Journal of Nonlinear Modeling and Analysis
Volume 6, Number 4, December 2024, 970–983

http://jnma-online.com
DOI:10.12150/jnma.2024.970

Bright-Dark Solitons, Kink Wave and Singular Periodic Wave Solutions for the Lonngren-Wave Equation

Ben Yang[1,†], Yunjia Song[1] and Xinxue Zhang[1]

Abstract In this article, exact solutions of the Lonngren-wave equation are investigated. Firstly, the equation is transformed into an ordinary differential equation by traveling wave transformation. Based on the homogeneous balance method, bright-solitons and singular periodic wave solutions of the equation are derived by applying the simple function expansion method and the Riccati equation method. Applying the $Exp(-\varphi(\varsigma))$ expansion method, we construct dark-solitons and kink wave solutions of the equation. Moreover, the 3-D, 2-D and density plots are drawn by choosing the appropriate parameters so that the properties of the solutions can be better studied. According to the Figures, the analysis of the dynamical behavior of the solutions is provided. This article enriches the diversity of the solutions of the equation.

Keywords Lonngren-wave equation, soliton solutions, kink wave solutions, singular periodic wave solutions

MSC(2010) 35A08, 35C08, 35Q51.

1. Introduction

Researchers have devoted more and more attention to the study of nonlinear partial differential equations (NLPDEs), which have appeared in various fields, such as physical engineering and biomedical science [1]. Hence, finding the solutions of the equations is significant. There are many methods for solving NLPDEs, for instance, the Jacobi elliptic function method [2], the inverse (G'/G)-expansion method [3], the $Exp(-\varphi(\varsigma))$ expansion method [4,5], the improved F-expansion method [6], the sine-Gordon expansion approach [7], the Lie symmetry method [8–10], the Riccati equation method [11, 12], the extended simple equation method [13, 14], the homogeneous balance method [15,16], the Hirota bilinear transformation [17,18], the modified tanh method [19,20], the generalized unified method [21,22], etc [23,24].

Consider the following Lonngren-wave equation [25]

$$\left(v_{xx} - \alpha v + \beta v^2\right)_{tt} + v_{xx} = 0, \tag{1.1}$$

in which α and β are real constants. Many scholars have studied the properties of the Lonngren-wave equation. Eq. (1.1) describes the propagation of the electrical signal in the tunnel diode [26]. In [27], Durur and Hülya acquired the traveling

†the corresponding author.
Email address:yangben09@163.com(B. Yang)
[1]School of Mathematical Sciences, Liaocheng University, Liaocheng, Shandong 252000, China

wave solutions of Eq. (1.1) by applying the generalized exponential rational function method, and concluded that the velocity is an important factor affecting the wave diffraction by parameter assignment. Duran [28] investigated the effect of conductivity on the electrical signal propagation and distribution by taking values to the parameters in the solutions. Baskonus et al. [29] constructed the hyperbolic function solutions of Eq. (1.1) by the sine-Gordon expansion method. Yokuş [30] succeeded in deriving the soliton solutions by the auxiliary equation method, and investigated the effect of bright and dark-solitons on the charge distribution. Barman et al. [31] employed the generalized Kudryashov method to yield the wave structure under different parameters.

The equation can be used to explain the transmission of electrical signals in semiconducting materials and the storage of energy in charged circuits, which has great practical significance. In addition, the exact solutions of the equation can help explain the intrinsic motivation, such as the problem of the dispersion of electrical signals, especially in semiconductors. In this paper, three methods are used to obtain the corresponding bright-solitons, dark-solitons, and kink wave solutions in [31], but with different functional expressions. Furthermore, the singular periodic wave solutions of Eq.(1.1) are derived which are of great interest.

This article is organized into the following sections. In Section 2, Eq. (1.1) is converted into an ODE by the traveling wave transformation. In Section 3, bright-solitons solutions are yielded by utilizing the simple equation expansion method. In Section 4, based on the $Exp(-\varphi(\varsigma))$ expansion method, dark-solitons, singular periodic wave and kink wave solutions are constructed. In Section 5, applying the Riccati equation method, singular periodic wave and bright-solitons solutions of Eq. (1.1) are derived. A brief discussion of the obtained graphs is given in Section 6. Section 7 gives the remark and comparisons. Finally, the results of the study are summarized in Section 8.

2. Preliminary

The key steps of the extended simple equation method, the $Exp(-\varphi(\varsigma))$ expansion method and the Riccati equation method are given. Applying the above three methods, we obtain bright-dark solitons, singular periodic wave and kink wave solutions.

Assume that the NLPDE takes the following form

$$\Psi\left(v, v_x, v_t, v_{xx}, v_{tt}, \cdots\right) = 0, \tag{2.1}$$

in which Ψ is a polynomial function of $v = v(x,t)$ and its partial derivatives.

Consider the following traveling wave transformation

$$v\left(x,t\right) = V\left(\varsigma\right), \varsigma = kx - wt, \tag{2.2}$$

in which k, w are constants. Putting (2.2) into Eq. (2.1), we have

$$\Phi\left(V, V', V'', \cdots\right) = 0, \tag{2.3}$$

where the superscript is denoted as the derivative with respect to ς.

Eq. (1.1) can be rewritten as the following ODE

$$w^2k^2V^{(4)} - \alpha w^2V'' + 2\beta w^2V'' + 2\beta w^2VV'' + k^2V'' = 0. \tag{2.4}$$

Integrating Eq. (2.4) twice with respect to ς, we get

$$k^2w^2\left(\frac{\beta V^2}{k^2}-\frac{\alpha V}{k^2}+\frac{V}{w^2}\right)+k^2w^2V''=0. \tag{2.5}$$

3. Application of extended simple equation method

The Eq. (2.5) has the solution of the form

$$V(\varsigma)=\sum_{i=-s}^{s}a_i\varphi(\varsigma), \tag{3.1}$$

where $a_i\,(i=-s,\cdots,-1,0,1,\cdots,s)$ are constants. Let φ satisfy the equation

$$\varphi'(\varsigma)=m_0+m_1\varphi+m_2\varphi^2+m_3\varphi^3, \tag{3.2}$$

in which m_0, m_1, m_2, m_3 are constants. The general solution of new simple ansatz (3.2) satisfies

$$\varphi(\varsigma)=-\frac{m_1-\sqrt{4m_0m_2-m_1^2}\tan\left(\frac{\sqrt{4m_0m_2-m_1^2}}{2}(\varsigma+\varsigma_0)\right)}{2m_2}.$$

Applying the homogeneous balance principle, $s=4$ can be determined. Thus, (3.1) can be rewritten as

$$V(\varsigma)=a_{-4}\varphi^{-4}+a_{-3}\varphi^{-3}+a_{-2}\varphi^{-2}+a_{-1}\varphi^{-1}+a_0+a_1\varphi+a_2\varphi^2+a_3\varphi^3+a_4\varphi^4. \tag{3.3}$$

By taking (3.3) and (3.2) into Eq. (2.5), and collecting the coefficients of φ^l, a set of algebraic equations about the parameters m_0, m_1, m_2, m_3, k, w, a_{-4}, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_4 are yielded. The values of these parameters can be found by Maple, and here are the following cases.

Case (1).

$$a_{-4}=0,\ a_{-3}=0,\ a_{-2}=0,\ a_{-1}=0, a_1=\frac{2\sqrt{-6a_0\beta w^2-3\alpha w^2+3k^2}m_2k}{w\beta},$$
$$a_2=-\frac{6k^2m_2^2}{\beta},\ a_3=0,\ a_4=0, m_0=\frac{1}{6}\frac{-a_0\beta w^2+\alpha w^2-k^2}{k^2b_2w^2},$$
$$m_1=-\frac{1}{3}\frac{\sqrt{6\beta w^2a_0-3\alpha w^2+3k^2}}{kw},\ m_2=m_2,\ m_3=0.$$

We can derive the singular periodic wave solution as follows

$$v_1(x,t)=a_0+\frac{1}{3}\frac{h\,(kwhm_2+3\tan lp)}{m_2kw^3\beta}-\frac{1}{6}\frac{(kwhm_2+3\tan lp)^2}{k^2m_2^2\beta w^4}, \tag{3.4}$$

where $h=\sqrt{-6\beta w^2a_0-3\alpha w^2+3k^2}$, $p=\sqrt{\alpha k^2w^4m_2^2-k^4w^2m_2^2}$, $l=\frac{1}{2}\frac{p\,(c_1+\varsigma)}{k^2m_2w^2}$ and k, w, α, β, m_2, a_0, are constants.

Case (2).

$$a_{-4}=0,\ a_{-3}=0,\ a_{-2}=0,\ a_{-1}=0,\ a_0=\frac{1}{24}\frac{36\alpha w^2k^2b_2^2-\beta^2w^2a_1^2-36k^4b_2^2}{\beta k^2w^2b_2^2},$$

$$a_1=a_1,\ a_2=-\frac{6k^2m_2^2}{\beta},\ a_3=0,\ a_4=0,$$

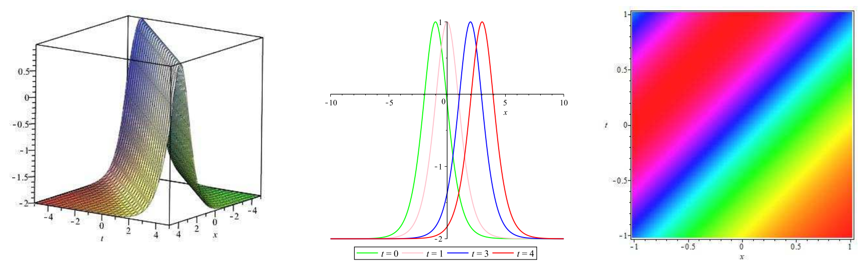
$$m_0=-\frac{1}{144}\frac{36\alpha w^2k^2m_2^2-\beta^2w^2a_1^2-36k^4m_2^2}{k^4w^2m_2^3},\ m_1=-\frac{1}{6}\frac{\beta a_1}{k^2m_2},$$

$$m_2=m_2, m_3=0.$$

The following solution is yielded

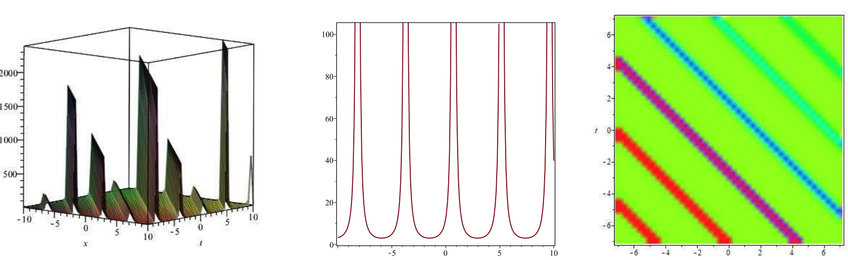
$$v_2(x,t)=\frac{1}{24}\frac{36\alpha k^2w^2m_2^2-\beta w^2a_1^2-36k^4m_2^2}{\beta k^2w^2m_2^2}+\frac{1}{12}\frac{a_1\left(f-6\tan\left(\frac{1}{2}\frac{r\left(c_1+\varsigma\right)}{k^4w^2m_2^3}\right)\right)}{k^4w^2m_2^4}$$
$$-\frac{1}{24}\frac{\left(f-6\tanh\left(\frac{1}{2}\frac{r\left(c_1+\varsigma\right)}{k^4w^2m_2^3}\right)r\right)^2}{k^6\beta m_2^6w^4}, \tag{3.5}$$

in which $r=\sqrt{\alpha k^6w^4m_2^6-k^8w^2m_2^6}$, $f=\beta k^2w^2a_1m_2^2$ and a_1, m_2, k, w, α, β are constants.



(a) (b) (c)

Figure 1. Bright-soliton solution (3.4) for $\alpha=-1$, $\beta=1$, $\omega=1$, $k=1$, $b_2=1$, $a_0=-1$, $c_1=1$. (a). 3-D plot; (b). The way of wave propagation along the x-axis at different time; (c). Density plot.



(a) (b) (c)

Figure 2. Singular periodic wave solution (3.5) for $\alpha=-1$, $\beta=-1$, $\omega=1$, $k=-1$, $b_2=1$, $a_1=2$, $c_1=-0.5$. (a).3-D plot; (b). the way of wave propagation along the x-axis; (c). Density plot.

4. Application of the $Exp\left(-\varphi\left(\varsigma\right)\right)$ expansion method

Assume that Eq. (2.5) has an exact solution of the following form

$$V\left(\varsigma\right) = a_i\left(\exp\left(-\varphi\left(\varsigma\right)\right)\right)^i + \cdots, \left(i = 1, 2, 3, \cdots\right), \tag{4.1}$$

where $a_i \neq 0$. We can acquire the value of i by the homogeneous balance principle. The φ satisfies the equation

$$\varphi'\left(\varsigma\right) = \exp\left(-\upsilon\left(\varsigma\right)\right) - \mu\exp\left(\upsilon\left(\varsigma\right)\right) + \kappa, \tag{4.2}$$

where μ, κ are constants.

The solutions of (4.2) are as follows:
When $\kappa^2 - 4\mu > 0$, $\mu \neq 0$,

$$\varphi\left(\varsigma\right) = \ln\left(\frac{-\sqrt{\kappa^2 - 4\mu}\tanh\left(\frac{\sqrt{\kappa^2 - 4\mu}}{2}\left(\varsigma + \varepsilon_0\right)\right) - \kappa}{2\mu}\right), \tag{4.3}$$

When $\kappa^2 - 4\mu < 0$,

$$\varphi\left(\varsigma\right) = \ln\left(\frac{\sqrt{4\mu - \kappa^2}\tan\left(\frac{\sqrt{4\mu - \kappa^2}}{2}\left(\varsigma + \varepsilon_0\right)\right) - \kappa}{2\mu}\right). \tag{4.4}$$

By balancing Eq. (2.5), we get $s = 2$, the solution can be rewritten as

$$V\left(\varsigma\right) = a_0 + a_1\exp\left(-\varphi\left(\varsigma\right)\right) + a_2\left(\exp\left(-\varphi\left(\varsigma\right)\right)\right)^2, \tag{4.5}$$

in which a_0, a_1, a_2 are constants and can be identified later. Putting (4.5) along with (4.2) into Eq. (2.5), a system of parametric algebraic equations about a_0, a_1, a_2, w, k can be yielded. With Maple software, and the values of parameters can be derived as follows.

Family I.

$$a_1 = \frac{a_0\kappa}{\mu},\ a_2 = \frac{a_0}{\mu},\ k = \frac{1}{6}\frac{\sqrt{-6\beta\mu a_0}}{\mu},\ w = \frac{\sqrt{-\left(\beta\kappa^2 a_0 - 4\beta\mu a_0 + 6\alpha\mu\right)\beta a_0}}{\beta\kappa^2 a_0 - 4\beta\mu a_0 + 6\alpha\mu}.$$

Case (1).
When $\kappa^2 - 4\mu > 0$, $\mu \neq 0$,

$$\begin{aligned} v_3\left(x, t\right) = & a_0 - \frac{2a_0\kappa}{\tanh\left(\frac{1}{2}C_1\sqrt{\kappa^2 - 4\mu} + \frac{1}{2}\varsigma\sqrt{\kappa^2 - 4\mu}\right)\sqrt{\kappa^2 - 4\mu} - \kappa} \\ & + \frac{4a_0\mu}{\left(-\tanh\left(\frac{1}{2}C_1\sqrt{\kappa^2 - 4\mu} + \frac{1}{2}\varsigma\sqrt{\kappa^2 - 4\mu}\right)\sqrt{\kappa^2 - 4\mu} - \kappa\right)^2}. \end{aligned} \tag{4.6}$$

Case (2).

When $\kappa^2 - 4\mu < 0$,

$$\begin{aligned} v_4\left(x,t\right) =& a_0 + \frac{2a_0\kappa}{\tan\left(\frac{1}{2}C_1\sqrt{4\mu-\kappa^2}+\frac{1}{2}\varsigma\sqrt{4\mu-\kappa^2}\right)\sqrt{4\mu-\kappa^2}-\kappa} \\ &+ \frac{4a_0\mu}{\left(\tan\left(\frac{1}{2}C_1\sqrt{4\mu-\kappa^2}+\frac{1}{2}\varsigma\sqrt{4\mu-\kappa^2}\right)\sqrt{4\mu-\kappa^2}-\kappa\right)^2}. \end{aligned} \tag{4.7}$$

Family II.

$$a_1 = \frac{6a_0\kappa}{\kappa^2+2\mu},\ a_2 = \frac{6a_0}{\kappa^2+2\mu},\ k = \frac{\sqrt{-\left(\kappa^2+2\mu\right)\beta a_0}}{\kappa^2+2\mu},$$
$$w = \frac{\sqrt{-\left(-\beta\kappa^2 a_0+\alpha\kappa^2+4\beta\mu a_0+2\alpha\mu\right)\beta a_0}}{-\beta\kappa^2 a_0+\alpha\kappa^2+4\beta\mu a_0+2\alpha\mu}.$$

Case (1).

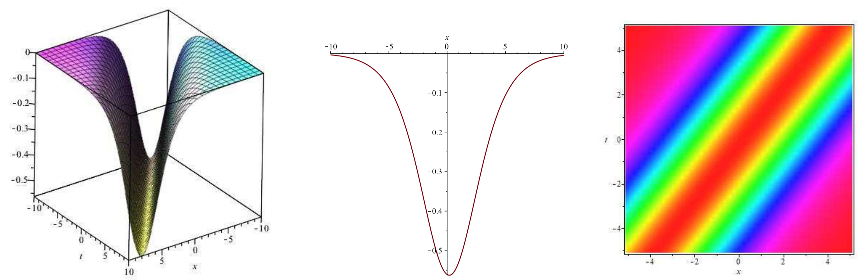
When $\kappa^2 - 4\mu > 0$, $\mu \neq 0$,

$$\begin{aligned} v_5\left(x,t\right) =& \frac{12a_0\kappa\mu}{\left(\kappa^2+2\mu\right)\left(-\tanh\left(\frac{1}{2}C1\sqrt{\kappa^2-4\mu}+\frac{1}{2}\varsigma\sqrt{\kappa^2-4\mu}\right)\sqrt{\kappa^2-4\mu}-\kappa\right)} \\ &+ \frac{24a_0\mu^2}{\left(\kappa^2+2\mu\right)\left(-\tanh\left(\frac{1}{2}C1\sqrt{\kappa^2-4\mu}+\frac{1}{2}\varsigma\sqrt{\kappa^2-4\mu}\right)\sqrt{\kappa^2-4\mu}-\kappa\right)^2} \\ &+ a_0. \end{aligned} \tag{4.8}$$

Case (2).

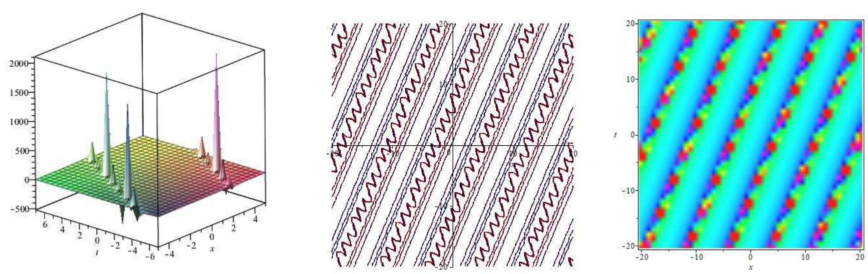
When $\kappa^2 - 4\mu < 0$,

$$\begin{aligned} v_6\left(x,t\right) =& a_0 + \frac{12a_0\kappa\mu}{\left(\kappa^2+2\mu\right)\left(\tan\left(\frac{1}{2}C1\sqrt{4\mu-\kappa^2}+\frac{1}{2}\varsigma\sqrt{4\mu-\kappa^2}\right)\sqrt{4\mu-\kappa^2}-\kappa\right)} \\ &+ \frac{24a_0\mu^2}{\left(\kappa^2+2\mu\right)\left(\tan\left(\frac{1}{2}C1\sqrt{4\mu-\kappa^2}+\frac{1}{2}\varsigma\sqrt{4\mu-\kappa^2}\right)\sqrt{4\mu-\kappa^2}-\kappa\right)^2}. \end{aligned} \tag{4.9}$$



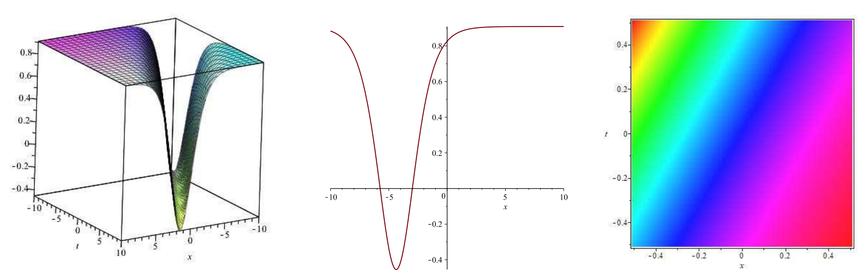
(a) (b) (c)

Figure 3. Dark-soliton solution (4.6) for $\alpha = 2$, $\beta = -1$, $\kappa = 2.5$, $\mu = 1$, $a_0 = 1$, $c_1 = -1$. (a). 3-D plot; (b). The way of wave propagation along the x-axis; (c). Density plot.



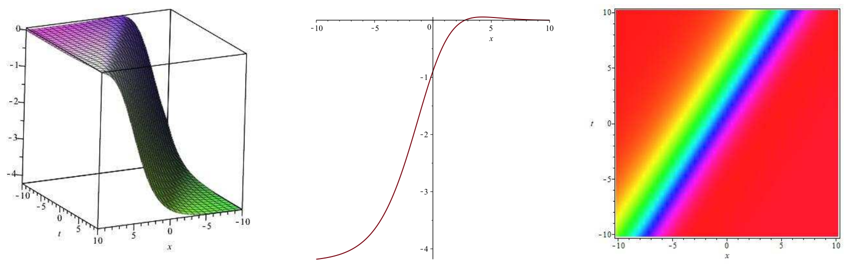
(a) (b) (c)

Figure 4. Singular periodic wave solution (4.7) for $\alpha = 5$, $\beta = -1$, $\kappa = 2.5$, $\mu = 2$, $a_0 = 2$, $c_1 = 2$. (a). 3-D plot; (b). Contour plot; (c). Density plot.



(a) (b) (c)

Figure 5. Dark-soliton solution (4.8) for $\alpha = 2$, $\beta = -1$, $\kappa = 3$, $\mu = 1$, $a_0 = 2$, $c_1 = 1$. (a). 3-D plot; (b). The way of wave propagation along the x-axis; (c). Density plot.



(a) (b) (c)

Figure 6. Kink wave solution (4.9) for $\alpha = 2$, $\beta = -1$, $\kappa = 2.5$, $\mu = 1$, $a_0 = 1$, $c_1 = -1$. (a). 3-D plot; (b). The way of wave propagation along the x-axis; (c). Density plot.

5. Application of Riccati equation method

Assume that the solution of Eq. (2.5) is as follows:

$$\chi = \sum_{i=0}^{s} a_i \phi^i, \tag{5.1}$$

in which $a_i (i = 0, 1, \cdots, s)$ are constants and ϕ satisfies the equation

$$\phi' = \phi^2 + \gamma, \tag{5.2}$$

where γ is a constant. The Eq. (5.2) has solutions of the forms:

$$\phi = \begin{cases} -\sqrt{-\gamma} \tanh\left(\sqrt{-\gamma}\varsigma\right), \gamma < 0, \\ -\dfrac{1}{\varsigma}, \gamma = 0, \\ \sqrt{\gamma} \tan\left(\sqrt{\gamma}\varsigma\right), \gamma > 0, \end{cases} \tag{5.3}$$

where $\varsigma = kx - wt$. By balancing Eq. (2.5), we get $s = 2$. Therefore, (5.1) can be rewritten as

$$\chi = a_0 + a_1 \phi + a_2 \phi^2. \tag{5.4}$$

Substitute (5.2) along with (5.4) into (2.5) and let the coefficients be zero, which leads a series of the algebraic equations about a_0, a_1, a_2, k, w. The values of the parameters can be calculated by using Maple software.

Family I.

$$k = \frac{\sqrt{(4w^2\gamma + 1)\,\alpha} w}{4w^2\gamma + 1}, w = w, a_0 = -\frac{2w^2\alpha\gamma}{(4w^2\gamma + 1)\,\beta}, a_1 = 0, a_2 = -\frac{6w^2\alpha}{(4w^2\gamma + 1)\,\beta}. \tag{5.5}$$

By putting (5.3) and (5.5) into (5.4) and replacing variables, the different types of solutions can be derived.

Case (1).

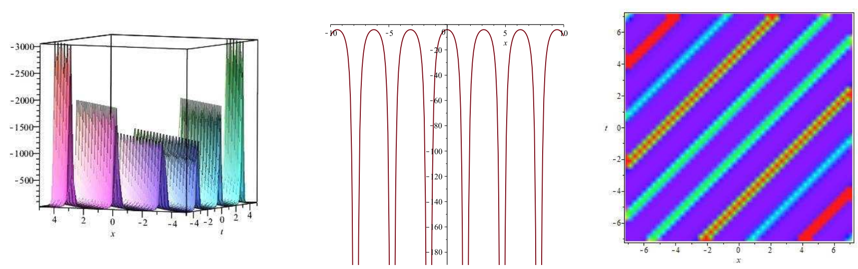
For $\gamma < 0$,

$$v_7\left(x,t\right) = \frac{2w^2\alpha\gamma\left(3\tanh\left(\frac{\sqrt{-\gamma}w\left(-4tw^2\gamma+\sqrt{\left(4w^2\gamma+1\right)\alpha}x-t\right)}{4m^2\gamma+1}\right)^2-1\right)}{\left(4w^2\gamma+1\right)\beta}. \tag{5.6}$$

Case (2).
For $\gamma > 0$,

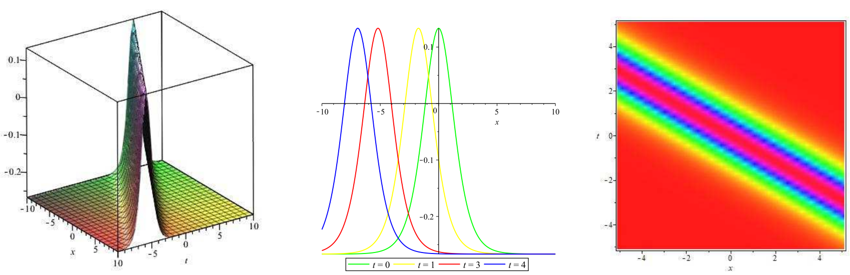
$$v_8\left(x,t\right) = -\frac{2\alpha rw^2\left(3\tan\left(\frac{\sqrt{r}w\left(-4trw^2+\sqrt{\left(4rw^2+1\right)\alpha x}-t\right)}{4rw^2+1}\right)+1\right)}{\left(4rw^2+1\right)\beta}, \tag{5.7}$$

in which w, α, γ are constants.



(a) (b) (c)

Figure 7. Singular periodic wave solution (5.6) for $\alpha = 5$, $\beta = 0.5$, $\gamma = 1$, $w = 1$. (a). 3-D plot; (b). The way of wave propagation along the x-axis; (c). Density plot.



(a) (b) (c)

Figure 8. Bright-soliton solution (5.7) for $\alpha = -1$, $\beta = 5$, $\gamma = -1$, $w = 1$. (a). 3-D plot; (b). The way of wave propagation along the x-axis; (c). Density plot.

6. Dynamical behavior analysis

In this part, the exact solutions obtained above are depicted graphically by mathematical software (maple). Through numerical simulations, the solutions of Eq.

(1.1) including bright-solitons, dark-solitons, kink wave, and singular periodic wave solutions are derived. Meanwhile, different values of the parameters are chosen to yield some important physical phenomena. A simple physical explanation of the graphs obtained is given in the following.

Fig. 1 and Fig. 8 respectively depict the physical structure of the bright-solitons (3.4), (5.7) when choosing $\alpha = -1$, $\beta = 1$, $\omega = 1$, $k = 1$, $b_2 = 1$, $a_0 = 1$, $c_1 = 1$, $\alpha = 1$, $\beta = 5$, $\gamma = -1$ and $\omega = 1$. The propagation tracks of the bright-solitons the x-axis are derived when $t = 0$, 1, 2, 4, respectively. The 3-D and density plots are drawn in (a) and (c).

Solutions (4.6) and (4.8) are connected with the tanh function, and the shapes of the dark-solitons are derived when the parameters are taken as $\alpha = 2$, $\beta = -1$, $\kappa = 2.5$, $\mu = 1$, $a_0 = 1$, $c_1 = -1$. $\alpha = 2$, $\beta = -1$, $\kappa = 3$, $\mu = 1$, $a_0 = 2$, and $c_1 = 1$, respectively, which correspond to the 3-D plots for the range of $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$ in Fig. 3 and Fig. 5. When $t = 0$, (b) reveals the propagation along the x-axis and (c) reflects the density plot of solitons.

Fig. 4 depicts the physical structure of the singular periodic wave solution (4.7) when the parameters are selected as $\alpha = 5$, $\beta = -1$, $\kappa = 2.5$, $\mu = 2$, $a_0 = 2$, $c_1 = 2$. Fig. 4(a) shows the 3-D plot of the periodic singular solution (4.7) for the range of $-6 \leq x \leq 6$ and $-6 \leq t \leq 6$. Fig. 4(b) and (c) display the corresponding contour and density plots.

The solution (4.9) depicts the structure of the kink wave solution with the form of tan function. When the parameters are taken as $\alpha = 2$, $\beta = -1$, $\kappa = 2.5$, $\mu = 1$, $a_0 = 1$, $c_1 = -1$, the 3-D structure and density plot are provided for the range of $-10 \leq x \leq 10$ and $-10 \leq t \leq 10$ in Fig. 6(a) and (c). The above wave propagation is from right to left. (b) represents the wave propagation path on the x-axis when $t = 0$.

The 3-D structure of the singular periodic wave solution corresponding to solutions (3.5) and (5.6) are obtained in Fig. 2 and Fig. 7 with the parameters $\alpha = -1$, $\beta = -1$, $\omega = 1$, $k = -1$, $b_2 = 1$, $a_1 = 2$, $c_1 = -0.5$, $\alpha = 5$, $\beta = 0.5$, $\gamma = 1$ and $\omega = 1$, respectively. (b) and (c) correspond to the motion trace and density plot of the singular periodic wave solution, respectively.

7. Remark and comparisons

The three methods we use are all based on the homogeneous balance principle, and we can know the forms of the solutions in advance. Many types of solutions (hyperbolic function solutions, trigonometric function solutions, rational function solutions, periodic function solutions, etc) can be constructed. These methods are less arithmetic, simple, direct and reliable, and are efficient methods for constructing soliton solutions. However, these methods may not be suitable for solving arbitrary nonlinear partial differential equations (e.g., equations that do not have the highest order derivative term and nonlinear terms), and need to reply on the auxiliary equation to achieve.

By comparing with other achievements of the model, the $(1/G')$-expansion method applied by Duran [32] only constructed three solutions which were in the forms of hyperbolic function solutions while the methods we use obtain more solutions including hyperbolic function solutions, trigonometric function solutions and rational function solutions. Comparing with the sine-Gordon method adapted in [33], the methods selected derive more types of the solutions which contain dark-solitons, bright-solitons, kink, an periodic-singular wave solutions. But the three methods need to achieve with the help of the auxiliary equation.

In addition, researchers all extracted the kink, dark-soliton, bright-soliton, with periodic solutions in Refs. [34, 35] which are the same results as ours, but the solutions they obtained are more than ours. Sivasundaram et al. [36] provided seven sets of solutions including logarithmic, hyperbolic, complex function solutions by the sine-Gordon expansion method and the rational sine-Gordon expansion method which don't reply on the auxiliary equations while our results contain dark-solitons, bright-solitons, kink, periodic-singular wave solutions, enriching the diversity of the exact solutions of the equation.

8. Conclusions

In summary, exact solutions of the Lonngren-wave equation are efficiently yielded by applying the extended simple equation method, the $Exp(-\varphi(\varsigma))$ expansion method and the Riccati equation method. Solitons, singular periodic wave and kink wave solutions are established. Then, the solutions obtained are graphically presented by setting the appropriate parameters. The construction of these solutions helps us understand the physical interpretation and phenomena of the equation. The results prove that the above three methods are very simple and effective in solving NLPDEs.

Acknowledgements

The authors are grateful to the reviewers and editors for their valuable suggestions that have helped us improve our paper.

References

[1] S. Kumar, A. Kumar, B. Samet, and J.F. Gómez-Aguilar, *A chaos study of tumor and effector cells in fractional tumor-immune model for cancer treatment*, Chaos, Solitons & Fractals, 2020, 141, 110321, 14 pages.

[2] I. Samir, N. Badra, A. R. Seadawy, H. M. Ahmed and A. H. Arnous, *Exact wave solutions of the fourth order non-linear partial differential equation of optical fiber pulses by using different methods*, Optik, 2021, 230, 166313, 10 pages.

[3] M. Niwas and S. Kumar, *Multi-peakons, lumps, and other solitons solutions for the (2+1)-dimensional generalized Benjamin-Ono equation: an inverse (G'/G)-expansion method and real-world applications*, Nonlinear Dynamics, 2023, 1–14.

[4] S. Kumar, M. Niwas, M. S. Osman and M. A. Abdou, *Abundant different types of exact soliton solution to the (4+1)-dimensional Fokas and (2+1)-dimensional breaking soliton equations*, Communications in Theoretical Physics, 2021, 73(10), 105007.

[5] M. S. Osman and B. Ghanbari, *New optical solitary wave solutions of Fokas-Lenells equation in presence of perturbation terms by a novel approach*, Optik, 2018, 175, 328–333.

[6] M. Ali Akbar and N. H. M. Ali, *The improved F-expansion method with Riccati equation and its applications in mathematical physics*, Cogent Mathematics, 2017, 4(1), 1282577, 19 pages.

[7] M. A. Abdou, L. Ouahid and S Kumar, *Plenteous specific analytical solutions for new extended deoxyribonucleic acid (DNA) model arising in mathematical biology*, Modern Physics Letters B, 2023, 37(34), 2350173.

[8] S. Kumar, W. X. Ma, S. K. Dhiman and A. Chauhan, *Lie group analysis with the optimal system, generalized invariant solutions, and an enormous variety of different wave profiles for the higher-dimensional modified dispersive water wave system of equations*, The European Physical Journal Plus, 2023, 138(5), 434, 13 pages.

[9] D. S. Wang and Y. Yin, *Symmetry analysis and reductions of the two-dimensional generalized Benney system via geometric approach*, Computers & Mathematics with Applications, 2016, 71(3), 748–757.

[10] S. Kumar, W. X. Ma and A. Kumar, *Lie symmetries, optimal system and group-invariant solutions of the (3+1)-dimensional generalized KP equation*, Chinese Journal of Physics, 2021, 69, 1–23.

[11] I. Hamid and S. Kumar, *Symbolic computation and Novel solitons, traveling waves and soliton-like solutions for the highly nonlinear (2+1)-dimensional Schrödinger equation in the anomalous dispersion regime via newly proposed modified approach*, Optical and Quantum Electronics, 2023, 55(9), 755, 23 pages.

[12] S. Kumar, I. Hamid and M. A. Abdou, *Dynamic frameworks of optical soliton solutions and soliton-like formations to Schrädinger-Hirota equation with parabolic law non-linearity using a highly efficient approach*, Optical and Quantum Electronics, 2023, 55(14), 1261, 31 pages.

[13] A. Ali, A. R. Seadawy and D. Lu, *Dispersive solitary wave soliton solutions of (2+1)-dimensional Boussineq dynamical equation via extended simple equation method*, Journal of King Saud University-Science, 2019, 31(4), 653–658.

[14] M. O, Al-Amr and S. El-Ganaini, *New exact traveling wave solutions of the (4+1)-dimensional Fokas equation*, Computers & Mathematics with Applications, 2017, 74(6), 1274–1287.

[15] Z. I. Dimitrova and K. N. Vitanov, *Homogeneous balance method and auxiliary equation method as particular cases of simple equations method (SEsM)AIP Conference Proceedings*, AIP Publishing LLC, 2021, 2321(1), 030004, 10 pages.

[16] C. Chen and Z. Wang, New exact solutions, *dynamical and chaotic behaviors for the fourth-order nonlinear generalized Boussinesq water wave equation*, Advances in Mathematical Physics, 2021, 2021, 1–13.

[17] S. J. Chen, W. X. Ma and X. L, *Bäcklund transformation, exact solutions and interaction behaviour of the (3+1)-dimensional Hirota-Satsuma-Ito-like equation*, Communications in Nonlinear Science and Numerical Simulation, 2020, 83, 105135, 12 pages.

[18] M. S. Osman and A. M. Wazwaz, *A general bilinear form to generate different wave structures of solitons for a (3+1)-dimensional Boiti-Leon-Manna-Pempinelli equation*, Mathematical Methods in the Applied Sciences, 2019, 42(18), 6277–6283.

[19] S. Rani, S. Kumar and N. Mann, *On the dynamics of optical soliton solutions, modulation stability, and various wave structures of a (2+1)-dimensional complex modified Korteweg-de-Vries equation using two integration mathematical methods*, Optical and Quantum Electronics, 2023, 55(8), 731, 27 pages.

[20] M. Adel, D. Baleanu, U. Sadiya, M. A. Arefin, M. H. Uddin, M. A. Elamin and M. S. Osman, *Inelastic soliton wave solutions with different geometrical structures to fractional order nonlinear evolution equations*, Results in Physics, 2022, 105661.

[21] M. S. Osman, *One-soliton shaping and inelastic collision between double solitons in the fifth-order variable-coefficient Sawada-Kotera equation*, Nonlinear-Dynamics, 2019, 96(2), 1491–1496.

[22] M. S. Osman and J. A. T, Machado, *The dynamical behavior of mixed-type soliton solutions described by (2+1)-dimensional Bogoyavlensky-Konopelchenko equation with variable coefficients*, Journal of Electromagnetic Waves and Applications, 2018, 32(11), 1457–1464.

[23] S. Kumar and M. Niwas, *Analyzing multi-peak and lump solutions of the variable-coefficient Boiti-Leon-Manna-Pempinelli equation: a comparative study of the Lie classical method and unified method with applications*, Nonlinear Dynamics, 2023, 1–19.

[24] S. Kumar, N, Mann, H. Kharbanda and M. Inc, *Dynamical behavior of analytical soliton solutions, bifurcation analysis, and quasi-periodic solution to the (2+1)-dimensional Konopelchenko-Dubrovsky (KD) system*, Analysis and Mathematical Physics, 2023, 13(3), 40, 30 pages.

[25] S. Akcaği and T. Aydemir, *Comparison between the* (G'/G)*-expansion method and the modified extended tanh method*, Open Physics, 2016, 14(1), 88–94.

[26] K. E. Lonngren, H. C. S. Hsuan and W. F. Ames, *On the soliton, invariant, and shock solutions of a fourth-order nonlinear equation*, Journal of Mathematical Analysis and Applications, 1975, 52(3), 538–545.

[27] H. Durur, *Energy-carrying wave simulation of the Lonngren-wave equation in semiconductor materials*, International Journal of Modern Physics B, 2021,35(21), 2150213.

[28] S. Duran, *Travelling wave solutions and simulation of the Lonngren wave equation for tunnel diode*, Optical and Quantum Electronics, 2021, 53(8), 1–9.

[29] H. M. Baskonus, H. Bulut and T. A. Sulaiman, *New complex hyperbolic structures to the lonngren-wave equation by using sine-gordon expansion method*, Applied Mathematics and Nonlinear Sciences, 2019, 4(1), 129–138.

[30] A. Yokuş, *Simulation of bright-dark soliton solutions of the Lonngren wave equation arising the model of transmission lines*, Modern Physics Letters B, 2021, 35(32), 2150484.

[31] H. K. Barman, A. R. Seadawy, R. Roy, M. A. Akbar and M. H. Raddadi, *Rational closed form soliton solutions to certain nonlinear evolution equations ascend in mathematical physics*, Results in Physics, 2021, 27, 104450, 12 pages.

[32] S. Duran, *Travelling wave solutions and simulation of the Lonngren wave equation for tunnel diode*, Optical and Quantum Electronics, 2021, 53(8), 458, 9 pages.

[33] H. M. Baskonus, H. Bulut and T. A. Sulaiman, *New complex hyperbolic structures to the lonngren-wave equation by using sine-gordon expansion method*, Applied Mathematics and Nonlinear Sciences, 2019, 4(1), 129–138.

[34] A. Kumar and S. Kumar, *Dynamic nature of analytical soliton solutions of the (1+1)-dimensional Mikhailov-Novikov-Wang equation using the unified approach*, International Journal of Mathematics and Computer in Engineering, 2023, 1(2), 217–228.

[35] S. Sivasundaram, A. Kumar and R. K. Singh, *On the complex properties to the first equation of the Kadomtsev-Petviashvili hierarchy*, International Journal of Mathematics and Computer in Engineering, 2024, 2(1), 71–84.

[36] A. Hussain, H. Ali, F. Zaman and N. Abbas, *New closed form solutions of some nonlinear pseudo-parabolic models via a new extended direct algebraic method*, International Journal of Mathematics and Computer in Engineering, 2024, 2(1), 35–58.