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Numerical Analysis for Fractional Riccati Differential Equations Based on Finite Difference Method

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Abstract The fractional Riccati differential equation has a wide application in various areas, for instance, economics and the description of solar activity. In this paper, we focus on the numerical approach of the fractional Riccati differential equations. Two different types of fractional operators are considered under the Riemann-Liouville and Caputo senses. From the numerical simulations, we observe that the explicit finite difference method is not stable. Instead, we employ the implicit finite difference methods to discretize the complicated systems such that stability can be guaranteed. We also exhibit the total error estimations for our algorithms to ensure good approximations. Compared with the other polynomial numerical methods, we can properly extend the model into a larger domain with a large terminal time, which can be verified by numerical examples. Further, we discuss some complex numerical examples to demonstrate the performance of our methods and indicate that our approaches are applicable and tractable to other fractional Riccati equations.

Keywords Fractional Riccati differential equations, finite difference method, implicit method, numerical examples

MSC(2010) 34A08, 65M06, 65N06.

1. Introduction

Riccati differential equation has been used widely in the stochastic control [13, 14, 16, 50, 53] and physics [2, 48, 54]. There are various types of Riccati differential equations and generally, one may see some quadratic forms as in [3, 49]:

$$\frac{dy(t)}{dt} = A(t)y(t) + B(t)y^2(t) + C(t),$$

where $A(\cdot)$, $B(\cdot)$, and $C(\cdot)$ are smooth functions. Its numerical and analytical solutions have been well studied. In [7], the authors established an analytic solution and a reliable numerical approximation of the Riccati equation by using Adomian's decomposition method. [22] presented a method for the computation of the periodic

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nonnegative definite stabilizing solution of the periodic Riccati equation. Algebraic Riccati equations have been discussed in [5, 20, 21].

As an extension of such a model, a fractional differential operator is employed since the modified equations significantly improve its application in practice (cf. [29, 32, 42, 52]). We consider the following fractional Riccati differential equation (cf. [6]):

$$D^{\alpha}y(t) = A(t)y^2(t) + B(t)y(t) + C(t), \quad 0 < \alpha \leq 1, \quad 0 \leq t \leq 1, \tag{1.1}$$

with a given initial condition, which generally has no analytical solutions (cf. [38, 39]). It is natural to establish alternative numerical methods to study the solution profile.

Several existing methods have been proposed for solving the fractional Riccati differential equation numerically. [26] proposed a modified variational iteration method based on Adomian polynomials. In [31], the authors used a fractional-order Legendre operational matrix. Homotopy perturbation technique and B-spline operational matrix were discussed in [27, 28]. [18] used the hyperbolic-NILT method to solve the fractional differential equations. A new modified Atangana-Baleanu was proposed in [46, 55]. Additionally, the theoretical results regarding the stability were presented in [10, 11]. Other efficient literature can be found in [1, 15, 24, 25, 35, 40, 42, 56].

However, there are some potential constraints within these existing approaches. One may see that we have the limitation for variable t in the equation (1.1), and the polynomial approximation will blow up with $t > 1$. Our major contribution can be summarized as the following: First, we consider a new discretization based on the finite difference method, which expands the original domain for larger t as an extension:

$$D^{\alpha}y(t) = A(t)y^2(t) + B(t)y(t) + C(t), \quad 0 < \alpha \leq 1, \quad t > 1. \tag{1.2}$$

Second, we establish the error estimation under some mild assumptions, which can be verified by its corresponding numerical results. Moreover, our analysis is based on two different fractional operator definitions: Caputo's fractional definition and Riemann-Liouville's fractional definition. Other fractional definitions may be conducted in a similar fashion, and we omit the details for brevity. Last but not least, we consider an implicit finite difference method instead of an explicit method, which loses the stability property in general. However, from our numerical experiments, we observe that the implicit method guarantees the stability of the system, and the graph fits well. Meanwhile, a quadratic solution is provided to manage the implicit component.

The rest of the paper is organized as follows. In Section 2, we present the basic model with two different definitions of fractional operator: Caputo and Riemann-Liouville. Section 3 provides error estimation based on the implicit finite difference method, and the theoretical results indicate stability. Then, Section 4 exhibits the numerical examples to support our theoretical analysis. We draw the conclusions in the Section 5.

2. Basic models

2.1. Caputo fractional derivative

There are various types of fractional derivatives given in the literature [8, 33, 34, 43, 45]. We intend to consider Caputo's fractional derivative in this section.

Definition 2.1. The fractional integral operator with the order α in the Caputo sense (see [1, 17]) is defined as

$$ {}_0^C D_x^\alpha y(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} y^{(m)}(t)dt, \quad \alpha > 0, \quad x > 0. $$

for $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, and $\Gamma(\cdot)$ is the gamma function.

One observes that in our model (1.1) $\alpha < 1$, which yields that $m = 1$ in the above definition. Therefore, the fractional operator can be further written as:

$$ {}_0^C D_x^\alpha y(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-t)^{-\alpha} y'(t)dt, \tag{2.1} $$

where $\alpha, x > 0$. This enables us to utilize the finite difference method with a proper mesh size in the direction of the independent variable, which further implies

$$ \begin{aligned} {}_0^C D^\alpha y(t_j) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} \frac{y'(t)}{(t_j - t)^\alpha} dt \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{y'(t)}{(t_j - t)^\alpha} dt \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{y(t_{k+1}) - y(t_k)}{(t_j - t_k)^\alpha}. \end{aligned} $$

Therefore, our fractional model (1.2) has the discretization scheme:

$$ A(t_{j+1})y^2(t_{j+1}) + B(t_{j+1})y(t_{j+1}) + (C(t_{j+1}) - C_A(t_0, \cdots, t_j)) = 0, \tag{2.2} $$

where the last term depends on all history values:

$$ C_A(t_0, \cdots, t_j) \equiv \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{y(t_{k+1}) - y(t_k)}{(t_j - t_k)^\alpha}. $$

Notice that our method distinguishes from the finite difference methods for general non-fractional equations involving only several past data within each iteration. Here, (2.2) includes all the history prior to the time t_{j+1}. Hence, it is straightforward to solve the implicit scheme in terms of t_{j+1} with the quadratic formula, where we may pick its positive value

$$ y(t_{j+1}) = \frac{-B(t_{j+1}) + \sqrt{(B(t_{j+1}))^2 - 4A(t_{j+1})(C(t_{j+1}) - C_A)}}{2A(t_{j+1})}. \tag{2.3} $$

Here, the discriminant should be guaranteed to be positive so that no complex values can be generated. For brevity, we postpone the analysis of discriminant to Sections 3 and 4.

2.2. Riemann-Liouville fractional derivative

The Riemann-Liouville (R-L) definition is another widely used fractional differentiation for reality modeling (cf. [23, 30, 36, 41, 47]). We employ the initial definition of the R-L fractional derivative proposed in [19] and state in the following definition for clarity.

Definition 2.2. The R-L fractional integral operator with the order $0 < \alpha \leq 1$ is defined as

$$ {}_{0}^{RL}D_{x}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_0^x (x-t)^{m-\alpha-1} f(t)dt, $$

for $x > 0$ with $m - 1 < \alpha \leq m$, and $m \in \mathbb{N}$.

The special case for $\alpha = 1$ can be reduced to the ordinary first-order derivative. Thus, in our Riccati model, we assume that $\alpha < 1$, which leads to $m = 1$ in the above definition. Therefore, the fractional operator can be further written as:

$$ D^{\alpha} y(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{-\alpha} y(t)dt, \tag{2.4} $$

where $x > 0$ and $0 < \alpha \leq 1$. This enables us to employ the finite difference method with a mesh size in the direction of the independent variable.

The discretization of our fractional Riccati equation is the literature (cf. [4, 12, 19, 37]). To the best of our knowledge, it is the first time to consider an implicit relationship and we intend to formulate a quadratic equation. To this end, we define the new function $F(\cdot)$:

$$ F(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y(\tau)d\tau. $$

The Riemann-Liouville fractional operator (2.4) renders that

$$ D^{\alpha} y(x) = \frac{d}{dt} F(t) = \frac{F(t_{j+1}) - F(t_j)}{\Delta t}, \tag{2.5} $$

where the discretization is given by

$$ \begin{aligned} F(t_{j+1}) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} (t_{j+1}-\tau)^{-\alpha} y(\tau)d\tau, \\ F(t_j) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} (t_j-\tau)^{-\alpha} y(\tau)d\tau. \end{aligned} \tag{2.6} $$

We consider a two-point approximation of the integration in (2.6) as follows:

$$\begin{aligned}
&F(t_{j+1})\\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} (t_{j+1}-\tau)^{-\alpha} y(\tau) d\tau\\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} (t_{j+1}-\tau)^{-\alpha} \frac{y(t_{k+1})+y(t_k)}{2} d\tau\\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j} \frac{y(t_{k+1})+y(t_k)}{2} \left(-\frac{1}{1-\alpha}\right) \left[(t_{j+1}-t_{k+1})^{-\alpha+1} - (t_{j+1}-t_k)^{-\alpha+1}\right]\\
&= -\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \frac{y(t_{k+1})+y(t_k)}{2} \left[(t_{j+1}-t_{k+1})^{-\alpha+1} - (t_{j+1}-t_k)^{-\alpha+1}\right].
\end{aligned} \tag{2.7}$$

The last equality holds since the properties of gamma function: $\Gamma(z+1) = z\Gamma(z)$ and we have $(1-\alpha)\Gamma(1-\alpha) = \Gamma(2-\alpha)$. Similarly, we can derive the approximation of $F(t_j)$ as follows:

$$\begin{aligned}
F(t_j) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_j} (t_j-\tau)^{-\alpha} y(\tau) d\tau\\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} (t_j-\tau)^{-\alpha} \frac{y(t_{k+1})+y(t_k)}{2} d\tau\\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{y(t_{k+1})+y(t_k)}{2} \left(-\frac{1}{1-\alpha}\right) \left[(t_j-t_{k+1})^{-\alpha+1} - (t_j-t_k)^{-\alpha+1}\right]\\
&= -\frac{1}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \frac{y(t_{k+1})+y(t_k)}{2} \left[(t_j-t_{k+1})^{-\alpha+1} - (t_j-t_k)^{-\alpha+1}\right].
\end{aligned} \tag{2.8}$$

Substituting them into the expression of $F'(t)$ in (2.5), we have the discretized estimation:

$$\begin{aligned}
\frac{d}{dt}F(t) = \frac{1}{\Delta t \Gamma(2-\alpha)} \Bigg\{ &- \sum_{k=0}^{j} \frac{y(t_{k+1})+y(t_k)}{2} [(t_{j+1}-t_{k+1})^{-\alpha+1} - (t_{j+1}-t_k)^{-\alpha+1}]\\
&+ \sum_{k=0}^{j-1} \frac{y(t_{k+1})+y(t_k)}{2} \left[(t_j-t_{k+1})^{-\alpha+1} - (t_j-t_k)^{-\alpha+1}\right] \Bigg\} + R_{\alpha,j},
\end{aligned} \tag{2.9}$$

where $R_{\alpha,j}$ is the error term defined as (cf. [4]):

$$R_{\alpha,j} = \frac{1}{\Delta t \Gamma(2-\alpha)} \left\{ \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} \frac{y(s)-y(t_{k+1})}{(t_{j+1}-s)^{\alpha}} ds - \sum_{k=0}^{j-1} \int_{t_k}^{t_{k+1}} \frac{y(s)-y(t_{k+1})}{(t_j-s)^{\alpha}} ds \right\}.$$

Therefore, our fractional model (1.2) obtains the discretization scheme:

$$\frac{F(t_{j+1})-F(t_j)}{\Delta t} = A(t_{j+1}) y^2(t_{j+1}) + B(t_{j+1}) y(t_{j+1}) + C(t_{j+1}), \tag{2.10}$$

where the left-hand side can be computed using (2.9). It is worth mentioning that we tend to use the implicit scheme as in the right-hand side of (2.10), which further yields quadratic equation in terms of the value at t_{j+1} with some simple algebraic manipulations as follows:

$$NA(t_{j+1})y^2(t_{j+1}) + (NB(t_{j+1}) - 1)y(t_{j+1}) + NC(t_{j+1}) + M = 0, \tag{2.11}$$

where N and M are quantities depending on the index j and the past values of function y at $t_0, \cdots, t_j$, and they are defined as follows:

$$\begin{aligned} N &= -\frac{1}{a_{jj}} \Delta t^\alpha \Gamma(2-\alpha), \\ M &= -y(t_j) + \frac{1}{a_{jj}} \left(-\sum_{k=0}^{j-1} (y(t_{k+1}) + y(t_k)) a_{jk} + \sum_{k=0}^{j-1} (y(t_{k+1}) + y(t_k)) b_{jk} \right). \end{aligned} \tag{2.12}$$

Further, the scalers are defined as

$$\begin{aligned} a_{jk} &= (j-k)^{1-\alpha} - (j+1-k)^{1-\alpha}, \\ b_{jk} &= (j-k-1)^{1-\alpha} - (j-k)^{1-\alpha}. \end{aligned} \tag{2.13}$$

For the specific $j \equiv k$, we have $a_{jj} = b_{j,j-1} = -1, a_{j,j-1} = 1 - 2^{1-\alpha}$. Hence, it is straightforward to solve the implicit scheme with the quadratic formula (positive value),

$$y(t_{j+1}) = \frac{-(NB(t_{j+1}) - 1) + \sqrt{(NB(t_{j+1}) - 1)^2 - 4NA(t_{j+1})(NC(t_{j+1}) + M)}}{2NA(t_{j+1})}. \tag{2.14}$$

The discriminant should be guaranteed to be positive so that no complex values will be generated. For brevity, we postpone the analysis of discriminant to Sections 3 and 4.

3. Error estimations

In this section, we focus on the error estimation for the fractional Riccati differential equation. To simplify the calculation, we take the specific equation as an example :

$$D^{1/2}y(t) + y(t) + y^2(t) = g(t), \tag{3.1}$$

where $g(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + t^2 + t^4$. The exact solution is $y(t) = t^2$, and the initial value is $y(0) = 0$. Our theorems are divided into two settings: one is based on the Caputo fractional definition, and the other is according to the R-L fractional definition.

Theorem 3.1. *In the Caputo fractional Riccati in Section 2.1, we consider the scheme starting at the origin with initial condition $y(t_1 = 0) = 0$. Then we have the basic estimation for the solution at each point $y(t_n) = ((n-1)\Delta t)^2 + O(\Delta t^\lambda)$, where Δt is the mesh size and λ is the higher order number.*

Proof. We simplify this equation with an implicit finite difference method,

$$\begin{aligned} g(t_{j+1}) &= \frac{1}{\Gamma(1-\alpha)}\int_0^{t_j}\frac{y'(s)}{(t_j-s)^\alpha}ds + y(t_{j+1}) + y^2(t_{j+1}) \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{j-1}\int_{t_i}^{t_{i+1}}\frac{y'(s)}{(t_j-s)^\alpha}ds + y(t_{j+1}) + y^2(t_{j+1}) \\ &= \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{j-1}\frac{y(t_{i+1})-y(t_i)}{(t_j-t_i)^\alpha} + y(t_{j+1}) + y^2(t_{j+1}). \end{aligned} \tag{3.2}$$

We further introduce the quantity

$$C_A(t_0,\cdots,t_j) = \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{j-1}\frac{y(t_{i+1})-y(t_i)}{(t_j-t_i)^\alpha}, \tag{3.3}$$

which is independent of $y(t_{j+1})$ but depending on the past values of y at $t_1,\cdots,t_j$. Then we have the modified equation:

$$y^2(t_{j+1}) + y(t_{j+1}) + (C_A(t_0,\cdots,t_j) - g(t_{j+1})) = 0, \tag{3.4}$$

which has the solution through the quadratic formula:

$$y(t_{j+1}) = \frac{-1+\sqrt{1-4(C_A(t_0,\cdots,t_j)-g(t_{j+1}))}}{2}. \tag{3.5}$$

Take the example for $g(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + t^2 + t^4$ with $t \geq 0$ into consideration, which implies

$$g(t_{j+1}) = \frac{8(jh)^{3/2}}{3\sqrt{\pi}} + t_{j+1}^2 + t_{j+1}^4 \approx \frac{8(h)^{3/2}}{3\sqrt{\pi}} + (jh)^2 + (jh)^4.$$

Notice that

$$C_A(t_0,\cdots,t_j) = \frac{1}{\Gamma(1-\alpha)}\sum_{i=1}^{j-1}[(j-i)h]^{-\alpha}\left(y(t_{i+1})-y(t_i)\right).$$

Now it is crucial to estimate the discriminant $\sqrt{1-4(C_A(t_0,\cdots,t_j)-g(t_{j+1}))}$. We may approximate it using the mesh size h as follows:

$$\begin{aligned} &\sqrt{1-4(C_A-g(t_{j+1}))} \\ &\approx \sqrt{(2(jh)^2+1)^2 + 4\left(\frac{8(jh)^{3/2}}{3\sqrt{\pi}} - \frac{1}{\sqrt{\pi}}\sum_{i=1}^{j-1}(y(t_{i+1})-y(t_i))(j-i)^{-1/2}h^{-1/2}\right)} \\ &\approx 2(jh)^2 + 1. \end{aligned} \tag{3.6}$$

This estimation indicates that our solution $y(t_{j+1})$ at time t_{j+1} can be approximated using $(jh)^2$, namely $y(t_{j+1}) \approx (jh)^2$, which coincides with our analytical solution of the Caputo fraction Riccati equation. Notice that we may assume $y_i = y(t_i) \approx (i-1)^2h^2$. The summation part can be thought of as $Mh^{3/2}$ where M is a constant, and it is negligible in our estimation. □

Theorem 3.2. *In the R-L fractional Riccati defined in Section 2.2, we consider* $A(t) = -1, B(t) = -1, C(t) = g(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + t^2 + t^4$ *for all* $t \in [0, T]$. *Then* $y(t_{j+1})$ *can be approximated by* $(jh)^2$ *for small* h, *where* $h > 0$ *is the mesh size of our iteration.*

Proof. We substitute the coefficients A, B, and C into the equation (2.14) to obtain

$$y(t_{j+1}) = \frac{N + 1 - \sqrt{(N+1)^2 + 4N(t_{j+1})(NC(t_{j+1}) + M)}}{-2N}, \tag{3.7}$$

where $N = \Delta t^{\alpha}\Gamma(2-\alpha)$ is a positive number, and

$$\begin{aligned} M &= -y(t_j) - \left(-\sum_{k=0}^{j-1}(y(t_{k+1}) + y(t_k))a_{jk} + \sum_{k=0}^{j-1}(y(t_{k+1}) + y(t_k))b_{jk} \right) \\ &= -y(t_j) + \sum_{k=0}^{j-1}[y(t_{k+1}) + y(t_k)](a_{jk} - b_{jk}). \end{aligned}$$

Here we intend to show that the numerator of (3.7) can be estimated using some scaled $(jh)^2$ as follows:

$$N + 1 - \sqrt{(N+1)^2 + 4N(t_{j+1})(NC(t_{j+1}) + M)} \approx -2N(jh)^2, \tag{3.8}$$

If we square both sides of (3.8), we obtain

$$(N+1)^2 + 4N(t_{j+1})(NC(t_{j+1}) + M) \approx 4N^2(jh)^4 + (N+1)^2 + 4N(N+1)(jh)^2.$$

With some easy algebraic manipulations, we further obtain

$$N(t_{j+1})(NC(t_{j+1}) + M) \approx N^2(jh)^4 + N(N+1)(jh)^2.$$

Then we substitute the function $C(\cdot)$ in our case to derive

$$NM + N^2\frac{8(jh)^{3/2}}{3\sqrt{\pi}} \approx N(jh)^2.$$

We are left to estimate the value of M as follows:

$$\begin{aligned} M &= -y(t_j) - \left(-\sum_{k=0}^{j-1}(y(t_{k+1}) + y(t_k))a_{jk} + \sum_{k=0}^{j-1}(y(t_{k+1}) + y(t_k))b_{jk} \right) \\ &= -y(t_j) + \sum_{k=0}^{j-1}[y(t_{k+1}) + y(t_k)](a_{jk} - b_{jk}) \\ &\approx -((j-1)h)^2 + \sum_{k=0}^{j-1}\left[(kh)^2 + (k-1)^2h^2)\right](a_{jk} - b_{jk}) \\ &= h^2\left(-(j-1)^2 + \sum_{k=0}^{j-1}(k^2 + (k-1)^2)(a_{jk} - b_{jk}) \right) \\ &\approx (jh)^2. \end{aligned}$$

It is straightforward to obtain that the last approximation converges as j tends to infinity by Matlab, namely, if we set the task function:

$$f(j) = \frac{1}{j^2}\left(-(j-1)^2 + \sum_{k=0}^{j-1}(k^2+(k-1)^2)(a_{jk}-b_{jk})\right),$$

one can easily deduce that $f(j)$ converges to 1 as j tends to infinity. This completes the proof. □

4. Numerical examples

In this section, we give three examples to exhibit our numerical methods in estimating the R-L and Caputo fractional Riccati equations. Meanwhile, we use a complex function in Example 4.3 to demonstrate the performance of our methods compared with polynomial approximations in terms of unusual complicated models with large terminal time.

Example 4.1. We consider the fractional Riccati equation

$$D^{1/2}y(t) + y(t) + y^2(t) = g(t), \tag{4.1}$$

where $g(t) = \frac{8t^{3/2}}{3\sqrt{\pi}} + t^2 + t^4$ with initial value $y(0) = 0$. The exact solution is $y(t) = t^2$. We plot the Figure 1 based on R-L fractional definition and Figure 2 for Caputo definition. The absolute error is shown in Figures 3 and 4. Moreover, the relative errors with different mesh sizes and terminal times are given in Table 1.

Table 1: Relative errors for R-L and Caputo Riccati

	$h=0.1, T=10$	$h=0.2, T=10$	$h=0.5, T=10$
R-L	0.006693708	0.006289459	0.005233233
Caputo	0.001817859	0.002459315	0.003605995
	$h=0.1, T=20$	$h=0.2, T=20$	$h=0.5, T=20$
R-L	0.00141319	0.001349044	0.001182568
Caputo	0.000333145	0.000454120	0.000678630
	$h=0.1, T=50$	$h=0.2, T=50$	$h=0.5, T=50$
R-L	0.000169318	0.000163655	0.000149126
Caputo	3.443717×10^{-5}	4.715865×10^{-5}	7.129121×10^{-5}

We find that even though the finite difference method of the R-L and Caputo fractional Riccati equations can generate good approximations of the analytical solution t^2 for different mesh sizes and terminal times (see Figures 1 and 2), Caputo definition provides less relative errors. However, one may notice fluctuations of absolute errors at the beginning of the iteration (see Figure 4), and they eventually tend to zero at terminal times. Both definitions in our finite difference method render decent approximations in this example. Further, our algorithms provide

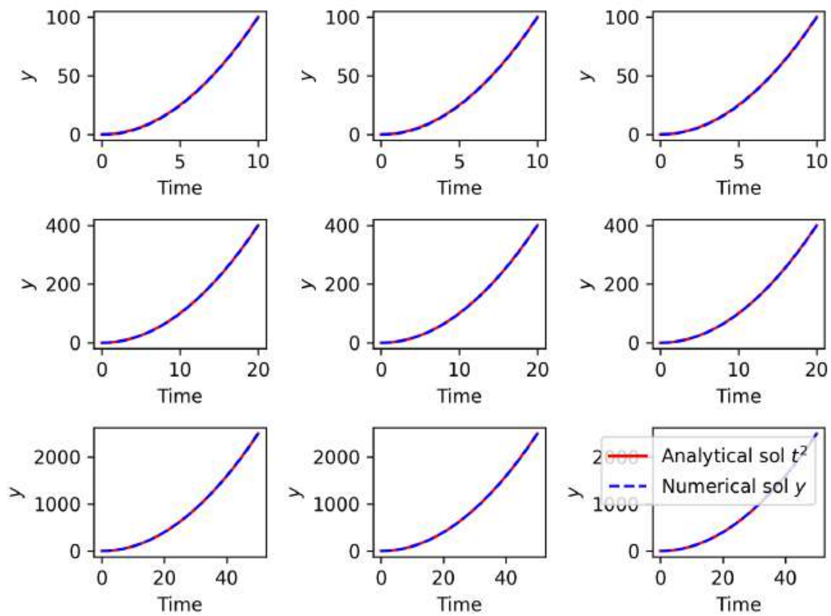

Figure 1. R-L fractional Riccati (4.1) with different mesh sizes and terminal times

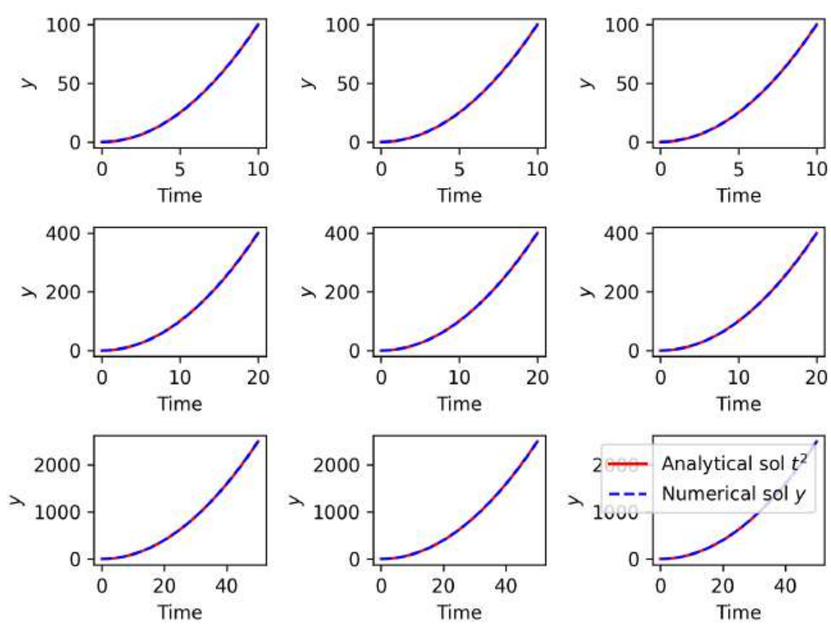

Figure 2. Caputo fractional Riccati (4.1) with different mesh sizes and terminal times

the accurate approximation solution at large terminal times T, and they still yield better approximation compared with polynomial approximation methods (see [6, 15, 31, 56]).

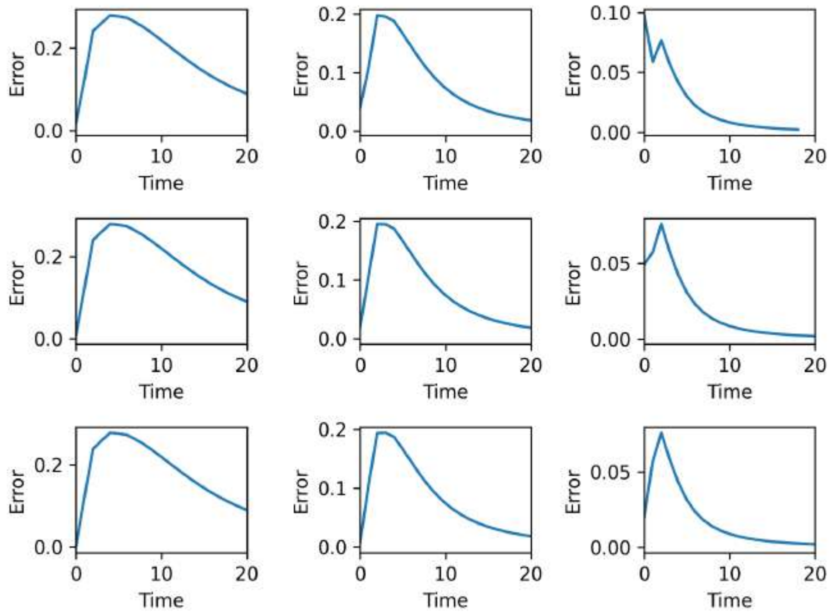

Figure 3. Absolute errors of R-L fractional Riccati (4.1) with different mesh sizes and terminal times

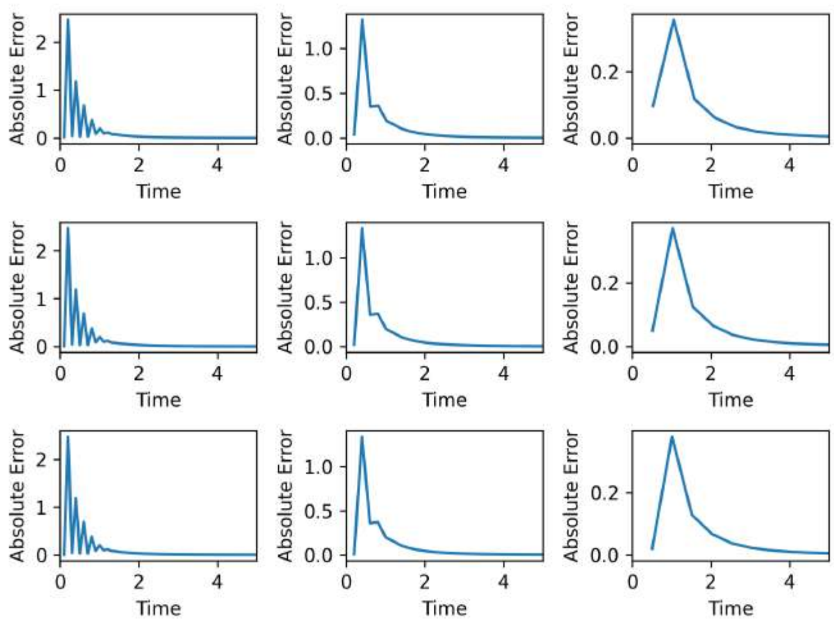

Figure 4. Absolute errors of Caputo fractional Riccati (4.1) with different mesh sizes and terminal times

Remark 4.1. We can also apply the Taylor expansion method to analyze such a model. The details are presented in Appendix A.

Example 4.2. For the second example, we consider the fractional Riccati equation

$$D^{1/2}y(t) + y^2(t) = h(t), \tag{4.2}$$

where $h(t) = \frac{1}{\sqrt{\pi}}\frac{32}{10}t^{5/2} + t^6$ with initial value $y(0) = 0$. The exact solution is $y(t) = t^3$. We plot Figure 5 based on the R-L fractional definition and Figure 6 Caputo definition. The absolute error is shown in Figures 7 and 8. Furthermore, the relative errors with different mesh sizes and terminal times are given in Table 2.

Table 2: Relative errors for R-L and Caputo Riccati

	$h = 0.1, T = 10$	$h = 0.2, T = 10$	$h = 0.5, T = 10$
R-L	0.000971738	0.000869276	0.000635862
Caputo	0.000976501	0.000877992	0.000709786
	$h = 0.1, T = 20$	$h = 0.2, T = 20$	$h = 0.5, T = 20$
R-L	0.000101877	9.381754380×10^{-5}	7.549694872×10^{-5}
Caputo	0.000139692	0.000131738	0.000117240
	$h = 0.1, T = 50$	$h = 0.2, T = 50$	$h = 0.5, T = 50$
R-L	4.861499585×10^{-6}	4.579302659×10^{-6}	3.946757902×10^{-6}
Caputo	9.784884134×10^{-6}	9.520875324×10^{-6}	9.021742935×10^{-6}

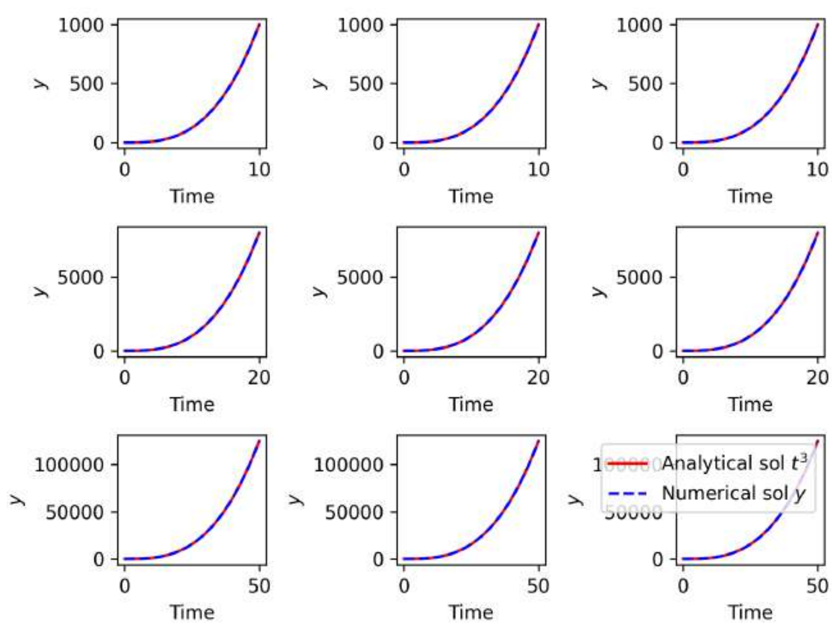

Figure 5. R-L fractional Riccati (4.2) with different mesh sizes and terminal times

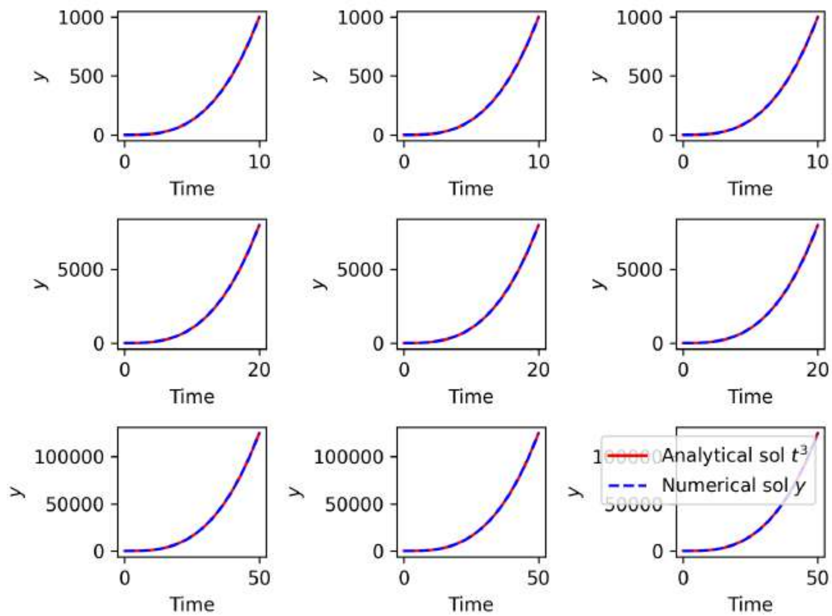

Figure 6. Caputo fractional Riccati (4.2) with different mesh sizes and terminal times

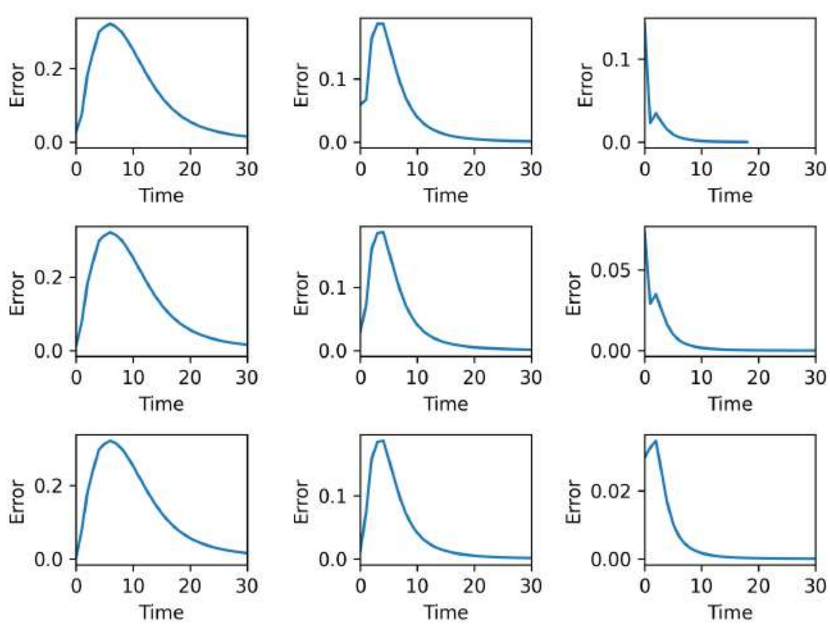

Figure 7. Absolute errors of R-L fractional Riccati (4.2) with different mesh sizes and terminal times

In this example, we can draw a similar conclusion as the previous example, which shows that both R-L and Caputo fractional Riccati provide a good approximation of the analytical solution t^3 in Example 4.2 by using the finite difference method.

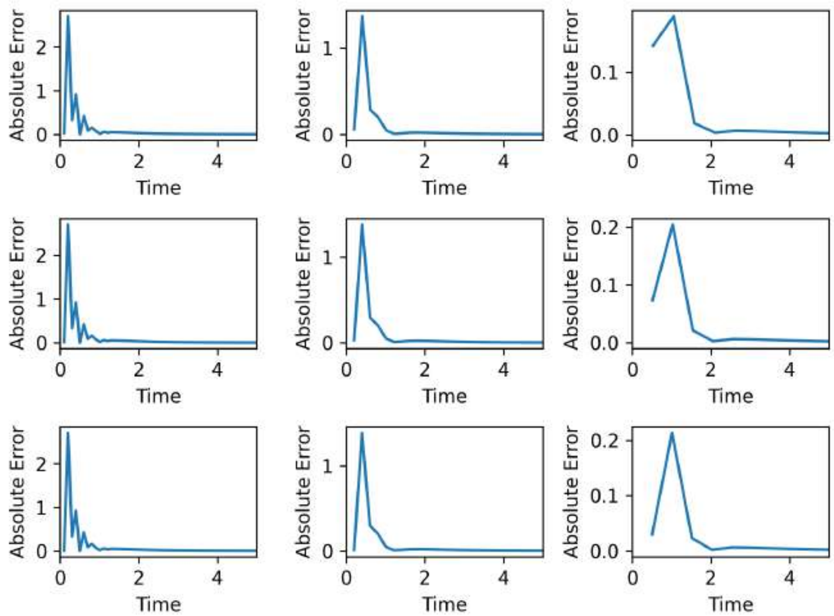

Figure 8. Aboslute errors of Caputo fractional Riccati (4.2) with different mesh sizes and terminal times

The relative errors are acceptable numerically. Figures 7 and 8 indicate that the absolute errors tend to zero regardless of mesh sizes and terminal times so that our methods enable large terminal-time approximations.

Example 4.3. We consider the fractional Riccati equation

$$D^{1/2}y(t) + y^2(t) = h(t), \tag{4.3}$$

where

$$h(t) = \frac{e}{\sqrt{\pi}}\left(\frac{1}{\sqrt{t}} - 2F(\sqrt{t})\right) + e^{2-2t}, \tag{4.4}$$

where F function is the Dowson integral [9, 51] with initial value $y(0) = e$. The exact solution is $y(t) = e^{1-t}$. Figure 9 demonstrates the approximation of the R-L Riccati in the setting of (4.3).

Even though the function $h(\cdot)$ in the fractional Riccati (4.3) takes an unusual Dowson integration, the simulation yields a nice approximation to the R-L Riccati equation (4.3) in Figure 9. This example indicates the power of our method, which is applicable and tractable in complex cases regardless of terminal times (see (4.3)).

Remark 4.2. It is worth mentioning that some limitations exist for our proposed method. Firstly, its accuracy is still uncertain when dealing with the corresponding partial differential equation, which is a more complicated system. Moreover, its practical applications are moot. It is also natural to see some connections between the implicit method with machine learning techniques as a function approximation.

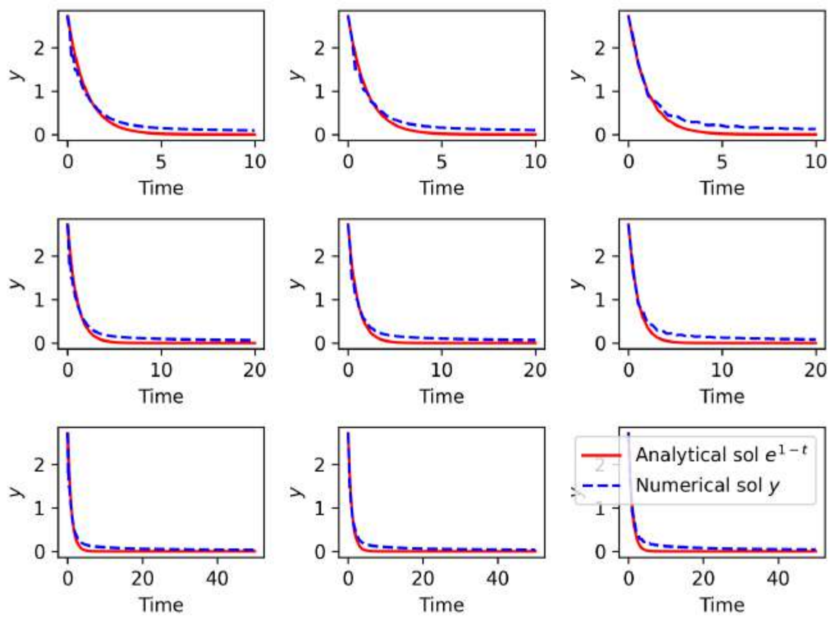

Figure 9. R-L fractional Riccati (4.3) with different mesh sizes and terminal times

5. Conclusion

In this paper, we focus on the fractional Riccati differential equation with two different definitions: Caputo and Riemann-Liouville. We discretize the system with the normal finite difference method and find the iteration through a non-trivial implicit scheme, which is different from the previous discussion. The implicit method can guarantee the stability of the system and make the solution smooth. The solution can be estimated by solving a quadratic equation. Further, the error estimations demonstrate the accuracy of the algorithm, and numerical simulations fit well in the numerical examples. Moreover, our methods provide the possibility to track large terminal time and render good approximations at the tails. There are two interesting paths we can follow in our further work: (i) We may extend the similar method to other fractional differential equations proposed in physics and engineering; (ii) In this paper, we only consider the time-dependent fractional operator. One may consider the space-dependent fractional operator, which can be challenging from the numerical perspective due to the past-dependent phenomenon. It remains unknown if one can draw connections with the Mittag-Leffler function (cf. [44]).

A. Taylor expansion

We aim to solve the fractional differential equation problem by proper Taylor expansions of the task functions. Consider Example 4.1:

$$D^{1/2}y(t) + y(t) + y^2(t) = g(t). \tag{A.1}$$

The normal Taylor expansion of $y(s)$ with respect to variable t can be given as follows:

$$y(s) = y(t) + y'(t)(s-t) + \frac{1}{2}(s-t)^2 y''(t) + O((s-t)^3).$$

Then, the first derivative of $y(s)$ can be written as

$$y'(s) = y'(t) + (s-t)y''(t) + O((s-t)^3).$$

If we substitute it into (A.1) under the Caputo definition, we have

$$\frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{y'(t) + (s-t)y''(t)}{(t-s)^{1/2}} ds + y(t) + y^2(t) = g(t).$$

The basic algebra indicates that

$$\int_0^t (t-s)^{-1/2} ds = 2\sqrt{t},$$
$$\int_0^t \frac{s}{(t-s)^{1/2}} ds = \frac{4}{3} t^{3/2}.$$

Thus we derive that

$$\int_0^t \frac{y'(t) + (s-t)y''(t)}{(t-s)^{1/2}} ds = 2\sqrt{t} y'(t) - \frac{2}{3} t^{3/2} y''(t).$$

Since we have the analytical solution $y(t) = t^2$, one can observe that the above equation becomes

$$2\sqrt{t}(2t) - \frac{2}{3} t^{3/2}(2) = \frac{8}{3} t^{3/2},$$

which is the first part of the right-hand side of $g(t)$.

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