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Critical Point Theorems of Non-smooth Functionals without the Palais-Smale Condition

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Abstract This paper introduces some new variants of abstract critical point theorems that do not rely on any compactness condition of Palais Smale type. The focus is on locally Lipschitz continuous functional $\Phi : E \to R$, where E is a reflexive banach space. The theorems are established through the utilization of the least action principle, the perturbation argument, the reduction method, and the properties of sub-differential and generalized gradients in the sense of F.H. Clarke. These approaches have been instrumental in advancing the theory of critical points, providing a new perspective that eliminates the need for traditional compactness constraints. The implications of these results are far-reaching, with potential applications in optimization, control theory, and partial differential equations.

Keywords Critical point, minimax theorems, locally Lipschitz functional, the least action principle, perturbation argument

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1. Introduction

Critical point theory has long been a cornerstone in the study of the existence and multiplicity of solutions for various classes of nonlinear problems. Traditionally, this domain has relied heavily on specific compactness assumptions, notably the Palais-Smale condition [3,15,23]. Notwithstanding, these presumptions are not universally applicable, particularly concerning certain substantial classes of functionals. This underscores the importance of developing innovative critical point theorems that are free from these constraints for broader applicability [2,10,16,21,22,25].

The Palais-Smale condition is a classical compactness assumption that is often used to prove the existence of critical points for functionals on Banach spaces. However, this condition may not hold for some important classes of functionals, such as non-smooth or non-convex ones. Therefore, many researchers have tried to develop new critical point theorems that can overcome this limitation, see for example [6,9], or apply to more general settings, see for example [1,5,19,26].

However, the Palais-Smale condition proved to be too restrictive for many modern variational problems that involve more complicated functionals. In response, researchers looked for more flexible frameworks that could handle non-compact situations, leading to the emergence of critical point theories that do not depend on

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this traditional compactness constraint.

The study in question has its roots to the early and mid-20th century, during which mathematicians like Marston Morse [17] and R. Palais [20] established the fundamental principles of critical point theory by building upon the variational methods developed by Euler and Lagrange. Morse theory, for instance, unveiled the correlation between the topology of manifolds and the critical points of smooth functions.

This study focuses on the category of locally Lipschitz continuous functionals $\Phi : E \to \mathbb{R}$, where E is a reflexive Banach space. We present novel variations of abstract critical point theorems that eliminate the need for any compactness criterion of Palais-Smale condition. The primary methodologies we employ include the least action principle, the perturbation argument, the reduction approach, and the characteristics of sub-differential and generalized gradients as defined by F.H. Clarke [11]. These strategies enable us to acquire critical points for functionals that may exhibit non-smooth or non-convex components, and to handle different sorts of restrictions and boundary conditions. By eliminating the need for traditional compactness constraints, these new critical point theorems offer far-reaching implications, with potential applications in optimization, control theory, and partial differential equations. The generalizations presented in this paper provide a new perspective on critical point theorems, making them applicable to a wider range of functionals and settings, and eliminating the need for traditional compactness constraints.

A.C. Lazer, E.M. Landesman and D.R. Meyers in [13, Theorem 1] established the existence and uniqueness of critical points for certain functional without the compactness conditions. This theorem provided conditions under which a real valued function defined on a real Hilbert space has a unique minimizer. The theorem uses a variational argument and the saddle point principle to demonstrate the existence and uniqueness of the critical point.

Bates and Ekeland [4], Manasevich [14] and A.C. Lazer [13] generalized [13, Theorem 1] to the case where X and Y are not necessarily finite dimensional or by weakening the conditions:

$$(D^2\Phi(u)h, h) \leq -m_1\|h\|^2, \tag{1.1}$$

$$(D^2\Phi(u)k, k) \geq m_2\|k\|^2 \tag{1.2}$$

for all $u \in H, h \in X$ and $k \in Y$.

Moussaoui and Gossez [18] generalized Theorem 1 in [13] by relaxing conditions (1.1) and (1.2) to conditions of coercivity and concavity and they supposed that Φ is of class $\mathcal{C}^1$ instead of $\mathcal{C}^2$. They proved the following theorem:

Theorem 1.1. *Let H be a Hilbert space such that : $H = V \oplus W$, where V is finite dimensional subspace of H and W its orthogonal space. Consider a functional $\Phi : H \to \mathbb{R}$ that satisfies the following conditions:*
(i) Φ is of class $\mathcal{C}^1$.
(ii) Φ is coercive on W.
(iii) For a fixed $w \in W$, the mapping $v \mapsto \Phi(v + w)$ is concave on V.
(iv) For a fixed $w \in W$, $\Phi(v + w) \to -\infty$ when $\|v\| \to +\infty$, $v \in V$; and this

convergence is uniform on bounded subsets of W.
(v) For all $v \in V$, Φ is weakly lower semicontinuous on $W + v$.
Under these conditions, the functional Φ admits a critical point in H.

We can also find the proof of this theorem in [12]. By the proof of Theorem 1.1, we conclude that the critical point, the existence of which has been proven, is of the minimax form. Then with assymptions of Theorem 1.1, there exists at least one critical point $u_0 = x_0 + w_0$ such that

$$\Phi(u_0) = \max_{x \in V} \Phi(x + w_0) = \min_{w \in W} \max_{x \in V} \Phi(x + w).$$

On the other hand, by the proof of Theorem 1.1, we can deduce easily that it's dual version is also available. We present its dual of Theorem 1.1

Theorem 1.2. *Let H be a Hilbert space such that $H = V \oplus W$, where V is a finite-dimensional subspace of H and W its orthogonal space. Consider a functional $\Phi : H \to \mathbb{R}$ that satisfies the following conditions:*
(i) Φ is of class $\mathcal{C}^1$.
(ii) Φ is anticoercive on W.
(iii) For a fixed $w \in W$, the mapping $v \mapsto \Phi(v + w)$ is convexe on V.
(iv) For a fixed $w \in W$, $\Phi(v + w) \to +\infty$ when $\|v\| \to +\infty$, for $v \in V$; and this convergence is uniform on bounded subsets of W.
(v) For all $v \in V$, Φ is weakly upper semicontinuous on $W + v$.
Under these conditions, the functional Φ admits a critical point $u_0 = x_0 + w_0$ such that

$$\Phi(u_0) = \min_{x \in V} \Phi(x + w_0) = \max_{w \in W} \min_{x \in V} \Phi(x + w).$$

In this paper, we will generalize Theorem 1.2. We assume a less restrictive smoothness condition for functionals. We assume that the functional Φ is locally Lipschitz continuous. We present four variants of [13, Theorem 1]. In the first theorem, we assume V has a finite dimension. In the other theorems , we remove the requirement that V be finite-dimensional. Our results generalize our previous theorems in [7] and [8] and the work of Chun-Lei Tang and Xing-Ping Wu in [24].

2. Preliminaries

Let X be a Banach space. We recall some properties for local Lipschitz functionals:

Definition 2.1. Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz functional. For any $x, v \in X$, the generalized directional derivative of Φ at x in the direction v is defined as:

$$\Phi^o(x, v) = \limsup_{y \to x, t \to 0^+} \frac{\Phi(y + tv) - \Phi(y)}{t}. \tag{2.1}$$

Definition 2.2. The generalized gradient (Clarke subdifferential) of Φ at a point x, denoted by $\partial\Phi(x)$, is the set of all vectors $\xi \in X^*$ such that: $\Phi^o(x, v) \geq \langle \xi, v \rangle$ for all $v \in X$.

Definition 2.3. Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz functional. For each $x, v \in X$, the directional derivative of Φ at x in the direction v is

$$\Phi'(x, v) = \lim_{t \to 0} \frac{\Phi(x + tv) - \Phi(x)}{t}.$$

Let $\Phi : X \to \mathbb{R}$ be a locally Lipschitz functional with rank k near x. The following properties of the generalized directional derivative and the generalized gradient may be proved in [11].

- $\Phi^o(x, v)$ is finite, positively homogeneous, subadditive on X and satisfies
$$|\Phi^o(x, v)| \leq K\|v\|.$$
- $\Phi^o(x, -v) = (-\Phi)^o(x, v)$.
- $(x, v) \to \Phi^o(x, v)$ is upper semicontinuous.
- For each $x \in X$, $\partial\Phi(x)$ is non-empty, convex, weak* compact subset of X^*.
- For every v in X, one has
$$\Phi^o(x, v) = max\{(x^*, v); \quad x^* \in \partial\Phi(x)\}.$$
- Let $\Phi, \Psi : X \to R$ be a locally Lipschitz function. Then
$$\partial(\Phi + \Psi)(x) \subset \partial\Phi(x) + \partial\Psi(x).$$
- If Φ is convex, then $\partial\Phi(x)$ coincides with the subdifferential of Φ in the sense of convex analysis. In fact
$$\partial\Phi(x) = \{x^* \in X^*; \quad \Phi(x') - \Phi(x) \geq (x^*, x' - x), \quad \forall x' \in X\}.$$
- Consider U as an open convex subset of X. When $f : U \to \mathbb{R}$ is both convex on U and Lipschitz near x, $\partial f(x)$ is equivalent to the subdifferential at x according to convex analysis, and $f^o(x, v)$ coincides with the directional derivative $f'(x, v)$ for each v.

3. Main results

According to our first theorem, we generalize Theorem 1.2 by assuming that $\Phi : E \to \mathbb{R}$ is a locally Lipschitz continuous functional. However, rather than convexity, we assume that $\Phi(. + w)$ is strictly convex for all $w \in W$.

Theorem 3.1. *Consider a reflexive Banach space E that can be decomposed as the direct sum of two closed subspaces V and W, where V is finite dimensional. Let $\Phi : E \to \mathbb{R}$ be a function that is locally Lipschitz continuous. Φ is weakly upper semi-continuous on $W + x$ for all $x \in V$, and Φ is anti-coercive on W, meaning that*
$$\Phi(w) \to -\infty \quad as \quad \|w\| \to +\infty.$$
Assume that $\Phi(. + w) : V \to R$ is strictly convex for all $w \in W$ and
$$\Phi(x + w) \to +\infty \quad as \quad \|x\| \to +\infty, \tag{3.1}$$
and the convergence is uniform on bounded subsets of W. Then Φ has at least one critical point $u_0 = x_0 + w_0$ such that
$$\Phi(u_0) = \min_{x\in V} \Phi(x + w_0) = \max_{w\in W} \min_{x\in V} \Phi(x + w).$$

The proof of Theorem 3.1 relies on the reduction method and the least action principle. We reiterate this principle, as seen in [15].

The least action principle : Let V be a reflexive Banach space and $\Psi : V \to \mathbb{R}$ be weakly lower semi-continuous. Assume that Ψ is coercive, that is,

$$\Psi(x) \to +\infty \quad \text{as } \|x\| \to +\infty.$$

Then Ψ has at least one minimum.

In the proof of Theorem 3.1, we will need the following Lemma.

Lemma 3.1. *The set*

$$V(w) = \{x \in V : \varphi(w) = \Phi(x+w) = \min_{g \in V} \Phi(g+w)\}$$

is a singleton and $\varphi : W \to \mathbb{R}$ is bounded above and achieves its maximum at some $w_0 \in W$.

Proof. The functional $\Phi(.+w) : V \to \mathbb{R}$ is strictly convex for all $w \in W$ and it is continuous, so it is weakly lower semi-continuous. Moreover, $\Phi(.+w) : V \to \mathbb{R}$ is coercive, then according to the least action principle, it has a minimum on $V+w$ for all $w \in W$. Then the set $V(w)$ is nonempty. We confirm that $V(w)$ is a singleton. Otherwise, we suppose that there exist x_1 and x_2 such that

$$\Phi(x_1+w) = \Phi(x_2+w) = \min_{g \in V} \Phi(g+w).$$

Let $x_\lambda = \lambda x_1 + (1-\lambda)x_2$ and $0 < \lambda < 1$. As $\Phi(.+w)$ is strictly convex on V for all $w \in W$, then

$$\Phi(x_\lambda + w) < \lambda\Phi(x_1+w) + (1-\lambda)\Phi(x_2+w) = \Phi(x_1+w),$$

which contradicts $x_1 \in V(w)$.

There exists a sequence $u_n = x_n + w_n$ such that $\Phi(u_n) \to \sup_W \varphi = b$ with $w_n \in W$ and $x_n \in V(w_n)$. We claim that

$$\|w_n\| \leq c.$$

Otherwise, as

$$\Phi(u_n) = \Phi(x_n + w_n) \leq \Phi(w_n),$$

and by the anti-coercivity of Φ on W, we deduce that $\Phi(w_n) \to -\infty$. Then $\Phi(u_n) \to -\infty$, which contradicts the convergence of $\Phi(u_n)$ to b. Hence, there exists a subsequence, still denoted by w_n such that $w_n \rightharpoonup w$. For all x in V, we have

$$\Phi(x+w) \geq \limsup_{n \to +\infty} \Phi(x+w_n) \geq \limsup_{n \to +\infty} \Phi(x_n+w_n) = b,$$

for all $x \in V$, so in particular for $x \in V(w)$. Then, φ is bounded above and achieves its maximum at some point $w_0 \in W$.

Proof of Theorem 3.1. Let $w_0 \in W$ and $u = x_0 + w_0$ such that $x_0 \in V(w_0)$ and φ attains its maximum on W at w_0. We will prove that u is a critical point of Φ. Let's prove that $\Phi^0(u, x) = 0$ for all $x \in V$. We have

$$\Phi(u + tx) - \Phi(u) \geq 0, \quad \forall t \in \mathbb{R}, \quad \forall x \in V.$$

So for $t > 0$, we have

$$\frac{\Phi(w_0 + x_0 + tx) - \Phi(x_0 + w_0)}{t} \geq 0.$$

Since Φ is convex on V, by passing to the limit, where t converges vers 0, we obtain that

$$\Phi'(x_0 + w_0, x) \geq 0, \quad \forall x \in V.$$

Then

$$\Phi'(x_0 + w_0, x) = 0, \quad \forall x \in V.$$

Hence

$$\Phi^0(u, x) = 0, \quad \forall x \in V. \tag{3.2}$$

Let $w_t = w_0 + th$ for $h \in W$ and $|t| \leq 1$. By Lemma 3.1, for each $0 < |t| \leq 1$, there exists $v_t \in V(w_t)$. Since $\|w_t\| \leq \|w_0\| + \|h_1\|$, by (3.1), there exists a constant $A > 0$ such that:

$$\Phi(x + w_t) > \max_W \Phi \geq \Phi(w_t), \tag{3.3}$$

for $x \in V, \|x\| \geq A$ and $|t| \leq 1$. (Since Φ is anticoercive and weakly upper continuous in the reflexive space W, it reaches its maximum.) We claim that:

$$\|x_t\| \leq A.$$

Otherwise, we would have

$$\Phi(x_t + w_t) > \Phi(w_t),$$

which contradicts $x_t \in V(w_t)$. Since V is reflexive, there exists a subsequence $t_n \to 0$ and $t_n \geq 0$ such that $x_{t_n} \to x_1 \in V$ (V is of finite dimension). We have

$$\Phi(x_{t_n} + w_{t_n}) \leq \Phi(x + w_{t_n}), \quad \forall x \in V.$$

By passing to the limit, we obtain that

$$\Phi(x_1 + w_0) \leq \Phi(x + w_0), \quad \forall x \in V.$$

We deduce that $x_1 \in V(w_0)$. Hence, by Lemma 3.1, we conclude that $x_1 = x_0$ and for $t_n < 0$, we have:

$$\frac{\Phi(w_{t_n} + x_{t_n}) - \Phi(x_{t_n} + w_0)}{t_n} \geq \frac{\Phi(w_{t_n} + x_{t_n}) - \Phi(x_0 + w_0)}{t_n} \geq 0.$$

Then, by Lebourg mean value theorem, there exists $\theta_n \in]0, 1[$ and

$$y_n^* \in \partial(-\Phi)(w_0 + x_{t_n} + \theta_n(-t_n)(-h))$$

such that

$$\frac{-\Phi(w_{t_n} + x_{t_n}) - (-\Phi(x_{t_n} + w_0))}{-t_n} = \langle y_n^*, -h \rangle .$$

Then $\langle y_n^*, -h\rangle \geq 0$. Hence $(-\Phi)^0(w_0+x_{t_n}+\theta_n(-t_n)(-h), -h) \geq 0$. Since $(-\Phi)^0(.,.)$ is upper semi-continuous, then

$$(-\Phi)^0(w_0 + x_0, -h) \geq 0, \quad \forall h \in W.$$

Then

$$\Phi^0(w_0 + x_0, h) \geq 0, \quad \forall h \in W.$$

For all $y \in E, \exists x \in V$ and $h \in W$ such that $y = x + h$. We have

$$\Phi^0(u, h) = \Phi^0(u, y - x) \leq \Phi^0(u, y) + \Phi^0(u, -x).$$

So

$$\Phi^0(u, y) \geq \Phi^0(u, h) - \Phi^0(u, -x).$$

Hence

$$\Phi^0(u, y) \geq 0 \quad \forall y \in E.$$

Then $0 \in \partial\Phi(u)$ and u is a critical point of Φ.

In the following theorem, we remove the condition that V is finite dimensional. This result generalizes some theorems in [12] and our Theorem 2.1 in [8]. Our theorem also generalizes Theorem 2.1 of Tang and Wu in [24] by assuming that $\Phi : E \to \mathbb{R}$ is a locally Lipschitz continuous functional instead of being of class $\mathcal{C}^1$. Moreover, Tang and Wu proved the existence of a criticlal point $u_0 = x_0 + w_0$ such that:

$$\Phi(u_0) = \inf_{x\in V} \Phi(x + w_0) = \sup_{w\in W} \inf_{x\in V} \Phi(x + w),$$

or, in the following theorem, our critical point is attained:

$$\Phi(u_0) = \min_{x\in V} \Phi(x + w_0) = \max_{w\in W} \min_{x\in V} \Phi(x + w).$$

Theorem 3.2. *Consider a reflexive Banach space E that may be decomposed as the direct sum of two closed subspaces V and W. Suppose that $\Phi : E \to \mathbb{R}$ is a locally Lipshitz continuous function, Φ is weakly upper semi-continuous on $W + v$ for all $v \in V$ and Φ is anti-coercive on W, that is,*

$$\Phi(w) \to -\infty \quad \text{as} \quad \|w\| \to +\infty.$$

Assume that there exists $a > 0$ such that for all $\xi \in \partial\Phi(v_1+w)$ and $\xi' \in \partial\Phi(v_2+w)$, we have

$$(\xi - \xi', v_1 - v_2) \geq a\|v_1 - v_2\|^2, \tag{3.4}$$

for all $w \in W, v_1, v_2 \in V$. Then Φ has at least one critical point $u_0 = v_0 + w_0$ such that :

$$\Phi(u_0) = \min_{v\in V} \Phi(v + w_0) = \max_{w\in W} \min_{v\in V} \Phi(v + w).$$

Remark 3.1. Condition (3.4) indicates that $\partial\Phi$ exhibits strong monotonicity, which in turn requires strict convexity of Φ.

Prior to presenting the proof of Theorem 3.2, we shall establish the following lemmas.

Lemma 3.2. *The set*

$$V(w) = \{v \in V : \varphi(w) = \Phi(v+w) = \min_{g \in V} \Phi(g+w)\}$$

is a singleton and $\varphi : W \to \mathbb{R}$ is bounded above and achieves its maximum at some $w_0 \in W$.

Proof. We first show that Φ is coercive on V. $\partial\Phi$ is strongly monotone which implies that $\Phi - \frac{1}{2}a\|v\|^2$ is convex. So

$$\Phi(v+w) - \frac{1}{2}a\|v\|^2 \geq \Phi(w) + (\eta, v), \quad \forall \eta \in \partial\Phi(w).$$

Then

$$\Phi(v+w) \geq \frac{1}{2}a\|v\|^2 - \|\eta\|\|v\| + \Phi(w).$$

Since $\eta \in \partial\Phi(w)$, $\exists k > 0$ such that $\|\eta\| \leq K$. So

$$\Phi(v+w) \to +\infty \quad \text{as} \quad \|v\| \to +\infty.$$

Since the fact that $\partial\Phi$ is strongly monotone implies that Φ is strictly convex, the rest of the proof is the same as in Lemma 3.1

Lemma 3.3. *The mapping $f : W \to V$ such that $f(w) \in V(w)$ is continuous.*

Proof. From Lemma 3.1, $V(w)$ is a singleton and we suppose that f is not continuous, then there exists $\gamma > 0$ and a sequence (w_n) converging to $w \in W$ and an integer N large enough such that

$$\|f(w_n) - f(w)\| \geq \gamma, \quad \forall n \geq N.$$

Let P be the projection of H onto V defined by $P(v+w) = v$, and let P^* be the operator adjoint of P. Let $\xi_n \in \partial\Phi(w_n + f(w))$ and $\xi'_n \in \partial\Phi(w_n + f(w_n))$. Then for each n we obtain

$$\begin{aligned}(\xi_n, f(w_n) - f(w)) &= (\xi_n, P(f(w_n) - f(w))) \\ &= (P^*(\xi_n), f(w_n) - f(w)).\end{aligned}$$

Therefore

$$\|P^*\xi_n\|\|f(w_n) - f(w)\| \geq -(\xi_n, f(w_n) - f(w)).$$

Since $\xi'_n \in \partial\Phi(w_n + f(w_n))$, then we have

$$\Phi^0(w_n + f(w_n), h) \geq (\xi'_n, h) \quad \forall h \in V.$$

By the proof of Theorem 3.1, we obtain

$$\Phi^0(w_n + f(w_n)), h) \leq 0 \quad \forall h \in V.$$

So

$$(\xi'_n, h) \leq 0 \quad \forall h \in V.$$

Then

$$\begin{aligned}\|P^*\xi_n\|\|f(w_n) - f(w)\| &\geq (\xi'_n - \xi_n, v_1 - v_2) \\ &\geq a\|f(w_n) - f(w)\|^2.\end{aligned}$$

For n large enough, we deduce that

$$\|P^*\xi_n\| \geq a\gamma, \tag{3.5}$$

and $\xi_n \in \partial\Phi(w_n + f(w))$ implies

$$\Phi^0(w_n + f(w), h) \geq (\xi_n, h) \quad \forall h \in V.$$

Since Φ^0 is upper continuous, we deduce that

$$\limsup_{n\to\infty} \Phi(\xi_n, h) \leq \Phi^0(w + f(w), h) \quad \forall h \in V.$$

So

$$\limsup_{n\to\infty} \Phi(\xi_n, h) \leq 0 \quad \forall h \in V.$$

Then

$$\limsup_{n\to\infty} (P^*\xi_n, h) \leq 0 \quad \forall x \in X.$$

This contradicts the inequality (3.5).

Proof of Theorem 3.2. Let $w_0 \in W$ and $u = v_0 + w_0$ such that $v_0 \in V(w_0)$ and φ attains its maximum on W at w_0. Then, we demonstrate that u is a critical point of Φ by employing the identical procedures outlined in Theorem 3.1. Let $w_t = w + th$ for $|t| \leq 1$ and $h \in W$. For each t such that $0 < |t| \leq 1$, there exists a unique $v_t \in V(w_t)$ and by Lemma 3.2, we conclude that v_{t_n} converges to v_0 and that $v_0 \in V(w_0)$. The rest of the proof is the same as in Theorem 3.1.

Remark 3.2. We can prove Theorem 3.2 under condition of convexity weaker than condition (3.4). Instead of condition (3.4), We suppose that there exists a strictly increasing function $g : [0, +\infty[\to [0, +\infty[$ such that for all $\xi \in \partial\Phi(v_1 + w)$ and $\xi' \in \partial\Phi(v_2 + w)$, we have

$$(\xi - \xi', v_1 - v_2) \geq g(\|v_1 - v_2\|)\|v_1 - v_2\|, \tag{3.6}$$

for all $w \in W, v_1, v_2 \in V$. And moreover, Φ must be coercive on $V + w$, for all $w \in W$.

We obtain the following corollary.

Corollary 3.1. *Let E be a reflexive Banach space such that $E = V \oplus W$, where V and W are two closed subspaces of E. Suppose that $\Phi : E \to \mathbb{R}$ is a locally Lipschitz continuous function. Φ is weakly upper semi-continuous on $W + v$ for all $v \in V$ and Φ is anti-coercive on W, that is,*

$$\Phi(w) \to -\infty \quad \text{as} \quad \|w\| \to +\infty.$$

There exists a strictly increasing function $g : [0, +\infty[\to [0, +\infty[$ such that for all $\xi \in \partial\Phi(v_1 + w)$ and $\xi' \in \partial\Phi(v_2 + w)$, we have

$$(\xi - \xi', v_1 - v_2) \geq g(\|v_1 - v_2\|)\|v_1 - v_2\|,$$

for all $w \in W, v_1, v_2 \in V$. Moreover Φ is coercive on $V + w$, $\forall w \in W$, that is,

$$\Phi(w + v) \to +\infty \quad \text{as} \quad \|v\| \to +\infty.$$

Then Φ has at least one critical point $u_0 = v_0 + w_0$ such that :

$$\Phi(u_0) = \min_{v\in V} \Phi(v + w_0) = \max_{w\in W} \min_{v\in V} \Phi(v + w).$$

The proof of this corollary is the same as that of Theorem 3.2.

In the following theorem, as in Theorem 3.2, we drop the condition that V is of finite dimension. We assume that the functional Φ is strictly convex on $V+w$ for all $w \in W$, but we add a new assumption:

$$\Phi^0(.,v) \quad \text{is weakly upper semi-continuous for all} \quad v \in V.$$

Theorem 3.3. *Consider a reflexive Banach space E that can be decomposed as the direct sum of two closed subspaces V and W. Let $\Phi : E \to \mathbb{R}$ be a function that is locally Lipschitz continuous such that $\Phi^0(.,x)$ is weakly upper semi-continuous for all $x \in E$. Moreover, Φ is weakly upper semi-continuous on $W+x$ for all $x \in V$, and Φ is anti-coercive on W, meaning that*

$$\Phi(w) \to -\infty \quad \textit{as} \quad \|w\| \to +\infty.$$

Assume that $\Phi(.+w) : V \to R$ is strictly convex for all $w \in W$ and

$$\Phi(x+w) \to +\infty \quad \textit{as} \quad \|x\| \to +\infty, \tag{3.7}$$

and the convergence is uniform on bounded subsets of W. Then Φ has at least one critical point $u_0 = x_0 + w_0$ such that

$$\Phi(u_0) = \min_{x \in V} \Phi(x+w_0) = \max_{w \in W} \min_{x \in V} \Phi(x+w).$$

Proof of Theorem 3.3. Lemma 3.1 is still valid for Theorem 3.3. Assume that φ attains its maximum on W at w_0. Let $w_t = w_0 + th$ for $h \in W$ and $|t| \leq 1$. As in the proof of Theorem 3.1 , we prove that there exists a subsequence $x_{t_n} \in V(w_{t_n})$ such that x_{t_n} converges weakly to $v_0 \in V$ when $t_n \to 0$ and $t_n \geq 0$.

Let's prove that $v_0 \in V(w_0)$. Since $\Phi^0(.,v)$ is weakly upper semi-continuous for all $v \in V$, we obtain

$$\limsup_{n \to +\infty} \phi^0(w_{t_n} + x_{t_n}, v) \leq \phi^0(w_0 + v_0, v) \quad \forall v \in V.$$

By the proof of Theorem 3.1, we have

$$\phi^0(w_n + f(w_n), v) = 0.$$

We deduce that

$$\phi^0(w_0 + v_0, v) \geq 0 \quad \forall v \in V.$$

By the convexity of Φ on $w_0 + V$, we have

$$\partial \Phi(w_0 + v_0) = \{x^* \in V^*; \quad \Phi(w_0 + v) - \Phi(w_0 + v_0) \geq (x^*, v - v_0), \quad \forall v \in V\}.$$

Then $0 \in \partial \Phi(w_0 + v_0)$, and we conclude that

$$\Phi(w_0 + v_0) = \min_{g \in V} \Phi(w_0 + g),$$

and because $f(w_0)$ is the unique element of V such that

$$\Phi(w_0 + f(w_0)) = \min_{g \in V} \Phi(g + w_0),$$

we deduce that $v_0 \in V(w_0)$.

For $t_n < 0$, we have

$$\frac{\Phi(w_{t_n} + x_{t_n}) - \Phi(x_{t_n} + w_0)}{t_n} \geq \frac{\Phi(w_{t_n} + x_{t_n}) - \Phi(v_0 + w_0)}{t_n} \geq 0.$$

Then, by Lebourg mean value theorem, there exists $\theta_n \in]0,1[$ and

$$y_n^* \in \partial(-\Phi)(w_0 + x_{t_n} + \theta_n(-t_n)(-h))$$

such that

$$\frac{-\Phi(w_{t_n} + x_{t_n}) - (-\Phi(x_{t_n} + w_0))}{-t_n} = (y_n^*, -h).$$

Then $(y_n^*, -h) \geq 0$. Hence $(-\Phi)^0(w_0 + x_{t_n} + \theta_n(-t_n)(-h), -h) \geq 0$. Since $(-\Phi)^0(w_0 + x_{t_n} + \theta_n(-t_n)(-h), -h) = \Phi^0(w_0 + x_{t_n} + \theta_n(-t_n)(-h), h)$ and $\Phi^0(., h)$ is weakly upper semi-continuous, then

$$\Phi^0(w_0 + v_0, h) \geq 0, \quad \forall h \in W.$$

The rest of the proof is the same as the proof of Theorem 3.1. By using Theorem 3.2 and employing the perturbation argument, we derive the subsequent theorem that extends the scope of precedent theorems. In the following theorem, we assume a less requirement:

$$\Phi(. + w) : V \to \mathbb{R} \quad \text{is convex,}$$

but we suppose that

$$\Phi^0(.,.) \quad \text{is weakly upper semi-continuous}$$

instead of

$$\Phi^0(., x) \quad \text{is weakly upper semi-continuous for all} \quad x \in E,$$

which is a necessary assymption for Theorem 3.3 when $\Phi(. + w)$ is strictly convex on V.

Theorem 3.4. *Let E be a reflexive Banach space such that $E = V \oplus W$, where V and W are two closed subspaces of E. Suppose that $\Phi : E \to \mathbb{R}$ is a locally Lipschitz continuous function, Φ is weakly upper semi-continuous on $W + v$ for all $v \in V$ and Φ is anti-coercive on W, that is,*

$$\Phi(w) \to -\infty \quad \text{as} \quad \|w\| \to +\infty.$$

Assume that $\Phi(. + w)$ is convex for all $w \in W$, $\Phi^0(.,.)$ is weakly upper semi-continuous. Moreover

$$\Phi(v + w) \to +\infty \quad \text{as} \quad \|v\| \to +\infty,$$

and the convergence is uniform on bounded subsets of W.

Then Φ has at least one critical point.

Theorem 3.4 generalizes Theorem 1.1 in [24] and our Theorem 3.1 in [7] by assuming that $\Phi : V \oplus W \to \mathbb{R}$ is a locally Lipschitz continuous functional instead of being of class $\mathcal{C}^1$. Also, Theorem 3.4 deals with a more general functional Φ, but in our Theorem 3.1 in [7], we proved the existence of a critical point for a particular class of functionals Φ, which should satisfy the following condition:

$$\Phi = \Phi_1 + \Phi_2,$$

such that $\Phi_1(v+w) = \Phi_1(v)+\Phi_1(w)$, for $v \in V, w \in W$ and Φ_2 is weakly continuous on E.

Proof of Theorem 3.4. For $n \in \mathbf{N}$, we define φ_n as follows:

$$\varphi_n(v+w) = \Phi(v+w) + \frac{1}{2n}\|v\|^2. \tag{3.8}$$

Since $\Phi(.+w)$ is convex for all $w \in W$, $\partial\varphi_n(.+w)$ is $\frac{1}{n}$ strongly monotone on V. Since Φ and $v \to \|v\|^2$ are locally Lipschitz functions, φ_n is a locally Lipschitz function. For all $v \in V$, φ_n is weakly upper semi-continuous on $W+v$, and it is anticoercive on W. Hence by Theorem 3.2 , φ_n has at least one critical point v_n+w_n such that:

$$\varphi_n(v_n+w_n) = \min_{v\in V}\varphi_n(v+w_n) = \max_{w\in W}\min_{v\in V}\varphi_n(v+w). \tag{3.9}$$

By (3.8) and (3.9), we deduce that

$$\Phi(w_n) = \varphi_n(w_n) \geq \varphi_n(v_n+w_n),$$

and

$$\varphi_n(v_n+w_n) \geq \min_{v\in V}\varphi_n(v) \geq \min_{v\in V}\Phi(v).$$

Since $\min_{v\in V}\Phi(v)$ is finite and Φ is anticoercive on W, we conclude that (w_n) is bounded. Let's prove that (v_n) is bounded too. It follows from (3.8) and (3.9) that

$$\Phi(v_n+w_n) \leq \varphi_n(v_n+w_n) \leq \varphi_n(w_n)$$

and

$$\varphi_n(w_n) = \Phi(w_n) \leq \sup_n \Phi(w_n).$$

Moreover, $\sup_n \Phi(w_n) < +\infty$. Otherwise, w_n has a subsequence which is denoted by w_n such that

$$\Phi(w_n) \to +\infty \quad \text{as} \quad n \to +\infty.$$

Since w_n is bounded in the reflexive space W, there exists a subsequence which is denoted by w_n such that $w_n \rightharpoonup w_0$. Hence,

$$\limsup_{n\to\infty}\Phi(w_n) \leq \Phi(w_0).$$

So $\Phi(v_n+w_n) < +\infty$ and by the coercivity of $\Phi(.+w)$ on V with uniform convergence on bounded subsets of W, we conclude that v_n is bounded. Since X is reflexive, there exists a subsequence of v_n+w_n, still denoted by v_n+w_n, which weakly converges to some v_0+w_0.

Let $g(v_n + w_n) = \frac{1}{2}\|v_n\|^2$. So it follows that

$$\varphi_n^0(v_n + w_n, u) \leq \Phi^0(v_n + w_n, u) + \frac{1}{n} g^0(v_n + w_n, u).$$

For all $u \in E$, there exists $k > 0$ such that $g^0(v_n + w_n, u) \leq K\|u\|$, so

$$\varphi_n^0(v_n + w_n, u) \leq \Phi^0(v_n + w_n, u) + \frac{1}{n} K\|u\|.$$

As $\varphi_n^0(v_n + w_n, u) \geq 0, \quad \forall u \in E$, and $\Phi^0(.,.)$ is weakly upper semicontinuous on E, we conclude that

$$\Phi^0(v_0 + w_0, u) \geq 0, \quad \forall u \in X.$$

Therefore, $0 \in \partial\Phi(0)$ and $v_0 + w_0$ is a critical point of Φ.

4. Conclusion

In this paper, we state new abstract nature critical point theorems for locally Lipschitz continuous functionals defined on reflexive Banach spaces. We formulate new variants in this respect in order to avoid the traditional compactness condition of the Palais-Smale type. The scope of the critical point theory has been enlarged to a much wider class of functionals, including non-smooth and non-convex ones, by the use of techniques such as the least action principle, perturbation arguments, the reduction approach, the properties of sub-differentials, and the generalized gradients in the sense of F.H. Clarke.

The results are of paramount importance to various branches, including optimization, control theory, and partial differential equations. There are several open problems and further directions for investigations.

- **Applications to specific class of PDEs:**
  Application of such theorems to that particular class of partial differential equations, namely, elliptic, parabolic, or hyperbolic equations, can result in valuable information and a solution to the boundary and initial value problems.
- **Exploration of other variational principles:**
  It seems likely that similar research on other variational principles, for example, the Ekeland variational principle or the mountain pass theorem in the context of functionals which are not smooth and do not satisfy the Palais-Smale condition, may spark new theoretical advances in the future.
- **Higher-order critical point theorems:** More rigorous is the process of developing higher-order critical point theorems, resulting in the aforementioned conditions for the existence of not only critical points but also higher-order critical points through further sophisticated mathematical techniques.

Generalizations and novel perspectives presented here establish a firm foundation for current research and possible future advances of this vibrant area of mathematical analysis.

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