Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 1, February 2025, 135–177 DOI:10.12150/jnma.2025.135

Some New Discrete Hermite-Hadamard Inequalities and Their Generalizations

Xiaoyue Han[1] and Run Xu[2,†]

Abstract This article mainly studies some new discrete Hermite-Hadamard inequalities for integer order and fractional order. For this purpose, the definitions of h-convexity and preinvexity for a real-valued function f defined on a set of integers $\mathbb{Z}$ are introduced. Under these two new definitions, some new discrete Hermite-Hadamard inequalities for integer order related to the endpoints and the midpoint $\frac{a+b}{2}$ based on the substitution rules are proposed, and they are generalized to fractional order forms. In addition, for the h-convex function on the time scale $\mathbb{Z}$, two new discrete Hermite-Hadamard inequalities for integer order by dividing the time scale differently are obtained.

Keywords Discrete fractional calculus, h-convex functions, preinvex functions, Hermite-Hadamard inequalities, times scales

MSC(2010) 26B25, 26A33, 26D10, 26D15.

1. Introduction

Convex theory has always been an important component of mathematical theory, which is often used to solve many problems in economics, optimization, engineering, and other fields [12,24,25]. Scholars obtained many classical inequalities by utilizing the different convexities of functions, such as Schur inequalities, Hermite-Hadamard (H-H) inequalities, and Ostrowski type inequalities [8,28,31].

Currently, many scholars are committed to studying the H-H inequality. In 1893, Hermite and Hadamard [14] proved the classical H-H inequality first, which gave estimates of the upper and lower bounds of the integral mean of any convex function, as follows:

If $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function in I and $u, v \in I$, where $u \le v$, then

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{v-u}\int_u^v f(\delta)\mathrm{d}\delta \le \frac{f(u)+f(v)}{2}. \tag{1.1}$$

In 2015, Noor et al. [22] obtained a new H-H integral inequality for h-convex

†the corresponding author.
Email address:xiaoyue991231@163.com(X. Han), xurun2005@163.com(R. Xu)
[1]School of Mathematical Sciences, Yangzhou University, 225002 Yangzhou, China
[2]School of Mathematical Sciences, Qufu Normal University, 273165 Qufu, China

functions by inserting a segmentation point $\frac{u+v}{2}$ on interval $[u,v]$:

$$\begin{aligned}\frac{1}{4\left[h\left(\frac{1}{2}\right)\right]^2} f\left(\frac{u+v}{2}\right) \leq & \frac{1}{4h\left(\frac{1}{2}\right)}\left[f\left(\frac{3u+v}{4}\right)+f\left(\frac{u+3v}{4}\right)\right] \leq \frac{1}{v-u}\int_u^v f(\delta)\mathrm{d}\delta \\ \leq & \left[\frac{f(u)+f(v)}{2}+f\left(\frac{u+v}{2}\right)\right]\int_0^1 h(t)\mathrm{d}t \\ \leq & \left\{[f(u)+f(v)]\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\right\}\int_0^1 h(t)\mathrm{d}t.\end{aligned}$$

It is an important method to generalize H-H inequality by dividing intervals differently. More researches on this aspect can be found in references [10, 30, 32].

Due to the importance of fractional operators in pure mathematics and applied mathematics, scholars defined various fractional integral operators from different directions. It has become one of the hot research topics to establish and study H-H inequalities based on generalized fractional order integrals.

In 2013, Sarikaya et al. [26] extended inequality (1.1) using Riemann-Liouville (R-L) fractional order integrals as follows:

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(v-u)^\alpha}\left[J_{u^+}^\alpha f(v)+J_{v^-}^\alpha f(u)\right] \leq \frac{f(u)+f(v)}{2}.$$

In 2014, Sarikaya et al. [27] established a midpoint type H-H inequality for R-L fractional order integrals based on the midpoint of the interval $[u,v]$:

$$f\left(\frac{u+v}{2}\right) \leq \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(v-u)^\alpha}\left[J_{\left(\frac{u+v}{2}\right)^+}^\alpha f(v)+J_{\left(\frac{u+v}{2}\right)^-}^\alpha f(u)\right] \leq \frac{f(u)+f(v)}{2}.$$

In 2017, Agarwal et al. [3] established some H-H inequalities via generalized k-fractional integrals. In 2020, Mehreen et al. [21] established H-H inequalities for p-convex functions via conformable fractional integrals. In 2023, Tariq et al. [33] presented a new version of the H-H inequalities for preinvex functions via non-conformable fractional integrals. For more research on fractional order H-H integral inequalities, please refer to references [7, 16, 18, 29, 36].

The theory of dynamic equations on time scales is a new field in mathematical science. In the past few years, some integral inequalities used for dynamic equations on time scales have attracted the attention of many scholars. Discrete calculus is a calculus theory on time scales, which is of great significance for describing the discontinuity of certain time variables. In recent years, with the development of discrete calculus, research on H-H inequalities for integer order and fractional order has gradually increased.

In 2016, Atıcı and Yaldız [2] defined the convexity of real functions on any time scale, and established discrete H-H inequalities for integer order and fractional order via convex functions on $\mathbb{Z}$. The main results are as follows:

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{2(v-u)}\left[\int_u^v f(\xi)\Delta\xi+\int_u^v f(\xi)\nabla\xi\right] \leq \frac{f(u)+f(v)}{2},$$

and

$$f\left(\frac{u+v}{2}\right) \leq \frac{\Gamma(\varepsilon)}{2\Lambda(v-u)}\left[\Delta_{v-1}^{-\varepsilon} f(u-\varepsilon)+{}_{u+1}\nabla^{-\varepsilon} f(v)\right] \leq \frac{f(u)+f(v)}{2},$$

where $\varepsilon > 0$ and

$$\Lambda = \int_{\mathbb{T}_{[u,v]}} \left((v-u)\xi + \varepsilon - 1\right)^{\overline{\varepsilon-1}} \hat{\Delta}\xi.$$

In 2022, Wang and Xu [37] established the discrete H-H inequality for integer order of the midpoint type via convex function on $\mathbb{Z}$:

$$f\left(\frac{u+v}{2}\right) \le \frac{1}{2(v-u)}\left[\int_u^{\frac{u+v}{2}} f(\xi)\Delta\xi + \int_{\frac{u+v}{2}}^v f(\xi)\nabla\xi\right] \le \frac{f(u)+f(v)}{2}.$$

Their result was equally extended to the discrete H-H inequality for fractional order:

$$f\left(\frac{u+v}{2}\right) \le \frac{\Gamma(\varepsilon)}{2\Lambda(v-u)}\left[{}_{\frac{u+v}{2}}\nabla^{-\varepsilon} f(v) + \nabla^{-\varepsilon}_{\frac{u+v}{2}} f(u)\right] \le \frac{f(u)+f(v)}{2},$$

where $\varepsilon > 0$ and

$$\Lambda = \int_{\mathbb{T}_{\left[\frac{u+v}{2},v\right]}} \left(\frac{v-u}{2}\xi + 1\right)^{\overline{\varepsilon-1}} \hat{\Delta}\xi.$$

In 2017, Yaldız and Agarwal [38] gave the discrete H-H inequalities for integer order and fractional order via s-convex functions on $\mathbb{Z}$. For more research on H-H inequalities on time scales, please refer to [4, 11, 19, 20].

So far, there are no discrete H-H inequalities for integer order and fractional order via h-convex functions and preinvex functions on $\mathbb{Z}$. Based on our new definitions of h-convex functions and preinvex functions on $\mathbb{Z}$, we establish discrete H-H inequalities for integer order and fractional order. Two new discrete H-H inequalities for h-convex functions on $\mathbb{Z}$ are also established by dividing the time scale.

The work of this paper is as follows. In Section 2, we review some basic definitions, theorems and give the concepts of h-convex functions and preinvex functions on $\mathbb{Z}$. In Section 3, we establish discrete H-H inequalities for integer order and fractional order about h-convex functions on $\mathbb{Z}$ related to the interval endpoints and the interval midpoint, respectively. We also obtain two new discrete H-H inequalities for integer order via h-convex functions on $\mathbb{Z}$ through dividing the time scales differently. In addition, we also establish the above two kinds of inequalities for preinvex functions on $\mathbb{Z}$ related to the interval endpoints and the interval midpoint, respectively. In Section 4, we provide a summary and point out relevant issues that can be further studied in the future.

2. Preliminaries

Let $\mathbb{Z}$ be the set of integers and $a, b \in \mathbb{Z}$ with $a < b$, $[a,b]_{\mathbb{Z}} = [a,b] \cap \mathbb{Z}$. We define

$$\mathbb{T}_{[a,b]} = \left\{ u \mid u = \frac{b-t}{b-a} \quad for \quad t \in [a,b]_{\mathbb{Z}} \right\};$$

$$\mathbb{T}_{[a,\frac{a+b}{2}]} = \left\{ u \mid u = \frac{2(t-a)}{b-a} \quad for \quad t \in \left[a, \frac{a+b}{2}\right]_{\mathbb{Z}} \right\};$$

$$\mathbb{T}_{[\frac{a+b}{2},b]} = \left\{ u \mid u = \frac{2(b-t)}{b-a} \quad for \quad t \in \left[\frac{a+b}{2}, b\right]_{\mathbb{Z}} \right\};$$

$$\mathbb{T}_{[a,\frac{3a+b}{4}]} = \left\{ u \mid u = \frac{4(t-a)}{b-a} \quad for \quad t \in \left[a, \frac{3a+b}{4}\right]_{\mathbb{Z}} \right\};$$

$$\mathbb{T}_{[\frac{3a+b}{4},\frac{a+3b}{4}]} = \left\{ u \mid u = \frac{4t-(3a+b)}{2(b-a)} \quad for \quad t \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]_{\mathbb{Z}} \right\};$$

$$\mathbb{T}_{[\frac{a+3b}{4},b]} = \left\{ u \mid u = \frac{4(b-t)}{b-a} \quad for \quad t \in \left[\frac{a+3b}{4}, b\right]_{\mathbb{Z}} \right\}.$$

It is easy to see that these sets are all subsets of $[0,1]$. In addition, $\mathbb{N}_{a,h}$ and ${}_{b,h}\mathbb{N}$ are special time scales: $\mathbb{N}_{a,h} = \{a, a+h, a+2h, \cdots\}$, ${}_{b,h}\mathbb{N} = \{\cdots, b-2h, b-h, b\}$, where $h > 0$ is the step size of the time scale. Specifically, when $h = 1$, $\mathbb{N}_{a,h}$ and ${}_{b,h}\mathbb{N}$ are represented as $\mathbb{N}_a$ and ${}_b\mathbb{N}$, respectively.

Next, we review several definitions which will be further used in this article.

For all $t \in \mathbb{Z}$, $\sigma(t) = t+1, \rho(t) = t-1, \mu(t) = \sigma(t) - t = 1, \upsilon(t) = t - \rho(t) = 1$, which are known as the forward jump, the backward jump, the forward graininess and the backward graininess operators, respectively. For a function $f : \mathbb{Z} \to \mathbb{R}$, the nabla and delta differences of f are given by

$$f^{\nabla}(t) = f(t+1) - f(t) \quad and \quad f^{\Delta}(t) = f(t) - f(t-1).$$

The nabla and delta sums of f are given by

$$\int_a^b f(s)\nabla s = \sum_{k=a+1}^{b} f(k) \quad and \quad \int_a^b f(s)\Delta s = \sum_{k=a}^{b-1} f(k),$$

where $a, b \in \mathbb{Z}$.

And the nabla and delta sums of f on the time scale $\mathbb{T}_{[a,b]}$ are respectively represented by the following symbols:

$$\int_{\mathbb{T}_{[a,b]}} f(s)\widetilde{\nabla}s \quad and \quad \int_{\mathbb{T}_{[a,b]}} f(s)\widetilde{\Delta}s,$$

where $\int_{\mathbb{T}_{[a,b]}} 1\widetilde{\nabla}s = \int_{\mathbb{T}_{[a,b]}} 1\widetilde{\Delta}s = 1$.

Assume that t, ε are arbitrary real numbers, and the rising and falling factorial functions are defined as follows [13]:

$$t^{\bar{\varepsilon}} = \frac{\Gamma(t+\varepsilon)}{\Gamma(t)}, \quad t, t+\varepsilon \in \mathbb{R} \setminus \mathbb{N}^-,$$

$$t^{(\varepsilon)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\varepsilon)}, \quad t+1, t+1-\varepsilon \in \mathbb{R} \setminus \mathbb{N}^-,$$

where $\Gamma(t) = \int_0^\infty \delta^{t-1} e^{-\delta} \mathrm{d}\delta$.

Remark 2.1. From the above definitions, we have $(k+\alpha-\sigma(t))^{(\alpha-1)} = (k-\rho(t))^{\overline{\alpha-1}}$.

Definition 2.1. ([1]) Let the real function f and $\alpha > 0$ be given. Then

(i) the delta left and right fractional sums are defined by:

$$ {}_a\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s), \quad t \in \mathbb{N}_{a+\alpha}, \tag{2.1} $$

$$ \Delta_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s)-t)^{(\alpha-1)} f(s), \quad t \in_{b-\alpha} \mathbb{N}; \tag{2.2} $$

(ii) the nabla left and right fractional sums are defined by:

$$ {}_a\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1}, \tag{2.3} $$

$$ \nabla_b^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s)-t)^{\overline{\alpha-1}} f(s), \quad t \in_{b-1} \mathbb{N}. \tag{2.4} $$

Remark 2.2. There is an equality between the delta right fractional sum and the nabla right fractional sum:

$$ \Delta_{b-1}^{-\alpha} f(a-\alpha) = \nabla_b^{-\alpha} f(a). $$

Definition 2.2. ([23]) The function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$ f(\chi t_1 + (1-\chi)t_2) \leq \chi f(t_1) + (1-\chi) f(t_2), \forall\ t_1, t_2 \in I \textit{ and } \chi \in [0,1]. \tag{2.5} $$

Definition 2.3. ([15]) Let s be given in $(0,1]$. The function $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ is said to be s-convex in the second sense, if

$$ f(\chi t_1 + (1-\chi)t_2) \leq \chi^s f(t_1) + (1-\chi)^s f(t_2), \forall\ t_1, t_2 \in I \textit{ and } \chi \in [0,1]. \tag{2.6} $$

Definition 2.4. ([34]) Let $h : (0,1) \to \mathbb{R}$ be a nonnegative function, $h \neq 0$. The nonnegative function $f : I \subset \mathbb{R} \to \mathbb{R}$ is said to be h-convex, if

$$ f(\chi t_1 + (1-\chi)t_2) \leq h(\chi) f(t_1) + h(1-\chi) f(t_2), \forall\ t_1, t_2 \in I \textit{ and } \chi \in [0,1]. \tag{2.7} $$

Remark 2.3. If the inequality signs in inequalities (2.5)-(2.7) are reversed, then f becomes the concave function, s-concave function in the second sense, and h-concave function, respectively.

Definition 2.5. ([6]) $Y \subset \mathbb{R}^n$ is invex with respect to $\psi(\cdot,\cdot)$, if

$$ t_1 + \chi\psi(t_2, t_1) \in Y, \quad \forall\ t_1, t_2 \in Y \textit{ and } \chi \in [0,1]. $$

Definition 2.6. ([35]) Let $Y \neq \varnothing \subset \mathbb{R}$ be an invex set with respect to $\psi : Y \times Y \neq \varnothing \to \mathbb{R}$. Then, the function $f : Y \to \mathbb{R}$ is said to be preinvex with respect to ψ, if

$$ f(t_1 + \chi\psi(t_2, t_1)) \leq \chi f(t_2) + (1-\chi) f(t_1), \forall\ t_1, t_2 \in Y \textit{ and } \chi \in [0,1]. $$

Proposition 2.1. ([17]) *Let $Y \subset \mathbb{R}^n$ be an open invex with respect to $\psi : Y \times Y \neq \varnothing \to \mathbb{R}$. For any $t_1, t_2 \in Y$ and $\chi \in [0,1]$,*

$$\psi(t_1, t_1 + \chi\psi(t_2, t_1)) = -\chi\psi(t_2, t_1)$$

and

$$\psi(t_2, t_1 + \chi\psi(t_2, t_1)) = (1-\chi)\psi(t_2, t_1).$$

According to the above equations, for any $t_1, t_2 \in Y$ and $\chi_1, \chi_2 \in [0,1]$, we have

$$\psi(t_1 + \chi_2\psi(t_2, t_1), t_1 + \chi_1\psi(t_2, t_1)) = (\chi_2 - \chi_1)\psi(t_2, t_1).$$

Definition 2.7. ([2]) The function $f : \mathbb{Z} \subset \mathbb{R} \to \mathbb{R}$ is said to be convex on $\mathbb{Z}$, if

$$f(\chi t_1 + (1-\chi)t_2) \leq \chi f(t_1) + (1-\chi)f(t_2), \forall\ t_1, t_2 \in \mathbb{Z},\ \textit{and}\ \chi \in \mathbb{T}_{[t_1,t_2]}. \tag{2.8}$$

Definition 2.8. ([38]) Let s be given in $(0,1]$. The function $f : I \subset \mathbb{Z}^+ \to \mathbb{R}$ is said to be s-convex in the second sense on $\mathbb{Z}$, if

$$f(\chi t_1 + (1-\chi)t_2) \leq \chi^s f(t_1) + (1-\chi)^s f(t_2), \forall\ t_1, t_2 \in I,\ \textit{and}\ \chi \in \mathbb{T}_{[t_1,t_2]}. \tag{2.9}$$

Now, we introduce two new types of convex function defined on $\mathbb{Z}$.

Definition 2.9. Let $h : (0,1) \to \mathbb{R}$ be a non-negative function, $h \neq 0$. The nonnegative function $f : I \subset \mathbb{Z} \to \mathbb{R}$ is said to be h-convex on $\mathbb{Z}$, if

$$f(\chi t_1 + (1-\chi)t_2) \leq h(\chi)f(t_1) + h(1-\chi)f(t_2), \forall\ t_1, t_2 \in I,\ \textit{and}\ \chi \in \mathbb{T}_{[t_1,t_2]}. \tag{2.10}$$

If the inequality is reversed, then f is an h-concave function on $\mathbb{Z}$.

Definition 2.10. Let $Y \neq \varnothing \subset \mathbb{Z}$ be an invex set with respect to $\psi : Y \times Y \neq \varnothing \to \mathbb{Z}$. Then, the function $f : Y \to \mathbb{R}$ is said to be preinvex with respect to ψ on $\mathbb{Z}$, if

$$f(t_1 + \chi\psi(t_2, t_1)) \leq \chi f(t_2) + (1-\chi)f(t_1), \forall\ t_1, t_2 \in Y,\ \textit{and}\ \chi \in \mathbb{T}_{[t_1,t_2]}.$$

To prove our conclusions, we also need the following substitution rules on time scale $\mathbb{T}$.

Theorem 2.1. ([5]) *Assume that $\omega : \mathbb{T} \to \mathbb{R}$ is strictly increasing and differentiable with rd-continuous derivative, $\widetilde{\mathbb{T}} := \omega(\mathbb{T})$ is a time scale. If $\phi : \mathbb{T} \to \mathbb{R}$ is an rd-continuous derivative, then for $u, v \in \mathbb{T}$, we have*

$$\int_{\omega(u)}^{\omega(v)} (\psi \circ \omega^{-1})(\xi)\widetilde{\Delta}(\xi) = \int_u^v \phi(\zeta)\omega^{\Delta}(\zeta)\Delta\zeta \tag{2.11}$$

or

$$\int_{\omega(u)}^{\omega(v)} (\psi \circ \omega^{-1})(\xi)\widetilde{\nabla}(\xi) = \int_u^v \phi(\zeta)\omega^{\nabla}(\zeta)\nabla\zeta. \tag{2.12}$$

Theorem 2.2. ([9]) *Assume that $\omega : \mathbb{T} \to \mathbb{R}$ is strictly decreasing and differentiable with rd-continuous derivative, $\widetilde{\mathbb{T}} := \omega(\mathbb{T})$ is a time scale. If $\phi : \mathbb{T} \to \mathbb{R}$ is an rd-continuous derivative, then for $u, v \in \mathbb{T}$, we have*

$$\int_{\omega(v)}^{\omega(u)} (\psi \circ \omega^{-1})(\xi)\widetilde{\nabla}(\xi) = \int_u^v \phi(\zeta)(-\omega^{\Delta})(\zeta)\Delta\zeta \tag{2.13}$$

or

$$\int_{\omega(v)}^{\omega(u)} (\psi \circ \omega^{-1})(\xi)\widetilde{\Delta}(\xi) = \int_u^v \phi(\zeta)(-\omega^{\nabla})(\zeta)\nabla\zeta. \tag{2.14}$$

3. Main results

3.1. Discrete Hermite-Hadamard inequalities for h-convex functions and its generalization

First, we provide the discrete H-H inequalities for integer order via h-convex functions on $\mathbb{Z}$ related to the endpoints.

Theorem 3.1. *Suppose that $f : [a,b]_{\mathbb{Z}} \to \mathbb{R}$ is an h-convex function with $a < b$, $a, b, \frac{a+b}{2} \in \mathbb{Z}$, and $h\left(\frac{1}{2}\right) \neq 0$. Then we have*

$$\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \leq M\big[f(a)+f(b)\big], \quad (3.1)$$

where $M = \int_{\mathbb{T}_{[a,b]}} \big[h(t)+h(1-t)\big]\widetilde{\Delta}t$.

Proof. Fixing $t \in \mathbb{T}_{[a,b]} \setminus \{0,1\}$, we define

$$x = ta + (1-t)b, y = (1-t)a + tb.$$

It is easy to see that $x, y \in [a,b]_{\mathbb{Z}}$ and $\frac{x+y}{2} = \frac{a+b}{2} \in \mathbb{Z}$. Since f is an h-convex function on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)\big[f(x)+f(y)\big].$$

This implies that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq h\left(\frac{1}{2}\right)\big[f(ta+(1-t)b) + f((1-t)a+tb)\big] \\ &\leq h\left(\frac{1}{2}\right)\big[h(t)f(a) + h(1-t)f(b) + h(1-t)f(a) + h(t)f(b)\big] \\ &= h\left(\frac{1}{2}\right)\big[h(t)+h(1-t)\big]\big[f(a)+f(b)\big]. \end{aligned}$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{[a,b]}$, we have

$$\begin{aligned} \int_{\mathbb{T}_{[a,b]}} f\left(\frac{a+b}{2}\right)\widetilde{\Delta}t &\leq h\left(\frac{1}{2}\right)\int_{\mathbb{T}_{[a,b]}} \big[f(ta+(1-t)b) + f((1-t)a+tb)\big]\widetilde{\Delta}t \\ &\leq h\left(\frac{1}{2}\right) M\big[f(a)+f(b)\big], \end{aligned} \quad (3.2)$$

where $M = \int_{\mathbb{T}_{[a,b]}} \big[h(t)+h(1-t)\big]\widetilde{\Delta}t$.

Define

$$l_1 = \int_{\mathbb{T}_{[a,b]}} f(ta+(1-t)b)\widetilde{\Delta}t, l_2 = \int_{\mathbb{T}_{[a,b]}} f((1-t)a+tb)\widetilde{\Delta}t.$$

Calculate l_1 and l_2 separately below.

First, we assert that $l_1 = \frac{1}{b-a}\int_a^b f(\tau)\nabla\tau$.

Let $k_1(t) : [a,b]_{\mathbb{Z}} \to \mathbb{T}_{[a,b]}$ be defined by $k_1(t) = \frac{b-t}{b-a}$ with $t \in [a,b]_{\mathbb{Z}}$. Then $k_1(t)$ is decreasing, $k_1^{-1}(t) = ta + (1-t)b$ and $(-k_1^{\nabla})(t) = \frac{1}{b-a}$.

According to Theorem 2.2, we obtain

$$\begin{aligned} l_1 &= \int_{0=k_1(b)}^{1=k_1(a)} (f \circ k_1^{-1})(t)\widetilde{\Delta} t = \int_a^b f(\tau)(-k_1^{\nabla})(\tau)\nabla\tau \\ &= \frac{1}{b-a}\int_a^b f(\tau)\nabla\tau. \end{aligned}$$

Next, we prove that $l_2 = \frac{1}{b-a}\int_a^b f(\tau)\Delta\tau$.

Let $k_2(t) : [a,b]_{\mathbb{Z}} \to \mathbb{T}_{[a,b]}$ be defined by $k_2(t) = \frac{t-a}{b-a}$ with $t \in [a,b]_{\mathbb{Z}}$. Then $k_2(t)$ is increasing, $k_2^{-1}(t) = (1-t)a + tb$ and $k_2^{\Delta}(t) = \frac{1}{b-a}$.

Using Theorem 2.1, we get

$$\begin{aligned} l_2 &= \int_{0=k_2(a)}^{1=k_2(b)} (f \circ k_2^{-1})(t)\widetilde{\Delta} t = \int_a^b f(\tau)k_2^{\Delta}(\tau)\Delta\tau \\ &= \frac{1}{b-a}\int_a^b f(\tau)\Delta\tau. \end{aligned}$$

Inserting l_1 and l_2 into (3.2), we find that

$$f\left(\frac{a+b}{2}\right) \le \frac{h\left(\frac{1}{2}\right)}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \le h\left(\frac{1}{2}\right) M\big[f(a)+f(b)\big]. \quad (3.3)$$

Therefore, the inequalities (3.1) hold. □

Corollary 3.1. *Suppose that $f : [a,b]_{\mathbb{Z}} \to \mathbb{R}$ is an h-concave function with $a < b$, $a, b, \frac{a+b}{2} \in \mathbb{Z}$, and $h\left(\frac{1}{2}\right) \neq 0$. Then we have*

$$\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \ge \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \ge M\big[f(a)+f(b)\big], \quad (3.4)$$

where M is given in Theorem 3.1.

Similar to the proof of Theorem 3.1, changing direction with every inequality sign, this result is obtained.

Remark 3.1. If the special functions are taken in Theorem 3.1, the corresponding discrete H-H inequalities for integer order related to the endpoints can be obtained:

(1) If $h(x) = x^s$, then (3.1) becomes an inequality for s-convex functions in the second sense on $\mathbb{Z}$:

$$\begin{aligned} 2^s f\left(\frac{a+b}{2}\right) &\le \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \\ &\le \big[f(a)+f(b)\big]\int_{\mathbb{T}_{[a,b]}} \big[t^s + (1-t)^s\big]\widetilde{\Delta} t. \end{aligned} \quad (3.5)$$

(2) If $h(x) = x$, then (3.1) becomes an inequality for convex functions on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{2(b-a)}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \le \frac{f(a)+f(b)}{2}. \quad (3.6)$$

(3) If $h(x)=1$, then (3.1) becomes an inequality for P-functions on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \leq 2\big[f(a)+f(b)\big]. \tag{3.7}$$

(4) If f is an h-concave function on $\mathbb{Z}$, the three discrete H-H inequalities for integer order in the above special cases of (1)-(3) can be obtained from Corollary 3.1, where we just reverse the inequality signs in (3.5)-(3.7).

Next, we prove the discrete H-H inequalities for integer order via h-convex functions on $\mathbb{Z}$ relating to the midpoint.

Theorem 3.2. *Suppose that $f : [a,b]_{\mathbb{Z}} \to \mathbb{R}$ is an h-convex function with $a<b$, $a, b, \frac{a+b}{2} \in \mathbb{Z}$, and $h\left(\frac{1}{2}\right) \neq 0$. Then we have*

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \leq& \frac{2}{b-a}\left[\int_{\frac{a+b}{2}}^b f(\tau)\nabla\tau + \int_a^{\frac{a+b}{2}} f(\tau)\Delta\tau\right] \\ \leq& N\big[f(a)+f(b)\big],\end{aligned} \tag{3.8}$$

where $N = \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left[h\left(\frac{t}{2}\right) + h\left(1-\frac{t}{2}\right)\right] \widetilde{\Delta} t$.

Proof. Fixing $t \in \mathbb{T}_{\left[\frac{a+b}{2},b\right]} \setminus \{0,1\}$, we define

$$x = \frac{t}{2}a + \frac{2-t}{2}b, y = \frac{2-t}{2}a + tb,$$

then $x, y \in [a,b]_{\mathbb{Z}}$ and $\frac{x+y}{2} = \frac{a+b}{2} \in \mathbb{Z}$. Since f is an h-convex function on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)\big[f(x)+f(y)\big].$$

This implies that

$$\begin{aligned}f\left(\frac{a+b}{2}\right) \leq& h\left(\frac{1}{2}\right)\left[f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right] \\ \leq& h\left(\frac{1}{2}\right)\left[h\left(\frac{t}{2}\right)f(a) + h\left(\frac{2-t}{2}\right)f(b)\right. \\ &\left. + h\left(\frac{2-t}{2}\right)f(a) + h\left(\frac{t}{2}\right)f(b)\right] \\ =& h\left(\frac{1}{2}\right)\left[h\left(\frac{t}{2}\right) + h\left(\frac{2-t}{2}\right)\right]\big[f(a)+f(b)\big].\end{aligned}$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{\left[\frac{a+b}{2},b\right]}$, then we have

$$\begin{aligned}&\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left(\frac{a+b}{2}\right)\widetilde{\Delta} t \\ &\quad\leq h\left(\frac{1}{2}\right)\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left[f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right]\widetilde{\Delta} t \\ &\quad\leq h\left(\frac{1}{2}\right) N\big[f(a)+f(b)\big],\end{aligned} \tag{3.9}$$

where $N = \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left[h\left(\frac{t}{2}\right) + h\left(1-\frac{t}{2}\right)\right] \widetilde{\Delta} t$.

Define

$$r_1 = \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \widetilde{\Delta} t, r_2 = \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \widetilde{\Delta} t,$$

then $r_1 = \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau$, $r_2 = \frac{2}{b-a}\int_{a}^{\frac{a+b}{2}} f(\tau)\Delta\tau$ (see [37]).

Inserting r_1 and r_2 into (3.9), we find that

$$f\left(\frac{a+b}{2}\right) \leq \frac{2h\left(\frac{1}{2}\right)}{b-a}\left[\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau + \int_{a}^{\frac{a+b}{2}} f(\tau)\Delta\tau\right] \leq h\left(\frac{1}{2}\right) N\big[f(a)+f(b)\big].$$

Therefore, the inequality (3.8) hold. □

Corollary 3.2. *Suppose that $f : [a,b]_{\mathbb{Z}} \to \mathbb{R}$ is an h-concave function with $a < b$, $a, b, \frac{a+b}{2} \in \mathbb{Z}$, and $h\left(\frac{1}{2}\right) \neq 0$. Then we have*

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) \geq \frac{2}{b-a}&\left[\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau + \int_{a}^{\frac{a+b}{2}} f(\tau)\Delta\tau\right] \\ &\geq N\big[f(a)+f(b)\big],\end{aligned} \tag{3.10}$$

where N is given in Theorem 3.2.

Similar to the proof of Theorem 3.2, changing direction with every inequality sign, this result is obtained.

Remark 3.2. If the special functions are taken in Theorem 3.2, the corresponding discrete H-H inequalities for integer order related to the midpoint can be obtained:

(1) If $h(x) = x^s$, then (3.8) becomes an inequality for s-convex functions in the second sense on $\mathbb{Z}$:

$$\begin{aligned}2^s f\left(\frac{a+b}{2}\right) \leq& \frac{2}{b-a}\left[\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau + \int_{a}^{\frac{a+b}{2}} f(\tau)\Delta\tau\right] \\ \leq& \big[f(a)+f(b)\big]\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left[\left(\frac{t}{2}\right)^s + \left(\frac{2-t}{2}\right)^s\right] \widetilde{\Delta} t.\end{aligned} \tag{3.11}$$

(2) If $h(x) = x$, then (3.8) becomes an inequality for convex functions on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\left[\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau + \int_{a}^{\frac{a+b}{2}} f(\tau)\Delta\tau\right] \leq \frac{f(a)+f(b)}{2}. \tag{3.12}$$

(3) If $h(x) = 1$, then (3.8) becomes an inequality for P-functions on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2}{b-a}\left[\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau + \int_{a}^{\frac{a+b}{2}} f(\tau)\Delta\tau\right] \leq 2\big[f(a)+f(b)\big]. \tag{3.13}$$

(4) If f is an h-concave function on $\mathbb{Z}$, the three discrete H-H inequalities for integer order in the above special cases of (1)-(3) can be obtained from Corollary 3.2, where we just reverse the inequality signs in (3.11)-(3.13).

Because $[a,b] = \left[a, \frac{a+b}{2}\right] \cup \left[\frac{a+b}{2}, b\right]$, we can also obtain the following discrete H-H inequalities for integer order.

Theorem 3.3. *Suppose that* $f : [a,b]_{\mathbb{Z}} \to \mathbb{R}$ *is an* h*-convex function with* $a < b$, $a, b, \frac{a+b}{2}, \frac{3a+b}{4}, \frac{a+3b}{4} \in \mathbb{Z}$, *and* $h\left(\frac{1}{2}\right) \neq 0$. *Then we have*

$$\begin{aligned} \frac{1}{2h^2\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) &\leq \Omega_1 \\ &\leq \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \leq \Omega_2 \qquad (3.14) \\ &\leq \left[\frac{1}{2} + h\left(\frac{1}{2}\right)\right]\left[f(a)+f(b)\right]\int_0^1 \left[h(t)+h(1-t)\right]\widetilde{\Delta}t, \end{aligned}$$

where

$$\begin{aligned} \Omega_1 &= \frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right], \\ \Omega_2 &= \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right]\int_0^1 \left[h(t)+h(1-t)\right]\widetilde{\Delta}t. \end{aligned}$$

Proof. Fixing $t \in \mathbb{T}_{[a,b]} \setminus \{0,1\}$, we define

$$x = ta + (1-t)b, y = (1-t)a + tb.$$

It is easy to see that $x, y \in [a,b]_{\mathbb{Z}}$ and $\frac{x+y}{2} = \frac{a+b}{2} \in \mathbb{Z}$. Since f is an h-convex function on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f(x)+f(y)\right].$$

For $\left[a, \frac{a+b}{2}\right]_{\mathbb{Z}}$, fixing $t \in \mathbb{T}_{\left[a, \frac{a+b}{2}\right]} \setminus \{0,1\}$, we define

$$x = at + \frac{a+b}{2}(1-t), y = a(1-t) + \frac{a+b}{2}t,$$

then we can get

$$\begin{aligned} f\left(\frac{3a+b}{4}\right) =& f\left(\frac{ta + (1-t)\frac{a+b}{2} + (1-t)a + t\frac{a+b}{2}}{2}\right) \\ \leq& h\left(\frac{1}{2}\right)\left[f\left(ta + (1-t)\frac{a+b}{2}\right) + f\left((1-t)a + t\frac{a+b}{2}\right)\right] \\ \leq& h\left(\frac{1}{2}\right)\left[h(t)f(a) + h(1-t)f\left(\frac{a+b}{2}\right)\right. \\ &\left. + h(1-t)f(a) + h(t)f\left(\frac{a+b}{2}\right)\right] \\ =& h\left(\frac{1}{2}\right)\left[f(a) + f\left(\frac{a+b}{2}\right)\right]\left[h(t)+h(1-t)\right]. \end{aligned}$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{\left[a,\,\frac{a+b}{2}\right]}$, then we have

$$\begin{aligned}
&\int_{\mathbb{T}_{\left[a,\,\frac{a+b}{2}\right]}} f\left(\frac{3a+b}{2}\right)\widetilde{\Delta}t \\
&\qquad\le h\left(\frac{1}{2}\right)\left[\int_{\mathbb{T}_{\left[a,\,\frac{a+b}{2}\right]}} f\left(ta+(1-t)\frac{a+b}{2}\right)\widetilde{\Delta}t\right. \\
&\qquad\quad\left.+\int_{\mathbb{T}_{\left[a,\,\frac{a+b}{2}\right]}} f\left((1-t)a+t\frac{a+b}{2}\right)\widetilde{\Delta}t\right] \\
&\qquad\le h\left(\frac{1}{2}\right)\left[f(a)+f\left(\frac{a+b}{2}\right)\right]\int_{\mathbb{T}_{\left[a,\,\frac{a+b}{2}\right]}}\left[h(t)+h(1-t)\right]\widetilde{\Delta}t.
\end{aligned} \tag{3.15}$$

Define

$$\begin{aligned}
p_1 &= \int_{\mathbb{T}_{\left[a,\,\frac{a+b}{2}\right]}} f\left(ta+(1-t)\frac{a+b}{2}\right)\widetilde{\Delta}t, \\
p_2 &= \int_{\mathbb{T}_{\left[a,\,\frac{a+b}{2}\right]}} f\left((1-t)a+t\frac{a+b}{2}\right)\widetilde{\Delta}t.
\end{aligned}$$

Calculate p_1 and p_2 separately below.

First, we assert that $p_1 = \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(\tau)\nabla\tau$.

Let $q_1(t) = \frac{a+b-2t}{b-a}$, $t \in \left[a, \frac{a+b}{2}\right]_{\mathbb{Z}}$. Then $q_1(t)$ is decreasing, $q_1^{-1}(t) = ta + (1-t)\frac{a+b}{2}$ and $(-q_1^{\nabla})(t) = \frac{2}{b-a}$.

According to Theorem 2.2, we obtain

$$\begin{aligned}
p_1 &= \int_{0=q_1\left(\frac{a+b}{2}\right)}^{1=q_1(a)} (f\circ q_1^{-1})(t)\widetilde{\Delta}t = \int_a^{\frac{a+b}{2}} f(\tau)(-q_1^{\nabla})(\tau)\nabla\tau \\
&= \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(\tau)\nabla\tau.
\end{aligned}$$

Next, we prove that $p_2 = \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(\tau)\Delta\tau$.

Let $q_2(t) = \frac{2(t-a)}{b-a}$, $t \in \left[a, \frac{a+b}{2}\right]_{\mathbb{Z}}$. Then $q_2(t)$ is increasing, $q_2^{-1}(t) = (1-t)a + t\frac{a+b}{2}$ and $q_2^{\Delta}(t) = \frac{2}{b-a}$.

Using Theorem 2.1, we get

$$\begin{aligned}
p_2 &= \int_{0=q_2(a)}^{1=q_2\left(\frac{a+b}{2}\right)} (f\circ q_2^{-1})(t)\widetilde{\Delta}t = \int_a^{\frac{a+b}{2}} f(\tau)q_2^{\Delta}(\tau)\Delta\tau \\
&= \frac{2}{b-a}\int_a^{\frac{a+b}{2}} f(\tau)\Delta\tau.
\end{aligned}$$

Inserting p_1 and p_2 into (3.15), we find that

$$\begin{aligned} f\left(\frac{3a+b}{4}\right) \leq& \frac{2h\left(\frac{1}{2}\right)}{b-a}\left[\int_a^{\frac{a+b}{2}} f(\tau)\nabla\tau + \int_a^{\frac{a+b}{2}} f(\tau)\Delta\tau\right] \\ \leq& h\left(\frac{1}{2}\right)\left[f(a)+f\left(\frac{a+b}{2}\right)\right]\int_{\mathbb{T}_{\left[a,\,\frac{a+b}{2}\right]}}\big[h(t)+h(1-t)\big]\widetilde{\Delta}t. \end{aligned} \tag{3.16}$$

Similarly, for $\left[\frac{a+b}{2}, b\right]_{\mathbb{Z}}$, fixing $t \in \mathbb{T}_{\left[\frac{a+b}{2},b\right]} \setminus \{0,1\}$, we define

$$x = t\frac{a+b}{2} + (1-t)b, y = (1-t)\frac{a+b}{2} + tb,$$

then we can get

$$\begin{aligned} f\left(\frac{a+3b}{4}\right) =& f\left(\frac{t\frac{a+b}{2}+(1-t)b+(1-t)\frac{a+b}{2}+tb}{2}\right) \\ \leq& h\left(\frac{1}{2}\right)\left[f\left(t\frac{a+b}{2}+(1-t)b\right)+f\left((1-t)\frac{a+b}{2}+tb\right)\right] \\ \leq& h\left(\frac{1}{2}\right)\left[h(t)f\left(\frac{a+b}{2}\right)+h(1-t)f(b)\right. \\ &\left.+h(1-t)f\left(\frac{a+b}{2}\right)+h(t)f(b)\right] \\ =& h\left(\frac{1}{2}\right)\left[f\left(\frac{a+b}{2}\right)+f(b)\right]\big[h(t)+h(1-t)\big]. \end{aligned}$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{\left[\frac{a+b}{2},b\right]}$, then we have

$$\begin{aligned} &\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left(\frac{a+3b}{4}\right)\widetilde{\Delta}t \\ \leq& h\left(\frac{1}{2}\right)\left[\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left(t\frac{a+b}{2}+(1-t)b\right)\widetilde{\Delta}t\right. \\ &\left.+\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left((1-t)\frac{a+b}{2}+tb\right)\widetilde{\Delta}t\right] \\ \leq& h\left(\frac{1}{2}\right)\left[f\left(\frac{a+b}{2}\right)+f(b)\right]\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}}\big[h(t)+h(1-t)\big]\widetilde{\Delta}t. \end{aligned} \tag{3.17}$$

Define

$$\begin{aligned} p_3 =& \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left(t\frac{a+b}{2}+(1-t)b\right)\widetilde{\Delta}t, \\ p_4 =& \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left((1-t)\frac{a+b}{2}+tb\right)\widetilde{\Delta}t. \end{aligned}$$

Calculate p_3 and p_4 separately below.

First, we assert that $p_3 = \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau$.

Let $q_3(t) = \frac{2(b-t)}{b-a}$, $t \in \left[\frac{a+b}{2}, b\right]_{\mathbb{Z}}$. Then $q_3(t)$ is decreasing, $q_3^{-1}(t) = t\frac{a+b}{2} + (1-t)b$ and $(-q_3^{\nabla})(t) = \frac{2}{b-a}$.

According to Theorem 2.2, we obtain

$$\begin{aligned} p_3 &= \int_{0=q_3(b)}^{1=q_3\left(\frac{a+b}{2}\right)} (f \circ q_3^{-1})(t)\widetilde{\Delta}t = \int_{\frac{a+b}{2}}^{b} f(\tau)(-q_3^{\nabla})(\tau)\nabla\tau \\ &= \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau. \end{aligned}$$

Next, we prove that $p_4 = \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(\tau)\Delta\tau$.

Let $q_4(t) = \frac{2t-(a+b)}{b-a}$, $t \in \left[\frac{a+b}{2}, b\right]_{\mathbb{Z}}$. Then $q_4(t)$ is increasing, $q_4^{-1}(t) = (1-t)\frac{a+b}{2} + tb$ and $q_4^{\Delta}(t) = \frac{2}{b-a}$.

Using Theorem 2.1, we get

$$\begin{aligned} p_4 &= \int_{0=q_4\left(\frac{a+b}{2}\right)}^{1=q_4(b)} (f \circ q_4^{-1})(t)\widetilde{\Delta}t = \int_{\frac{a+b}{2}}^{b} f(\tau)q_4^{\Delta}(\tau)\Delta\tau \\ &= \frac{2}{b-a}\int_{\frac{a+b}{2}}^{b} f(\tau)\Delta\tau. \end{aligned}$$

Inserting p_3 and p_4 into (3.17), we find that

$$\begin{aligned} f\left(\frac{a+3b}{4}\right) \leq & \frac{2h\left(\frac{1}{2}\right)}{b-a}\left[\int_{\frac{a+b}{2}}^{b} f(\tau)\nabla\tau + \int_{\frac{a+b}{2}}^{b} f(\tau)\Delta\tau\right] \\ \leq & h\left(\frac{1}{2}\right)\left[f\left(\frac{a+b}{2}\right) + f(b)\right]\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left[h(t) + h(1-t)\right]\widetilde{\Delta}t. \end{aligned} \tag{3.18}$$

Adding (3.16) and (3.18), we get

$$\begin{aligned} & f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \\ & \quad \leq \frac{2h\left(\frac{1}{2}\right)}{b-a}\left[\int_{a}^{b} f(\tau)\nabla\tau + \int_{a}^{b} f(\tau)\Delta\tau\right] \\ & \quad \leq h\left(\frac{1}{2}\right)\left[f(a) + f\left(\frac{a+b}{2}\right)\right]\int_{\mathbb{T}_{\left[a,\frac{a+b}{2}\right]}} \left[h(t) + h(1-t)\right]\widetilde{\Delta}t \\ & \qquad + h\left(\frac{1}{2}\right)\left[f\left(\frac{a+b}{2}\right) + f(b)\right]\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left[h(t) + h(1-t)\right]\widetilde{\Delta}t \\ & \quad \leq h\left(\frac{1}{2}\right)\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\int_{0}^{1} \left[h(t) + h(1-t)\right]\widetilde{\Delta}t, \end{aligned}$$

namely,

$$\begin{aligned}\Omega_1 =&\frac{1}{2h\left(\frac{1}{2}\right)}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\ \leq&\frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau+\int_a^b f(\tau)\Delta\tau\right]\\ \leq&\frac{1}{2}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]\int_0^1\left[h(t)+h(1-t)\right]\widetilde{\Delta}t\\ =&\Omega_2.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\frac{1}{2h^2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\\ &\quad=\frac{1}{2h^2\left(\frac{1}{2}\right)}f\left(\frac{1}{2}\frac{3a+b}{4}+\frac{1}{2}\frac{a+3b}{4}\right)\\ &\quad\leq\frac{1}{2h^2\left(\frac{1}{2}\right)}\left[h\left(\frac{1}{2}\right)f\left(\frac{3a+b}{4}\right)+h\left(\frac{1}{2}\right)f\left(\frac{a+3b}{4}\right)\right]\\ &\quad=\Omega_1\\ &\quad\leq\frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau+\int_a^b f(\tau)\Delta\tau\right]\\ &\quad\leq\Omega_2\\ &\quad\leq\left[\frac{f(a)+f(b)}{2}+h\left(\frac{1}{2}\right)\big(f(a)+f(b)\big)\right]\int_0^1\left[h(t)+h(1-t)\right]\widetilde{\Delta}t\\ &\quad=\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\left[f(a)+f(b)\right]\int_0^1\left[h(t)+h(1-t)\right]\widetilde{\Delta}t,\end{aligned}$$

which completes the proof. □

Corollary 3.3. *Suppose that $f:[a,b]_{\mathbb{Z}}\to\mathbb{R}$ is an h-concave function with $a<b$, $a,b,\frac{a+b}{2},\frac{3a+b}{4},\frac{a+3b}{4}\in\mathbb{Z}$, and $h\left(\frac{1}{2}\right)\neq 0$. Then we have*

$$\begin{aligned}\frac{1}{2h^2\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)&\geq\Omega_1\\ &\geq\frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau+\int_a^b f(\tau)\Delta\tau\right]\geq\Omega_2 \qquad (3.19)\\ &\geq\left[\frac{1}{2}+h\left(\frac{1}{2}\right)\right]\left[f(a)+f(b)\right]\int_0^1\left[h(t)+h(1-t)\right]\widetilde{\Delta}t,\end{aligned}$$

where Ω_1,Ω_2 are given in Theorem 3.3.

Similar to the proof of Theorem 3.3, changing direction with every inequality sign, this result is obtained.

Remark 3.3. If the special functions are taken in Theorem 3.3, the corresponding discrete H-H inequalities for integer order can be obtained:

(1) If $h(x) = x^s$, then (3.14) becomes an inequality for s-convex functions in the second sense on $\mathbb{Z}$:

$$\begin{aligned} 2^{2s-1} f\left(\frac{a+b}{2}\right) &\le 2^{s-1}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\le \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \\ &\le \frac{1}{2}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right]\int_0^1 \left[t^s + (1-t)^s\right]\widetilde{\Delta}t \\ &\le \left[\frac{1}{2} + \frac{1}{2^s}\right]\left[f(a) + f(b)\right]\int_0^1 \left[t^s + (1-t)^s\right]\widetilde{\Delta}t. \end{aligned} \tag{3.20}$$

(2) If $h(x) = x$, then (3.14) becomes an inequality for convex functions on $\mathbb{Z}$:

$$\begin{aligned} 2f\left(\frac{a+b}{2}\right) \le& f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right) \\ \le& \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \\ \le& \frac{1}{2}\left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b)\right] \le f(a) + f(b). \end{aligned} \tag{3.21}$$

(3) If $h(x) = 1$, then (3.14) becomes an inequality for P-functions on $\mathbb{Z}$:

$$\begin{aligned} \frac{1}{2} f\left(\frac{a+b}{2}\right) \le& \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ \le& \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \\ \le& f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \le 3\left[f(a) + f(b)\right]. \end{aligned} \tag{3.22}$$

(4) If f is an h-concave function on $\mathbb{Z}$, the three discrete H-H inequalities for integer order in the above special cases of (1)-(3) can be obtained from Corollary 3.3, where we just reverse the inequality signs in (3.20)-(3.22).

Based on the above theorem, new interval piecewise points $t = \frac{3a+b}{4}$ and $t = \frac{a+3b}{4}$ are given, and we obtain a new estimate of the discrete H-H inequalities for integer order.

Theorem 3.4. *Suppose that* $f : [a,b]_{\mathbb{Z}} \to \mathbb{R}$ *is an* h*-convex function with* $a < b$, $a, b, \frac{a+b}{2}, \frac{3a+b}{4}, \frac{a+3b}{4}, \frac{7a+b}{8}, \frac{a+7b}{8} \in \mathbb{Z}$, *and* $h\left(\frac{1}{2}\right) \neq 0$. *Then we have*

$$\begin{aligned} &\frac{1}{4h\left(\frac{1}{2}\right)}\left[\frac{1}{h\left(\frac{1}{2}\right)} + 2\right] f\left(\frac{a+b}{2}\right) \\ &\qquad \le \Omega_3 \le \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \\ &\qquad \le \Omega_4 \le \left\{\frac{1}{4} + \frac{3}{4}\left[h\left(\frac{1}{4}\right) + h\left(\frac{3}{4}\right)\right]\right\}\left[f(a) + f(b)\right] \end{aligned} \tag{3.23}$$

$$\times \int_0^1 \left[h(t) + h(1-t)\right] \widetilde{\Delta} t,$$

where

$$\Omega_3 = \frac{1}{4h\left(\frac{1}{2}\right)} \left[f\left(\frac{7a+b}{8}\right) + 2f\left(\frac{a+b}{2}\right) + f\left(\frac{a+7b}{8}\right) \right],$$

$$\Omega_4 = \left[\frac{f(a)+f(b)}{4} + \frac{3}{4} f\left(\frac{3a+b}{4}\right) + \frac{3}{4} f\left(\frac{a+3b}{4}\right) \right] \int_0^1 \left[h(t) + h(1-t)\right] \widetilde{\Delta} t.$$

Proof. Fixing $t \in \mathbb{T}_{[a,b]} \setminus \{0,1\}$, we define

$$x = ta + (1-t)b, y = (1-t)a + tb.$$

It is easy to see that $x, y \in [a,b]_{\mathbb{Z}}$ and $\frac{x+y}{2} = \frac{a+b}{2} \in \mathbb{Z}$. Since f is an h-convex function on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\left(\frac{x+y}{2}\right) \le h\left(\frac{1}{2}\right) \left[f(x) + f(y)\right].$$

For $\left[a, \frac{3a+b}{4}\right]_{\mathbb{Z}}$, fixing $t \in \mathbb{T}_{\left[a, \frac{3a+b}{4}\right]} \setminus \{0,1\}$, we define

$$x = ta + (1-t)\frac{3a+b}{4}, y = (1-t)a + t\frac{3a+b}{4},$$

then we can get

$$\begin{aligned}
f\left(\frac{7a+b}{8}\right) =& f\left(\frac{ta + (1-t)\frac{3a+b}{4} + (1-t)a + t\frac{3a+b}{4}}{2}\right) \\
\le& h\left(\frac{1}{2}\right) \left[f\left(ta + (1-t)\frac{3a+b}{4}\right) + f\left((1-t)a + t\frac{3a+b}{4}\right) \right] \\
\le& h\left(\frac{1}{2}\right) \left[h(t)f(a) + h(1-t) f\left(\frac{3a+b}{4}\right) \right. \\
& \left. + h(1-t)f(a) + h(t) f\left(\frac{3a+b}{4}\right) \right] \\
=& h\left(\frac{1}{2}\right) \left[f(a) + f\left(\frac{3a+b}{4}\right) \right] \left[h(t) + h(1-t)\right].
\end{aligned}$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{\left[a, \frac{3a+b}{4}\right]}$, then we have

$$\begin{aligned}
&\int_{\mathbb{T}_{\left[a, \frac{3a+b}{4}\right]}} f\left(\frac{7a+b}{8}\right) \widetilde{\Delta} t \\
&\le h\left(\frac{1}{2}\right) \left[\int_{\mathbb{T}_{\left[a, \frac{3a+b}{4}\right]}} f\left(ta + (1-t)\frac{3a+b}{4}\right) \widetilde{\Delta} t \right. \\
&\quad \left. + \int_{\mathbb{T}_{\left[a, \frac{3a+b}{4}\right]}} f\left((1-t)a + t\frac{3a+b}{4}\right) \widetilde{\Delta} t \right] \\
&\le h\left(\frac{1}{2}\right) \left[f(a) + f\left(\frac{3a+b}{4}\right) \right] \int_{\mathbb{T}_{\left[a, \frac{3a+b}{4}\right]}} \left[h(t) + h(1-t)\right] \widetilde{\Delta} t.
\end{aligned} \tag{3.24}$$

Define

$$u_1 = \int_{\mathbb{T}_{\left[a,\, \frac{3a+b}{4}\right]}} f\left(ta + (1-t)\frac{3a+b}{4}\right) \widetilde{\Delta} t,$$
$$u_2 = \int_{\mathbb{T}_{\left[a,\, \frac{3a+b}{4}\right]}} f\left((1-t)a + t\frac{3a+b}{4}\right) \widetilde{\Delta} t.$$

Calculate u_1 and u_2 separately below.

First, we assert that $u_1 = \frac{4}{b-a} \int_a^{\frac{3a+b}{4}} f(\tau)\nabla\tau$.

Let $w_1(t) = \frac{3a+b-4t}{b-a}$, $t \in \left[a, \frac{3a+b}{4}\right]_{\mathbb{Z}}$. Then $w_1(t)$ is decreasing, $w_1^{-1}(t) = ta + (1-t)\frac{3a+b}{4}$ and $(-w_1^{\nabla})(t) = \frac{4}{b-a}$.

According to Theorem 2.2, we obtain

$$u_1 = \int_{0=w_1\left(\frac{3a+b}{4}\right)}^{1=w_1(a)} (f \circ w_1^{-1})(t) \widetilde{\Delta} t = \int_a^{\frac{3a+b}{4}} f(\tau)(-w_1^{\nabla})(\tau)\nabla\tau$$
$$= \frac{4}{b-a} \int_a^{\frac{3a+b}{4}} f(\tau)\nabla\tau.$$

Next, we prove that $u_2 = \frac{4}{b-a} \int_a^{\frac{3a+b}{4}} f(\tau)\Delta\tau$.

Let $w_2(t) = \frac{4(t-a)}{b-a}$, $t \in \left[a, \frac{3a+b}{4}\right]_{\mathbb{Z}}$. Then $w_2(t)$ is increasing, $w_2^{-1}(t) = (1-t)a + t\frac{3a+b}{4}$ and $w_2^{\Delta}(t) = \frac{4}{b-a}$.

Using Theorem 2.1, we get

$$u_2 = \int_{0=w_2(a)}^{1=w_2\left(\frac{3a+b}{4}\right)} (f \circ w_2^{-1})(t) \widetilde{\Delta} t = \int_a^{\frac{3a+b}{4}} f(\tau) w_2^{\Delta}(\tau)\Delta\tau$$
$$= \frac{4}{b-a} \int_a^{\frac{3a+b}{4}} f(\tau)\Delta\tau.$$

Inserting u_1 and u_2 into (3.24), we find that

$$\begin{aligned} f\left(\frac{7a+b}{8}\right) \leq & \frac{4h\left(\frac{1}{2}\right)}{b-a} \left[\int_a^{\frac{3a+b}{4}} f(\tau)\nabla\tau + \int_a^{\frac{3a+b}{4}} f(\tau)\Delta\tau\right] \\ \leq & h\left(\frac{1}{2}\right) \left[f(a) + f\left(\frac{3a+b}{4}\right)\right] \int_{\mathbb{T}_{\left[a,\, \frac{3a+b}{4}\right]}} \left[h(t) + h(1-t)\right] \widetilde{\Delta} t. \end{aligned} \tag{3.25}$$

Similarly, for $\left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]_{\mathbb{Z}}$, fixing $t \in \mathbb{T}_{\left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]} \setminus \{0, 1\}$, we define

$$x = t\frac{3a+b}{4} + (1-t)\frac{a+3b}{4}, y = (1-t)\frac{3a+b}{4} + t\frac{a+3b}{4},$$

then we can get

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) =& f\left(\frac{t\frac{3a+b}{4}+(1-t)\frac{a+3b}{4}+(1-t)\frac{3a+b}{4}+t\frac{a+3b}{4}}{2}\right)\\
\leq& h\left(\frac{1}{2}\right)\left[f\left(t\frac{3a+b}{4}+(1-t)\frac{a+3b}{4}\right)+f\left((1-t)\frac{3a+b}{4}+t\frac{a+3b}{4}\right)\right]\\
\leq& h\left(\frac{1}{2}\right)\left[h(t)f\left(\frac{3a+b}{4}\right)+h(1-t)f\left(\frac{a+3b}{4}\right)\right.\\
&\left.+h(1-t)f\left(\frac{3a+b}{4}\right)+h(t)f\left(\frac{a+3b}{4}\right)\right]\\
=& h\left(\frac{1}{2}\right)\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\big[h(t)+h(1-t)\big].
\end{aligned}
$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}$, then we have

$$
\begin{aligned}
&\int_{\mathbb{T}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}} f\left(\frac{a+b}{2}\right)\widetilde{\Delta}t\\
&\quad\leq h\left(\frac{1}{2}\right)\left[\int_{\mathbb{T}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}} f\left(t\frac{3a+b}{4}+(1-t)\frac{a+3b}{4}\right)\widetilde{\Delta}t\right.\\
&\qquad\left.+\int_{\mathbb{T}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}} f\left((1-t)\frac{3a+b}{4}+t\frac{a+3b}{4}\right)\widetilde{\Delta}t\right]\\
&\quad\leq h\left(\frac{1}{2}\right)\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\
&\qquad\times\int_{\mathbb{T}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}}\big[h(t)+h(1-t)\big]\widetilde{\Delta}t.
\end{aligned}
\tag{3.26}
$$

Define

$$
\begin{aligned}
u_3 &= \int_{\mathbb{T}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}} f\left(t\frac{3a+b}{4}+(1-t)\frac{a+3b}{4}\right)\widetilde{\Delta}t,\\
u_4 &= \int_{\mathbb{T}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}} f\left((1-t)\frac{3a+b}{4}+t\frac{a+3b}{4}\right)\widetilde{\Delta}t.
\end{aligned}
$$

Calculate u_3 and u_4 separately below.

First, we assert that $u_3 = \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(\tau)\nabla\tau$.

Let $w_3(t) = \frac{a+3b-4t}{2(b-a)}$, $t \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]_{\mathbb{Z}}$. Then $w_3(t)$ is decreasing, $w_3^{-1}(t) = t\frac{3a+b}{4}+(1-t)\frac{a+3b}{4}$ and $(-w_3^{\nabla})(t) = \frac{2}{b-a}$.

According to Theorem 2.2, we obtain

$$\begin{aligned} u_3 &= \int_{0=w_3\left(\frac{a+3b}{4}\right)}^{1=w_3\left(\frac{3a+b}{4}\right)} (f \circ w_3^{-1})(t)\widetilde{\Delta} t = \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(\tau)(-w_3^{\nabla})(\tau)\nabla\tau \\ &= \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(\tau)\nabla\tau. \end{aligned}$$

Next, we prove that $u_4 = \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(\tau)\Delta\tau$.

Let $w_4(t) = \frac{4t-(3a+b)}{2(b-a)}$, $t \in \left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]_{\mathbb{Z}}$. Then $w_4(t)$ is increasing, $w_4^{-1}(t) = (1-t)\frac{3a+b}{4} + t\frac{a+3b}{4}$ and $w_4^{\Delta}(t) = \frac{2}{b-a}$.

Using Theorem 2.1, we get

$$\begin{aligned} u_4 &= \int_{0=w_4\left(\frac{3a+b}{4}\right)}^{1=w_4\left(\frac{a+3b}{4}\right)} (f \circ w_4^{-1})(t)\widetilde{\Delta} t = \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(\tau)w_4^{\Delta}(\tau)\Delta\tau \\ &= \frac{2}{b-a}\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(\tau)\Delta\tau. \end{aligned}$$

Inserting u_3 and u_4 into (3.26), we find that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \le& \frac{2h\left(\frac{1}{2}\right)}{b-a}\left[\int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(\tau)\nabla\tau + \int_{\frac{3a+b}{4}}^{\frac{a+3b}{4}} f(\tau)\Delta\tau\right] \\ \le& h\left(\frac{1}{2}\right)\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\times \int_{\mathbb{T}_{\left[\frac{3a+b}{4}, \frac{a+3b}{4}\right]}} \left[h(t) + h(1-t)\right]\widetilde{\Delta} t. \end{aligned} \tag{3.27}$$

Similarly, for $\left[\frac{a+3b}{4}, b\right]_{\mathbb{Z}}$, fixing $t \in \mathbb{T}_{\left[\frac{a+3b}{4}, b\right]} \setminus \{0, 1\}$, we define

$$x = t\frac{a+3b}{4} + (1-t)b, y = (1-t)\frac{a+3b}{4} + tb,$$

then we can get

$$\begin{aligned} f\left(\frac{a+7b}{8}\right) =& f\left(\frac{t\frac{a+3b}{4} + (1-t)b + (1-t)\frac{a+3b}{4} + tb}{2}\right) \\ \le& h\left(\frac{1}{2}\right)\left[f\left(t\frac{a+3b}{4} + (1-t)b\right) + f\left((1-t)\frac{a+3b}{4} + tb\right)\right] \\ \le& h\left(\frac{1}{2}\right)\left[h(t)f\left(\frac{a+3b}{4}\right) + h(1-t)f(b)\right. \\ &\left. + h(1-t)f\left(\frac{a+3b}{4}\right) + h(t)f(b)\right] \\ \le& h\left(\frac{1}{2}\right)\left[f\left(\frac{a+3b}{4}\right) + f(b)\right]\left[h(t) + h(1-t)\right]. \end{aligned}$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}$, then we have

$$\begin{aligned}
&\int_{\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}} f\left(\frac{a+7b}{8}\right)\widetilde{\Delta}t \\
&\quad\le h\left(\frac{1}{2}\right)\left[\int_{\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}} f\left(t\frac{a+3b}{4}+(1-t)b\right)\widetilde{\Delta}t\right. \\
&\qquad\left.+\int_{\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}} f\left((1-t)\frac{a+3b}{4}+tb\right)\widetilde{\Delta}t\right] \\
&\quad\le h\left(\frac{1}{2}\right)\left[f\left(\frac{a+3b}{4}\right)+f(b)\right]\int_{\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}}\big[h(t)+h(1-t)\big]\widetilde{\Delta}t.
\end{aligned} \tag{3.28}$$

Define

$$\begin{aligned}
u_5 &= \int_{\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}} f\left(t\frac{a+3b}{4}+(1-t)b\right)\widetilde{\Delta}t, \\
u_6 &= \int_{\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}} f\left((1-t)\frac{a+3b}{4}+tb\right)\widetilde{\Delta}t.
\end{aligned}$$

Calculate u_5 and u_6 separately below.

First, we assert that $u_5 = \frac{4}{b-a}\int_{\frac{a+3b}{4}}^{b} f(\tau)\nabla\tau$.

Let $w_5(t) = \frac{4(b-t)}{b-a}$, $t \in \left[\frac{a+3b}{4}, b\right]_{\mathbb{Z}}$. Then $w_5(t)$ is decreasing, $w_5^{-1}(t) = t\frac{a+3b}{4} + (1-t)b$ and $(-w_5^{\nabla})(t) = \frac{4}{b-a}$.

According to Theorem 2.2, we obtain

$$\begin{aligned}
u_5 &= \int_{0=w_5(b)}^{1=w_5\left(\frac{a+3b}{4}\right)} (f\circ w_5^{-1})(t)\widetilde{\Delta}t = \int_{\frac{a+3b}{4}}^{b} f(\tau)(-w_5^{\nabla})(\tau)\nabla\tau \\
&= \frac{4}{b-a}\int_{\frac{a+3b}{4}}^{b} f(\tau)\nabla\tau.
\end{aligned}$$

Next, we prove that $u_6 = \frac{4}{b-a}\int_{\frac{a+3b}{4}}^{b} f(\tau)\Delta\tau$.

Let $w_6(t) = \frac{4t-(a+3b)}{b-a}$, $t \in \left[\frac{a+3b}{4}, b\right]_{\mathbb{Z}}$. Then $w_6(t)$ is increasing, $w_6^{-1}(t) = (1-t)\frac{a+3b}{4} + tb$ and $w_6^{\Delta}(t) = \frac{4}{b-a}$.

Using Theorem 2.1, we get

$$\begin{aligned}
u_6 &= \int_{0=w_6\left(\frac{a+3b}{4}\right)}^{1=w_6(b)} (f\circ w_6^{-1})(t)\widetilde{\Delta}t = \int_{\frac{a+3b}{4}}^{b} f(\tau)w_6^{\Delta}(\tau)\Delta\tau \\
&= \frac{4}{b-a}\int_{\frac{a+3b}{4}}^{b} f(\tau)\Delta\tau.
\end{aligned}$$

Inserting u_5 and u_6 into (3.28), we find that

$$\begin{aligned} f\left(\frac{a+7b}{8}\right) \leq & \frac{4h\left(\frac{1}{2}\right)}{b-a}\left[\int_{\frac{a+3b}{4}}^{b} f(\tau)\nabla\tau + \int_{\frac{a+3b}{4}}^{b} f(\tau)\Delta\tau\right] \\ \leq & h\left(\frac{1}{2}\right)\left[f\left(\frac{a+3b}{4}\right)+f(b)\right]\int_{\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}}\left[h(t)+h(1-t)\right]\widetilde{\Delta}t. \end{aligned} \tag{3.29}$$

Letting (3.25) + 2 × (3.27) + (3.29), we have

$$\begin{aligned} f\left(\frac{7a+b}{8}\right) & +2f\left(\frac{a+b}{2}\right)+f\left(\frac{a+7b}{8}\right) \\ \leq & \frac{4h\left(\frac{1}{2}\right)}{b-a}\left[\int_{a}^{b} f(\tau)\nabla\tau + \int_{a}^{b} f(\tau)\Delta\tau\right] \\ \leq & h\left(\frac{1}{2}\right)\left[f(a)+f\left(\frac{3a+b}{4}\right)\right]\int_{\mathbb{T}_{\left[a,\frac{3a+b}{4}\right]}}\left[h(t)+h(1-t)\right]\widetilde{\Delta}t \\ & + 2h\left(\frac{1}{2}\right)\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\ & \times \int_{\mathbb{T}_{\left[\frac{3a+b}{4},\frac{a+3b}{4}\right]}}\left[h(t)+h(1-t)\right]\widetilde{\Delta}t \\ & + h\left(\frac{1}{2}\right)\left[f\left(\frac{a+3b}{4}\right)+f(b)\right]\int_{\mathbb{T}_{\left[\frac{a+3b}{4},b\right]}}\left[h(t)+h(1-t)\right]\widetilde{\Delta}t \\ \leq & h\left(\frac{1}{2}\right)\left[f(a)+3f\left(\frac{3a+b}{4}\right)+3f\left(\frac{a+3b}{4}\right)+f(b)\right] \\ & \times \int_{0}^{1}\left[h(t)+h(1-t)\right]\widetilde{\Delta}t, \end{aligned}$$

namely,

$$\begin{aligned} \Omega_3 = & \frac{1}{4h\left(\frac{1}{2}\right)}\left[f\left(\frac{7a+b}{8}\right)+2f\left(\frac{a+b}{2}\right)+f\left(\frac{a+7b}{8}\right)\right] \\ \leq & \frac{1}{b-a}\left[\int_{a}^{b} f(\tau)\nabla\tau + \int_{a}^{b} f(\tau)\Delta\tau\right] \\ \leq & \frac{1}{4}\left[f(a)+3f\left(\frac{3a+b}{4}\right)+3f\left(\frac{a+3b}{4}\right)+f(b)\right]\int_{0}^{1}\left[h(t)+h(1-t)\right]\widetilde{\Delta}t \\ = & \Omega_4. \end{aligned}$$

Hence, we have

$$\begin{aligned} & \frac{1}{4h\left(\frac{1}{2}\right)}\left[\frac{1}{h\left(\frac{1}{2}\right)}+2\right]f\left(\frac{a+b}{2}\right) \\ = & \frac{1}{4h^2\left(\frac{1}{2}\right)}\left[f\left(\frac{1}{2}\cdot\frac{7a+b}{8}+\frac{1}{2}\cdot\frac{a+7b}{8}\right)+2h\left(\frac{1}{2}\right)f\left(\frac{a+b}{2}\right)\right] \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{4h^2\left(\frac{1}{2}\right)}\left[h\left(\frac{1}{2}\right)f\left(\frac{7a+b}{8}\right)+2h\left(\frac{1}{2}\right)f\left(\frac{a+b}{2}\right)+h\left(\frac{1}{2}\right)f\left(\frac{a+7b}{8}\right)\right]\\
&=\Omega_3\\
&\leq \frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau+\int_a^b f(\tau)\Delta\tau\right]\\
&\leq \Omega_4\\
&\leq \left\{\frac{1}{4}+\frac{3}{4}\left[h\left(\frac{1}{4}\right)+h\left(\frac{3}{4}\right)\right]\right\}\left[f(a)+f(b)\right]\int_0^1\left[h(t)+h(1-t)\right]\widetilde{\Delta}t,
\end{aligned}
$$

which completes the proof. □

Corollary 3.4. *Suppose that $f:[a,b]_{\mathbb{Z}}\to\mathbb{R}$ is an h-concave function with $a<b$, $a,b,\frac{a+b}{2},\frac{3a+b}{4},\frac{a+3b}{4},\frac{7a+b}{8},\frac{a+7b}{8}\in\mathbb{Z}$, and $h\left(\frac{1}{2}\right)\neq 0$. Then we have*

$$
\begin{aligned}
&\frac{1}{4h\left(\frac{1}{2}\right)}\left[\frac{1}{h\left(\frac{1}{2}\right)}+2\right]f\left(\frac{a+b}{2}\right)\\
&\qquad\geq\Omega_3\geq\frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau+\int_a^b f(\tau)\Delta\tau\right]\\
&\qquad\geq\Omega_4\geq\left\{\frac{1}{4}+\frac{3}{4}\left[h\left(\frac{1}{4}\right)+h\left(\frac{3}{4}\right)\right]\right\}\left[f(a)+f(b)\right]\\
&\qquad\times\int_0^1\left[h(t)+h(1-t)\right]\widetilde{\Delta}t,
\end{aligned}
\tag{3.30}
$$

where Ω_3,Ω_4 are given in Theorem 3.4.

Similar to the proof of Theorem 3.4, changing direction with every inequality sign, this result is obtained.

Remark 3.4. If the special functions are taken in Theorem 3.4, the corresponding discrete H-H inequalities for integer order can be obtained:

(1) If $h(x)=x^s$, then (3.23) becomes an inequality for s-convex functions in the second sense on $\mathbb{Z}$:

$$
\begin{aligned}
&\left[2^{2s-2}+2^{s-1}\right]f\left(\frac{a+b}{2}\right)\\
&\qquad\leq 2^{s-2}\left[f\left(\frac{7a+b}{8}\right)+2f\left(\frac{a+b}{2}\right)+f\left(\frac{a+7b}{8}\right)\right]\\
&\qquad\leq\frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau+\int_a^b f(\tau)\Delta\tau\right]\\
&\qquad\leq\frac{1}{4}\left[f(a)+3f\left(\frac{3a+b}{4}\right)+3f\left(\frac{a+3b}{4}\right)+f(b)\right]\\
&\qquad\times\int_0^1\left[t^s+(1-t)^s\right]\widetilde{\Delta}t\\
&\qquad\leq\left\{\frac{1}{4}+\frac{3}{4}\left[\left(\frac{1}{4}\right)^s+\left(\frac{3}{4}\right)^s\right]\right\}\left[f(a)+f(b)\right]
\end{aligned}
\tag{3.31}
$$

$$\times \int_0^1 \left[t^s + (1-t)^s\right] \widetilde{\Delta} t.$$

(2) If $h(x) = x$, then (3.23) becomes an inequality for convex functions on $\mathbb{Z}$:

$$\begin{aligned} &2 \cdot f\left(\frac{a+b}{2}\right) \\ \leq &\frac{1}{2}\left[f\left(\frac{7a+b}{8}\right) + 2f\left(\frac{a+b}{2}\right) + f\left(\frac{a+7b}{8}\right)\right] \\ \leq &\frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \\ \leq &\frac{1}{4}\left[f(a) + 3f\left(\frac{3a+b}{4}\right) + 3f\left(\frac{a+3b}{4}\right) + f(b)\right] \\ \leq &f(a) + f(b). \end{aligned} \tag{3.32}$$

(3) If $h(x) = 1$, then (3.23) becomes an inequality for P-functions on $\mathbb{Z}$:

$$\begin{aligned} &\frac{3}{4} \cdot f\left(\frac{a+b}{2}\right) \\ \leq &\frac{1}{4}\left[f\left(\frac{7a+b}{8}\right) + 2f\left(\frac{a+b}{2}\right) + f\left(\frac{a+7b}{8}\right)\right] \\ \leq &\frac{1}{b-a}\left[\int_a^b f(\tau)\nabla\tau + \int_a^b f(\tau)\Delta\tau\right] \\ \leq &\frac{1}{2}\left[f(a) + 3f\left(\frac{3a+b}{4}\right) + 3f\left(\frac{a+3b}{4}\right) + f(b)\right] \\ \leq &\frac{7}{2}\left[f(a) + f(b)\right]. \end{aligned} \tag{3.33}$$

(4) If f is an h-concave function on $\mathbb{Z}$, the three discrete H-H inequalities for integer order in the above special cases of (1)-(3) can be obtained from Corollary 3.4, where we just reverse the inequality signs in (3.31)-(3.33).

The following theorem extends the discrete H-H inequalities for integer order in Theorem 3.1 to fractional forms involving the nabla fractional sums.

Theorem 3.5. *Suppose that $f : [a,b]_{\mathbb{Z}} \to \mathbb{R}$ is an h-convex function with $a < b$, $a, b, \frac{a+b}{2} \in \mathbb{Z}$, and $h\left(\frac{1}{2}\right) \neq 0$. Then for $\alpha > 0$, we have*

$$\frac{\gamma_1}{h\left(\frac{1}{2}\right)} \cdot f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{b-a}\left[{}_a\nabla^{-\alpha} f(b) + \nabla_b^{-\alpha} f(a)\right] \leq \left[f(a) + f(b)\right] \cdot \gamma_2, \tag{3.34}$$

where

$$\begin{aligned} \gamma_1 &= \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} \widetilde{\Delta} t, \\ \gamma_2 &= \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} \left[h(t) + h(1-t)\right] \widetilde{\Delta} t. \end{aligned}$$

Proof. Fixing $t \in \mathbb{T}_{[a,b]} \setminus \{0,1\}$, we define

$$x = ta + (1-t)b, y = (1-t)a + tb.$$

It is easy to see that $x, y \in [a,b]_{\mathbb{Z}}$ and $\frac{x+y}{2} = \frac{a+b}{2} \in \mathbb{Z}$. Since f is an h-convex function on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)\big[f(x) + f(y)\big].$$

This implies that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq h\left(\frac{1}{2}\right)\big[f(ta+(1-t)b) + f((1-t)a+tb)\big] \\ &\leq h\left(\frac{1}{2}\right)\big[h(t)+h(1-t)\big]\big[f(a)+f(b)\big]. \end{aligned}$$

Multiplying each term by $\big((b-a)t+1\big)^{\overline{\alpha-1}}$ and integrating with respect to t over $\mathbb{T}_{[a,b]}$, then we have

$$\begin{aligned} &\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} f\left(\frac{a+b}{2}\right)\widetilde{\Delta}t \\ \leq& h\left(\frac{1}{2}\right)\left[\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} f(ta+(1-t)b)\widetilde{\Delta}t \right. \\ &\left. + \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} f((1-t)a+tb)\widetilde{\Delta}t\right] \\ \leq& h\left(\frac{1}{2}\right)\big[f(a)+f(b)\big]\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}}\big[h(t)+h(1-t)\big]\widetilde{\Delta}t, \end{aligned}$$

namely,

$$\begin{aligned} \gamma_1 \cdot f\left(\frac{a+b}{2}\right) \leq& h\left(\frac{1}{2}\right)\left[\int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} f(ta+(1-t)b)\widetilde{\Delta}t \right. \\ &\left. + \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} f((1-t)a+tb)\widetilde{\Delta}t\right] \\ \leq& h\left(\frac{1}{2}\right)\big[f(a)+f(b)\big]\cdot \gamma_2, \end{aligned}$$

where

$$\begin{aligned} \gamma_1 &= \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}}\widetilde{\Delta}t, \\ \gamma_2 &= \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}}\big[h(t)+h(1-t)\big]\widetilde{\Delta}t. \end{aligned}$$

Define

$$\begin{aligned} l_3 &= \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} f(ta+(1-t)b)\widetilde{\Delta}t, \\ l_4 &= \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} f((1-t)a+tb)\widetilde{\Delta}t. \end{aligned}$$

Calculate l_3 and l_4 separately below.

First, we assert that $l_3 = \frac{\Gamma(\alpha)}{b-a}{}_a\nabla^{-\alpha}f(b)$.

Let $k_3(t):[a,b]_{\mathbb{Z}} \to \mathbb{T}_{[a,b]}$ be defined by $k_3(t)=\frac{b-t}{b-a}$ with $t\in[a,b]_{\mathbb{Z}}$. Then $k_3(t)$ is decreasing and $k_3^{-1}(t)=ta+(1-t)b$, $(-k_3^{\nabla})(t)=\frac{1}{b-a}$.

In addition, letting $g_1(t)=(b-t+1)^{\overline{\alpha-1}}$, and $F_1(t)=g_1(t)f(t)$, then we obtain

$$F_1(k_3^{-1}(t)) = g_1(k_3^{-1}(t))f(k_3^{-1}(t)) = \big((b-a)t+1\big)^{\overline{\alpha-1}} f\big(ta+(1-t)b\big).$$

So, $l_3 = \int_{\mathbb{T}_{[a,b]}} F_1(k_3^{-1}(t))\widetilde{\Delta}t$.

Making use of Theorem 2.2, we have

$$\begin{aligned} l_3 &= \int_{0=k_3(b)}^{1=k_3(a)} (F_1\circ k_3^{-1})(t)\widetilde{\Delta}t = \int_a^b F_1(\tau)(-k_3^{\nabla})(\tau)\nabla\tau \\ &= \frac{1}{b-a}\int_a^b (b-\tau+1)^{\overline{\alpha-1}} f(\tau)\nabla\tau \\ &= \frac{\Gamma(\alpha)}{b-a}{}_a\nabla^{-\alpha}f(b). \end{aligned}$$

Next, we assert that $l_4 = \frac{\Gamma(\alpha)}{b-a}\nabla_b^{-\alpha}f(a)$.

Let $k_4(t):[a,b]_{\mathbb{Z}} \to \mathbb{T}_{[a,b]}$ be defined by $k_4(t)=\frac{t-a}{b-a}$ with $t\in[a,b]_{\mathbb{Z}}$. Then $k_4(t)$ is increasing and $k_4^{-1}(t)=(1-t)a+tb$, $k_4^{\Delta}(t)=\frac{1}{b-a}$.

In addition, letting $g_2(t)=(t-a+1)^{\overline{\alpha-1}}$, and $F_2(t)=g_2(t)f(t)$, then we obtain

$$F_2(k_4^{-1}(t)) = g_2(k_4^{-1}(t))f(k_4^{-1}(t)) = \big((b-a)t+1\big)^{\overline{\alpha-1}} f\big((1-t)a+tb\big).$$

So, $l_4 = \int_{\mathbb{T}_{[a,b]}} F_2(k_4^{-1}(t))\widetilde{\Delta}t$.

Using Theorem 2.1, we get

$$\begin{aligned} l_4 &= \int_{0=k_4(a)}^{1=k_4(b)} (F_2\circ k_4^{-1})(t)\widetilde{\Delta}t = \int_a^b F_2(\tau)k_4^{\Delta}(\tau)\Delta\tau \\ &= \frac{1}{b-a}\int_a^b (\tau-a+1)^{\overline{\alpha-1}} f(\tau)\Delta\tau \\ &= \frac{\Gamma(\alpha)}{b-a}\nabla_b^{-\alpha}f(a). \end{aligned}$$

Thus, we obtain

$$\gamma_1\cdot f\left(\frac{a+b}{2}\right) \le \frac{h\left(\frac12\right)\Gamma(\alpha)}{b-a}\left[{}_a\nabla^{-\alpha}f(b)+\nabla_b^{-\alpha}f(a)\right] \le h\left(\frac12\right)\left[f(a)+f(b)\right]\cdot\gamma_2,$$

which means inequality (3.34) hold. □

Corollary 3.5. *Suppose that $f:[a,b]_{\mathbb{Z}}\to\mathbb{R}$ is an h-concave function with $a<b$, $a,b,\frac{a+b}{2}\in\mathbb{Z}$, and $h\left(\frac12\right)\neq 0$. Then for $\alpha>0$, we have*

$$\frac{\gamma_1}{h\left(\frac12\right)}\cdot f\left(\frac{a+b}{2}\right) \ge \frac{\Gamma(\alpha)}{b-a}\left[{}_a\nabla^{-\alpha}f(b)+\nabla_b^{-\alpha}f(a)\right] \ge \left[f(a)+f(b)\right]\cdot\gamma_2, \quad (3.35)$$

where γ_1,γ_2 are given in Theorem 3.5.

Similar to the proof of Theorem 3.5, changing direction with every inequality sign, this result is obtained.

Remark 3.5. Concerning the above discrete H-H inequalities for fractional order, we obtain

(1) For $\alpha = 1$, Theorem 3.5 reduces to Theorem 3.1.

(2) According to the relationship between the delta right fractional sum and the nabla right fractional sum: $\Delta_{b-1}^{-\alpha} f(a-\alpha) = \nabla_b^{-\alpha} f(a)$, the following inequality are equivalent to inequality (3.34):

$$\begin{aligned} \frac{\gamma_1}{h\left(\frac{1}{2}\right)} \cdot f\left(\frac{a+b}{2}\right) \leq & \frac{\Gamma(\alpha)}{b-a}\left[{}_a\nabla^{-\alpha} f(b) + \Delta_{b-1}^{-\alpha} f(a-\alpha)\right] \\ \leq & \big[f(a)+f(b)\big] \cdot \gamma_2, \end{aligned} \tag{3.36}$$

where γ_1, γ_2 are given in Theorem 3.5.

Remark 3.6. If the special functions are taken in Theorem 3.5, the corresponding discrete H-H inequalities for fractional order related to the endpoint can be obtained:

(1) If $h(x) = x^s$, then (3.34) becomes an inequality for s-convex functions in the second sense on $\mathbb{Z}$:

$$\begin{aligned} 2^s \gamma_1 \cdot f\left(\frac{a+b}{2}\right) \leq & \frac{\Gamma(\alpha)}{b-a}\left[{}_a\nabla^{-\alpha} f(b) + \nabla_b^{-\alpha} f(a)\right] \\ \leq & \big[f(a)+f(b)\big] \\ & \times \int_{\mathbb{T}_{[a,b]}} \big((b-a)t+1\big)^{\overline{\alpha-1}} \big[t^s + (1-t)^s\big] \widetilde{\Delta} t, \end{aligned} \tag{3.37}$$

where γ_1 is given in Theorem 3.5.

(2) If $h(x) = x$, then (3.34) becomes an inequality for convex functions on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{2\gamma_1(b-a)}\left[{}_a\nabla^{-\alpha} f(b) + \nabla_b^{-\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{3.38}$$

where γ_1 is given in Theorem 3.5.

(3) If $h(x) = 1$, then (3.34) becomes an inequality for P-functions on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{\gamma_1(b-a)}\left[{}_a\nabla^{-\alpha} f(b) + \nabla_b^{-\alpha} f(a)\right] \leq 2\big[f(a)+f(b)\big], \tag{3.39}$$

where γ_1 is given in Theorem 3.5.

(4) If f is an h-concave function on $\mathbb{Z}$, the three discrete H-H inequalities for fractional order in the above special cases of (1)-(3) can be obtained from Corollary 3.5, where we just reverse the inequality signs in (3.37)-(3.39).

The following theorem extends the discrete H-H inequalities for integer order in Theorem 3.2 to fractional forms involving the nabla fractional sums.

Theorem 3.6. *Suppose that $f : [a,b]_{\mathbb{Z}} \to \mathbb{R}$ is an h-convex function with $a < b$, $a, b, \frac{a+b}{2} \in \mathbb{Z}$, and $h\left(\frac{1}{2}\right) \neq 0$. Then for $\alpha > 0$, we have*

$$\frac{\gamma_3}{2h\left(\frac{1}{2}\right)} \cdot f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{b-a}\left[{}_{\frac{a+b}{2}}\nabla^{-\alpha} f(b) + \nabla^{-\alpha}_{\frac{a+b}{2}} f(a)\right] \leq \frac{f(a)+f(b)}{2} \cdot \gamma_4, \quad (3.40)$$

where

$$\gamma_3 = \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}} \widetilde{\Delta} t,$$

$$\gamma_4 = \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}} \left[h\left(\frac{t}{2}\right) + h\left(1-\frac{t}{2}\right)\right] \widetilde{\Delta} t.$$

Proof. Fixing $t \in \mathbb{T}_{\left[\frac{a+b}{2},b\right]} \setminus \{0,1\}$, we define

$$x = \frac{t}{2}a + \frac{2-t}{2}b, y = \frac{2-t}{2}a + tb,$$

then $x, y \in [a,b]_{\mathbb{Z}}$ and $\frac{x+y}{2} = \frac{a+b}{2} \in \mathbb{Z}$. Since f is an h-convex function on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\left(\frac{x+y}{2}\right) \leq h\left(\frac{1}{2}\right)\left[f(x)+f(y)\right],$$

that is

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq h\left(\frac{1}{2}\right)\left[f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) + f\left(\frac{2-t}{2}a + \frac{t}{2}b\right)\right] \\ &\leq h\left(\frac{1}{2}\right)\left[h\left(\frac{t}{2}\right) + h\left(\frac{2-t}{2}\right)\right]\left[f(a)+f(b)\right]. \end{aligned}$$

Multiplying each term by $\left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}}$ and integrating with respect to t over $\mathbb{T}_{\left[\frac{a+b}{2},b\right]}$, then we have

$$\begin{aligned} &\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} f\left(\frac{a+b}{2}\right)\left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}} \widetilde{\Delta} t \\ \leq & h\left(\frac{1}{2}\right)\left[\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}} f\left(\frac{t}{2}a + \frac{2-t}{2}b\right) \widetilde{\Delta} t \right. \\ &\left. + \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}} f\left(\frac{2-t}{2}a + \frac{t}{2}b\right) \widetilde{\Delta} t\right] \\ \leq & h\left(\frac{1}{2}\right)\left[f(a)+f(b)\right] \\ &\times \int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}} \left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}} \left[h\left(\frac{t}{2}\right) + h\left(1-\frac{t}{2}\right)\right] \widetilde{\Delta} t, \end{aligned} \quad (3.41)$$

namely,

$$\begin{aligned}\gamma_3\cdot f\left(\frac{a+b}{2}\right)\leq & h\left(\frac{1}{2}\right)\left[\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}}\left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\widetilde{\Delta}t\right.\\ &\left.+\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}}\left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\widetilde{\Delta}t\right]\\ \leq & h\left(\frac{1}{2}\right)\left[f(a)+f(b)\right]\cdot\gamma_4,\end{aligned}\tag{3.42}$$

where

$$\begin{aligned}\gamma_3&=\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}}\left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}}\widetilde{\Delta}t,\\ \gamma_4&=\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}}\left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}}\left[h\left(\frac{t}{2}\right)+h\left(1-\frac{t}{2}\right)\right]\widetilde{\Delta}t.\end{aligned}$$

Define

$$\begin{aligned}r_3&=\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}}\left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}}f\left(\frac{t}{2}a+\frac{2-t}{2}b\right)\widetilde{\Delta}t,\\ r_4&=\int_{\mathbb{T}_{\left[\frac{a+b}{2},b\right]}}\left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}}f\left(\frac{2-t}{2}a+\frac{t}{2}b\right)\widetilde{\Delta}t.\end{aligned}$$

Then $r_3=\frac{2\Gamma(\alpha)}{b-a}\,{}_{\frac{a+b}{2}}\nabla^{-\alpha}f(b)$, $r_4=\frac{2\Gamma(\alpha)}{b-a}\nabla^{-\alpha}_{\frac{a+b}{2}}f(a)$ (see [37]).

Thus, we have

$$\begin{aligned}\gamma_3\cdot f\left(\frac{a+b}{2}\right)\leq&\frac{2h\left(\frac{1}{2}\right)\Gamma(\alpha)}{b-a}\left[{}_{\frac{a+b}{2}}\nabla^{-\alpha}f(b)+\nabla^{-\alpha}_{\frac{a+b}{2}}f(a)\right]\\ \leq& h\left(\frac{1}{2}\right)\left[f(a)+f(b)\right]\cdot\gamma_4,\end{aligned}\tag{3.43}$$

which means inequality (3.40) holds. □

Corollary 3.6. *Suppose that $f:[a,b]_{\mathbb{Z}}\to\mathbb{R}$ is an h-concave function with $a<b$, $a,b,\frac{a+b}{2}\in\mathbb{Z}$, and $h\left(\frac{1}{2}\right)\neq 0$. Then for $\alpha>0$, we have*

$$\frac{\gamma_3}{2h\left(\frac{1}{2}\right)}\cdot f\left(\frac{a+b}{2}\right)\geq\frac{\Gamma(\alpha)}{b-a}\left[{}_{\frac{a+b}{2}}\nabla^{-\alpha}f(b)+\nabla^{-\alpha}_{\frac{a+b}{2}}f(a)\right]\geq\frac{f(a)+f(b)}{2}\cdot\gamma_4,\tag{3.44}$$

where γ_3,γ_4 are given in Theorem 3.6.

Similar to the proof of Theorem 3.6, changing direction with every inequality sign, this result is obtained.

Remark 3.7. Concerning the above discrete H-H inequalities for fractional order, we obtain

(1) For $\alpha = 1$, Theorem 3.6 reduces to Theorem 3.2.

(2) According to the relationship between the delta right fractional sum and the nabla right fractional sum: $\Delta_{b-1}^{-\alpha} f(a-\alpha) = \nabla_b^{-\alpha} f(a)$, the following inequality are equivalent to inequality (3.40):

$$\begin{aligned} \frac{\gamma_3}{2h\left(\frac{1}{2}\right)} \cdot f\left(\frac{a+b}{2}\right) \leq & \frac{\Gamma(\alpha)}{b-a}\left[{}_{\frac{a+b}{2}}\nabla^{-\alpha} f(b) + \Delta_{\frac{a+b}{2}-1}^{-\alpha} f(a-\alpha)\right] \\ \leq & \frac{f(a)+f(b)}{2} \cdot \gamma_4, \end{aligned} \tag{3.45}$$

where γ_3, γ_3 are given in Theorem 3.6.

Remark 3.8. If the special functions are taken in Theorem 3.6, the corresponding discrete H-H inequalities for fractional order related to the midpoint can be obtained:

(1) If $h(x) = x^s$, then (3.40) becomes an inequality for s-convex functions in the second sense on $\mathbb{Z}$:

$$\begin{aligned} & 2^{s-1}\gamma_3 \cdot f\left(\frac{a+b}{2}\right) \\ & \leq \frac{\Gamma(\alpha)}{b-a}\left[{}_{\frac{a+b}{2}}\nabla^{-\alpha} f(b) + \nabla_{\frac{a+b}{2}}^{-\alpha} f(a)\right] \\ & \leq \frac{f(a)+f(b)}{2} \\ & \quad \times \int_{\mathbb{T}_{\left[\frac{a+b}{2}, b\right]}} \left(\frac{b-a}{2}t+1\right)^{\overline{\alpha-1}} \left[\left(\frac{t}{2}\right)^s + \left(\frac{2-t}{2}\right)^s\right] \widetilde{\Delta} t, \end{aligned} \tag{3.46}$$

where γ_3 is given in Theorem 3.6.

(2) If $h(x) = x$, then (3.40) becomes an inequality for convex functions on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{\gamma_3(b-a)}\left[{}_{\frac{a+b}{2}}\nabla^{-\alpha} f(b) + \nabla_{\frac{a+b}{2}}^{-\alpha} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{3.47}$$

where γ_3 is given in Theorem 3.6.

(3) If $h(x) = 1$, then (3.40) becomes an inequality for P-functions on $\mathbb{Z}$:

$$f\left(\frac{a+b}{2}\right) \leq \frac{2\Gamma(\alpha)}{\gamma_3(b-a)}\left[{}_{\frac{a+b}{2}}\nabla^{-\alpha} f(b) + \nabla_{\frac{a+b}{2}}^{-\alpha} f(a)\right] \leq 2\left[f(a)+f(b)\right], \tag{3.48}$$

where γ_3 is given in Theorem 3.6.

(4) If f is an h-concave function on $\mathbb{Z}$, the three discrete H-H inequalities for fractional order in the above special cases of (1)-(3) can be obtained from Corollary 3.6, where we just reverse the inequality signs in (3.46)-(3.48).

3.2. Discrete Hermite-Hadamard inequalities for preinvex functions and its generalization

First, we prove the discrete H-H inequalities for integer order via preinvex functions on $\mathbb{Z}$ related to the endpoints.

Theorem 3.7. *Suppose that* $f : [a, a + \psi(b, a)]_{\mathbb{Z}} \to \mathbb{R}$ *is a preinvex function with* $\psi(b, a) > 0$, $a, b, \frac{\psi(b,a)}{2} \in \mathbb{Z}$*, and satisfies Proposition* 2.1*. Then we have*

$$\begin{aligned} f\left(a + \frac{1}{2}\psi(b, a)\right) \leq & \frac{1}{2\psi(b, a)} \left[\int_{a}^{a+\psi(b,a)} f(\tau)\nabla\tau + \int_{a}^{a+\psi(b,a)} f(\tau)\Delta\tau\right] \\ \leq & \frac{f(a) + f(b)}{2}. \end{aligned} \tag{3.49}$$

Proof. Fixing $t \in \mathbb{T}_{[a,a+\psi(b,a)]} \setminus \{0, 1\}$, we define

$$x = a + t\psi(b, a), y = a + (1 - t)\psi(b, a).$$

Obviously, we can get $x, y \in [a, a + \psi(b, a)]_{\mathbb{Z}}$ and $\frac{x+y}{2} \in \mathbb{Z}$. Since f is a preinvex function on $[x, y]_{\mathbb{Z}}$ (or $[y, x]_{\mathbb{Z}}$), we have

$$f\left(x + \frac{1}{2}\psi(y, x)\right) \leq \frac{1}{2}\big[f(x) + f(y)\big].$$

This implies that

$$\begin{aligned} & f\left(a + t\psi(b, a) + \frac{1}{2}\psi\big(a + (1 - t)\psi(b, a), a + t\psi(b, a)\big)\right) \\ \leq & \frac{1}{2}\big[f(a + t\psi(b, a)) + f(a + (1 - t)\psi(b, a))\big] \\ \leq & \frac{1}{2}\big[tf(b) + (1 - t)f(a) + (1 - t)f(b) + tf(a)\big] \\ \leq & \frac{f(a) + f(b)}{2}. \end{aligned}$$

According to Proposition 2.1, we have

$$f\left(a + \frac{1}{2}\psi(b, a)\right) \leq \frac{1}{2}\big[f(a + t\psi(b, a)) + f(a + (1 - t)\psi(b, a))\big] \leq \frac{f(a) + f(b)}{2}.$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{[a,a+\psi(b,a)]}$, then we have

$$\begin{aligned} & \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} f\left(a + \frac{1}{2}\psi(b, a)\right) \widetilde{\Delta}t \\ \leq & \frac{1}{2} \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big[f(a + t\psi(b, a)) + f(a + (1 - t)\psi(b, a))\big] \widetilde{\Delta}t \\ \leq & \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \frac{f(a) + f(b)}{2} \widetilde{\Delta}t. \end{aligned} \tag{3.50}$$

Define

$$\begin{aligned} l_1 &= \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} f(a + (1 - t)\psi(b, a))\widetilde{\Delta}t, \\ l_2 &= \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} f(a + t\psi(b, a))\widetilde{\Delta}t. \end{aligned}$$

Calculate l_1 and l_2 separately below.

First, we assert that $l_1 = \frac{1}{\psi(b,a)} \int_a^{a+\psi(b,a)} f(\tau)\nabla\tau$.

Let $k_1(t) : [a, a+\psi(b,a)]_{\mathbb{Z}} \to \mathbb{T}_{[a,a+\psi(b,a)]}$ be defined by $k_1(t) = 1 - \frac{t-a}{\psi(b,a)}$ with $t \in [a, a+\psi(b,a)]_{\mathbb{Z}}$. Then $k_1(t)$ is decreasing, $k_1^{-1}(t) = a + (1-t)\psi(b,a)$ and $(-k_1^{\nabla})(t) = \frac{1}{\psi(b,a)}$.

According to Theorem 2.2, we obtain

$$\begin{aligned} l_1 &= \int_{0=k_1(a+\psi(b,a))}^{1=k_1(a)} (f \circ k_1^{-1})(t)\widetilde{\Delta}t = \int_a^{a+\psi(b,a)} f(\tau)(-k_1^{\nabla})(\tau)\nabla\tau \\ &= \frac{1}{\psi(b,a)} \int_a^{a+\psi(b,a)} f(\tau)\nabla\tau. \end{aligned}$$

Next, we prove that $l_2 = \frac{1}{\psi(b,a)} \int_a^{a+\psi(b,a)} f(\tau)\Delta\tau$.

Let $k_2(t) : [a, a+\psi(b,a)]_{\mathbb{Z}} \to \mathbb{T}_{[a,a+\psi(b,a)]}$ be defined by $k_2(t) = \frac{t-a}{\psi(b,a)}$ with $t \in [a, a+\psi(b,a)]_{\mathbb{Z}}$. Then $k_2(t)$ is increasing, $k_2^{-1}(t) = a + t\psi(b,a)$ and $k_2^{\Delta}(t) = \frac{1}{\psi(b,a)}$.

Using Theorem 2.1, we get

$$\begin{aligned} l_2 &= \int_{0=k_2(a)}^{1=k_2(a+\psi(b,a))} (f \circ k_2^{-1})(t)\widetilde{\Delta}t = \int_a^{a+\psi(b,a)} f(\tau)k_2^{\Delta}(\tau)\Delta\tau \\ &= \frac{1}{\psi(b,a)} \int_a^{a+\psi(b,a)} f(\tau)\Delta\tau. \end{aligned}$$

Inserting l_1 and l_2 into (3.50), we find that

$$\begin{aligned} f\left(a + \frac{1}{2}\psi(b,a)\right) &\leq \frac{1}{2\psi(b,a)} \left[\int_a^{a+\psi(b,a)} f(\tau)\nabla\tau + \int_a^{a+\psi(b,a)} f(\tau)\Delta\tau\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.51}$$

Therefore, the inequality (3.49) hold. □

Remark 3.9. If we choose $\psi(b,a) = b - a$ in Theorem 3.7, the inequality (3.49) reduce to inequality (3.6).

Next, we prove the discrete H-H inequalities for integer order via preinvex functions in $\mathbb{Z}$ related to the midpoint.

Theorem 3.8. *Suppose that $f : [a, a+\psi(b,a)]_{\mathbb{Z}} \to \mathbb{R}$ is a preinvex function with $\psi(b,a) > 0$, $a, b, \frac{\psi(b,a)}{2} \in \mathbb{Z}$, and satisfies Proposition 2.1. Then we have*

$$\begin{aligned} f\left(a + \frac{\psi(b,a)}{2}\right) &\leq \frac{1}{\psi(b,a)} \left[\int_{a+\frac{\psi(b,a)}{2}}^{a+\psi(b,a)} f(\tau)\nabla\tau + \int_a^{a+\frac{\psi(b,a)}{2}} f(\tau)\Delta\tau\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{3.52}$$

Proof. Fixing $t \in \mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]} \setminus \{0,1\}$, we define

$$x = a + \frac{t}{2}\psi(b,a), y = a + \frac{2-t}{2}\psi(b,a),$$

then $x, y \in [a, a+\psi(b,a)]_{\mathbb{Z}}$ and $\frac{x+y}{2} \in \mathbb{Z}$. Since f is a preinvex function on $[x, y]_{\mathbb{Z}}$ (or $[y, x]_{\mathbb{Z}}$), we have

$$f\left(x + \frac{1}{2}\psi(y, x)\right) \leq \frac{1}{2}\big[f(x) + f(y)\big].$$

This implies that

$$\begin{aligned}
&f\left(a + \frac{t}{2}\psi(b,a) + \frac{1}{2}\psi\left(a + \frac{2-t}{2}\psi(b,a), a + \frac{t}{2}\psi(b,a)\right)\right) \\
&\leq \frac{1}{2}\left[f\left(a + \frac{2-t}{2}\psi(b,a)\right) + f\left(a + \frac{t}{2}\psi(b,a)\right)\right] \\
&\leq \frac{1}{2}\left[\frac{2-t}{2}f(b) + \frac{t}{2}f(a) + \frac{t}{2}f(b) + \frac{2-t}{2}f(a)\right] \\
&= \frac{f(a)+f(b)}{2}.
\end{aligned}$$

According to Proposition 2.1, we have

$$\begin{aligned}
f\left(a + \frac{1}{2}\psi(b,a)\right) \leq& \frac{1}{2}\left[f\left(a + \frac{2-t}{2}\psi(b,a)\right) + f\left(a + \frac{t}{2}\psi(b,a)\right)\right] \\
\leq& \frac{f(a)+f(b)}{2}.
\end{aligned}$$

Integrating the above inequalities with respect to t over $\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]}$, then we have

$$\begin{aligned}
&\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]}} f\left(a + \frac{1}{2}\psi(b,a)\right)\widetilde{\Delta}t \\
&\leq \frac{1}{2}\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]}} \left[f\left(a + \frac{2-t}{2}\psi(b,a)\right) + f\left(a + \frac{t}{2}\psi(b,a)\right)\right]\widetilde{\Delta}t \qquad (3.53) \\
&\leq \frac{1}{2}\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]}} \big[f(a) + f(b)\big]\widetilde{\Delta}t.
\end{aligned}$$

Define

$$\begin{aligned}
r_1 &= \int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]}} f\left(a + \frac{2-t}{2}\psi(b,a)\right)\widetilde{\Delta}t, \\
r_2 &= \int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]}} f\left(a + \frac{t}{2}\psi(b,a)\right)\widetilde{\Delta}t.
\end{aligned}$$

Calculate r_1 and r_2 separately below.

First, we assert that $r_1 = \frac{2}{\psi(b,a)}\int_{a+\frac{\psi(b,a)}{2}}^{a+\psi(b,a)} f(\tau)\nabla\tau$.

Let $s_1(t) : \left[a + \frac{\psi(b,a)}{2}, a+\psi(b,a)\right]_{\mathbb{Z}} \to \mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]}$ be defined by $s_1(t) = 2 - \frac{2(t-a)}{\psi(b,a)}$ with $t \in \left[a + \frac{\psi(b,a)}{2}, a+\psi(b,a)\right]_{\mathbb{Z}}$. Then $s_1(t)$ is decreasing, $s_1^{-1}(t) = a + \frac{2-t}{2}\psi(b,a)$ and $(-s_1^{\nabla})(t) = \frac{2}{\psi(b,a)}$.

According to Theorem 2.2, we obtain

$$\begin{aligned} r_1 &= \int_{0=s_1(a+\psi(b,a))}^{1=s_1\left(a+\frac{\psi(b,a)}{2}\right)} (f\circ s_1^{-1})(t)\widetilde{\Delta}t = \int_{a+\frac{\psi(b,a)}{2}}^{a+\psi(b,a)} f(\tau)(-s_1^{\nabla})(\tau)\nabla\tau \\ &= \frac{2}{\psi(b,a)}\int_{a+\frac{\psi(b,a)}{2}}^{a+\psi(b,a)} f(\tau)\nabla\tau. \end{aligned}$$

Next, we prove that $r_2 = \frac{2}{\psi(b,a)}\int_a^{a+\frac{\psi(b,a)}{2}} f(\tau)\Delta\tau$.

Assume that $t = \frac{2\left(a+\psi(b,a)-\tau\right)}{\psi(b,a)}$ with $\tau \in \left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]_{\mathbb{Z}}$.

Setting $\hat{\tau} = 2a+\psi(b,a)-\tau$ and $\hat{t} = \frac{2(\hat{\tau}-a)}{\psi(b,a)}$, we have $\hat{\tau} \in \left[a, a+\frac{\psi(b,a)}{2}\right]_{\mathbb{Z}}$ and $\hat{t} \in \mathbb{T}_{\left[a,a+\frac{\psi(b,a)}{2}\right]}$. Hence

$$\begin{aligned} r_2 &= \int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}} f\left(a+\frac{t}{2}\psi(b,a)\right)\widetilde{\Delta}t \\ &= \int_{\mathbb{T}_{\left[a,a+\frac{\psi(b,a)}{2}\right]}} f\left(a+\frac{\hat{t}}{2}\psi(b,a)\right)\widetilde{\Delta}\hat{t}. \end{aligned}$$

Let $s_2(t) : \left[a, a+\frac{\psi(b,a)}{2}\right]_{\mathbb{Z}} \to \mathbb{T}_{\left[a,a+\frac{\psi(b,a)}{2}\right]}$ be defined by $s_2(t) = \frac{2(t-a)}{\psi(b,a)}$ with $t \in \left[a, a+\frac{\psi(b,a)}{2}\right]_{\mathbb{Z}}$. Then $s_2(t)$ is increasing, $s_2^{-1}(t) = a+\frac{t}{2}\psi(b,a)$ and $s_2^{\Delta}(t) = \frac{2}{\psi(b,a)}$. Using Theorem 2.1, we get

$$\begin{aligned} r_2 &= \int_{0=s_2(a)}^{1=s_2\left(a+\frac{\psi(b,a)}{2}\right)} (f\circ s_2^{-1})(\hat{t})\widetilde{\Delta}\hat{t} = \int_a^{a+\frac{\psi(b,a)}{2}} f(\tau)s_2^{\Delta}(\tau)\Delta\tau \\ &= \frac{2}{\psi(b,a)}\int_a^{a+\frac{\psi(b,a)}{2}} f(\tau)\Delta\tau. \end{aligned}$$

Inserting r_1 and r_2 into (3.53), we find that

$$\begin{aligned} f\left(a+\frac{\psi(b,a)}{2}\right) &\leq \frac{1}{\psi(b,a)}\left[\int_{a+\frac{\psi(b,a)}{2}}^{a+\psi(b,a)} f(\tau)\nabla\tau + \int_a^{a+\frac{\psi(b,a)}{2}} f(\tau)\Delta\tau\right] \\ &\leq \frac{f(a)+f(b)}{2}. \end{aligned}$$

Therefore, the inequality (3.52) hold. □

Remark 3.10. If we choose $\psi(b,a) = b-a$ in Theorem 3.8, the inequality (3.52) reduce to inequality (3.12).

The following theorem extends the H-H inequalities for integer order in Theorem 3.7 to fractional forms involving the nabla fractional sums.

Theorem 3.9. *Suppose that* $f : [a, a + \psi(b, a)]_{\mathbb{Z}} \to \mathbb{R}$ *is a preinvex function with* $\psi(b, a) > 0$, $a, b, \frac{\psi(b,a)}{2} \in \mathbb{Z}$, *and satisfies Proposition* 2.1. *Then for* $\alpha > 0$, *we have*

$$\begin{aligned} f\left(a + \frac{1}{2}\psi(b, a)\right) \leq & \frac{\Gamma(\alpha)}{2\eta_1 \psi(b, a)} \left[{}_a\nabla^{-\alpha} f(t)|_{t=a+\psi(b,a)} + \nabla^{-\alpha}_{a+\psi(b,a)} f(t)|_{t=a} \right] \\ \leq & \frac{f(a) + f(b)}{2}, \end{aligned} \tag{3.54}$$

where

$$\eta_1 = \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \left(\psi(b, a) + 1\right)^{\overline{\alpha-1}} \widetilde{\Delta} t.$$

Proof. Fixing $t \in \mathbb{T}_{[a,a+\psi(b,a)]} \setminus \{0, 1\}$, we define

$$x = a + t\psi(b, a), y = a + (1 - t)\psi(b, a).$$

Obviously, we can get that $x, y \in [a, a + \psi(b, a)]_{\mathbb{Z}}$ and $\frac{x+y}{2} \in \mathbb{Z}$. Since f is a preinvex function on $[x, y]_{\mathbb{Z}}$ (or $[y, x]_{\mathbb{Z}}$), we have

$$f\left(x + \frac{1}{2}\psi(y, x)\right) \leq \frac{1}{2}\big[f(x) + f(y)\big].$$

This implies that

$$\begin{aligned} & f\left(a + t\psi(b, a) + \frac{1}{2}\psi\big(a + (1 - t)\psi(b, a), a + t\psi(b, a)\big)\right) \\ \leq & \frac{1}{2}\big[f(a + t\psi(b, a)) + f(a + (1 - t)\psi(b, a))\big] \\ \leq & \frac{1}{2}\big[tf(b) + (1 - t)f(a) + (1 - t)f(b) + tf(a)\big] \\ \leq & \frac{f(a) + f(b)}{2}. \end{aligned}$$

According to Proposition 2.1, we have

$$f\left(a + \frac{1}{2}\psi(b, a)\right) \leq \frac{1}{2}\big[f(a + t\psi(b, a)) + f(a + (1 - t)\psi(b, a))\big] \leq \frac{f(a) + f(b)}{2}.$$

Multiplying each term by $\big(\psi(b, a)t + 1\big)^{\overline{\alpha-1}}$ and integrating with respect to t over $\mathbb{T}_{[a,a+\psi(b,a)]}$, then we have

$$\begin{aligned} & \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b, a)t + 1\big)^{\overline{\alpha-1}} f\left(a + \frac{1}{2}\psi(b, a)\right) \widetilde{\Delta} t \\ \leq & \frac{1}{2}\left[\int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b, a)t + 1\big)^{\overline{\alpha-1}} f(a + t\psi(b, a)) \widetilde{\Delta} t \right. \\ & \left. + \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b, a)t + 1\big)^{\overline{\alpha-1}} f(a + (1 - t)\psi(b, a))\big] \widetilde{\Delta} t \right] \\ \leq & \frac{f(a) + f(b)}{2} \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b, a)t + 1\big)^{\overline{\alpha-1}} \widetilde{\Delta} t, \end{aligned}$$

namely,

$$\eta_1 \cdot f\left(a+\frac{1}{2}\psi(b,a)\right) \leq \frac{1}{2}\left[\int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b,a)t+1\big)^{\overline{\alpha-1}} f(a+t\psi(b,a))\widetilde{\Delta}t \right. \\ \left. + \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b,a)t+1\big)^{\overline{\alpha-1}} f(a+(1-t)\psi(b,a))\big]\widetilde{\Delta}t\right] \\ \leq \frac{f(a)+f(b)}{2}\cdot\eta_1,$$

where

$$\eta_1 = \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b,a)t+1\big)^{\overline{\alpha-1}}\widetilde{\Delta}t.$$

Define

$$l_3 = \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b,a)t+1\big)^{\overline{\alpha-1}} f(a+(1-t)\psi(b,a))\big]\widetilde{\Delta}t,$$
$$l_4 = \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} \big(\psi(b,a)t+1\big)^{\overline{\alpha-1}} f(a+t\psi(b,a))\widetilde{\Delta}t.$$

Calculate l_3 and l_4 separately below.

First, we assert that $l_3 = \frac{\Gamma(\alpha)}{\psi(b,a)}\, {}_a\nabla^{-\alpha} f(t)|_{t=a+\psi(b,a)}$.

Let $k_3(t) : [a, a+\psi(b,a)]_{\mathbb{Z}} \to \mathbb{T}_{[a,a+\psi(b,a)]}$ be defined by $k_3(t) = 1 - \frac{t-a}{\psi(b,a)}$ with $t \in [a, a+\psi(b,a)]_{\mathbb{Z}}$. Then $k_3(t)$ is decreasing, $k_3^{-1}(t) = a+(1-t)\psi(b,a)$ and $(-k_3^{\nabla})(t) = \frac{1}{\psi(b,a)}$.

In addition, let $g_1(t) = (a+\psi(b,a)-t+1)^{\overline{\alpha-1}}$, and $F_1(t) = g_1(t)f(t)$. Then we obtain

$$F_1(k_3^{-1}(t)) = g_1(k_3^{-1}(t))f(k_3^{-1}(t)) = \big(t\psi(b,a)+1\big)^{\overline{\alpha-1}} f\big(a+(1-t)\psi(b,a)\big),$$

so, $l_3 = \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} F_1(k_3^{-1}(t))\widetilde{\Delta}t$.

Making use of Theorem 2.2, we have

$$l_3 = \int_{0=k_3(a+\psi(b,a))}^{1=k_3(a)} (F_1\circ k_3^{-1})(t)\widetilde{\Delta}t = \int_a^{a+\psi(b,a)} F_1(\tau)(-k_3^{\nabla})(\tau)\nabla\tau \\ = \frac{1}{\psi(b,a)}\int_a^{a+\psi(b,a)} (a+\psi(b,a)-\tau+1)^{\overline{\alpha-1}} f(\tau)\nabla\tau \\ = \frac{\Gamma(\alpha)}{\psi(b,a)}\, {}_a\nabla^{-\alpha} f(t)|_{t=a+\psi(b,a)}.$$

Next, we assert that $l_4 = \frac{\Gamma(\alpha)}{\psi(b,a)}\nabla^{-\alpha}_{a+\psi(b,a)} f(t)|_{t=a}$.

Let $k_4(t) : [a, a+\psi(b,a)]_{\mathbb{Z}} \to \mathbb{T}_{[a,a+\psi(b,a)]}$ be defined by $k_4(t) = \frac{t-a}{\psi(b,a)}$ with $t \in [a, a+\psi(b,a)]_{\mathbb{Z}}$. Then $k_4(t)$ is increasing, $k_4^{-1}(t) = a+t\psi(b,a)$ and $k_4^{\Delta}(t) = \frac{1}{\psi(b,a)}$.

In addition, let $g_2(t) = (t-a+1)^{\overline{\alpha-1}}$, and $F_2(t) = g_2(t)f(t)$. Then we obtain

$$F_2(k_4^{-1}(t)) = g_2(k_4^{-1}(t))f(k_4^{-1}(t)) = \big(\psi(b,a)t+1\big)^{\overline{\alpha-1}} f\big(a+t\psi(b,a)\big),$$

so, $l_4 = \int_{\mathbb{T}_{[a,a+\psi(b,a)]}} F_2(k_4^{-1}(t))\widetilde{\Delta}t$.

Using Theorem 2.1, we get

$$\begin{aligned} l_4 &= \int_{0=k_4(a)}^{1=k_4(a+\psi(b,a))} (F_2 \circ k_4^{-1})(t)\widetilde{\Delta}t = \int_a^{a+\psi(b,a)} F_2(\tau)k_4^{\Delta}(\tau)\Delta\tau \\ &= \frac{1}{\psi(b,a)} \int_a^{a+\psi(b,a)} (\tau - a + 1)^{\overline{\alpha-1}} f(\tau)\Delta\tau \\ &= \frac{\Gamma(\alpha)}{\psi(b,a)} \nabla_{a+\psi(b,a)}^{-\alpha} f(t)|_{t=a}. \end{aligned}$$

Thus, we obtain

$$\begin{aligned} \eta_1 \cdot f\left(a + \frac{1}{2}\psi(b,a)\right) \leq & \frac{\Gamma(\alpha)}{2\psi(b,a)} \left[{}_a\nabla^{-\alpha} f(t)|_{t=a+\psi(b,a)} + \nabla_{a+\psi(b,a)}^{-\alpha} f(t)|_{t=a}\right] \\ \leq & \frac{f(a)+f(b)}{2} \cdot \eta_1, \end{aligned}$$

which means inequality (3.54) hold. □

Remark 3.11. Concerning the above discrete H-H inequalities for fractional order, we obtain

(1) For $\alpha = 1$, Theorem 3.9 reduces to Theorem 3.7.

(2) According to the relationship between the delta right fractional sum and the nabla right fractional sum: $\Delta_{b-1}^{-\alpha} f(a-\alpha) = \nabla_b^{-\alpha} f(a)$, the following inequality are equivalent to inequality (3.54):

$$\begin{aligned} & f\left(a + \frac{1}{2}\psi(b,a)\right) \\ \leq & \frac{\Gamma(\alpha)}{2\eta_1\psi(b,a)} \left[{}_a\nabla^{-\alpha} f(t)|_{t=a+\psi(b,a)} + \Delta_{a+\psi(b,a)-1}^{-\alpha} f(t)|_{t=a-\alpha}\right] \\ \leq & \frac{f(a)+f(b)}{2}, \end{aligned} \tag{3.55}$$

where η_1 is given in Theorem 3.9.

Remark 3.12. If we choose $\psi(b,a) = b - a$ in Theorem 3.9, the inequality (3.54) reduce to inequality (3.38).

The following theorem extends the H-H inequalities for integer order in Theorem 3.8 to fractional forms involving the nabla fractional sums.

Theorem 3.10. *Suppose that $f : [a, a+\psi(b,a)]_{\mathbb{Z}} \to \mathbb{R}$ is a preinvex function with $\psi(b,a) > 0$, $a, b, \frac{\psi(b,a)}{2} \in \mathbb{Z}$, and satisfies Proposition 2.1. Then for $\alpha > 0$, we have*

$$\begin{aligned} & f\left(a + \frac{1}{2}\psi(b,a)\right) \\ \leq & \frac{\Gamma(\alpha)}{\eta_2\psi(b,a)} \left[{}_{a+\frac{\psi(b,a)}{2}}\nabla^{-\alpha} f(t)|_{t=a+\psi(b,a)} + \nabla_{a+\frac{\psi(b,a)}{2}}^{-\alpha} f(t)|_{t=a}\right] \\ \leq & \frac{f(a)+f(b)}{2}, \end{aligned} \tag{3.56}$$

where

$$\eta_2=\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}+1\right)^{\overline{\alpha-1}}\widetilde{\Delta}t.$$

Proof. Fixing $t\in\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}\setminus\{0,1\}$, we define

$$x=a+\frac{t}{2}\psi(b,a),y=a+\frac{2-t}{2}\psi(b,a),$$

then $x,y\in[a,a+\psi(b,a)]_{\mathbb{Z}}$ and $\frac{x+y}{2}\in\mathbb{Z}$. Since f is a preinvex function on $[x,y]_{\mathbb{Z}}$ (or $[y,x]_{\mathbb{Z}}$), we have

$$f\left(x+\frac{1}{2}\psi(y,x)\right)\le\frac{1}{2}\big[f(x)+f(y)\big],$$

that is

$$\begin{aligned}
&f\left(a+\frac{t}{2}\psi(b,a)+\frac{1}{2}\psi\left(a+\frac{2-t}{2}\psi(b,a),a+\frac{t}{2}\psi(b,a)\right)\right)\\
&\le\frac{1}{2}\left[f\left(a+\frac{2-t}{2}\psi(b,a)\right)+f\left(a+\frac{t}{2}\psi(b,a)\right)\right]\\
&\le\frac{1}{2}\left[\frac{2-t}{2}f(b)+\frac{t}{2}f(a)+\frac{t}{2}f(b)+\frac{2-t}{2}f(a)\right]\\
&=\frac{f(a)+f(b)}{2}.
\end{aligned}$$

According to Proposition 2.1, we have

$$\begin{aligned}
f\left(a+\frac{1}{2}\psi(b,a)\right)\le&\frac{1}{2}\left[f\left(a+\frac{2-t}{2}\psi(b,a)\right)+f\left(a+\frac{t}{2}\psi(b,a)\right)\right]\\
\le&\frac{f(a)+f(b)}{2}.
\end{aligned}$$

Multiplying each term by $\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}}$ and integrating with respect to t over $\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}$, then we have

$$\begin{aligned}
&\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}f\left(a+\frac{1}{2}\psi(b,a)\right)\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}}\widetilde{\Delta}t\\
\le&\frac{1}{2}\left[\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}}f\left(a+\frac{2-t}{2}\psi(b,a)\right)\widetilde{\Delta}t\right.\\
&\left.+\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}}f\left(a+\frac{t}{2}\psi(b,a)\right)\widetilde{\Delta}t\right]\\
\le&\frac{f(a)+f(b)}{2}\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}}\widetilde{\Delta}t,
\end{aligned}\tag{3.57}$$

namely,

$$
\begin{aligned}
&\eta_2 \cdot f\left(a+\frac{1}{2}\psi(b,a)\right)\\
\leq&\frac{1}{2}\left[\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}} f\left(a+\frac{2-t}{2}\psi(b,a)\right)\widetilde{\Delta}t\right.\\
&\left.+\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}} f\left(a+\frac{t}{2}\psi(b,a)\right)\widetilde{\Delta}t\right]\\
\leq&\frac{f(a)+f(b)}{2}\cdot\eta_2,
\end{aligned}
\tag{3.58}
$$

where

$$\eta_2=\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}}\widetilde{\Delta}t.$$

Define

$$
\begin{aligned}
r_3&=\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}} f\left(a+\frac{2-t}{2}\psi(b,a)\right)\widetilde{\Delta}t,\\
r_4&=\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}\left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}} f\left(a+\frac{t}{2}\psi(b,a)\right)\widetilde{\Delta}t.
\end{aligned}
$$

Calculate r_3 and r_4 separately below.

First, we assert that $r_3=\frac{2\Gamma(\alpha)}{\psi(b,a)}{}_{a+\frac{\psi(b,a)}{2}}\nabla^{-\alpha}f(t)|_{t=a+\psi(b,a)}$.

Let $s_3(t):\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]_{\mathbb{Z}}\to\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}$ be defined by $s_3(t)=2-\frac{2(t-a)}{\psi(b,a)}$ with $t\in\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]_{\mathbb{Z}}$. Then $s_3(t)$ is decreasing, $s_3^{-1}(t)=a+\frac{2-t}{2}\psi(b,a)$ and $(-s_3^{\nabla})(t)=\frac{2}{\psi(b,a)}$.

In addition, let $g_3(t)=(a+\psi(b,a)-t+1)^{\overline{\alpha-1}}$, and $F_3(t)=g_3(t)f(t)$. Then we obtain

$$F_3(s_3^{-1}(t))=g_3(s_3^{-1}(t))f(s_3^{-1}(t))=\left(\frac{t}{2}\psi(b,a)+1\right)^{\overline{\alpha-1}} f\left(a+\frac{2-t}{2}\psi(b,a)\right),$$

so, $r_3=\int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2},a+\psi(b,a)\right]}}F_3(s_3^{-1}(t))\widetilde{\Delta}t$.

According to Theorem 2.2, we obtain

$$
\begin{aligned}
r_3&=\int_{0=s_3(a+\psi(b,a))}^{1=s_3\left(a+\frac{\psi(b,a)}{2}\right)}(F_3\circ s_3^{-1})(t)\widetilde{\Delta}t=\int_{a+\frac{\psi(b,a)}{2}}^{a+\psi(b,a)}F_3(\tau)(-s_3^{\nabla})(\tau)\nabla\tau\\
&=\frac{2}{\psi(b,a)}\int_{a+\frac{\psi(b,a)}{2}}^{a+\psi(b,a)}(a+\psi(b,a)-\tau+1)^{\overline{\alpha-1}}f(\tau)\nabla\tau\\
&=\frac{2\Gamma(\alpha)}{\psi(b,a)}{}_{a+\frac{\psi(b,a)}{2}}\nabla^{-\alpha}f(t)|_{t=a+\psi(b,a)}.
\end{aligned}
$$

Next, we claim that $r_4 = \frac{2\Gamma(\alpha)}{\psi(b,a)} \nabla^{-\alpha}_{a+\frac{\psi(b,a)}{2}} f(t)|_{t=a}$.

Assume that $t = \frac{2(a+\psi(b,a)-\tau)}{\psi(b,a)}$ with $\tau \in \left[a + \frac{\psi(b,a)}{2}, a+\psi(b,a)\right]_{\mathbb{Z}}$.

Setting $\hat{\tau} = 2a + \psi(b,a) - \tau$ and $\hat{t} = \frac{2(\hat{\tau}-a)}{\psi(b,a)}$, we have $\hat{\tau} \in \left[a, a + \frac{\psi(b,a)}{2}\right]_{\mathbb{Z}}$ and $\hat{t} \in \mathbb{T}_{\left[a,a+\frac{\psi(b,a)}{2}\right]}$. Hence

$$\begin{aligned} r_4 &= \int_{\mathbb{T}_{\left[a+\frac{\psi(b,a)}{2}, a+\psi(b,a)\right]}} \left(\frac{\psi(b,a)}{2}t+1\right)^{\overline{\alpha-1}} f\left(a + \frac{t}{2}\psi(b,a)\right) \widetilde{\Delta} t \\ &= \int_{\mathbb{T}_{\left[a,a+\frac{\psi(b,a)}{2}\right]}} \left(\frac{\psi(b,a)}{2}\hat{t}+1\right)^{\overline{\alpha-1}} f\left(a + \frac{\hat{t}}{2}\psi(b,a)\right) \widetilde{\Delta} \hat{t}. \end{aligned}$$

Let $s_4(t) : \left[a, a+\frac{\psi(b,a)}{2}\right]_{\mathbb{Z}} \to \mathbb{T}_{\left[a,a+\frac{\psi(b,a)}{2}\right]}$ be defined by $s_4(t) = \frac{2(t-a)}{\psi(b,a)}$ with $t \in \left[a, a+\frac{\psi(b,a)}{2}\right]_{\mathbb{Z}}$. Then $s_4(t)$ is increasing, $s_4^{-1}(t) = a + \frac{t}{2}\psi(b,a)$ and $s_4^{\Delta}(t) = \frac{2}{\psi(b,a)}$.

In addition, let $g_4(t) = (t-a+1)^{\overline{\alpha-1}}$, and $F_4(t) = g_4(t)f(t)$. Then we obtain

$$F_4(s_4^{-1}(t)) = g_4(s_4^{-1}(t))f(s_4^{-1}(t)) = \left(\frac{t}{2}\psi(b,a)+1\right)^{\overline{\alpha-1}} f\left(a+\frac{t}{2}\psi(b,a)\right),$$

so, $r_4 = \int_{\mathbb{T}_{\left[a,a+\frac{\psi(b,a)}{2}\right]}} F_4(s_4^{-1}(\hat{t}))\widetilde{\Delta}\hat{t}$.

Using Theorem 2.1, we get

$$\begin{aligned} r_4 &= \int_{0=s_4(a)}^{1=s_4\left(a+\frac{\psi(b,a)}{2}\right)} (F_4 \circ s_4^{-1})(\hat{t})\widetilde{\Delta}\hat{t} = \int_a^{a+\frac{\psi(b,a)}{2}} F_4(\tau)s_4^{\Delta}(\tau)\Delta\tau \\ &= \frac{2}{\psi(b,a)} \int_a^{a+\frac{\psi(b,a)}{2}} (\tau-a+1)^{\overline{\alpha-1}} f(\tau)\Delta\tau \\ &= \frac{2\Gamma(\alpha)}{\psi(b,a)} \nabla^{-\alpha}_{a+\frac{\psi(b,a)}{2}} f(t)|_{t=a}. \end{aligned}$$

Thus, we have

$$\begin{aligned} &\eta_2 \cdot f\left(a + \frac{1}{2}\psi(b,a)\right) \\ &\quad \le \frac{\Gamma(\alpha)}{\psi(b,a)} \left[{}_{a+\frac{\psi(b,a)}{2}}\nabla^{-\alpha} f(t)|_{t=a+\psi(b,a)} + \nabla^{-\alpha}_{a+\frac{\psi(b,a)}{2}} f(t)|_{t=a} \right] \\ &\quad \le \frac{f(a)+f(b)}{2} \cdot \eta_2, \end{aligned} \tag{3.59}$$

which means inequality (3.56) hold. □

Remark 3.13. Concerning the above discrete H-H inequalities for fractional order, we obtain

(1) For $\alpha = 1$, Theorem 3.10 reduces to Theorem 3.8.

(2) According to the relationship between the delta right fractional sum and the nabla right fractional sum: $\Delta_{b-1}^{-\alpha} f(a-\alpha) = \nabla_b^{-\alpha} f(a)$, the following inequality are equivalent to inequality (3.56):

$$\begin{aligned} &f\left(a+\frac{1}{2}\psi(b,a)\right) \\ \leq &\frac{\Gamma(\alpha)}{\eta_2\psi(b,a)}\left[{}_{a+\frac{\psi(b,a)}{2}}\nabla^{-\alpha} f(t)|_{t=a+\psi(b,a)} + \Delta_{a+\frac{\psi(b,a)}{2}-1}^{-\alpha} f(t)|_{t=a-\alpha}\right] \\ \leq &\frac{f(a)+f(b)}{2}, \end{aligned} \tag{3.60}$$

where η_2 is given in Theorem 3.10.

Remark 3.14. If we choose $\psi(b,a) = b-a$ in Theorem 3.10, the inequality (3.56) reduce to inequality (3.47).

4. Conclusion

We define two new convex functions: h-convex and preinvex on the time scale $\mathbb{Z}$. Based on these two new definitions, some new discrete Hermite-Hadamard inequalities for integer order and fractional order are obtained. Our results generalize the discrete Hermite-Hadamard inequalities for P-functions, convex functions, and s-convex functions in the second sense on the time scale $\mathbb{Z}$. In addition, by dividing the defined intervals differently, two new generalized discrete Hermite-Hadamard inequalities for h-convex on the time scale $\mathbb{Z}$ are obtained, which is another important innovation of this paper. These results play an important role in studying the qualitative properties of difference equations. Some inequalities involving other time scales or sum operators also can be provided in the future.

Acknowledgements

The authors are very grateful to the anonymous referees for their valuable suggestions and comments, which helped to improve the quality of the paper.

References

[1] T. Abdeljawad, *On Delta and Nabla Caputo fractional differences and dual identities*, Discrete Dynamics in Nature and Society, 2013. DOI: https://doi.org/10.1155/2013/406910

[2] F. M. Atıcı and H. Yaldız, *Convex functions on discrete time domains*, Canadian Mathematical Bulletin, 2016, 59(2), 225–233.

[3] P. Agarwal, M. Jleli and M. Tomar, *Certain Hermite-Hadamard type inequalities via generalized k-fractional integrals*, Journal of Inequalities and Applications, 2017, 2017, 1–10.

[4] F. M. Atıcı and H. Yaldız, *Refinements on the discrete Hermite-Hadamard inequality*, Arabian Journal of Mathematics, 2018, 7(2), 175–183.

[5] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Springer Science & Business Media, 2001.

[6] A. Barani, A. G. Ghazanfari and S. S. Dragomir, *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, Journal of Inequalities and Applications, 2012, 2012(1), 1–9.

[7] T. Du, J. Liao, L. Chen and M. U. Awan, *Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized* (α, m)*-preinvex functions*, Journal of Inequalities and Applications, 2016, 2016(1), 1–24.

[8] S. S. Dragomir, *Generalization of Ostrowski inequality for convex functions*, Mathematica Slovaca, 2018, 68(5), 1017–1040.

[9] P. W. Eloe, Q. Sheng and J. Henderson, *Notes on crossed symmetry solutions of the two-point boundary value problems on time scales*, Journal of Difference Equations and Applications, 2003, 9(1), 29–48.

[10] A. El Farissi, Z. Latreuch and B. Belaïdi, *Hadamard-type inequalities for twice differentiable functions*, RGMIA Research Report collection, 2009, 12(1).

[11] B. O. Fagbemigun, A. A. Mogbademu and J. O. Olaleru, *Hermite-Hadamard inequality for a certain class of convex functions on time scales*, Honam Mathematical Journal, 2022, 44(1), 17–25.

[12] J. Green and W. P. Heller, *Mathematical analysis and convexity with applications to economics*, Handbook of Mathematical Economics, 1981, 1, 15–52.

[13] C. Goodrich and A. C. Peterson, *Discrete Fractional Calculus*, Cham: Springer, 2015.

[14] J. Hadamard, *Étude sur les propriétés des fonctions entières et en particulier d'une fonction considérée par Riemann*, Journal de Mathématiques Pures et Appliquées, 1893, 9, 171–215.

[15] H. Hudzik and L. Maligranda, *Some remarks on s-convex functions*, Aequationes Mathematicae, 1994, 48, 100–111.

[16] M. Adil Khan, Y. Khurshid, T. S. Du and Y. M. Chu, *Generalization of Hermite-Hadamard type inequalities via conformable fractional integrals*, Journal of Function Spaces, 2018, 2018, 1–12.

[17] S. R. Mohan and S. K. Neogy, *On invex sets and preinvex functions*, Journal of Mathematical Analysis and Applications, 1995, 189(3), 901–908.

[18] P. O. Mohammed and M. Z. Sarikaya, *Hermite-Hadamard type inequalities for* F*-convex function involving fractional integrals*, Journal of Inequalities and Applications, 2018, 2018, 1–33.

[19] P. O. Mohammed, T. Abdeljawad, M. A. Alqudah and F. Jarad, *New discrete inequalities of Hermite-Hadamard type for convex functions*, Advances in Difference Equations, 2021, 2021(1), 1–10.

[20] P. O. Mohammed, C. S. Ryoo, A. Kashuri, Y. S. Hamed and K. M. Abualnaja, *Some Hermite-Hadamard and Opial dynamic inequalities on time scales*, Journal of Inequalities and Applications, 2021, 2021(1), 1–11.

[21] N. Mehreen and M. Anwar, *Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for* p*-convex functions via conformable fractional integrals*, Journal of Inequalities and Applications, 2020, 2020, 1–18.

[22] M. A. Noor, K. I. Noor and M. U. Awan, *A new Hermite-Hadamard type inequality for* h*-convex functions*, Creative Mathematics and Informatics, 2015, 24(2), 191–197.

[23] J. E. Peajcariaac and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, Academic Press, 1992.

[24] T. Pennanen, *Convex duality in stochastic optimization and mathematical finance*, Mathematics of Operations Research, 2011, 36(2), 340–362.

[25] J. Pelczyński, *Application of the theory of convex sets for engineering structures with uncertain parameters*, Applied Sciences, 2020, 10(19), 6864.

[26] M. Z. Sarikaya, E. Set, H. Yaldiz and N. Başak, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Mathematical and Computer Modelling, 2013, 57(9-10), 2403–2407.

[27] M. Z. Sarikaya and H. Yaldiz, *On generalized Hermite-Hadamard type integral inequalities involving Riemann-Liouville fractional integrals*, Nihonkai Mathematical Journal, 2014, 25(2), 93–104.

[28] M. B. Sun, X. P. Li, S. F. Tang and Z. Y. Zhang, *Schur convexity and inequalities for a multivariate symmetric function*, Journal of Function Spaces, 2020, 2020, 1-10.

[29] S. K. Sahoo, H. Ahmad, M. Tariq, B. Kodamasingh, H. Aydi and M. De la Sen, *Hermite-Hadamard type inequalities involving k-fractional operator for $(\bar{h}, m)$-convex functions*, Symmetry, 2021, 13(9), 1686.

[30] T. Saeed, W. Afzal, M. Abbas, S. Treanţă and M. De la Sen, *Some new generalizations of integral inequalities for harmonical cr-(h_1, h_2)-Godunova-Levin functions and applications*, Mathematics, 2022, 10(23), 4540.

[31] M. Samraiz, M. Malik, S. Naheed, G. Rahman and K. Nonlaopon, *Hermite-Hadamard-type inequalities via different convexities with applications*, Journal of Inequalities and Applications, 2023, 2023(1), 70.

[32] T. Tunc, H. Budak, F. Usta and M. Z. Sarikaya, *New Hermite-Hadamard type inequalities on fractal set*, International Journal of Nonlinear Analysis and Applications, 2021, 12(1), 782–789.

[33] M. Tariq, S. K. Ntouyas and A. A. Shaikh, *New variant of Hermite-Hadamard, Fejér and Pachpatte-type inequality and its refinements pertaining to fractional integral operator*, Fractal and Fractional, 2023, 7(5), 405.

[34] S. Varošanec, *On h-convexity*, Journal of Mathematical Analysis and Applications, 2007, 326(1), 303–311.

[35] T. Weir and B. Mond, *Pre-invex functions in multiple objective optimization*, Journal of Mathematical Analysis and Applications, 1988, 136(1), 29–38.

[36] X. Wu, J. Wang and J. Zhang, *Hermite-Hadamard-type inequalities for convex functions via the fractional integrals with exponential kernel*, Mathematics, 2019, 7(9), 845.

[37] Q. Wang and R. Xu, *Generalized Hermite-Hadamard inequalities on discrete time scales*, Fractal and Fractional, 2022, 6(10), 563.

[38] H. Yaldız and P. Agarwal, *s-convex functions on discrete time domains*, Analysis, 2017, 37(4), 179–184.