]\right|$$
$$\leq\frac{\alpha(b-a)}{2}\left\{(\Omega_1(\alpha,0))^{1-\frac{1}{q}}\left[\left(\Omega_5(\alpha,0)\left|f'(b)\right|^{q}+(\Omega_1(\alpha,0)-\Omega_5(\alpha,0))\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.+\left(\Omega_5(\alpha,0)\left|f'(a)\right|^{q}+(\Omega_1(\alpha,0)-\Omega_5(\alpha,0))\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$+(\Omega_2(\alpha,0))^{1-\frac{1}{q}}\left[\left(\Omega_6(\alpha,0)\left|f'(b)\right|^{q}+(\Omega_2(\alpha,0)-\Omega_6(\alpha,0))\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Omega_6(\alpha,0)\left|f'(a)\right|^{q}+(\Omega_2(\alpha,0)-\Omega_6(\alpha,0))\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$+(\Omega_3(\alpha,0))^{1-\frac{1}{q}}\left[\left(\Omega_7(\alpha,0)\left|f'(b)\right|^{q}+(\Omega_3(\alpha,0)-\Omega_7(\alpha,0))\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\Omega_7(\alpha,0)\left|f'(a)\right|^{q}+(\Omega_3(\alpha,0)-\Omega_7(\alpha,0))\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]$$
$$+(\Omega_4(\alpha,0))^{1-\frac{1}{q}}\left[\left(\Omega_8(\alpha,0)\left|f'(b)\right|^{q}+(\Omega_4(\alpha,0)-\Omega_8(\alpha,0))\left|f'(a)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$\left.\left.+\left(\Omega_8(\alpha,0)\left|f'(a)\right|^{q}+(\Omega_4(\alpha,0)-\Omega_8(\alpha,0))\left|f'(b)\right|^{q}\right)^{\frac{1}{q}}\right]\right\},$$

which is given in [14, Theorem 6].

Remark 2.6. Consider $\lambda = 0$ and $\alpha = 1$ in Theorem 2.3. Then, we have the Euler-Maclaurin-type inequality

$$\begin{aligned}&\left|\frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right)+2f\left(\frac{a+b}{2}\right)+3f\left(\frac{a+5b}{6}\right)\right]-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\\&\leq\frac{(b-a)}{72}\left[\left(\frac{|f'(b)|^q+8|f'(a)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{|f'(a)|^q+8|f'(b)|^q}{9}\right)^{\frac{1}{q}}+\left(\frac{17}{8}\right)^{1-\frac{1}{q}}\right.\\&\quad\left.\times\left[\left(\frac{361|f'(b)|^q+863|f'(a)|^q}{576}\right)^{\frac{1}{q}}+\left(\frac{361|f'(a)|^q+863|f'(b)|^q}{576}\right)^{\frac{1}{q}}\right]\right].\end{aligned}$$

This coincides with [14, Corollary 3].

3. Summary and concluding remarks

Some Euler-Maclaurin-type inequalities are given for the case of differentiable convex functions by using tempered fractional integrals. More precisely, Euler-Maclaurin-type inequalities are obtained by taking advantage of the convexity, the Hölder inequality, and the power mean inequality. Moreover, the results are given by using special cases of the established theories.

The results and methods presented in this study regarding Euler–Maclaurin-type inequalities using tempered fractional integrals could provide new opportunities for further exploration by mathematicians in this field. In addition to this,one can try to generalize our results by utilizing a different version of convex function classes or another type of fractional integral operators. Finally, one can obtain likewise Euler-Maclaurin-type inequalities by tempered fractional integrals for convex functions by using quantum calculus.

References

[1] Budak H., Hezenci F., Kara H, *On parametrized inequalities of Ostrowski and Simpson type for convex functions via generalized fractional integral*, Math. Methods Appl. Sci., 44(30), 12522-12536, 2021.

[2] Budak H., Hezenci F., and Kara H, *On generalized Ostrowski, Simpson and Trapezoidal type inequalities for co-ordinated convex functions via generalized fractional integrals*, Advances in Difference Equations, 2021, 1-32, 2021.

[3] R.G. Buschman, *Decomposition of an integral operator by use of Mikusinski calculus*, SIAM J. Math. Anal. 3 (1972) 83-85.

[4] Davis P.J., Rabinowitz P., *Methods of numerical integration*, Academic Press, New York-San Francisco-London, 1975.

[5] Dedić Lj., Matić M., Pečarić J., *Euler-Maclaurin formulae*, Math. Inequal. Appl., 6(2) 247-275, 2003.

[6] Dedić L.J., Matić M., Pečarić J., and Vukelic, A, *On Euler-Simpson 3/8 formulae.* Nonlinear Studies, 18(1), 1-26, 2011.

[7] Dragomir S.S., *On Simpson's quadrature formula for mappings of bounded variation and applications*, Tamkang J. of Math. 30, 53-58, 1999.

[8] Erden S., Iftikhar S., Kumam P., and Awan M. U, *Some Newton's like inequalities with applications*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales. Serie A. Matematicas, 114(4), 1-13, 2020.

[9] Franjić I., Pečarić J., *Corrected Euler-Maclaurin's formulae*, Rend. Circ. Mat. Palermo 54, 259-272, 2005.

[10] Franjić I., Pečarić J., *On corrected Euler-Simpson's* 3/8 *formulae*, Nonlinear Studies, 13(4), 309-319, 2006.

[11] Gao S. and Shi W., *On new inequalities of Newton's type for functions whose second derivatives absolute values are convex*, International Journal of Pure and Applied Mathematics, 74(1), 33-41, 2012.

[12] Gorenflo R., Mainardi F., *Fractional calculus: Integral and differential equations of fractional order*, Springer Verlag, Wien, 1997.

[13] Hezenci F., Budak H., and Kara H., *New version of Fractional Simpson type inequalities for twice differentiable functions*, Advances in Difference Equations, 2021(460), 2021.

[14] Hezenci F., Budak H., *Maclaurin-type inequalities for Riemann-Liouville fractional integrals*, Annales Universitatis Mariae Curie-Sklodowska, sectio A-Mathematica, 76(2), 15-32, 2023.

[15] Hezenci F., Budak H., Kosem P., *A new version of Newton's inequalities for Riemann-Liouville fractional integrals*, Rocky Mountain Journal of Mathematics, 53(1), 49-64.

[16] Hezenci F., Budak H., *Some Perturbed Newton type inequalities for Riemann-Liouville fractional integrals*, Rocky Mountain Journal of Mathematics, 53(4), 1117-1127.

[17] Iftikhar S., Kumam P. and Erden S., *Newton's-type integral inequalities via local fractional integrals*, Fractals, 28(03), 2050037, 2020.

[18] Kilbas A. A., Srivastava H.M., Trujillo J.J., *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

[19] C. Li, W. Deng and L. Zhao, Well-posedness and numerical algorithm for the tempered fractional ordinary differential equations, *Discret. Cont. Dyn-B* 24, 2019, 1989-2015.

[20] M.M. Meerschaert and A. Sikorskii, *Stochastic Models for Fractional Calculus*, *De Gruyter Studies in Mathematics*, 43, 2012.

[21] M.M. Meerschaert, F. Sabzikar, and J. Chen, *Tempered fractional calculus*, J. Comput. Phys. 293, 2015, 14-28.

[22] P.O. Mohammed, M.Z. Sarikaya, and D. Baleanu, *On the Generalized Hermite-Hadamard Inequalities via the Tempered Fractional Integrals*, Symmetry, 2020; 12(4):595.

[23] Noor M.A., Noor K.I., Iftikhar S., *Some Newton's type inequalities for harmonic convex functions*, J. Adv. Math. Stud., 9(1), 07-16, 2016.

[24] Pečarić J. E., Proschan F., Tong Y. L., *Convex Functions, Partial Orderings and Statistical Applications*, Academic Press, Boston, 1992.

[25] S. Samko, A. Kilbas, and O. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, London, 1993.

[26] Sitthiwirattham T., Nonlaopon K., Ali M.A., Budak H., *Riemann-Liouville fractional Newton's type inequalities for differentiable convex functions* Fractal Fract., 6(3), 175, 2022.

[27] H.M. Srivastava and R.G. Buschman, *Convolution Integral Equations with Special Function Kernels*, John Wiley & Sons, New York, 1977.

[28] Mohammed P. O. and Fernandez A., (2023). *Integral inequalities in fractional calculus with general analytic kernels*. Filomat, 37(11), 3659-3669.

[29] Artion Kashuri, Soubhagya Kumar Sahoo, Pshtiwan Othman Mohammed, Eman Al-Sarairah, Nejmeddine Chorfi, *Novel inequalities for subadditive functions via tempered fractional integrals and their numerical investigations*, AIMS Mathematics, 2024, 9(5): 13195-13210. doi: 10.3934/math.2024643

[30] Mehmood S, Mohammed PO, Kashuri A, Chorfi N, Mahmood SA, Yousif MA, *Some New Fractional Inequalities Defined Using cr-Log-h-Convex Functions and Applications*. Symmetry. 2024; 16(4):407. https://doi.org/10.3390/sym16040407

[31] Srivastava HM, Sahoo SK, Mohammed PO, Kashuri A, Chorfi N., *Results on Minkowski-Type Inequalities for Weighted Fractional Integral Operators*, Symmetry. 2023; 15(8):1522. https://doi.org/10.3390/sym15081522