Journal of Nonlinear Modeling and Analysis http://jnma-online.com
Volume 6, Number 4, December 2024, 949–969 DOI:10.12150/jnma.2024.949

Controllability of Neutral Fractional Functional Differential Equations with Two Caputo Fractional Derivatives*

Qi Wang[1,†] and Shumin Zhu[2]

Abstract Multiple fractional derivatives enrich the dynamic properties of fractional differential equations. This paper concerns with neutral fractional functional differential equations with two Caputo fractional derivatives. By using the fixed point methods with the fractional integral inequalities, the existence results and controllability of the equations are considered in the cases of finite delay and infinite delay, respectively. An example is given to illustrate the main results.

Keywords Neutral Caputo fractional functional differential equations, controllability, fractional integral inequalities, fixed point theorems

MSC(2010) 34A08, 34K35.

1. Introduction

By using the fixed point method and other nonlinear analyses, the controllability of integer-order and fractional-order neutral functional differential equations is discussed widely, which involves Cauchy conditions and nonlocal conditions, abstract spaces and general spaces, non-impulsive and impulsive differential systems, as well as finite delay and infinite delay. At present, the approximate controllability [1, 2, 6, 11, 13, 15, 25, 27, 32, 35, 36, 38, 46, 60, 61], controllability [3–5, 8–10, 12, 18, 19, 21, 22, 24, 28–31, 33, 37, 39–45], exact controllability [17, 26], total controllability [43], numerical controllability [7, 23], relative controllability [16, 20], etc. al, of fractional differential equation with only one fractional derivative operator are widely considered. In [47], the author considered the existence results of impulsive Caputo fractional functional differential inclusions with variable times in the case of finite delay

$$\begin{cases} {}^C D^{\alpha}[{}^C D^{\beta} x(t) - g(t, x_t)] \in F(t, x_t), t \in J := [0, T]; \\ x(t^+) = I_k(x(t)),\ {}^C D^{\beta} x(t^+) = I_k^*(x(t)),\ t = \tau_k(x(t)); \\ x(t) = \phi(t),\ t \in [-\tau, 0];\ {}^C D^{\beta} x(0) = \mu \in R. \end{cases} \tag{1.1}$$

†the corresponding author.
Email address:wq200219971974@163.com(Q. Wang)
1School of Mathematical Sciences, Anhui University, 230601, Hefei, P. R. China
*The authors were supported by the project of the construction of the peak discipline of higher education in Anhui Province of China.

In [48], the finite-time stability of the following Caputo fractional functional differential system is considered

$$\begin{cases} {}^cD_0^\alpha x(t) = Ax(t) + Bx(t-\tau(t)) + Dw(t) + f(t, x(t), x(t-\tau(t)), w(t)) \\ \qquad\qquad + {}^cD_0^\mu x(t-\tau(t)), t \in [0, T], \\ x(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{1.2}$$

In [49], using the fixed point method of multi-valued maps, the properties of semigroup and the properties of generalized Clarke's sub-differentials, the authors considered the existence of optimal feedback control for Caputo fractional neutral evolution systems in Hilbert spaces. In [50], using the fixed point method with theory of fractional calculus and stochastic analysis, the existence result for stochastic integro-differential equations in Hilbert spaces is obtained. Fore more details of the existence and stability of neutral fractional functional differential equations, see [6, 51–55, 57–60].

Taking into account the influence of multiple fractional derivatives of fractional differential equations, the authors considered the existence, attractivity and stability of fractional differential equations [61–66]. Till now, there have been few papers on the controllability of fractional differential equations with multiple fractional derivatives, especially in the case of finite time delay. Inspired by the above literature, in this paper, we investigate the following fractional functional differential equation with two Caputo fractional derivatives

$$\begin{cases} {}^CD^\alpha[{}^CD^\beta x(t) - g(t, x_t)] = f(t, x_t), \ t \in J := [0, T]; \\ x(t) = \phi(t), \ t \in I; \\ {}^CD^\beta x(0) = \mu \in R^n, \end{cases} \tag{1.3}$$

and the controllability form

$$\begin{cases} {}^CD^\alpha[{}^CD^\beta x(t) - g(t, x_t)] = f(t, x_t) + Bu(t), \ t \in J := [0, T]; \\ x(t) = \phi(t), \ t \in I; \\ {}^CD^\beta x(0) = \mu \in R^n, \end{cases} \tag{1.4}$$

where ${}^CD^\alpha$ and ${}^CD^\beta$ denote the Caputo fractional derivative of α, β order, respectively. $\alpha, \beta \in (0, 1), \alpha + \beta \in (1, 2)$; $x = (x_1, \cdots, x_n)^T$, $f, g \in C(J \times \mathcal{D}, R^n)$, $\phi \in \mathcal{D}, I = (-\infty, 0]$ or $[-\tau, 0]$. For any function x defined on $I \cup J$ and any $t \in J$, we denote by x_t the element of $\mathcal{D}$ defined by $x_t = x(t + \theta), \theta \in I$, which represents the history of the state from time $t - \tau$ up to the present time t. The norm of $\mathcal{D}$ is $\|u\|_{\mathcal{D}} := \sup\{\|u(t)\| : t \in I\}$. The control function $u(t) \in L^\infty(J, R^m)$ or $u(t) \in L^2(J, R^m)$, B is an $n \times m$ matrix.

In this article, some sufficient conditions for the existence results of (1.3) and controllability results of (1.4) are established, respectively, by using fractional integral inequalities, nonlinear analysis, fixed point approach including the contraction mapping principle and the Schaefer fixed point theorem.

2. Preliminaries

Let $L^p(J, R^n)$ be the Banach space of all measurable functions from J into R^n, which are Lebesgue integrable with the norm $\|x\|_{L^p} = (\int_0^T \|x(t)\|^p dt)^{\frac{1}{p}}, 1 \le p < +\infty$. Let $C(J, R^n)$ be the Banach space of all continuous functions from J into R^n with the norm $\|u\| = \sup\{\|u(t)\| : t \in J\}$.

To consider the existence results and controllability results of (1.3) and (1.4), the space

$$C^\beta((-\infty, T], R^n) = \{u : I \to R^n : u(t)|_I \in \mathcal{D}; u(t)|_J, {}^C D^\beta u(t)|_J \in C(J, R^n)\},$$
$$C^\beta([-\tau, T], R^n) = \{u : I \to R^n : u(t)|_I \in \mathcal{D}; u(t)|_J, {}^C D^\beta u(t)|_J \in C(J, R^n)\},$$

is denoted as Banach space of all continuous functions from $(-\infty, T]$ or $[-\tau, T]$ into R^n, respectively, with the norm $\|u\|_\beta := \|\phi\|_{\mathcal{D}} + \|u\| + \|{}^C D^\beta u\|$.

Definition 2.1. [67, 68] For a function h given on the interval $[a, b]$, the Caputo fractional derivative of order α of h is defined by

$$({}^c D^\alpha_{a^+} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.2. [67, 68] The Riemann-Liouville integral of order $\gamma > 0$ for the function f defined on $[a, b]$ is given by

$$I^\gamma_a f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s) ds, t \in [a, b], \gamma > 0,$$

provided the right side is point-wise defined on $[a, b]$, where $\Gamma(\cdot)$ is the gamma function.

For more knowledge on fractional calculus, see [67, 68] for details.

Lemma 2.1 (Lemma 1 [69])**.** *Let u be a nonnegative continuous function defined on the interval $I = [a, b]$ and $p(t) : I \to (0, \infty)$ be a nondecreasing continuous function. Suppose that $q(t) : I \to [0, \infty)$ is a nondecreasing continuous function. If u satisfies the following inequality:*

$$u(t) \le p(t) + q(t) \sum_{i=1}^n (I^{\alpha_i}_{a^+} u)(t), \ \alpha_i > 0, \ t \in I, \tag{2.1}$$

then for every $k \in N$ such that $(k+1)\min\{\alpha_1, \alpha_2, \cdots, \alpha_n\} > 1$,

$$u(t) \le P_k(t) \exp\left(\int_a^t H_{k+1}(t, s) ds \right), \ t \in I, \tag{2.2}$$

where

$$P_k(t):=p(t)\left(1+\sum_{j=1}^{k}(q(t))^j\sum_{\substack{i_1+\cdot+i_n=j,\\ 0\le i_1,\cdots,i_n\le j}}\begin{pmatrix} j\\ i_1,\cdots,i_n\end{pmatrix}\frac{(t-a)^{i_1\alpha_1+\cdots+i_n\alpha_n}}{\Gamma(1+i_1\alpha_1+\cdots+i_n\alpha_n)}\right),$$

$$H_{k+1}(t,s):=(q(t))^{k+1}\sum_{\substack{j_1+\cdot+j_n=k+1,\\ 0\le j_1,\cdots,j_n\le k+1}}\begin{pmatrix} k+1\\ j_1,\cdots,j_n\end{pmatrix}\frac{(t-s)^{j_1\alpha_1+\cdots+j_n\alpha_n}}{\Gamma(1+j_1\alpha_1+\cdots+j_n\alpha_n)},$$

with $\begin{pmatrix} k+1\\ j_1,\cdots,j_n\end{pmatrix}$ *representing a combination number formula.*

We introduce the definition of the phase space $(\mathcal{D},\|\cdot\|_{\mathcal{D}})$ in [70].

Let X be a Banach space and a linear topological space of functions from $(-\infty,0]$ to X with the semi-norm $\|\cdot\|_{\mathcal{D}}$ is called an admissible phase space $\mathcal{D}$ if the following conditions hold

(A_1) If $x:(-\infty,T]\to X$ $x_0\in\mathcal{D}$, then for any $t\in[0,T)$, the following conditions hold.

(a) $x_t\in\mathcal{D}$;

(b) $\|x(t)\|\le H\|x_t\|_{\mathcal{D}}$ for some positive constant H;

(c) there are functions $K(t),M(t):[0,\infty)\to[1,\infty)$ such that

$$\|x_t\|_{\mathcal{B}}\le K(t)\sup\{\|x(s)\|:s\in[0,t]\}+M(t)\|x_0\|_{\mathcal{D}},$$

where K is continuous, M is locally bounded and H,K,M are independent of x with $K_T=\sup\{K(s):s\in[0,T]\}$, $M_T=\sup\{M(s):s\in[0,T]\}$.

(A_2) For the function $x(\cdot)$ in (A_1), the function $x_t\in\mathcal{D}$ and is continuous on the interval $[0,T]$.

(A_3) The space $\mathcal{D}$ is a Banach space.

3. Existence and controllability results

Definition 3.1. The function $x\in C^{\beta}((-\infty,T],R^n)$ or $C^{\beta}([-\tau,T],R^n)$ is a solution of (1.3) if and only if $x(t)$ satisfies the following integral equation:

$$x(t)=\begin{cases}\phi(0)+\dfrac{[\mu-g(0,\phi(0))]t^{\beta}}{\Gamma(\beta+1)}+\dfrac{\int_0^t(t-s)^{\beta-1}g(s,x_s)ds}{\Gamma(\beta)}\\ +\dfrac{\int_0^t(t-s)^{\alpha+\beta-1}[f(s,x_s)]ds}{\Gamma(\alpha+\beta)},t\in J,\\ x(t)=\phi(t),\ t\in I.\end{cases}\tag{3.1}$$

Definition 3.2. The function $x\in C^{\beta}((-\infty,T],R^n)$ or $C^{\beta}([-\tau,T],R^n)$ is a solution of (1.4) if and only if $x(t)$ satisfies the following integral equation:

$$x(t) = \begin{cases} \phi(0) + \frac{[\mu - g(0,\phi(0))]t^\beta}{\Gamma(\beta+1)} + \frac{\int_0^t (t-s)^{\beta-1} g(s,x_s)ds}{\Gamma(\beta)} \\ + \frac{\int_0^t (t-s)^{\alpha+\beta-1}[f(s,x_s)+Bu(s)]ds}{\Gamma(\alpha+\beta)}, t \in J, \\ x(t) = \phi(t),\ t \in I. \end{cases} \tag{3.2}$$

Definition 3.3. (1.4) is said to be controllable on the interval J if for every $\phi \in \mathcal{D}, x_T \in R^n$, there is a control function $u \in L^\infty(J, R^m)$, such that the mild solution of (3.1) satisfies $x(T) = x_T$ and $x_0 = \phi \in \mathcal{D}$.

We make the following assumptions throughout the paper.

(H_1) The functions $g, f \in C(J \times \mathcal{D}, R^n)$ and there exist $c_i(t), i = 1, 3$ such that

$$\begin{aligned} \|g(t,u) - g(t,v)\| &\le c_1(t)\|u - v\|_{\mathcal{D}}, \\ \|f(t,u) - f(t,v)\| &\le c_3(t)\|u - v\|_{\mathcal{D}}, \\ g(0,0) = f(0,0) &= 0, t \in J, u \in \mathcal{D}, \end{aligned}$$

with

$$\begin{aligned} \|g(t,u)\| &\le c_1(t)\|u\|_{\mathcal{D}} + c_2, \\ \|f(t,u)\| &\le c_3(t)\|u\|_{\mathcal{D}} + c_4, t \in J, u \in \mathcal{D}, \end{aligned}$$

where

$$\begin{aligned} c_2 = \sup\{\|g(t,0)\|, t \in J\}, c_4 = \sup\{\|f(t,0)\|, t \in J\}, \\ \hat{c}_1 = \sup\{c_1(t), t \in J\}, \hat{c}_3 = \sup\{c_3(t), t \in J\}. \end{aligned}$$

(H_1') For the functions $g, f \in C(J \times \mathcal{D}, R^n)$, there exist some bounded functions $c_i(t) \in L^{\frac{1}{q_i}}(J, R^n), i = 1, 3, q_1 \in [0, \beta), q_3 \in [0, \alpha)$, such that

$$\begin{aligned} \|g(t,u) - g(t,v)\| &\le c_1(t)\|u - v\|_{\mathcal{D}}, \\ \|f(t,u) - f(t,v)\| &\le c_3(t)\|u - v\|_{\mathcal{D}}, \\ g(0,0) = f(0,0) &= 0, t \in J, u \in \mathcal{D}. \end{aligned}$$

Furthermore

$$\begin{aligned} \|g(t,u)\| &\le c_1(t)\|u\|_{\mathcal{D}} + c_2, \\ \|f(t,u)\| &\le c_3(t)\|u\|_{\mathcal{D}} + c_4, t \in J, u \in \mathcal{D}, \end{aligned}$$

where $c_2 = \sup\{\|g(t,0)\|, t \in J\}$ and $c_4 = \sup\{\|f(t,0)\|, t \in J\}$.

(H_2)

$$\begin{aligned} \frac{K_T\hat{c}_1T^\beta}{\Gamma(\beta+1)} + \frac{K_T\hat{c}_3T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \frac{M_1M_2T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\left\{\frac{K_T\hat{c}_1T^\beta}{\Gamma(\beta+1)} + \frac{K_T\hat{c}_3T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\right\} < 1, \\ \hat{c}_1K_T + \frac{K_T\hat{c}_3T^\alpha}{\Gamma(\alpha+1)} + \frac{M_1M_2T^\alpha}{\Gamma(\alpha+1)}\left\{\frac{K_TT^\beta\hat{c}_1}{\Gamma(\beta+1)} + \frac{K_TT^{\alpha+\beta}\hat{c}_3}{\Gamma(\alpha+\beta+1)}\right\} < 1. \end{aligned}$$

(H_2')

$$\frac{K_T}{\Gamma(\beta)}\left(\frac{1-q_1}{\beta-q_1}\right)^{1-q_1}T^{\beta-q_1}\|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)}+\frac{K_T}{\Gamma(\alpha+\beta)}\left(\frac{1-q_2}{\alpha-q_2}\right)^{1-q_2}T^{\alpha-q_2}\|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)}$$
$$+\frac{M_1M_2}{\Gamma(\alpha+\beta)}\frac{T^{\alpha+\beta}}{\alpha+\beta}\left[\frac{K_T}{\Gamma(\beta)}\left(\frac{1-q_1}{\beta-q_1}\right)^{1-q_1}T^{\beta-q_1}\|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)}\right.$$
$$\left.+\frac{K_T}{\Gamma(\alpha+\beta)}\left(\frac{1-q_2}{\alpha-q_2}\right)^{1-q_2}T^{\alpha-q_2}\|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)}\right]<1,$$
$$c_1(t)+\left(\frac{1-q_2}{\alpha-q_2}\right)^{1-q_2}T^{\alpha-q_2}\|c_3\|_{L^{\frac{1}{q_2}}_J}+\frac{M_1M_2T^{\alpha}}{\Gamma(\alpha+1)}\left\{\frac{K_T}{\Gamma(\beta)}\left(\frac{1-q_1}{\beta-q_1}\right)^{1-q_1}T^{\beta-q_1}\|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)}\right.$$
$$\left.+\frac{K_T}{\Gamma(\alpha+\beta)}\left(\frac{1-q_2}{\alpha+\beta-q_2}\right)^{1-q_2}T^{\alpha+\beta-q_2}\|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)}\right\}<1,t\in J.$$

(H_3) The bounded operator $B: L^\infty(J,R^m)\to R^n$ is linear. The operator $\mathbf{W}_u: L^\infty(J,R^m)\to R^n$ is defined as $\mathbf{W}_u=\frac{1}{\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}Bu(s)ds$, $\mathbf{W}_u$ with inverse operator $\mathbf{W}_u^{-1}$ (in $L^\infty(J,R^m)/ker\mathbf{W}_u$) and there exist $M_i, i=1,2$ such that $\|B\|\le M_1, \|\mathbf{W}_u^{-1}\|\le M_2$.

(H_3') $B: L^\infty(J,R^m)\to R^n$ is a linear operator. The operator $\mathbf{W}_u: L^2(J,R^m)\to R^n$ is defined as $\mathbf{W}_u=\frac{1}{\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}Bu(s)ds$, $\mathbf{W}_u$ with inverse operator $\mathbf{W}_u^{-1}$ (in $L^2(J,R^m)/ker\mathbf{W}_u$) and there are two positive constants $M_i, i=1,2$ such that $\|B\|\le M_1, \|\mathbf{W}_u^{-1}\|\le M_2$.

Firstly, we consider the controllability results of (1.4) in the case of infinite delay.

Theorem 3.1. *If the assumptions* $(H_1),(H_2)$ *and* (H_3) *hold. Then problem* (1.4) *is controllable on* J.

Proof. Define the space $S(T)=\{u\in C(J,R^n): u(0)=\phi(0)\}$ with the norm $\|u\|=\sup\{\|u(t)\|, t\in J\}$. For any $x\in C^\beta((-\infty,T],R^n)$, define the control function $u_x(\cdot)$ as

$$u_x(t)=\mathbf{W}_u^{-1}\Big\{x_1-[\phi(0)+[\mu-g(0,\phi_0)]\frac{T^\beta}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}g(s,x_s)ds$$
$$+\frac{1}{\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}f(s,x_s)ds]\Big\},\ t\in J, \tag{3.3}$$

where x_1 denotes the final state of (1.4) at T .

Consider the space $\Omega=\{x\in C^\beta((-\infty,T],R^n); \|x\|_\beta\le r\}, r>0$. We define the operator $N:\Omega\to\Omega$ as

$$(Nx)(t)=\begin{cases}\phi(0)+[\mu-g(0,\phi(0))]\frac{t^\beta}{\Gamma(\beta+1)}+\frac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g(s,x_s)ds\\ \quad+\frac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}[f(s,x_s)+Bu_x(s)]ds,\ t\in J,\\ \phi(t),\ t\in I.\end{cases} \tag{3.4}$$

So the fixed point of N is the solution of (1.4).

1. Firstly, we show that $N\Omega\subseteq\Omega$. For any $x\in\Omega$, by (H_1), we get

$$\|(Nx)(t)\|=\|\phi(t)\|,\ \|(Nx)\|_\beta=\|\phi\|_{\mathcal{D}}\le r, t\in I.$$

By $(A_1(c))$ of the definition on the phase space $(\mathcal{D},\|\cdot\|_{\mathcal{D}})$ in [70], we have

$$\begin{aligned}\|x_t\|_{\mathcal{D}}&\le K(t)\sup\{\|x(s)\|: s\in[0,t]\}+M(t)\|\phi(0)\|_{\mathcal{D}}\\&\le K_T\sup\{\|x(s)\|: s\in[0,t]\}+M_T\|\phi\|_{\mathcal{D}}\\&\le K_Tr+M_T\|\phi\|_{\mathcal{D}}=\mathcal{A}, t\in J.\end{aligned} \tag{3.5}$$

By the control function, we get

$$\begin{aligned}
\|u_x\|_{L^\infty(J,R^m)} &\leq M_2[\|x_1\| + \|\phi(0)\|_{\mathcal{D}} + (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)}] \\
&\quad + M_2[\tfrac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}\|g(s,x_s)\|ds \\
&\quad + \tfrac{1}{\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}\|f(s,x_s)\|ds] \\
&\leq M_2[\|x_1\| + \|\phi(0)\|_{\mathcal{D}} + (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)}] \\
&\quad + \tfrac{1}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}(c_1(s)\|x_s\|_{\mathcal{D}} + c_2)ds \\
&\quad + \tfrac{1}{\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}(c_3(s)\|x_s\|_{\mathcal{D}} + c_4)ds \\
&\leq M_2[\|x_1\| + \|\phi(0)\|_{\mathcal{D}} + (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)}] \qquad (3.6)\\
&\quad + \tfrac{M_2}{\Gamma(\beta)}\int_0^T (T-s)^{\beta-1}c_1(s)\mathcal{A}ds + \tfrac{M_2c_2T^\beta}{\Gamma(\beta+1)} \\
&\quad + \tfrac{M_2}{\Gamma(\alpha+\beta)}\int_0^T (T-s)^{\alpha+\beta-1}c_3(s)\mathcal{A}ds + \tfrac{M_2c_4T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} \\
&\leq M_2[\|x_1\| + \|\phi(0)\|_{\mathcal{D}} + (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)}] \\
&\quad + \tfrac{M_2\hat{c}_1\mathcal{A}t^{(\beta+1)}}{\Gamma(\beta+1)} + \tfrac{M_2c_2T^\beta}{\Gamma(\beta+1)} + \tfrac{M_2\hat{c}_3\mathcal{A}T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} + \tfrac{M_2c_4T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}] \\
&\leq M_2\big[\|x_1\| + \|\phi(0)\|_{\mathcal{D}} + (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)} \\
&\quad + \tfrac{\hat{c}_1\mathcal{A}T^{(\beta+1)}}{\Gamma(\beta+1)} + \tfrac{c_2T^\beta}{\Gamma(\beta+1)} + \tfrac{\hat{c}_3\mathcal{A}T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} + \tfrac{c_4T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}\big] = \mathcal{B}, t \in J.
\end{aligned}$$

So we have

$$\begin{aligned}
\|(Nx)(t)\| &\leq (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}} + c_2)\tfrac{t^\beta}{\Gamma(\beta+1)} \\
&\quad + \tfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(c_1(s)\|x_s\|_{\mathcal{D}} + c_2)ds \\
&\quad + \tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(c_3(s)\|x_s\|_{\mathcal{D}} + c_4)ds \\
&\quad + \tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\|B\|\|u_x(s)\|ds \\
&\leq (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}} + 2c_2)\tfrac{t^\beta}{\Gamma(\beta+1)} \qquad (3.7)\\
&\quad + \tfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}c_1(s)\mathcal{A}ds + \tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}c_3(s)\mathcal{A}ds \\
&\quad + \tfrac{c_4t^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} + \tfrac{M_1t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}\|u_x\|_{L^\infty(J,R^m)} \\
&\leq (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}} + 2c_2)\tfrac{T^\beta}{\Gamma(\beta+1)} + \tfrac{\hat{c}_1T^\beta\mathcal{A}}{\Gamma(\beta+1)} \\
&\quad + \tfrac{\hat{c}_3\mathcal{A}T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} + \tfrac{c_4T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} + \tfrac{M_1T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}\mathcal{B} = \mathcal{C}, t \in J,
\end{aligned}$$

and

$$\begin{aligned}
\|{}^CD^\beta(Nx)(t)\| &= \|g(t,x_t) + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s,x_s) + Bu_x(s)]ds\| \\
&\leq \|g(t,x_t)\| + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_s)\|ds \\
&\quad + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|B\|\|u_x(s)\|ds \\
&\leq \hat{c}_1\|x_t\|_{\mathcal{D}} + c_2 + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(c_3(s)\|x_s\|_{\mathcal{D}} + c_4)ds \qquad (3.8)\\
&\quad + \tfrac{M_1t^\alpha}{\Gamma(\alpha+1)}\|u_x\|_{L^\infty(J,R^m)} \\
&\leq \hat{c}_1\mathcal{A} + c_2 + \tfrac{(\hat{c}_3\mathcal{A}+c_4)T^\alpha}{\Gamma(\alpha+1)} + \tfrac{M_1\mathcal{B}T^\alpha}{\Gamma(\alpha+1)} = \mathcal{D}, t \in J.
\end{aligned}$$

Then we get

$$\sup\{\|(Nx)(t)\|\} \leq \mathcal{C}, \quad \sup\{\|{}^CD^\beta(Nx)(t)\|\} \leq \mathcal{D}, t \in J,$$

and

$$\|(Nx)\|_\beta \leq \|\phi\|_{\mathcal{D}} + \mathcal{C} + \mathcal{D} \leq r,$$

i.e. $N\Omega \subseteq \Omega$.

2. N is a contraction mapping operator. Choosing any $x, y \in \Omega$ with $x_0(t) = y_0(t), t \in I$, it follows that

$$\begin{aligned}
&\|(Nx)(t) - (Ny)(t)\| \\
\leq\ & \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|g(s,x_s) - g(s,y_s)\| ds \\
&+ \tfrac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} [\|f(s,x_s) - f(s,y_s)\| + \|Bu_x(s) - Bu_y(s)\|] ds \\
\leq\ & \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c_1(s) K(s) \sup\{\|x(\theta) - y(\theta)\| : \theta \in [0,s]\} ds \\
&+ \tfrac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} c_3(s) K(s) \sup\{\|x(\theta) - y(\theta)\| : \theta \in [0,s]\} ds \\
&+ \tfrac{M_1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \|\mathbf{W}_u^{-1} \Big\{ \tfrac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \|g(s,x_s) - g(s,y_s)\| ds \\
&+ \tfrac{1}{\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} \|f(s,x_s) - f(s,y_s)\| ds \Big\} \| ds \\
\leq\ & \tfrac{K_T}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c_1(s) ds \|x-y\| + \tfrac{K_T}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} c_3(s) ds \|x-y\| \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \| \Big[\tfrac{1}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} \|g(s,x_s) - g(s,y_s)\| ds \\
&+ \tfrac{1}{\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} \|f(s,x_s) - f(s,y_s)\| ds \Big] \| ds \\
\leq\ & \tfrac{K_T}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c_1(s) ds \|x-y\| + \tfrac{K_T}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} c_3(s) ds \|x-y\| \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \Big[\tfrac{K_T}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} c_1(s) ds \\
&+ \tfrac{K_T}{\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} c_3(s) ds \Big] ds \|x-y\| \\
\leq\ & \Big[\tfrac{K_T \hat{c}_1 T^\beta}{\Gamma(\beta+1)} + \tfrac{K_T \hat{c}_3 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big] \|x-y\| \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} \Big[\tfrac{K_T \hat{c}_1 T^\beta}{\Gamma(\beta+1)} + \tfrac{K_T \hat{c}_3 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big] ds \|x-y\| \\
\leq\ & \Big\{ \tfrac{K_T \hat{c}_1 T^\beta}{\Gamma(\beta+1)} + \tfrac{K_T \hat{c}_3 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} + \tfrac{M_1 M_2 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big[\tfrac{K_T \hat{c}_1 T^\beta}{\Gamma(\beta+1)} + \tfrac{K_T \hat{c}_3 T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)} \Big] \Big\} \|x-y\|.
\end{aligned} \tag{3.9}$$

In the same way, we obtain

$$\begin{aligned}
&\|{}^C D^\beta (Nx)(t) - {}^C D^\beta (Ny)(t)\| \\
\leq\ & \|g(t,x_t) - g(t,y_t)\| + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,x_s) - f(s,y_s)\| ds \\
&+ \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|Bu_x(s) - Bu_y(s)\| ds \\
\leq\ & c_1(t) \|x-y\|_{\mathcal{D}} + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c_3(s) \|x_s - y_s\|_{\mathcal{D}} ds \\
&+ \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|B\| \|u_x(s) - u_y(s)\| ds \\
\leq\ & \hat{c}_1 K_T \|x-y\| + \tfrac{K_T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c_3(s) ds \|x-y\| \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\tfrac{K_T}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} c_1(s) ds \\
&+ \tfrac{K_T}{\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} c_3(s) ds] ds \|x-y\| \\
\leq\ & \Big\{ \hat{c}_1 K_T + \tfrac{K_T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c_3(s) ds \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\tfrac{K_T T^\beta \hat{c}_1}{\Gamma(\beta+1)} + \tfrac{K_T T^{\alpha+\beta} \hat{c}_3}{\Gamma(\alpha+\beta+1)}] ds \Big\} \|x-y\| \\
\leq\ & \Big\{ \hat{c}_1 K_T + \tfrac{K_T \hat{c}_3 T^\alpha}{\Gamma(\alpha+1)} + \tfrac{M_1 M_2 T^\alpha}{\Gamma(\alpha+1)} \Big[\tfrac{K_T T^\beta \hat{c}_1}{\Gamma(\beta+1)} + \tfrac{K_T T^{\alpha+\beta} \hat{c}_3}{\Gamma(\alpha+\beta+1)} \Big] \Big\} \|x-y\|, t \in J.
\end{aligned} \tag{3.10}$$

Thus we get

$$\|Nx - Ny\|_\beta < \|x - y\|_\beta,$$

i.e. N is a contraction operator and N has a fixed point. Thus problem (1.4) is controllable on J. □

Theorem 3.2. *Suppose that the assumptions $(H_1'), (H_2')$ and (H_3') hold with $\alpha > \frac{1}{2}$. Then problem (1.4) is controllable on J.*

Proof. Define the space $S(T)$ as in Theorem 3.1. For any $x \in C^\beta((-\infty, T], R^n)$, choose the control function $u_x(\cdot)$ as in (3.3). Consider $\Omega = \{x \in C^\beta((-\infty, T], R^n); \|x\|_\beta \leq r\}$, where $\|x\|_\beta := \|\phi\|_\mathcal{D} + \|x\| + \|^C D^\beta x\|$. Define the operator $N : \Omega \to \Omega$ as in (3.4). We show that N is a contraction mapping operator and has a fixed point.

1. $N\Omega \subseteq \Omega$. For any $x \in \Omega$, by (H_1'), we have

$$\|(Nx)(t)\| = \|\phi(t)\|, \ \|(Nx)\|_\beta = \|\phi\|_\mathcal{D} \leq r, t \in I.$$

By $(A_1(c))$ in the definition 3.4, as the proof in (3.5), we have

$$\begin{aligned}\|x_t\|_\mathcal{D} &\leq K(t) \sup\{\|x(s)\| : s \in [0, t]\} + M(t)\|\phi(0)\|_\mathcal{D} \\ &\leq K_T r + M_T\|\phi\|_\mathcal{D} = \mathcal{A}', t \in J.\end{aligned}$$

For the control function u_x, by Hölder inequality, we get

$$\begin{aligned}\|u_x\| \leq\, & M_2[\|x_1\| + \|\phi(0)\|_\mathcal{D} + (\|\mu\| + c_1(0)\|\phi(0)\|_\mathcal{D})\tfrac{T^\beta}{\Gamma(\beta+1)}] \\ & + M_2[\tfrac{1}{\Gamma(\beta)}\textstyle\int_0^T (T-s)^{\beta-1}\|g(s, x_s)\|ds \\ & + \tfrac{1}{\Gamma(\alpha+\beta)}\textstyle\int_0^T (T-s)^{\alpha+\beta-1}\|f(s, x_s)\|ds] \\ \leq\, & M_2[\|x_1\| + \|\phi(0)\|_\mathcal{D} + (\|\mu\| + c_1(0)\|\phi(0)\|_\mathcal{D})\tfrac{T^\beta}{\Gamma(\beta+1)}] \\ & + \tfrac{1}{\Gamma(\beta)}\textstyle\int_0^T (T-s)^{\beta-1}(c_1(s)\|x_s\|_\mathcal{D} + c_2)ds \\ & + \tfrac{1}{\Gamma(\alpha+\beta)}\textstyle\int_0^T (T-s)^{\alpha+\beta-1}(c_3(s)\|x_s\|_\mathcal{D} + c_4)ds \\ \leq\, & M_2[\|x_1\| + \|\phi(0)\|_\mathcal{D} + (\|\mu\| + c_1(0)\|\phi(0)\|_\mathcal{D})\tfrac{T^\beta}{\Gamma(\beta+1)}] \\ & + \tfrac{M_2\mathcal{A}}{\Gamma(\beta)}\textstyle\int_0^T (T-s)^{\beta-1}c_1(s)ds + \tfrac{M_2 c_2 T^\beta}{\Gamma(\beta+1)} \\ & + \tfrac{M_2\mathcal{A}}{\Gamma(\alpha+\beta)}\textstyle\int_0^T (T-s)^{\alpha+\beta-1}c_3(s)ds + \tfrac{M_2 c_4 T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} \\ \leq\, & M_2[\|x_1\| + \|\phi(0)\|_\mathcal{D} + (\|\mu\| + c_1(0)\|\phi(0)\|_\mathcal{D})\tfrac{T^\beta}{\Gamma(\beta+1)}] \\ & + \tfrac{M_2\mathcal{A}}{\Gamma(\beta)}\Big(\textstyle\int_0^T (T-s)^{\frac{\beta-1}{1-q_1}}ds\Big)^{1-q_1}\|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)} + \tfrac{M_2 c_2 T^\beta}{\Gamma(\beta+1)} \\ & + \tfrac{M_2\mathcal{A}}{\Gamma(\alpha+\beta)}\Big(\textstyle\int_0^T (T-s)^{\frac{\alpha+\beta-1}{1-q_2}}ds\Big)^{1-q_2}\|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} + \tfrac{M_2 c_4 T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}] \\ \leq\, & M_2\Big[\|x_1\| + \|\phi(0)\| + (\|\mu\| + c_1(0)\|\phi(0)\|)\tfrac{T^\beta}{\Gamma(\beta+1)}\Big] \\ & + \tfrac{M_2\mathcal{A}}{\Gamma(\beta)}\Big(\tfrac{1-q_1}{\beta-q_1}\Big)^{1-q_1}T^{\beta-q_1}\|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)} + \tfrac{M_2 c_2 T^\beta}{\Gamma(\beta+1)} \\ & + \tfrac{M_2\mathcal{A}}{\Gamma(\alpha+\beta)}\Big(\tfrac{1-q_2}{\alpha+\beta-q_2}\Big)^{1-q_2}T^{\alpha+\beta-q_2}\|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \\ & + \tfrac{M_2 c_4 T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} = \mathcal{B}', \ t \in J,\end{aligned}$$

and

$$\|u_x\|_{L^2(J,R^n)} \leq \mathcal{B}'\sqrt{T}.$$

So by (H_1'), we have

$$\begin{aligned}
\|(Nx)(t)\| &\leq (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}} + c_2)\tfrac{t^\beta}{\Gamma(\beta+1)} \\
&\quad + \tfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}(c_1(s)\|x_s\|_{\mathcal{D}} + c_2)ds \\
&\quad + \tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}(c_3(s)\|x_s\|_{\mathcal{D}} + c_4)ds \\
&\quad + \tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}\|B\|\|u_x(s)\|ds \\
&\leq (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}} + 2c_2)\tfrac{t^\beta}{\Gamma(\beta+1)} + \tfrac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}c_1(s)\mathcal{A}'ds \\
&\quad + \tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}c_3(s)\mathcal{A}'ds + \tfrac{c_4 t^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} \\
&\quad + \tfrac{M_1}{\Gamma(\alpha+\beta)}\sqrt{\tfrac{T^{2\alpha+2\beta-1}}{2\alpha+2\beta-1}}\|u_x\|_{L^2(J,R^n)} \\
&\leq (\|\mu\| + c_1(0)\|\phi(0)\|_{\mathcal{D}} + 2c_2)\tfrac{T^\beta}{\Gamma(\beta+1)} \\
&\quad + \tfrac{M_2\mathcal{A}'}{\Gamma(\beta)}\left(\tfrac{1-q_1}{\beta-q_1}\right)^{1-q_1} T^{\beta-q_1}\|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)} + \tfrac{M_2 c_2 T^\beta}{\Gamma(\beta+1)} \\
&\quad + \tfrac{M_2\mathcal{A}'}{\Gamma(\alpha+\beta)}\left(\tfrac{1-q_2}{\alpha+\beta-q_2}\right)^{1-q_2} T^{\alpha+\beta-q_2}\|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} + \tfrac{c_4 T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)} \\
&\quad + \tfrac{M_1}{\Gamma(\alpha+\beta)}\sqrt{\tfrac{T^{2\alpha+2\beta-1}}{2\alpha+2\beta-1}}\mathcal{B}'\sqrt{T} = \mathcal{C}', t \in J,
\end{aligned}$$

and

$$\begin{aligned}
&\|{}^C D^\beta (Nx)(t)\| \\
&= \|g(t,x_t) + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[f(s,x_s) + Bu_x(s)]ds\| \\
&\leq \|g(t,x_t)\| + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|f(s,x_s)\|ds \\
&\quad + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|B\|\|u_x(s)\|ds \\
&\leq \hat{c}_1\|x_t\|_{\mathcal{D}} + c_2 + \tfrac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(c_3(s)\|x_s\|_{\mathcal{D}} + c_4)ds \\
&\quad + \tfrac{M_1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\|u_x(s)\|ds \\
&\leq \hat{c}_1\mathcal{A} + c_2 + \tfrac{c_4 T^\alpha}{\Gamma(\alpha+1)} + \tfrac{\mathcal{A}'}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}c_3(s)ds \\
&\quad + \tfrac{M_1}{\Gamma(\alpha)}(\int_0^t (t-s)^{2\alpha-2}ds)^{\frac{1}{2}}\|u_x\|_{L^2(J,R^n)} \\
&\leq \hat{c}_1\mathcal{A}' + c_2 + \tfrac{c_4 T^\alpha}{\Gamma(\alpha+1)} + \tfrac{\mathcal{A}'}{\Gamma(\alpha)}\left(\tfrac{1-q_2}{\alpha-q_2}\right)^{1-q_2} T^{\alpha-q_2}\|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \\
&\quad + \tfrac{M_1}{\Gamma(\alpha)}\sqrt{\tfrac{T^{2\alpha-1}}{2\alpha-1}}\|u_x\|_{L^2(J,R^n)} \\
&\leq \hat{c}_1\mathcal{A}' + c_2 + \tfrac{c_4 T^\alpha}{\Gamma(\alpha+1)} + \tfrac{\mathcal{A}'}{\Gamma(\alpha)}\left(\tfrac{1-q_2}{\alpha-q_2}\right)^{1-q_2} T^{\alpha-q_2}\|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \\
&\quad + \tfrac{M_1}{\Gamma(\alpha)}\sqrt{\tfrac{T^{2\alpha-1}}{2\alpha-1}}\mathcal{C}' = \mathcal{D}', t \in J.
\end{aligned}$$

Thus it follows that

$$\sup\{\|(Nx)(t)\|\} \leq \mathcal{C}, \quad \sup\{\|{}^C D^\beta (Nx)(t)\|\} \leq \mathcal{D}, t \in J,$$

and

$$\|(Nx)\|_\beta \leq \|\phi\|_{\mathcal{D}} + \mathcal{C} + \mathcal{D} \leq r,$$

i.e. $N\Omega \subseteq \Omega$.

2. N is a contraction mapping operator. Choose any $x, y \in \Omega$ with $x_0 = y_0, t \in I$, as (3.9) and (3.10), we get

$$\begin{aligned}
&\|(Nx)(t) - (Ny)(t)\| \\
\le& \tfrac{K_T}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c_1(s) ds \|x-y\| + \tfrac{K_T}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} c_3(s) ds \|x-y\| \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \Big[\tfrac{K_T}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} c_1(s) ds \\
&+ \tfrac{K_T}{\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} c_3(s) ds \Big] ds \|x-y\| \\
\le& \Big[\tfrac{K_T}{\Gamma(\beta)} \Big(\tfrac{1-q_1}{\beta-q_1} \Big)^{1-q_1} T^{\beta-q_1} \|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)} \\
&+ \tfrac{K_T}{\Gamma(\alpha+\beta)} \Big(\tfrac{1-q_2}{\alpha-q_2} \Big)^{1-q_2} T^{\alpha-q_2} \|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \Big] \|x-y\| \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} \Big[\tfrac{K_T}{\Gamma(\beta)} \Big(\tfrac{1-q_1}{\beta-q_1} \Big)^{1-q_1} T^{\beta-q_1} \|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)} \\
&+ \tfrac{K_T}{\Gamma(\alpha+\beta)} \Big(\tfrac{1-q_2}{\alpha-q_2} \Big)^{1-q_2} T^{\alpha-q_2} \|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \Big] ds \|x-y\| \\
\le& \Big\{ \tfrac{K_T}{\Gamma(\beta)} \Big(\tfrac{1-q_1}{\beta-q_1} \Big)^{1-q_1} T^{\beta-q_1} \|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)} + \tfrac{K_T}{\Gamma(\alpha+\beta)} \Big(\tfrac{1-q_2}{\alpha-q_2} \Big)^{1-q_2} T^{\alpha-q_2} \|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha+\beta)} \tfrac{T^{\alpha+\beta}}{\alpha+\beta} \Big[\tfrac{K_T}{\Gamma(\beta)} \Big(\tfrac{1-q_1}{\beta-q_1} \Big)^{1-q_1} T^{\beta-q_1} \|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)} \\
&+ \tfrac{K_T}{\Gamma(\alpha+\beta)} \Big(\tfrac{1-q_2}{\alpha-q_2} \Big)^{1-q_2} T^{\alpha-q_2} \|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \Big] \Big\} \|x-y\|,
\end{aligned}$$

and

$$\begin{aligned}
&\|{}^C D^\beta (Nx)(t) - {}^C D^\beta (Ny)(t)\| \\
\le& \|g(t,x_t) - g(t,y_t)\| + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s,x_s) - f(s,y_s)\| ds \\
&+ \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|Bu_x(s) - Bu_y(s)\| ds \\
\le& c_1(t) \|x-y\|_{\mathcal{D}} + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c_3(s) \|x_s - y_s\|_{\mathcal{D}} ds \\
&+ \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|B\| \|u_x(s) - u_y(s)\| ds \\
\le& c_1(t) K_T \|x-y\| + \tfrac{K_T}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} c_3(s) ds \|x-y\| \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\tfrac{K_T}{\Gamma(\beta)} \int_0^T (T-s)^{\beta-1} c_1(s) ds \\
&+ \tfrac{K_T}{\Gamma(\alpha+\beta)} \int_0^T (T-s)^{\alpha+\beta-1} c_3(s) ds] ds \|x-y\| \\
\le& \Big\{ c_1(t) + \tfrac{K_T}{\Gamma(\alpha)} \Big(\tfrac{1-q_2}{\alpha-q_2} \Big)^{1-q_2} T^{\alpha-q_2} \|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \\
&+ \tfrac{M_1 M_2}{\Gamma(\alpha)} \tfrac{T^\alpha}{\alpha} \Big[\tfrac{K_T}{\Gamma(\beta)} \Big(\tfrac{1-q_1}{\beta-q_1} \Big)^{1-q_1} T^{\beta-q_1} \|c_1\|_{L^{\frac{1}{q_1}}(J,R^n)} \\
&+ \tfrac{K_T}{\Gamma(\alpha+\beta)} \Big(\tfrac{1-q_2}{\alpha+\beta-q_2} \Big)^{1-q_2} T^{\alpha+\beta-q_2} \|c_3\|_{L^{\frac{1}{q_2}}(J,R^n)} \Big] \Big\} \|x-y\|, t \in J.
\end{aligned}$$

In all, by (H_2'), it follows that

$$\|Nx - Ny\|_\beta < \|x-y\|_\beta,$$

i.e. N is a contraction mapping operator and has a fixed point. Thus (1.4) is controllable on the interval J. □

Remark 3.1. Using the Banach contraction mapping principle, we can get the existence result of the solution of (1.3) similar to the proof of Theorem 3.1 without the control function u. So we omit it.

Theorem 3.3. *If the assumptions* (H_1)-(H_2) *hold, then problem* (1.3) *has a unique solution on* J.

Secondly, we consider the controllability of problem (1.4) in the case of finite delay.

Theorem 3.4. *Suppose that the assumptions* (H_1)-(H_3) *hold, then problem* (1.4) *is controllable on* J.

Proof. For any $x \in C^\beta([-\tau, T], R^n)$, define the control function $u_x(\cdot)$ and N as in Theorem 3.2. We will show that N is continuous and completely continuous.

1. N is continuous. Let $\{x_n\}$ be a sequence in $C^\beta([-\tau, T], R^n)$ such that $\lim_{n\to\infty} x_n = x$. Then

$$\sup\{\|(Fx_n)(t) - (Fx)(t)\|\} = \sup\{\|\phi(t) - \phi(t)\|\} = 0, t \in [-\tau, 0].$$

When $t \in J$, we have

$$\begin{aligned} \|(Nx_n)(t) - (Nx)(t)\| \leq & \tfrac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|g(s, x_{ns}) - g(s, x_s)\| ds \\ & + \tfrac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \|f(s, x_{ns}) - f(s, x_s)\| ds \\ & + \tfrac{1}{\Gamma(\alpha+\beta)} \int_0^t (t-s)^{\alpha+\beta-1} \|Bu_{x_n}(s) - Bu_x(s)\| ds \to 0, \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} & \|{}^C D^\beta (Nx_n)(t) - {}^C D^\beta (Nx)(t)\| \leq \|g(t, x_{nt}) - g(t, x_t)\| \\ & + \tfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} [\|f(s, x_{ns}) - f(s, x_s)\| + \|Bu_{x_n}(s) - Bu_x(s)\|] ds \to 0. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), by the continuity of f, g and Lebesgue dominated convergence theorem, we get

$$\lim_{n\to\infty} \sup\{\|(Nx_n)(t) - (Nx)(t)\|\} = 0,$$
$$\lim_{n\to\infty} \sup\{\|{}^C D^\beta (Nx_n)(t) - {}^C D^\beta (Nx)(t)\|\} = 0, t \in J.$$

In all

$$\lim_{n\to\infty} \|F(x_n) - Fx\|_\beta = 0,$$

i.e. N is continuous.

2. The operator N maps a bounded set $B_r = \{x \in C^\beta([-\tau, T], R^n) \mid \|x\|_\beta \leq r\}, r > 0$ to a bounded set $NB_r = \{y : \|y\|_\beta \leq L\}, L > 0$. We get easily

$$\|(Nx)(t)\| = \|\phi(t)\|, \ \|(Nx)\|_\beta = \|\phi\|_\beta = \|\phi\|_{\mathcal{D}} \leq r, t \in [-\tau, 0].$$

From (H_1') and (H_2'), similar to the proof of Theorem 3.2, we get Nx and ${}^C D^\beta (Nx)$ all are bounded.

3. N maps $x \in B_r \subset C^\beta([-\tau, T], R^n)$ into equi-continuous, i.e. as $\hat{t}_1 \to \hat{t}_2$, $\hat{t}_1, \hat{t}_2 \in [-\tau, 0]$,

$$(Nx)(\hat{t}_2) - (Nx)(\hat{t}_1) = \phi(\hat{t}_2) - \phi(\hat{t}_1) \to 0.$$

Choose $\hat{t}_1, \hat{t}_2 \in J$ with $\hat{t}_1 \le \hat{t}_2$. As $\hat{t}_1 \to \hat{t}_2$, we have

$$
\begin{aligned}
& \|(Nx)(\hat{t}_2) - (Nx)(\hat{t}_1)\| \\
\le\ & \tfrac{1}{\Gamma(\beta)} \int_0^{\hat{t}_1} [(\hat{t}_1 - s)^{\beta-1} - (\hat{t}_2 - s)^{\beta-1}] \|g(s, x_s)\| ds \\
& + \tfrac{1}{\Gamma(\beta)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\beta-1} \|g(s, x_s)\| ds + \tfrac{1}{\Gamma(\alpha+\beta)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\alpha+\beta-1} \|f(s, x_s)\| ds \\
& + \tfrac{1}{\Gamma(\alpha+\beta)} \int_{\hat{t}_1}^{\hat{t}_2} [(\hat{t}_2 - s)^{\alpha+\beta-1} - (\hat{t}_1 - s)^{\alpha+\beta-1}] \|f(s, x_s)\| ds \\
& + \tfrac{1}{\Gamma(\alpha+\beta)} \int_0^{\hat{t}_1} (\hat{t}_2 - s)^{\alpha+\beta-1} \|Bu_x(s)\| ds \\
& + \tfrac{1}{\Gamma(\alpha+\beta)} \int_{\hat{t}_1}^{\hat{t}_2} [(\hat{t}_2 - s)^{\alpha+\beta-1} - (\hat{t}_1 - s)^{\alpha+\beta-1}] \|Bu_x(s)\| ds \\
\le\ & [\tfrac{1}{\Gamma(\beta)} \int_0^{\hat{t}_1} [(\hat{t}_1 - s)^{\beta-1} - (\hat{t}_2 - s)^{\beta-1}] (c_1(s) \|x_s\|_{\mathcal{D}} + c_2) ds \\
& + \tfrac{1}{\Gamma(\beta)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\beta-1} (c_1(s) \|x_s\|_{\mathcal{D}} + c_2) ds \\
& + \tfrac{1}{\Gamma(\alpha+\beta)} \int_0^{\hat{t}_1} [(\hat{t}_2 - s)^{\alpha+\beta-1} - (\hat{t}_1 - s)^{\alpha+\beta-1}] (c_3(s) \|x_s\|_{\mathcal{D}} + c_4) ds \\
& + \tfrac{1}{\Gamma(\alpha+\beta)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\alpha+\beta-1} (c_3(s) \|x_s\|_{\mathcal{D}} + c_4) ds + \tfrac{M_1(\hat{t}_2^{\alpha+\beta} - \hat{t}_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)} \|u_x\|_{L^\infty(J, R^m)} \\
\le\ & \tfrac{(\hat{c}_1 \mathcal{A} + c_2)}{\Gamma(\beta)} \int_0^{\hat{t}_1} [(\hat{t}_1 - s)^{\beta-1} - (\hat{t}_2 - s)^{\beta-1}] ds + \tfrac{(\hat{c}_1 \mathcal{A} + c_2)}{\Gamma(\beta)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\beta-1} ds \\
& + \tfrac{(\hat{c}_3 \mathcal{A} + c_4)}{\Gamma(\alpha+\beta)} \int_0^{\hat{t}_1} [(\hat{t}_2 - s)^{\alpha+\beta-1} - (\hat{t}_1 - s)^{\alpha+\beta-1}] ds \\
& + \tfrac{(\hat{c}_3 \mathcal{A} + c_4)}{\Gamma(\alpha+\beta)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\alpha+\beta-1} ds + \tfrac{M_1(\hat{t}_2^{\alpha+\beta} - \hat{t}_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)} \|u_x\|_{L^\infty(J, R^m)} \\
\le\ & \tfrac{(\hat{c}_1 \mathcal{A} + c_2)}{\Gamma(\beta+1)} (\hat{t}_1^\beta - \hat{t}_2^\beta + (\hat{t}_2 - \hat{t}_1)^\beta) + \tfrac{(\hat{c}_1 \mathcal{A} + c_2)(\hat{t}_2 - \hat{t}_1)^\beta}{\Gamma(\beta+1)} \\
& + \tfrac{(\hat{c}_3 \mathcal{A} + c_4)}{\Gamma(\alpha+\beta+1)} (\hat{t}_2^{\alpha+\beta} - (\hat{t}_2 - \hat{t}_1)^{\alpha+\beta} + \hat{t}_1^{\alpha+\beta}) \\
& + \tfrac{(\hat{c}_3 \mathcal{A} + c_4)}{\Gamma(\alpha+\beta+1)} (\hat{t}_2 - \hat{t}_1)^{\alpha+\beta} + \tfrac{M_1(\hat{t}_2^{\alpha+\beta} - \hat{t}_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)} \|u_x\|_{L^\infty(J, R^m)} \to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
& \|{}^C D^\beta [(Nx)(\hat{t}_2) - (Nx)(\hat{t}_1)]\| \\
\le\ & \|g(\hat{t}_2, x_{\hat{t}_2}) - g(\hat{t}_2, x_{\hat{t}_1})\| + \|g(\hat{t}_2, x_{\hat{t}_1}) - g(\hat{t}_1, x_{\hat{t}_1})\| \\
& + \tfrac{1}{\Gamma(\alpha)} \int_0^{\hat{t}_1} [(\hat{t}_2 - s)^{\alpha-1} - (\hat{t}_1 - s)^{\alpha-1}] \|f(s, x_s)\| ds \\
& + \tfrac{1}{\Gamma(\alpha)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\alpha-1} \|f(s, x_s)\| ds + \tfrac{1}{\Gamma(\alpha)} \int_0^{\hat{t}_1} (\hat{t}_2 - s)^{\alpha-1} \|Bu_x(s)\| ds \\
& + \tfrac{1}{\Gamma(\alpha)} \int_{\hat{t}_1}^{\hat{t}_2} [(\hat{t}_1 - s)^{\alpha-1} - (\hat{t}_2 - s)^{\alpha-1}] \|Bu_x(s)\| ds \\
\le\ & \|g(\hat{t}_2, x_{\hat{t}_2}) - g(\hat{t}_2, x_{\hat{t}_1})\| + \|g(\hat{t}_2, x_{\hat{t}_1}) - g(\hat{t}_1, x_{\hat{t}_1})\| \\
& + \tfrac{1}{\Gamma(\alpha)} \int_0^{\hat{t}_1} [(\hat{t}_1 - s)^{\alpha-1} - (\hat{t}_2 - s)^{\alpha-1}] \|f(s, x_s)\| ds \\
& + \tfrac{1}{\Gamma(\alpha)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\alpha-1} \|f(s, x_s)\| ds + \tfrac{M_1(\hat{t}_2^\alpha - \hat{t}_1^\alpha)}{\Gamma(\alpha+1)} \|u_x\|_{L^\infty(J, \mathrm{R}^n)} \\
\le\ & \|g(\hat{t}_2, x_{\hat{t}_2}) - g(\hat{t}_2, x_{\hat{t}_1})\| + \|g(\hat{t}_2, x_{\hat{t}_1}) - g(\hat{t}_1, x_{\hat{t}_1})\| \\
& + \tfrac{1}{\Gamma(\alpha)} \int_0^{\hat{t}_1} [(\hat{t}_1 - s)^{\alpha-1} - (\hat{t}_2 - s)^{\alpha-1}] (c_3(s) \|x_s\|_{\mathcal{D}} + c_4) ds \\
& + \tfrac{1}{\Gamma(\alpha)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\alpha-1} (c_3(s) \|x_s\|_{\mathcal{D}} + c_4) ds + \tfrac{M_1(\hat{t}_2^\alpha - \hat{t}_1^\alpha)}{\Gamma(\alpha+1)} \|u_x\|_{L^\infty(J, \mathrm{R}^n)} \\
\le\ & \|g(\hat{t}_2, x_{\hat{t}_2}) - g(\hat{t}_2, x_{\hat{t}_1})\| + \|g(\hat{t}_2, x_{\hat{t}_1}) - g(\hat{t}_1, x_{\hat{t}_1})\| + \tfrac{\mathcal{A}}{\Gamma(\alpha)} \int_{\hat{t}_1}^{\hat{t}_2} (\hat{t}_2 - s)^{\alpha-1} ds \\
& + \tfrac{\mathcal{A}}{\Gamma(\alpha)} \int_0^{\hat{t}_1} [(\hat{t}_1 - s)^{\alpha-1} - (\hat{t}_2 - s)^{\alpha-1}] ds + \tfrac{M_1(\hat{t}_2^{\alpha+\beta} - \hat{t}_1^{\alpha+\beta})}{\Gamma(\alpha+\beta+1)} \|u_x\|_{L^\infty(J, \mathrm{R}^n)} \to 0.
\end{aligned}
$$

So we get

$$\lim_{\hat{t}_2\to\hat{t}_1}\sup\{\|(Nx)(\hat{t}_2)-(Nx)(\hat{t}_1)\|\}=0,$$
$$\lim_{\hat{t}_2\to\hat{t}_1}\sup\{\|^C D^\beta[(Nx)(\hat{t}_2)-(Nx)(\hat{t}_1)]\|\}=0, t\in J.$$

In all, we can get

$$\lim_{\hat{t}_2\to\hat{t}_1}\|(Nx)(\hat{t}_2)-(Nx)(\hat{t}_1)\|_\beta=0, \hat{t}_1,\hat{t}_2\in[-\tau,T].$$

By steps 1-3 and Ascoli-Arzela theorem, it follows that the operator $N: C^\beta([-\tau,T], R^n)\to C^\beta([-\tau,T],R^n)$ is continuous and completely continuous.

4. The set $K=\{x\in C^\beta([-\tau,T],R^n): x=\lambda Nx, 0<\lambda<1\}$ is bounded. For any $x\in K$, $x=\lambda Nx, 0<\lambda<1$. So we get

$$\begin{aligned} x(t)=\lambda\Big[&\phi(0)+[\mu-g(0,\phi_0)]\tfrac{t^\beta}{\Gamma(\beta+1)}+\tfrac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}g(s,x_s)ds\\ &+\tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}[f(s,x_s)+Bu_x(s)]ds\Big], t\in J,\end{aligned} \tag{3.13}$$

and

$$^C D^\beta x(t)=\lambda[g(t,x_t)+\tfrac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,x_s)+Bu_x(s)]ds], t\in J. \tag{3.14}$$

By (H_1'), similar to the proof of the first step of theorem 3.2, we get

$$\begin{aligned}\|x(t)\|\le&(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}}+c_2)\tfrac{t^\beta}{\Gamma(\beta+1)}\\
&+\tfrac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}[c_1(s)\|x_s\|_{\mathcal{D}}+c_2]ds\\
&+\tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}[c_3(s)\|x_s\|_{\mathcal{D}}+c_4]ds\\
&+\tfrac{1}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\|B\|\|u_x(s)\|ds\\
\le&(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}}+c_2)\tfrac{t^\beta}{\Gamma(\beta+1)}\\
&+\tfrac{1}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}[c_1(s)(K(s)\sup\{\|x(\theta)\|:\theta\in[0,s]\}\\
&+M(s)\|\phi(0)\|_{\mathcal{D}})+c_2]ds\\
&+\int_0^t(t-s)^{\alpha+\beta-1}[c_3(s)(K(s)\sup\{\|x(\theta)\|:\theta\in[0,s]\}\\
&+M(s)\|\phi(0)\|_{\mathcal{D}})+c_4]ds+\tfrac{M_1T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}\|u_x\|_{L^\infty(J,R^n)}\\
\le&(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}}+c_2)\tfrac{t^\beta}{\Gamma(\beta+1)}+\tfrac{\hat{c}_1M_T\|\phi(0)\|_{\mathcal{D}}T^\beta+c_2}{\Gamma(\beta+1)}\\
&+\tfrac{\hat{c}_1K_T}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds+\tfrac{\hat{c}_3M_T\|\phi(0)\|_{\mathcal{D}}T^{(\alpha+\beta)}+c_4}{\Gamma(\alpha+\beta+1)}\\
&+\tfrac{\hat{c}_3K_T}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds\\
&+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}\Big\{\|\phi(0)\|_{\mathcal{D}}+(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)}\\
&+\tfrac{1}{\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\|g(s,x_s)\|ds+\tfrac{1}{\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\|f(s,x_s)\|ds\Big\}\\
\le&\tfrac{(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}}+c_2)t^\beta}{\Gamma(\beta+1)}+\tfrac{\hat{c}_1M_T\|\phi(0)\|_{\mathcal{D}}T^\beta+c_2}{\Gamma(\beta+1)}+\tfrac{\hat{c}_3M_T\|\phi(0)\|_{\mathcal{D}}T^{(\alpha+\beta)}+c_4}{\Gamma(\alpha+\beta+1)}\\
&+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}[\|\phi(0)\|+(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)}]\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}\left\{\tfrac{\int_0^T(T-s)^{\beta-1}[c_1(s)\|x_s\|_{\mathcal{D}}+c_2]ds}{\Gamma(\beta)}+\tfrac{\int_0^T(T-s)^{\alpha+\beta-1}[c_3(s)\|x_s\|_{\mathcal{D}}+c_4]ds}{\Gamma(\alpha+\beta)}\right\}\\
&+\tfrac{\hat{c}_1K_T}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds\\
&+\tfrac{\hat{c}_3K_T}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds\\
\le{}&(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}}+c_2)\tfrac{t^\beta}{\Gamma(\beta+1)}+\tfrac{\hat{c}_1M_T\|\phi(0)\|_{\mathcal{D}}T^\beta+c_2}{\Gamma(\beta+1)}\\
&+\tfrac{\hat{c}_3M_T\|\phi(0)\|_{\mathcal{D}}T^{(\alpha+\beta)}+c_4}{\Gamma(\alpha+\beta+1)}+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}[\|\phi(0)\|+(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)}]\\
&+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}[\tfrac{(\hat{c}_1M_T\|\phi(0)\|_{\mathcal{D}}+c_2)T^\beta}{\Gamma(\beta+1)}+\tfrac{(\hat{c}_3M_T\|\phi(0)\|_{\mathcal{D}}+c_4)T^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}]\\
&+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}[\tfrac{\hat{c}_1K_T}{\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds\\
&+\tfrac{\hat{c}_3K_T}{\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds]\\
&+\tfrac{\hat{c}_1K_T}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds\\
&+\tfrac{\hat{c}_3K_T}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds,t\in J.
\end{aligned}
\tag{3.15}
$$

Let

$$
\nu(t)=\sup\{\|x(s)\|:s\in[0,t]\},t\in J. \tag{3.16}
$$

Then we get

$$
\begin{aligned}
\nu(t)\le{}&(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}}+c_2)\tfrac{t^\beta}{\Gamma(\beta+1)}\\
&+\tfrac{\hat{c}_1M_T\|\phi(0)\|_{\mathcal{D}}T^\beta+c_2}{\Gamma(\beta+1)}+\tfrac{\hat{c}_3M_T\|\phi(0)\|_{\mathcal{D}}T^{(\alpha+\beta)}+c_4}{\Gamma(\alpha+\beta+1)}\\
&+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}[\|\phi(0)\|+(\|\mu\|+c_1(0)\|\phi(0)\|_{\mathcal{D}})\tfrac{T^\beta}{\Gamma(\beta+1)}]\\
&+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}\Big[\tfrac{(\hat{c}_3M_T\|\phi(0)\|_{\mathcal{D}}+c_4)T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}\\
&+\tfrac{(\hat{c}_1M_T\|\phi(0)\|_{\mathcal{D}}+c_2)T^\beta}{\Gamma(\beta+1)}\Big]+\tfrac{M_1M_2T^{(\alpha+\beta)}}{\Gamma(\alpha+\beta+1)}\Big[\tfrac{\hat{c}_1K_T}{\Gamma(\beta)}\int_0^T(T-s)^{\beta-1}\nu(s)ds\\
&+\tfrac{\hat{c}_3K_T}{\Gamma(\alpha+\beta)}\int_0^T(T-s)^{\alpha+\beta-1}\nu(s)ds\Big]+\tfrac{\hat{c}_1K_T}{\Gamma(\beta)}\int_0^t(t-s)^{\beta-1}\nu(s)ds\\
&+\tfrac{\hat{c}_3K_T}{\Gamma(\alpha+\beta)}\int_0^t(t-s)^{\alpha+\beta-1}\nu(s)ds,t\in J,
\end{aligned}
\tag{3.17}
$$

By Lemma 2.1 (Lemma 1 [69]), there exists a positive constant $\hat{L}$ such that $t\in J,\|x(t)\|\le\nu(t)\le\hat{L}$. Thus

$$
\begin{aligned}
\|^CD^\beta x(t)\|&=\|g(t,x_t)+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[f(s,x_s)+Bu_x(s)]ds\|\\
&\le c_1(t)\|x_t\|_{\mathcal{D}}+c_2+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[\|f(s,x_s)\|+\|Bu_x(s)\|]ds\\
&\le\hat{c}_1\|x_t\|_{\mathcal{D}}+c_2+\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[c_3(s)\|x_s\|_{\mathcal{D}}+c_4]ds\\
&\quad+\frac{M_1\|u_x(t)\|_{L^\infty(J,R^m)}T^\alpha}{\Gamma(\alpha+1)}\\
&\le\hat{c}_1K_T\sup\{\|x(\theta)\|:\theta\in[0,t]\}+\hat{c}_1M_T\|\phi(0)\|_{\mathcal{D}}+c_2\\
&\quad+\frac{\hat{c}_3K_T}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sup\{\|x(\theta)\|:\theta\in[0,s]\}ds\\
&\quad+\frac{(\hat{c}_3M_T+c_4)T\alpha}{\Gamma(\alpha+1)}+\frac{M\|u_x\|_{L^\infty(J,R^m)}T^\alpha}{\Gamma(\alpha+1)}=:L',t\in J,
\end{aligned}
$$

so $\|x(t)\| \leq \hat{L}, \|^C D^\beta x(t)\| \leq \hat{L}', t \in J$, and $\|x\|_\beta \leq \|\phi\|_{\mathcal{D}} + \hat{L} + \hat{L}'$, i.e K is a bounded set.

By using the Schaefer fixed point theorem, the operator N has a fixed point, which is the solution of problem (1.4). So (1.4) is controllable on the interval J. □

Remark 3.2. If $Bu(t) = 0$, as the proof of Theorem 3.4, we can get the existence of solution of problem (1.3) in the case of finite delay.

Example 3.1. Let $\mathcal{D} = \{y \in C(I, R) : \lim_{s \to -\infty} \exp(\gamma s) y(s) \text{ exists in } R, \gamma > 0\}$ with the norm $\|y\|_{\mathcal{D}} = \sup_{s \leq 0}\{\exp(\gamma s)\|y(s)\|\}$. Then, $\mathcal{D}$ satisfies the assumptions in [70] with $K(t) = M(t) = 1$.

Consider the following neutral fractional functional differential equations with two Caputo fractional derivaives as a special case of Eq.(1.4) with $n = 1$,

$$\begin{cases} {}^C D^{0.8}\left[{}^C D^{0.6} x(t) - \dfrac{\int_{-\infty}^{-t} \exp(\gamma\theta)\exp(t+\theta)x(t+\theta)d\theta}{100(1+t)(2+t)} \right] \\ = \dfrac{\int_{-\infty}^{-t} \exp(\gamma\theta)\exp(t+\theta)x(t+\theta)d\theta}{60(1+t^2)(2+t^2)} + Bu(t), t \in [0,1] = J, \\ x(t) = \phi(t) \in \mathcal{D},\ t \in (-\infty, 0] = I, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} g(t, x_t) &= \frac{\int_{-\infty}^{-t} \exp(\gamma\theta)\exp(t+\theta)x(t+\theta)d\theta}{100(1+t)(2+t)}, \\ f(t, x_t) &= \frac{\int_{-\infty}^{-t} \exp(\gamma\theta)\exp(t+\theta)x(t+\theta)d\theta}{60(1+t^2)(2+t^2)}, t \in [0,1] = J, x_t \in \mathcal{D}, \\ Bu(t) &= u(t), B = I. \end{aligned}$$

Then, we get

$$\begin{aligned} \|g(t,x) - g(t,y)\| &\leq \frac{\int_{-\infty}^{-t} \exp(\gamma\theta)\exp(t+\theta)\|x(t+\theta) - y(t+\theta)\|d\theta}{100(1+t)(2+t)} \\ &\leq \frac{\|x-y\|_{\mathcal{D}}}{100\gamma(1+t)(2+t)} \leq \frac{\|x-y\|_{\mathcal{D}}}{100\gamma}, \\ \|f(t,x) - f(t,y)\| &\leq \frac{\int_{-\infty}^{-t} \exp(\gamma\theta)\exp(t+\theta)\|x(t+\theta) - y(t+\theta)\|d\theta}{60(1+t^2)(2+t^2)} \\ &\leq \frac{\|x-y\|_{\mathcal{D}}}{60\gamma(1+t^2)(2+t^2)} \leq \frac{\|x-y\|_{\mathcal{D}}}{60\gamma}, \end{aligned}$$

where $c_1(t) = \frac{1}{100\gamma} \in L^2(J, R), c_3(t) = \frac{1}{60\gamma} \in L^2(J, R), i = 1, 3, q_1 = 0.5 \in [0, 0.6)$ and $q_3 = 0.5 \in [0, 0.8)$. Then, the conditions $(H_1), (H_2)$ and (H_3) in Theorem 3.2 are satisfied. So, (4.1) is controllable on the interval J by Theorem 3.2. Also, the existence of a unique solution of (4.1) without the control function u is obtained by Theorem 3.2.

Let $c_1(t) = \dfrac{1}{100\gamma(1+t)(2+t)}, c_3(t) = \dfrac{1}{60\gamma(1+t^2)(2+t^2)}$. Then $(H_1'), (H_2')$ and (H_3') in Theorem 3.3 are satisfied. So, (4.1) is controllable on the interval J by Theorem 3.3.

4. Conclusions

In this paper, the controllability of a class of neutral fractional functional differential equations with two derivatives of Caputo type on a bounded interval under Lipschitz conditions is considered. The methods in this paper are effective. We will consider the existence results of solutions and controllability results under non-Lipschitz conditions for fractional differential equations(including R-L type) in the future.

Acknowledgements

All authors are grateful to the anonymous referees for their helpful clarification, which contributed to the progress in the presentation of the research paper.

References

[1] R. Sakthivel, N. I. Mahmudov, J. H. Kim, *Approximate controllability of nonlinear impulsive differential systems*, Rep. Math. Phys, 2007, 60(1), 85-96.

[2] X. L. Fu, K. D. Mei, *Approximate controllability of semilinear partial functional differential systems*, Journal of Dynamical and Control Systems, 2009, 15(3): 425-443.

[3] Z. M. Yan, *Controllability of fractional-order partial neutral functional integrodifferential inclusions with infinite delay*, Journal of the Franklin Institute, 2011, 348(8), 2156-2173.

[4] A. Debbouche, D. Baleanu, *Controllability of fractional evolution nonlocal impulsive quasilinear delay integro-differential systems*, Comput. Math. Appl. 2011,62(3), 1442-1450.

[5] K. Balachandran, J. Kokila, *On the controllability of fractional dynamical systems*, Int. J. Appl. Math. Comput. Sci. 2012, 22(3), 523-531.

[6] Z. M. Yan, *Approximate controllability of fractional neutral integro-differential inclusions with state-dependent delay in Hilbert spaces*, IMA Journal of Mathematical Control and Information, 2013, 30(4), 443-462.

[7] K. Balachandran, V. Govindaraj, *Numerical controllability of fractional dynamical systems*, Optimization, 2014, 63(8), 1267-1279.

[8] Z. X. Tai, S. X. Lun, *On controllability of fractional impulsive neutral infinite delay evolution integro-differential systems in Banach spaces*, Applied Mathematics Letters, 2012, 25, 104-110.

[9] R. Sakthivel, N. I. Mahmudov, J. J. Nieto, *Controllability for a class of fractional-order neutral evolution control systems*, Applied Mathematics and Computation, 2012, 218, 10334-10340.

[10] X. F. Zhou, J. Wei, L. G. Hu, *Controllability of a fractional linear time-invariant neutral dynamical system*, Applied Math Letter, 2013, 26(4), 418-424.

[11] F. Mokkedem, X. L. Fu, *Approximate controllability of semi-linear neutral integro-differential systems with finite delay*, Applied Mathematics and Computation, 2014, 242, 202-215.

[12] V. Vijayakumar, A. Selvakumar, R. Murugesu, *Controllability for a class of fractional neutral integro-differential equations with unbounded delay*, Applied Mathematics and Computation, 2024, 232, 303-312.

[13] P. Balasubramaniam, P. Tamilalagan, *Approximate controllability of a class of fractional neutral stochastic integro-differential inclusions with infinite delay by using Mainardi's function*, Applied Mathematics and Computation, 2015, 256, 232-246.

[14] Z. M. Yan, F. X. Lu, *The optimal control of a new class of impulsive stochastic neutral evolution integro-differential equations with infinite delay*, International Journal of Control, 2016, 89(8), 1592-1612.

[15] K. Jeet, D. Bahuguna, *Approximate Controllability of Nonlocal Neutral Fractional Integro-Differential Equations with Finite Delay*, J Dyn Control Syst, 2016, 22,485-504.

[16] K. Balachandran, S. Divya, R. L. Germá, J. J. Trujillo, *Relative controllability of nonlinear neutral fractional integro-differential systems with distributed delays in control*, Math. Methods Appl. Sci.,2016, 39(2),214-224.

[17] H. P. Ma, L. Biu, *Exact controllability and continuos dependent of fractional neutral integro-differential equations with state-dependent delay*, Acta Mathematica Scientia, 2017,37B(1), 235-258.

[18] Y. Zhou, V. Vijayakumar, R. Murugesu, *Controllability results for fractional order neutral functional differential inclusions with infinite delay*, Fixed point theory, 2017, 18(2),773-798.

[19] B. S. Vadivoo, R. Ramachandran, J. Cao, et al, *Controllability analysis of nonlinear neutral-type fractional-order differential systems with state delay and impulsive effects*, International Journal of Control, Automation and Systems, 2018, 16,659-669.

[20] M. Li, A. Debbouche, J. R. Wang, *Relative controllability in fractional differential equations with pure delay*, Math. Methods Appl. Sci., 2018, 41(18),8906-8914.

[21] A. Kumar, M. Malik, R. Sakthivel, *Controllability of the second-order nonlinear differential equations with non-instantaneous impulses*, Dyn. Control Syst.,2018, 24(2),325-342.

[22] B. G. Priya and P. Muthukumar, *Controllability and minimum energy of fractional neutral delay syste control of fractional neutral delay system*, IFAC Papers, On Line 51-1 (2018) 592-597.

[23] V. Govindaraj, K. Balachandran, R. K. George, *Numerical approach for the controllability of composite fractional dynamical systems*, J. Appl. Nonlinear Dyn.,2018, 7(1), 59-72.

[24] M. Muslim, A. Kumar, *Controllability of fractional differential equation of order $\alpha \in (1,2]$ with noninstantaneous impulses*, Asian J. Control, 2018,20(2),935-942.

[25] S. Liu, A Debbouche, J. R. Wang, *ILC method for solving approximate controllability of fractional differential equations with noninstantaneous impulses*, J. Comput. Appl. Math.,2018, 339,343-355.

[26] C. Ravichandran, N. Valliammal, J. J. Nieto, *New results on exact controllability of a class of fractional neutral integro-differential systems with state-dependent delay in Banach spaces*, Journal of the Franklin Institute, 2019, 356(3), 1535-1565.

[27] V. Vijayakunnar, *Approximate controllability results for non-densely defined fractional neutral differential inclusions with Hille-Yosida operators*, International Journal of Control, 2019, 92(9),2210-2222.

[28] J. R. Wang, A. G. Ibrahim, M. Feckan, Y. Zhou, *Controllability of fractional non-instantaneous impulsive differential inclusions without compactness*, IMA J. Math. Control Inform., 2019, 36(2),443-460.

[29] Y. Huang, Z. Liu, *Controllability of nonlinear impulsive integro-differential fractional time-invariant systems*, J. Integral Equations Appl.,2019, 31(3),329-341.

[30] M. Malik, R. Dhayal, S. Abbas, A. Kumar, *Controllability of non-autonomous nonlinear differential system with non-instantaneous impulses*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2019, 113(1), 103-118.

[31] M. Malik, R. Dhayal, S. Abbas, *Exact controllability of a retarded fractional differential equation with non-instantaneous impulses*, Dyn. Contin. Discrete B, 2019, 26, 53-69.

[32] K. Jeet, N. Sukavanam, *Approximate controllability of nonlocal and impulsive neutral integro-differential equations using the resolvent operator theory and an approximating technique*, Applied Mathematics and Computation,2020, 364,124690.

[33] K. Kavitha, V. Vijayakumar, R. Udhayakumar, *Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness*, Chaos, Solitons & Fractals, 2020, 139, 110035.

[34] K. Kavitha, V. Vijayakumar, A. Shukla, et al, *Results on approximate controllability of Sobolev-type fractional neutral differential inclusions of Clarke subdifferential type*, Chaos, Solitons & Fractals, 2021, 151: 111264.

[35] K. Kavitha , V. Vijayakumar, K. S. Nisar, *A note on approximate controllability of the Hilfer fractional neutral differential inclusions with infinite delay*, Mathematical Methods in the Applied Sciences, 2021, 44(6), 4428-4447.

[36] K. S. Nisar, V. Vijayakumar, *Results concerning to approximate controllability of non-densely defined Sobolev-type Hilfer fractional neutral delay differential system*, Mathematical Methods in the Applied Sciences, 2021, 44(17),13615-13632.

[37] K. Jothimani, K. Kaliraj, S. K. Panda, et al, *Results on controllability of non-densely characterized neutral fractional delay differential system*, Evolution Equations & Control Theory, 2021, 10(3),619-631.

[38] C. Dineshkumar, R. Udhayakumar, K. S. Nisar, *A discussion on the approximate controllability of Hilfer fractional neutral stochastic integro-differential systems*, Chaos Solitons & Fractals, 2021, 142.

[39] K. Kavitha, V. Vijayakumar, *Discussion on the controllability results for fractional neutral impulsive Atangana-Baleanu delay integro-differential systems*, Mathematical Methods in the Applied Sciences. Oct 2021.

[40] K. Kavitha, V. Vijayakumar, R. Udhayakumar, *Results on controllability of Hilfer fractional neutral differential equations with infinite delay via measures of noncompactness*, Chaos, Solitons & Fractals, 2020, 139: 110035.

[41] P. Balasubramaniam, *Controllability of semilinear noninstantaneous impulsive ABC neutral fractional differential equations*, Chaos, Solitons and Fractals, 2021, 152,111276.

[42] K. S. Nisar, K. Jothimani, K. Kaliraj and C. Ravichandran, *An analysis of controllability results for nonlinear Hilfer neutral fractional derivatives with non-dense domain*, Chaos, Solitons and Fractals,2021, 146, 110915.

[43] V. Kumar, M. Malik, A. Debbouche, *Total controllability of neutral fractional differential equation with non-instantaneous impulsive effects*, Journal of Computational and Applied Mathematics, 2021, 383, 113158.

[44] P. Balasubramaniam, *Controllability of semilinear noninstantaneous impulsive ABC neutral fractional differential equations*, Chaos, Solitons and Fractals, 2021, 152, 111276.

[45] K. Kavitha, V. Vijayakumar, C. Ravichandran, *Results on controllability of Hilfer fractional differential equations with infinite delay via measures of noncompactness*, Asian Journal of Control, 2022, 24(3),1406-1415.

[46] C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, et al, *A note concerning to approximate controllability of Atangana-Baleanu fractional neutral stochastic systems with infinite delay*, Chaos, Solitons & Fractals, 2022, 157: 111916.

[47] H. Ergören, *Impulsive functional delay differential inclusions of fractional order at variable times*, Advances in Difference Equations, 2016,37.
DOI: https://doi.org/10.1186/s13662-016-0770-1.

[48] F. F. Du, J. G. Lu, *Finite-time stability of neutral fractional order time delay systems with Lipschitz nonlinearities*, Appl Math Comput, 2020, 375,125079.

[49] S. Vivek, V. Vijayakumar, *A note concerning to optimal feedback control for Caputo fractional neutral stochastic evolution systems*, Qual. Theory Dyn. Syst., 2023, 22, 155.
DOI: https://doi.org/10.1007/s12346-023-00855-3.

[50] S. Vivek, V. Vijayakumar, *New discussion on optimal feedback control for Caputo fractional neutral evolution systems governed by hemivariational inequalities*, Mathematical Methods in the Applied Sciences, 2024, 47(6), 3903-3920.

[51] Z. S. Aghayan, A. Alfi, J. T. Machado, *Robust stability analysis of uncertain fractional order neutral-type delay nonlinear systems with actuator saturation*, Appl Math Model 2021, 90,1035-1048.

[52] H. T. Tuan, H. D. Thai, G. Roberto, *An analysis of solutions to fractional neutral differential equations with delay*, Commun Nonlinear Sci Numer Simulat, 2021, 100, 105854.

[53] J. Ren, C. B. Zhai, *Stability analysis of generalized neutral fractional differential systems with time delays*, Applied Mathematics Letters. 2021, 116,106987.

[54] T. Ismail, N. M. Huseynov, *Analysis of positive fractional-order neutral time-delay systems*, Journal of the Franklin Institute, 2022, 359,294-330.

[55] Q. L. Han, *Stability of linear neutral systems with linear fractional norm-bounded uncertainty*, In: Proceedings of the American Control Conference, New York IEEE Press, 2005, 2827-2832.

[56] L. Hong, S. M. Zhong, H. B. Li, *Asymptotic stability analysis of fractional-order neutral systems with time delay*, Adv. Differ. Equ. 2015, 2015, 325-335.

[57] S. Liu, X. Wu, Y. J. Zhang, R. Yang, *Asymptotical stability of Riemann-Liouville fractional neutral systems*, Appl. Math. Lett. 2017, 69, 168-173.

[58] K. Kavitha, V. Vijayakumar, R. Udhayakumar, et al, *Results on the existence of Hilfer fractional neutral evolution equations with infinite delay via measures of noncompactness*, Mathematical Methods in the Applied Sciences, 2021, 44(2),1438-1455.

[59] N. I. Mahmudov, R. Murugesu, C. Ravichandran, et al, *Approximate controllability results for fractional semilinear integro-differential inclusions in Hilbert spaces*, Results Math, 2017, 71, 45-61.

[60] S. Vivek, V. Vijayakumar, *An analysis on the approximate controllability of neutral functional hemivariational inequalities with impulses*, Optimization, 2023, 1-24.
DOI: https://doi.org/10.1080/02331934.2023.2239851.

[61] J. Q. Ding, L. F. Ma, *Existence and uniqueness of solutions of initial value problems for nonlinear fractional differential equations*, Appl. Math. Letters, 2010, 23, 675-680.

[62] F. L. Chen, Y. Zhou, *Attractivity of fractional functional differential equations*, Computers and Mathematics with Applications, 2011, 312, 1359-1369.

[63] F. L Chen, J. J. Neito, Y. Zhou, *Global attractivity for nonlinear fractional differential equations*, Nonlinear Analysis: RWA, 13(2012) 287-298.

[64] G. T. Wang, S. Y. Liu, L. H. Zhang, *Neutral fractional integro-differential equation with nonlinear term depending on lower order derivative*, Journal of Computational and Applied Mathematics, 2014, 260, 167-172.

[65] J. Lossada, J. J. Nieto, E. Pourhadi, *On the attractivity of solutions for a class of multi-term fractional functional differential equations*, J. Comput. and Appl. Mathematics, 2017, 312, 2-12.

[66] O. Brandibur, E. Kaslik, *Stability analysis of multi-term fractional-differential equations with three fractional derivatives*, J. Math. Anal. Appl., 2021, 495, 124751.

[67] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Mathematics Studies, vol. 24, Elsevier Science B. V., Amsterdam, 2006.

[68] K. S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, Wiley, New York,1993.

[69] Y. Jalilian, *Fractional integral inequalities and their applications to fractional differential equations*, Acta Math. Sci. Ser. B(Eng. Ed.), 2016, 36, 1317-1330.

[70] Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Infinite Delay*, Berlin/Heidelberg: Springer, 1991.