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A General Korteweg-de Vries-Burgers Equation: Novel Ideas and Novel Results

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Abstract We consider the Cauchy problem for a general Korteweg-de Vries-Burgers equation and the Cauchy problem for the corresponding linear equation. We will couple together a few novel ideas, several existing ideas and existing results and use rigorous mathematical analysis to accomplish several very important and very interesting results for these Cauchy problems.

Keywords General Korteweg-de Vries-Burgers equation, global smooth solution, existence and uniqueness, sharp rate decay estimates, exact limits, optimal decay estimates

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1. Introduction

1.1. The mathematical model equations and known related results

Consider the Cauchy problem for the following general Korteweg-de Vries-Burgers equation

$$\frac{\partial}{\partial t}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial^3}{\partial x^3}u+\gamma\mathcal{H}\frac{\partial^2}{\partial x^2}u+\frac{\partial}{\partial x}\mathcal{N}(u)=f(x,t), \tag{1.1}$$

$$u(x,0)=u_0(x). \tag{1.2}$$

Also, consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v-\alpha\frac{\partial^2}{\partial x^2}v+\beta\frac{\partial^3}{\partial x^3}v+\gamma\mathcal{H}\frac{\partial^2}{\partial x^2}v=f(x,t), \tag{1.3}$$

$$v(x,0)=u_0(x). \tag{1.4}$$

In these equations, the positive constant $\alpha>0$ represents the diffusion coefficient, the real constants β and γ represent dispersion coefficients, the function $u_0=u_0(x)$ represents the initial function and the function $f=f(x,t)$ represents the external force. Note that the initial functions in both the nonlinear problem and the linear problem are the same, so are the external forces. The Hilbert operator $\mathcal{H}:L^2(\mathbb{R})\to L^2(\mathbb{R})$ is defined by the principal value of the following singular integral

$$[\mathcal{H}\phi](x)=\frac{1}{\pi}\text{ P. V. }\int_{\mathbb{R}}\frac{\phi(y)}{x-y}\mathrm{d}y, \qquad \phi\in L^2(\mathbb{R}).$$

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The Fourier transformation of the Hilbert operator $\mathcal{H}$ is given by

$$\widehat{\mathcal{H}\phi}(\xi) = \mathrm{i}\mathcal{S}(\xi)\widehat{\phi}(\xi),$$

for all $\phi \in L^2(\mathbb{R})$ and for all $\xi \in \mathbb{R}$, where $\mathcal{S} = \mathcal{S}(\xi)$ represents the standard sign function

$$\mathcal{S}(\xi) = -1 \text{ for all } \xi < 0, \qquad \mathcal{S}(0) = 0, \qquad \mathcal{S}(\xi) = +1 \text{ for all } \xi > 0.$$

Note that

$$\int_{\mathbb{R}} \phi(x)\mathcal{H}\phi(x)\mathrm{d}x = 0,$$

for all functions $\phi \in L^2(\mathbb{R})$.

The nonlinear function $\mathcal{N} = \mathcal{N}(u) \in C^\infty(\mathbb{R})$. There exists a positive constant $C > 0$, independent of u, such that

$$|\mathcal{N}(u)| \leq C(|u|^2 + |u|^5),$$

for all $u \in \mathbb{R}$. Suppose that there exists the limit

$$\lim_{u\to 0} \frac{\mathcal{N}(u)}{u^2} = \mathcal{L},$$

where $\mathcal{L} \in \mathbb{R}$ is some real constant.

Here are many examples of the nonlinear function

$$\mathcal{N}(u) = u^2, \qquad \mathcal{N}(u) = \sin(u^2), \qquad \mathcal{N}(u) = \arctan(u^2), \qquad \mathcal{N}(u) = \ln(1 + u^2),$$
$$\mathcal{N}(u) = u^2 + u^3, \qquad \mathcal{N}(u) = u^2 + u^3 + u^4, \qquad \mathcal{N}(u) = u^2 + u^3 + u^4 + u^5.$$

The model equation reduces to the nonlinear Korteweg-de Vries-Burgers equation

$$\frac{\partial}{\partial t}u + \frac{\partial^3}{\partial x^3}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}(u^2) = f(x,t),$$

if the nonlinear function $\mathcal{N}(u) = u^2$ and the dispersion coefficients $(\beta, \gamma) = (1, 0)$; it reduces to the nonlinear Benjamin-Ono-Burgers equation

$$\frac{\partial}{\partial t}u + \mathcal{H}\frac{\partial^2}{\partial x^2}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}(u^2) = f(x,t),$$

if the nonlinear function $\mathcal{N}(u) = u^2$ and the dispersion coefficients $(\beta, \gamma) = (0, 1)$; and it reduces to the Burgers equation

$$\frac{\partial}{\partial t}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}(u^2) = f(x,t),$$

if the nonlinear function $\mathcal{N}(u) = u^2$ and the dispersion coefficients $(\beta, \gamma) = (0, 0)$.

We allow the parameters $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}$ to be any real constants to include very general cases.

Here are many very important and very interesting questions.

1. Can we couple together simple ideas (such as the Fourier transformation, the Parseval's identity, Lebesgue's dominated convergence theorem, squeeze theorem) in an appropriate way (probably the best way) to establish very important and very interesting results?
2. Can we accomplish the existence and uniqueness of the global smooth solution of the general Korteweg-de Vries-Burgers equation (1.1), if we make very simple assumptions on the initial function and the external force?
3. Can we use the global smooth solution of the corresponding linear equation (1.3) to approximate the solution of the general Korteweg-de Vries-Burgers equation (1.1)?
4. Can we establish the sharp rate decay estimates for all order derivatives of the global smooth solution?
5. Can we accomplish the exact limits for all order derivatives of the global smooth solution, in terms of some known information, representing certain physical mechanisms?
6. Will the exact limits of the global smooth solution of the general Korteweg-de Vries-Burgers equation (1.1) reduces to the exact limits of the global smooth solution of the corresponding linear equation (1.3)?
7. Are the ratios of the exact limits of the global smooth solution of the general Korteweg-de Vries-Burgers equation (1.1) the same as the ratios of the exact limits of the solution of the corresponding linear equation (1.3), for each fixed order $m \geq 0$?
8. What are the influences of various physical mechanisms (represented by the diffusion coefficient, the integral of the initial function, the integral of the external force, and the order of the derivatives) on the exact limits?
9. Can we establish the optimal decay estimates for all order derivatives of the global smooth solution, so that the most important constants (represented by $\mathcal{A}$ and $\mathcal{C}$) are independent of any norm of any order derivatives of the initial function, the external force and the global solution, for all sufficiently large t? Other positive constants (represented by $\mathcal{B}$ and $\mathcal{D}$) in the estimates are much less important because $\mathcal{B}t^{-1}$ and $\mathcal{D}t^{-1}$ become arbitrarily small as $t \to \infty$. This kind of decay estimates may play a substantial role in long time, accurate numerical simulations.
10. For very similar equations, such as the Benjamin-Bona-Mahony-Burgers equation, can we apply the same ideas to establish very similar results?

Let us review known related results of the nonlinear Korteweg-de Vries-Burgers equation and the nonlinear Benjamin-Ono-Burgers equation. They will play various roles in the mathematical analysis.

First of all, consider the nonlinear Korteweg-de Vries-Burgers equation. If the initial function and the external force satisfy the following assumptions

$$
\begin{aligned}
&u_0 \in H^{2m}(\mathbb{R}),\\
&f \in C^\infty(\mathbb{R} \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R})) \cap L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})),
\end{aligned}
$$

for all positive constants $m > 0$, then there exists a global smooth solution

$$
u \in C^\infty(\mathbb{R} \times \mathbb{R}^+),
$$

$$u \in L^{\infty}(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \frac{\partial}{\partial x} u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})),$$

for all positive constants $m > 0$.

Moreover, if the initial function and the external force satisfy the following conditions

$$\begin{aligned}
&u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R}),\\
&f \in L^1(\mathbb{R} \times \mathbb{R}) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R})) \cap L^2(\mathbb{R}^+, H^2(\mathbb{R})),\\
&\int_{\mathbb{R}} u_0(x)\mathrm{d}x \neq 0, \qquad \int_0^{\infty} \int_{\mathbb{R}} f(x,t)\mathrm{d}x\mathrm{d}t \neq 0,\\
&\int_{\mathbb{R}} u_0(x)\mathrm{d}x + \int_0^{\infty} \int_{\mathbb{R}} f(x,t)\mathrm{d}x \neq 0,
\end{aligned}$$

then there hold the elementary decay estimates

$$\begin{aligned}
&\sup_{t>0} \left\{ t^{1/2} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \right\} < \infty,\\
&\sup_{t>0} \left\{ t^{3/2} \int_{\mathbb{R}} \left| \frac{\partial}{\partial x} u(x,t) \right|^2 \mathrm{d}x \right\} < \infty,\\
&\sup_{t>0} \left\{ t^{5/2} \int_{\mathbb{R}} \left| \frac{\partial^2}{\partial x^2} u(x,t) \right|^2 \mathrm{d}x \right\} < \infty,\\
&\sup_{t>0} \left\{ t \|u(\cdot,t)\|_{L^\infty}^2 \right\} < \infty,\\
&\sup_{t>0} \left\{ t^2 \left\| \frac{\partial}{\partial x} u(\cdot,t) \right\|_{L^\infty}^2 \right\} < \infty,\\
&\sup_{t>0} \left\{ t^3 \left\| \frac{\partial^2}{\partial x^2} u(\cdot,t) \right\|_{L^\infty}^2 \right\} < \infty.
\end{aligned}$$

Furthermore, if there exist real scalar smooth functions ϕ and ψ,

$$\phi \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}), \qquad \psi \in C^{\infty}(\mathbb{R} \times \mathbb{R}^+) \cap L^1(\mathbb{R} \times \mathbb{R}^+),$$

such that the initial function and the external force are given by

$$u_0(x) = \frac{\partial}{\partial x} \phi(x), \qquad f(x,t) = \frac{\partial}{\partial x} \psi(x,t),$$

for all $(x,t) \in \mathbb{R} \times \mathbb{R}^+$, then the decay rate would be faster, namely, there holds

$$\sup_{t>0} \left\{ t^{3/2} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \right\} < \infty.$$

See [1], [2], [3], [4], [9], [13], [16], [17], [21], [22], [23], [24], [25], [26], [27], [30], [31], [32], [36], [37], [38], [39], [40], [41] for the above results.

Remark 1.1. To obtain the decay estimate with the faster rate

$$\sup_{t>0} \left\{ t^{3/2} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \right\} < \infty,$$

from the estimate with the slower rate

$$\sup_{t>0}\left\{t^{1/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty,$$

we may apply the Fourier splitting method.

Secondly, consider the nonlinear Benjamin-Ono-Burgers equation. It is well known that there exists a unique global smooth solution

$$\begin{aligned}&u\in C^\infty(\mathbb{R}\times\mathbb{R}^+),\\&u\in L^\infty(\mathbb{R}^+,H^{2m}(\mathbb{R})),\qquad \frac{\partial}{\partial x}u\in L^2(\mathbb{R}^+,H^{2m}(\mathbb{R})),\qquad \forall m>0,\end{aligned}$$

under appropriate conditions on the initial function and the external force.

If the initial function and the external force satisfy the additional conditions

$$\begin{aligned}&u_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),\qquad f\in L^1(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R})),\\&\int_{\mathbb{R}}u_0(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}f(x,t)\mathrm{d}x\mathrm{d}t\neq 0,\end{aligned}$$

then there holds the following elementary decay estimate

$$\sup_{t>0}\left\{t^{1/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty.$$

If the initial function and the external force satisfy more additional assumptions, then the decay estimate may be improved

$$\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty.$$

See [8], [10], [11], [14], [15], [18], [19], [20], [28], [29], [34], [33], [35], [43] for the above results.

Let the initial function and the external force satisfy the following conditions

$$\begin{aligned}&u_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),\\&f\in C^\infty(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R})),\end{aligned}$$

Suppose that there exist real scalar smooth functions

$$\begin{aligned}&\phi\in C^1(\mathbb{R})\cap L^1(\mathbb{R}),\\&\psi\in C^\infty(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}\times\mathbb{R}^+),\end{aligned}$$

such that

$$u_0(x)=\frac{\partial}{\partial x}\phi(x),\qquad f(x,t)=\frac{\partial}{\partial x}\psi(x,t),$$

for all $(x,t)\in\mathbb{R}\times\mathbb{R}^+$.

For the simplest case

$$\mathcal{N}(u)=u^2,$$

for all $u\in\mathbb{R}$, the author has obtained the following results for the Benjamin-Ono-Burgers equation in [43].

(1) There exists a unique global smooth solution

$$u \in C^\infty(\mathbb{R} \times \mathbb{R}^+),$$
$$u \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \frac{\partial}{\partial x} u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \forall m > 0.$$

(2) There hold the following sharp rate decay estimates

$$\sup_{t>0} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \right\} < \infty,$$

for all positive constants $m > 0$, where the differential operator $(-\triangle)^m = (-\frac{\partial^2}{\partial x^2})^m$.

(3) The explicit representations of the following exact limits

$$\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \right\},$$
$$\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m [u(x,t) - v(x,t)]|^2 \mathrm{d}x \right\},$$

for all real constants $m \geq 0$, in terms of some physical mechanisms, have been accomplished, where $v = v(x,t)$ represents the global smooth solution of the Cauchy problem for the corresponding linear equations (1.3)-(1.4).

(4) There hold the following optimal decay estimates

$$t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \leq \mathcal{A}(\alpha,\delta,\varepsilon,m) + \mathcal{B}(\alpha,\delta,\varepsilon,m)t^{-1},$$
$$t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m [u(x,t) - v(x,t)]|^2 \mathrm{d}x \leq \mathcal{C}(\alpha,\delta,\varepsilon,m) + \mathcal{D}(\alpha,\delta,\varepsilon,m)t^{-1},$$

for all order derivatives of the global smooth solution, for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, for all sufficiently large t, where the positive constants are independent of any norm of any order derivatives of the initial function and the global solution u.

However, these problems have been open if the nonlinear function satisfy the condition

$$|\mathcal{N}(u)| \leq C(|u|^2 + |u|^5),$$

for all $u \in \mathbb{R}$.

Many very important and very interesting mathematical problems have been open. These open problems are very important and very interesting in applied mathematics. The author will study these important and interesting problems.

1.2. The main purposes

The main purposes of this paper are to accomplish:

(1) The existence and uniqueness of the global smooth solution

$$u \in C^\infty(\mathbb{R} \times \mathbb{R}^+),$$
$$u \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \frac{\partial}{\partial x} u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \forall m > 0.$$

(2) The sharp rate decay estimates

$$\sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\}<\infty,$$

for all positive constants $m>0$, where $(-\triangle)^m=(-\frac{\partial^2}{\partial x^2})^m$.

(3) The computations and explicit representations of the exact limits

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\},$$
$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m [u(x,t)-v(x,t)]|^2\mathrm{d}x\right\},$$

for all constants $m\geq 0$, in terms of some physical mechanisms.

(4) The optimal decay estimates

$$t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\leq \mathcal{A}(\alpha,\delta,\varepsilon,m)+\mathcal{B}(\alpha,\delta,\varepsilon,m)t^{-1},$$
$$t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m [u(x,t)-v(x,t)]|^2\mathrm{d}x\leq \mathcal{C}(\alpha,\delta,\varepsilon,m)+\mathcal{D}(\alpha,\delta,\varepsilon,m)t^{-1},$$

for all order derivatives of the global smooth solution.

We will couple together a few novel ideas, several classical ideas and existing results to accomplish these results. We will make complete use of the representations of the Fourier transformations of the global smooth solutions.

First of all, we will establish several elementary estimates and apply them to conduct comprehensive analysis. Then we will apply an elegant iteration technique to the comprehensive analysis to establish the sharp rate decay estimates. Then we will apply the elementary estimates and the sharp rate decay estimates to establish several fundamental limits. Then we will make complete use of the fundamental limits to accomplish the exact limits. We will represent the exact limits as explicit as possible. We will represent the exact limits in terms of some known information, such as the diffusion coefficient α, the order m of the derivative, and the following integrals

$$\int_{\mathbb{R}}\phi(x)\mathrm{d}x,\qquad \int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t,\qquad \int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t.$$

Moreover, we will make complete use of the comprehensive analysis and the exact limits to establish the optimal decay estimates for all order derivatives of the global smooth solution. The four positive constants

$$\mathcal{A}=\mathcal{A}(\alpha,\delta,\varepsilon,m),\ \ \mathcal{B}=\mathcal{B}(\alpha,\delta,\varepsilon,m),$$
$$\mathcal{C}=\mathcal{C}(\alpha,\delta,\varepsilon,m),\ \ \mathcal{D}=\mathcal{D}(\alpha,\delta,\varepsilon,m),$$

will be made as explicit as possible, because this will have a deep influence on long time accurate numerical simulations.

1.3. The mathematical assumptions

We make the following assumptions for the Cauchy problem for the general Korteweg-de Vries-Burgers equation (1.1)-(1.2).

(A1) Let the initial function and the external force satisfy the following assumptions

$$u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$
$$f \in C^\infty(\mathbb{R} \times \mathbb{R}^+) \cap L^1(\mathbb{R} \times \mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R})).$$

(A2) Suppose that there exist real scalar smooth functions

$$\phi \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}),$$
$$\psi \in C^\infty(\mathbb{R} \times \mathbb{R}^+) \cap L^1(\mathbb{R} \times \mathbb{R}^+),$$

such that

$$u_0(x) = \frac{\partial}{\partial x}\phi(x), \qquad f(x,t) = \frac{\partial}{\partial x}\psi(x,t),$$

for all $\mathbb{R} \times \mathbb{R}^+$.

(A3) Suppose that there exist the following limits

$$\lim_{t\to\infty}\left\{t^{m+3/2}\int_{\mathbb{R}} |(-\triangle)^m \psi(x,t)|\mathrm{d}x\right\} \equiv \mathcal{L}(m),$$

for all real constants $m \geq 0$.

Here is a slightly weaker assumption

$$\sup_{t>0}\left\{t^{m+3/2}\int_{\mathbb{R}} |(-\triangle)^m \psi(x,t)|\mathrm{d}x\right\} \equiv \mathcal{L}(m) < \infty,$$

for all real constants $m \geq 0$.

(A4) Suppose that the nonlinear function $\mathcal{N} = \mathcal{N}(u) \in C^\infty(\mathbb{R})$. Suppose that there exists a positive constant $C > 0$, independent of u, such that

$$|\mathcal{N}(u)| \leq C(|u|^2 + |u|^5),$$

for all $u \in \mathbb{R}$. Suppose that there exists the nonzero limit

$$\mathcal{L} \equiv \lim_{u\to 0}\frac{\mathcal{N}(u)}{u^2} \neq 0.$$

(A5) Suppose that there exists a global weak solution

$$u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R})), \qquad \frac{\partial}{\partial x}u \in L^2(\mathbb{R} \times \mathbb{R}^+).$$

(A6) Suppose that there holds the following elementary decay estimate

$$\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}} |u(x,t)|^2\mathrm{d}x\right\} < \infty.$$

1.4. The main results

There are two parts in the results: the main results (for the general Korteweg-de Vries-Burgers equation (1.1)) and the minor results (for the corresponding linear equation (1.3)). The main results contain five theorems.

Theorem 1.1. *There exists a unique global smooth solution*

$$u \in C^{\infty}(\mathbb{R} \times \mathbb{R}^+),$$
$$u \in L^{\infty}(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \frac{\partial}{\partial x} u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \forall m > 0,$$

to the Cauchy problem for the general Korteweg-de Vries-Burgers equation (1.1)-(1.2).

Theorem 1.2. *There hold the following sharp rate decay estimates*

$$\sup_{t>0} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \right\} < \infty,$$

for all order derivatives of the global smooth solution, where the differential operator $(-\triangle)^m = (-\frac{\partial^2}{\partial x^2})^m$.

Theorem 1.3. *There hold the following representations for the exact limits*

$$\begin{aligned}
&\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \right\} \\
&= \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \\
&\cdot \left\{ \int_{\mathbb{R}} \phi(x) \mathrm{d}x + \int_0^{\infty} \int_{\mathbb{R}} \psi(x,t) \mathrm{d}x\mathrm{d}t - \int_0^{\infty} \int_{\mathbb{R}} \mathcal{N}(u(x,t)) \mathrm{d}x\mathrm{d}t \right\}^2, \\
&\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m [u(x,t) - v(x,t)]|^2 \mathrm{d}x \right\} \\
&= \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \left\{ \int_0^{\infty} \int_{\mathbb{R}} \mathcal{N}(u(x,t)) \mathrm{d}x\mathrm{d}t \right\}^2,
\end{aligned}$$

for all real constants $m \geq 0$, *for the general Korteweg-de Vries-Burgers equation* (1.1).

Theorem 1.4. *The ratios of the exact limits of the global smooth solution of the general Korteweg-de Vries-Burgers equation* (1.1) *are the same as the ratios of the exact limits of the global smooth solution of the corresponding linear equation* (1.3), *for each fixed constant* m. *That is,*

$$\begin{aligned}
&\left\{ \lim_{t\to\infty} \left[t^{2m+5/2} \int_{\mathbb{R}} |(-\triangle)^{m+1/2} u(x,t)|^2 \mathrm{d}x \right] \right\} \\
&/ \left\{ \lim_{t\to\infty} \left[t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^{m} u(x,t)|^2 \mathrm{d}x \right] \right\} \\
&= \left\{ \lim_{t\to\infty} \left[t^{2m+5/2} \int_{\mathbb{R}} |(-\triangle)^{m+1/2} v(x,t)|^2 \mathrm{d}x \right] \right\} \\
&/ \left\{ \lim_{t\to\infty} \left[t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^{m} v(x,t)|^2 \mathrm{d}x \right] \right\} = \frac{4m+3}{4\alpha},
\end{aligned}$$

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}|(-\triangle)^{m+1}u(x,t)|^2\mathrm{d}x\right]\right\}\\
&/\ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}u(x,t)|^2\mathrm{d}x\right]\right\}\\
&=\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}|(-\triangle)^{m+1}v(x,t)|^2\mathrm{d}x\right]\right\}\\
&/\ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}v(x,t)|^2\mathrm{d}x\right]\right\}=\frac{(4m+3)(4m+5)}{(4\alpha)^2},
\end{aligned}$$

for all real constants $m \geq 0$.

Additionally, there hold

$$\begin{aligned}
&\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}|(-\triangle)^{m+1/2}[u(x,t)-v(x,t)]|^2\mathrm{d}x\right]\right\}\\
&/\ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}[u(x,t)-v(x,t)]|^2\mathrm{d}x\right]\right\}\\
&=\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}}|(-\triangle)^{m+1/2}v(x,t)|^2\mathrm{d}x\right]\right\}\\
&/\ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}v(x,t)|^2\mathrm{d}x\right]\right\}=\frac{4m+3}{4\alpha},\\
&\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}|(-\triangle)^{m+1}[u(x,t)-v(x,t)]|^2\mathrm{d}x\right]\right\}\\
&/\ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}[u(x,t)-v(x,t)]|^2\mathrm{d}x\right]\right\}\\
&=\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}}|(-\triangle)^{m+1}v(x,t)|^2\mathrm{d}x\right]\right\}\\
&/\ \left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}v(x,t)|^2\mathrm{d}x\right]\right\}=\frac{(4m+3)(4m+5)}{(4\alpha)^2},
\end{aligned}$$

for all real constants $m \geq 0$.

Theorem 1.5. *There hold the following optimal decay estimates*

$$\begin{aligned}
&t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}u(x,t)|^2\mathrm{d}x \leq \mathcal{A}(\alpha,\delta,\varepsilon,m)+\mathcal{B}(\alpha,\delta,\varepsilon,m)t^{-1},\\
&t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}[u(x,t)-v(x,t)]|^2\mathrm{d}x \leq \mathcal{C}(\alpha,\delta,\varepsilon,m)+\mathcal{D}(\alpha,\delta,\varepsilon,m)t^{-1},
\end{aligned}$$

for all order derivatives of the global smooth solution, for all positive constants $0 < \delta < 4$ *and* $0 < \varepsilon < 1$ *and for all sufficiently large* t.

The four positive constants are given by

$$\begin{aligned}
&\mathcal{A}(\alpha,\delta,\varepsilon,m)\\
&=\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}}|\phi(x)|\mathrm{d}x\right\}^2\\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^{\infty}\int_{\mathbb{R}}|\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{\int_0^\infty \int_{\mathbb{R}} |\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\mathcal{B}(\alpha,\delta,\varepsilon,m)\\
&= C(m)\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\mathcal{L}\left(m-\frac{1}{2}+\frac{1}{4}(1+\delta)\right)\\
&+ C(m)\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot \left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+1+\delta}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty\int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^4,\\
&\mathcal{C}(\alpha,\delta,\varepsilon,m)\\
&= \frac{2}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{\int_0^\infty \int_{\mathbb{R}} |\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\mathcal{D}(\alpha,\delta,\varepsilon,m)\\
&= C(m)\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot \left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+1+\delta}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty\int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^4.
\end{aligned}
$$

Let us make the main results clear. Let the nonlinear function $\mathcal{N}(u) = u^p$, where $p = 2, 3, 4, 5$, then all of the results in

(1) Theorem 1.1
(2) Theorem 1.2
(3) Theorem 1.3
(4) Theorem 1.4
(5) Theorem 1.5

are true. The results are also true for more complicated functions, such as

$$\mathcal{N}(u) = \sin(u^2), \qquad \arctan(u^2), \qquad \ln(1+u^2),$$

and the linear combinations of these functions.

Below are the minor results - they are for the global smooth solution of the Cauchy problem for the corresponding linear equation (1.3)-(1.4).

Theorem 1.6. *There hold the following representations for the exact limits*

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^m v(x,t)|^2\mathrm{d}x\right\}$$

$$= \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \left\{ \int_{\mathbb{R}} \phi(x) \mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t) \mathrm{d}x \mathrm{d}t \right\}^2,$$

for all order derivatives of the global smooth solution of the Cauchy problem for the corresponding linear equation (1.3)-(1.4).

Theorem 1.7. *The ratios of the exact limits of the global smooth solution of the corresponding linear equation* (1.3) *are given by*

$$\begin{aligned} & \left\{ \lim_{t\to\infty} \left[t^{2m+5/2} \int_{\mathbb{R}} |(-\triangle)^{m+1/2} v(x,t)|^2 \mathrm{d}x \right] \right\} \\ / & \left\{ \lim_{t\to\infty} \left[t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^{m} v(x,t)|^2 \mathrm{d}x \right] \right\} \\ = & \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+4} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \\ = & \frac{4m+3}{4\alpha}, \\ & \left\{ \lim_{t\to\infty} \left[t^{2m+7/2} \int_{\mathbb{R}} |(-\triangle)^{m+1} v(x,t)|^2 \mathrm{d}x \right] \right\} \\ / & \left\{ \lim_{t\to\infty} \left[t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^{m} v(x,t)|^2 \mathrm{d}x \right] \right\} \\ = & \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+6} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \\ = & \frac{(4m+3)(4m+5)}{(4\alpha)^2}, \end{aligned}$$

for all real constants $m \geq 0$.

2. The mathematical analysis and the proofs of the main results

Consider the Cauchy problem for the general Korteweg-de Vries-Burgers equation (1.1)-(1.2). To simplify the notations, let us represent the fractional order derivative by

$$(-\triangle)^m u = \left(-\frac{\partial^2}{\partial x^2} \right)^m u.$$

The main purposes of this section are to accomplish:

(1) The existence and uniqueness of the global smooth solution

$$\begin{aligned} & u \in C^\infty(\mathbb{R} \times \mathbb{R}^+), \\ & u \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \frac{\partial}{\partial x} u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \forall m > 0. \end{aligned}$$

(2) The sharp rate decay estimates

$$\sup_{t>0} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \right\} < \infty,$$

for all order derivatives of the global smooth solution.

(3) The computations and explicit representations of the exact limits

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\},$$
$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m[u(x,t)-v(x,t)]|^2\mathrm{d}x\right\},$$

for all order derivatives of the global smooth solution, in terms of certain physical mechanisms, such as the following integrals

$$\int_{\mathbb{R}}\phi(x)\mathrm{d}x,\qquad \int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t,\qquad \int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t.$$

(4) The optimal decay estimates

$$t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\le\mathcal{A}(\alpha,\delta,\varepsilon,m)+\mathcal{B}(\alpha,\delta,\varepsilon,m)t^{-1},$$
$$t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m[u(x,t)-v(x,t)]|^2\mathrm{d}x\le\mathcal{C}(\alpha,\delta,\varepsilon,m)+\mathcal{D}(\alpha,\delta,\varepsilon,m)t^{-1},$$

for all order derivatives of the global smooth solution, for all positive constants $0<\delta<4$ and $0<\varepsilon<1$ and for all sufficiently large t. The positive constants

$$\mathcal{A}=\mathcal{A}(\alpha,\delta,\varepsilon,m),\ \ \mathcal{B}=\mathcal{B}(\alpha,\delta,\varepsilon,m),$$
$$\mathcal{C}=\mathcal{C}(\alpha,\delta,\varepsilon,m),\ \ \mathcal{D}=\mathcal{D}(\alpha,\delta,\varepsilon,m),$$

will be derived explicitly in the mathematical analysis.

The Main Strategy: We will couple together a few novel ideas, several existing ideas (such as the Fourier transformation, the Parseval's identity, Lebesgue's dominated convergence theorem, squeeze theorem) and existing results (the existence of the global weak solution, the elementary decay estimate) to accomplish these results. Also we will make complete use of the representations of the Fourier transformations of the global smooth solutions.

Let us define

$$\mathcal{I}(m)=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta,$$
$$\mathcal{J}=\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2.$$

Let the positive constants $0<\delta<4$ and $0<\varepsilon<1$. Let $C(m)$, $C_1(m)$, $C_2(m)$, $C_3(m)$, $C_4(m)$, $C_5(m)$ represent any positive constants, independent of (x,t), u and its derivatives. They only depend on m.

2.1. The representations of the Fourier transformations of the global smooth solutions

First of all, let us review a few popular concepts.

Definition 2.1. Define the Fourier transformation

$$[\mathcal{F}\phi](\xi) \equiv \widehat{\phi}(\xi) \equiv \int_{\mathbb{R}} \exp(-\mathrm{i}x\xi)\phi(x)\mathrm{d}x, \qquad \xi \in \mathbb{R},$$

for all functions $\phi \in L^1(\mathbb{R})$, where $\mathrm{i} = \sqrt{-1}$.

Definition 2.2. Define the differential operator $(-\triangle)^m \equiv (-\frac{\partial^2}{\partial x^2})^m$ by using the Fourier transformation

$$\mathcal{F}[(-\triangle)^m\phi](\xi) \equiv \mathcal{F}\left\{\left(-\frac{\partial^2}{\partial x^2}\right)^m \phi\right\}(\xi) = |\xi|^{2m}\widehat{\phi}(\xi), \qquad \xi \in \mathbb{R},$$

where $\phi \in L^1(\mathbb{R})$ and $m > 0$ is a positive constant.

The Fourier transformations of the global smooth solutions of the two Cauchy problems (1.1)-(1.2) and (1.3)-(1.4) may be represented as

$$\begin{aligned}
&\widehat{u}(\xi,t) = \mathrm{i}\xi\exp[(-\alpha|\xi|^2 + \mathrm{i}\beta\xi^3 + \mathrm{i}\gamma|\xi|\xi)t]\widehat{\phi}(\xi)\\
&+ \mathrm{i}\xi\int_0^t \exp[(-\alpha|\xi|^2 + \mathrm{i}\beta\xi^3 + \mathrm{i}\gamma|\xi|\xi)(t-\tau)]\widehat{\psi}(\xi,\tau)\mathrm{d}\tau\\
&- \mathrm{i}\xi\int_0^t \exp[(-\alpha|\xi|^2 + \mathrm{i}\beta\xi^3 + \mathrm{i}\gamma|\xi|\xi)(t-\tau)]\widehat{\mathcal{N}(u)}(\xi,\tau)\mathrm{d}\tau,\\
&\widehat{v}(\xi,t) = \mathrm{i}\xi\exp[(-\alpha|\xi|^2 + \mathrm{i}\beta\xi^3 + \mathrm{i}\gamma|\xi|\xi)t]\widehat{\phi}(\xi)\\
&+ \mathrm{i}\xi\int_0^t \exp[(-\alpha|\xi|^2 + \mathrm{i}\beta\xi^3 + \mathrm{i}\gamma|\xi|\xi)(t-\tau)]\widehat{\psi}(\xi,\tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\xi,t) \in \mathbb{R}\times\mathbb{R}^+$.

Let us make the change of variables $\eta = t^{1/2}\xi$, where $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}$ and $t > 0$. Now the representations of the Fourier transformations become

$$\begin{aligned}
&t^{1/2}\widehat{u}(t^{-1/2}\eta,t) = \mathrm{i}\eta\exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\\
&+ \mathrm{i}\eta\int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&- \mathrm{i}\eta\int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&t^{1/2}\widehat{v}(t^{-1/2}\eta,t) = \mathrm{i}\eta\exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\\
&+ \mathrm{i}\eta\int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\eta,t) \in \mathbb{R}\times\mathbb{R}^+$.

The details of the mathematical analysis of the global smooth solution are very complicated. Let us introduce some notations to simplify the presentations of the main ideas and the main steps in the mathematical analysis.

Definition 2.3. Define the following auxiliary functions

$$\begin{aligned}
&\Lambda_1(\eta,t) = \exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\\
&+ \int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}
&\Lambda_2(\eta,t) = \exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\\
&+ \int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&- \int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Lambda_3(\eta,t) = \int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\eta,t) \in \mathbb{R}\times\mathbb{R}^+$.

Definition 2.4. Define the following auxiliary functions

$$\begin{aligned}
&\Gamma_1(\eta,t) = \exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\\
&+ \int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Gamma_2(\eta,t) = \exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\\
&+ \int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&- \int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Gamma_3(\eta,t) = \int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\eta,t) \in \mathbb{R}\times\mathbb{R}^+$, where the positive constant $0<\varepsilon<1$.

Remark 2.1. There hold the following relationships

$$\begin{aligned}
\Lambda_1(\eta,t) &= \Lambda_2(\eta,t) + \Lambda_3(\eta,t),\\
\Gamma_1(\eta,t) &= \Gamma_2(\eta,t) + \Gamma_3(\eta,t),
\end{aligned}$$

for all $(\eta,t) \in \mathbb{R}\times\mathbb{R}^+$.

Remark 2.2. There hold the following relationships

$$\begin{aligned}
\mathrm{i}\eta\Lambda_1(\eta,t) &= t^{1/2}\widehat{v}(t^{-1/2}\eta,t),\\
\mathrm{i}\eta\Lambda_2(\eta,t) &= t^{1/2}\widehat{u}(t^{-1/2}\eta,t),
\end{aligned}$$

for all $(\eta,t) \in \mathbb{R}\times\mathbb{R}^+$.

Remark 2.3. There hold the following relationships

$$\begin{aligned}
&\Lambda_1(\eta,t) - \Gamma_1(\eta,t)\\
&= \int_{(1-\varepsilon)t}^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Lambda_2(\eta,t) - \Gamma_2(\eta,t)\\
&= \int_{(1-\varepsilon)t}^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&- \int_{(1-\varepsilon)t}^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

$$\begin{aligned}&\Lambda_3(\eta,t)-\Gamma_3(\eta,t)\\ &=\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\end{aligned}$$

for all $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$.

Remark 2.4. There hold the following relationships

$$\begin{aligned}&\alpha|\eta|^2[\Lambda_1(\eta,t)-\Gamma_1(\eta,t)]\\ &=\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\ &\alpha|\eta|^2[\Lambda_2(\eta,t)-\Gamma_2(\eta,t)]\\ &=\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\ &-\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\ &\alpha|\eta|^2[\Lambda_3(\eta,t)-\Gamma_3(\eta,t)]\\ &=\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\end{aligned}$$

for all $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$.

Definition 2.5. Define

$$\begin{aligned}&I_1(m,t)=\int_{\mathbb{R}}|\eta|^{4m+2}\left|\exp(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\right|^2\mathrm{d}\eta,\\ &I_2(m,t)=\int_{\mathbb{R}}|\eta|^{4m+2}\\ &\cdot\left|\int_{0}^{(1-\varepsilon)t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,\\ &I_3(m,t)=\int_{\mathbb{R}}|\eta|^{4m+2}\\ &\cdot\left|\int_{0}^{(1-\varepsilon)t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,\\ &I_4(m,t)=\int_{\mathbb{R}}|\eta|^{4m+2}\\ &\cdot\left|\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,\\ &I_5(m,t)=\int_{\mathbb{R}}|\eta|^{4m+2}\\ &\cdot\left|\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta,\end{aligned}$$

for all real constants $m\geq 0$.

Remark 2.5. It is easy to show that

$$t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}|t^{1/2}\widehat{u}(t^{-1/2}\eta,t)|^2 \mathrm{d}\eta,$$

for all $m \geq 0$ and $t > 0$. Moreover

$$\int_{\mathbb{R}} |\eta|^{4m}\left|t^{1/2}\widehat{u}(t^{-1/2}\eta,t)\right|^2 \mathrm{d}\eta \leq 5\sum_{k=1}^{5} I_k(m,t),$$

for all $m \geq 0$ and for all $t > 0$.

We will see that there exist positive constants $C_k = C_k(m) > 0$, independent of t, for all $k = 1,2,3,4,5$, such that

$$\begin{aligned}
&\sup_{t>0} I_1(m,t) \leq C_1(m),\\
&\sup_{t>0} I_2(m,t) \leq C_2(m),\\
&\sup_{t>0} I_3(m,t) \leq C_3(m),\\
&\sup_{t>0} I_4(m,t) \leq C_4(m)t^{-1},\\
&\sup_{t>0} I_5(m,t) \leq C_5(m)t^{-1},
\end{aligned}$$

for all real constants $m \geq 0$ and for all $t > 0$.

For the integrals of the form

$$\int_{\mathbb{R}} |\eta|^{4m+2}\left|\int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\cdots\cdots \mathrm{d}\tau\right|^2 \mathrm{d}\eta,$$

there exists a positive lower bound for $\alpha(1-\frac{\tau}{t})$:

$$\alpha(1-\frac{\tau}{t}) \geq \alpha\varepsilon,$$

on the interval $[0,(1-\varepsilon)t]$. The exponential function $\exp(-2\alpha\varepsilon|\eta|^2)$ dominates $|\eta|^{4m+2}$, for all real constants $m \geq 0$. In another word, the following integrals

$$\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta, \qquad \int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta$$

exist, for all $m \geq 0$ and $0 < \varepsilon < 1$.

On the other hand, for the integrals of the form

$$\int_{\mathbb{R}} |\eta|^{4m+2}\left|\int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\cdots\cdots \mathrm{d}\tau\right|^2 \mathrm{d}\eta,$$

it is very complicated and difficult to control, because

$$\alpha(1-\frac{\tau}{t}) \geq 0,$$

on $[(1-\varepsilon)t,t]$. There exists no positive lower bound for $\alpha(1-\frac{\tau}{t})$ and there exists no available exponential function to control $|\eta|^{4m+2}$. That is why the elementary estimates for these integrals are very complicated and very difficult.

Definition 2.6. Define

$$\mathcal{I}(m) = \frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta,$$

$$\mathcal{J} = \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty\int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2.$$

2.2. The elementary estimates

First of all, let us establish a series of elementary estimates.

Lemma 2.1. *There hold the following elementary estimates*

$$\int_{\mathbb{R}} |\eta|^{4m+2}\left|\exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\right|^2 \mathrm{d}\eta$$
$$\le \int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}} |\phi(x)|\mathrm{d}x\right\}^2,$$

$$\int_{\mathbb{R}} |\eta|^{4m+2}\left|\int_0^{(1-\varepsilon)t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3 t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$\le \int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}} |\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2,$$

$$\int_{\mathbb{R}} |\eta|^{4m+2}\left|\int_0^{(1-\varepsilon)t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3 t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$\le \int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}} |\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\right\}^2.$$

These estimates are true for all positive constants $0<\varepsilon<1$, for all real constants $m\ge 0$, and for all $t>0$.

Proof. The proof of the elementary estimates are very simple and the details are skipped. □

Lemma 2.2. *There hold the following elementary estimates*

$$\int_{\mathbb{R}} |\eta|^{4m+2}\left|\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3 t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$\le \frac{1}{t}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}$$
$$\cdot\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[t^{m+1+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\psi(x,\tau)|\mathrm{d}x\right]\right\}^2,$$

$$\int_{\mathbb{R}} |\eta|^{4m+2}\left|\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3 t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$\le \frac{C(m)}{t}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[t^{3/2} \int_{\mathbb{R}} |u(x,\tau)|^2 \mathrm{d}x \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[t^{2m+1+\frac{1}{2}\delta} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} u(x,\tau)|^2 \mathrm{d}x \right] \right\}.$$

These estimates are true for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$, for all real constants $m \ge 0$ and for all $t > 0$.

Proof. The proof follows from the applications of simple properties of the Fourier transformation and the change of variables $\eta = t^{1/2}\xi$, where $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}$ and $t > 0$.

First of all, there hold the following estimates

$$\int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_{(1-\varepsilon)t}^{t} \exp\left\{ (-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1 - \frac{\tau}{t}) \right\} \widehat{\psi}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$= \int_{\mathbb{R}} \frac{1}{\alpha^2 |\eta|^{1+\delta}} \left| \int_{(1-\varepsilon)t}^{t} \alpha |\eta|^{2m+1+\frac{1}{2}(1+\delta)} \right.$$
$$\left. \cdot \ \exp\left\{ (-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1 - \frac{\tau}{t}) \right\} \widehat{\psi}(t^{-1/2}\eta, \tau) \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$= \int_{\mathbb{R}} \frac{1}{\alpha^2 |\eta|^{1+\delta}} \left| \int_{(1-\varepsilon)t}^{t} \alpha |\eta|^2 \exp\left\{ (-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1 - \frac{\tau}{t}) \right\} \right.$$
$$\left. \cdot \ \left\{ |\eta|^{2m-1+\frac{1}{2}(1+\delta)} \widehat{\psi}(t^{-1/2}\eta, \tau) \right\} \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$= \int_{\mathbb{R}} \frac{1}{\alpha^2 |\eta|^{1+\delta}} \left| \int_{(1-\varepsilon)t}^{t} \alpha |\eta|^2 \exp\left\{ (-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1 - \frac{\tau}{t}) \right\} \right.$$
$$\left. \cdot \ \left\{ t^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} \mathcal{F}\left[(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} \psi \right] (t^{-1/2}\eta, \tau) \right\} \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\le \int_{\mathbb{R}} \frac{1}{\alpha^2 |\eta|^{1+\delta}} \left| \int_{(1-\varepsilon)t}^{t} \alpha |\eta|^2 \exp\left\{ -\alpha|\eta|^2 (1 - \frac{\tau}{t}) \right\} \mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[t^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} \psi(x,\tau)| \mathrm{d}x \right] \right\}^2$$
$$\le \frac{1}{t} \left\{ \int_{\mathbb{R}} \frac{[1 - \exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2 |\eta|^{1+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t \le \tau \le t} \left[t^{m+1+\frac{1}{4}(1+\delta)} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} \psi(x,\tau)| \mathrm{d}x \right] \right\}^2,$$

for all real constants $m \ge 0$.

Secondly, let us make estimates for integrals involving the nonlinear function $\mathcal{N}(u)$. Let us use $\mathcal{N}(u) = u^2$ as a typical example. Other cases of $\mathcal{N}(u)$ may be treated similarly. We have

$$\left\{ t^{m+\frac{1}{2}+\frac{1}{4}(1+\delta)} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} \mathcal{N}(u(x,\tau))| \mathrm{d}x \right\}^2$$
$$= \left\{ t^{m+\frac{1}{2}+\frac{1}{4}(1+\delta)} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} [u(x,\tau)]^2| \mathrm{d}x \right\}^2$$

$$\leq C(m)\left\{t^{m+\frac{1}{2}+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}|u(x,t)||(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}u(x,\tau)|\mathrm{d}x\right\}^2$$
$$\leq \frac{C(m)}{t}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\right\}$$
$$\cdot\ \left\{t^{2m+1+\frac{1}{2}\delta}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}u(x,\tau)|^2\mathrm{d}x\right\},$$

where $(1-\varepsilon)t\leq\tau\leq t$.

Now we have the following computations and estimates

$$\int_{\mathbb{R}}|\eta|^{4m+2}\left|\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\int_{\mathbb{R}}\frac{1}{\alpha^2|\eta|^{1+\delta}}\left|\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^{2m+1+\frac{1}{2}(1+\delta)}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\right.$$
$$\left.\cdot\ \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\int_{\mathbb{R}}\frac{1}{\alpha^2|\eta|^{1+\delta}}\left|\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\right.$$
$$\left.\cdot\ \left\{|\eta|^{2m-1+\frac{1}{2}(1+\delta)}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\right\}\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$=\int_{\mathbb{R}}\frac{1}{\alpha^2|\eta|^{1+\delta}}\left|\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\right.$$
$$\left.\cdot\ \left\{t^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\mathcal{F}[(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\mathcal{N}(u)](t^{-1/2}\eta,\tau)\right\}\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$\leq\int_{\mathbb{R}}\frac{1}{\alpha^2|\eta|^{1+\delta}}\left|\frac{1}{t}\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{-\alpha|\eta|^2(1-\frac{\tau}{t})\right\}\mathrm{d}\tau\right|^2\mathrm{d}\eta$$
$$\cdot\ \left\{t^{m+\frac{1}{2}+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\mathcal{N}(u(x,\tau))|\mathrm{d}x\right\}^2$$
$$\leq\frac{C(m)}{t}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}$$
$$\cdot\ \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[t^{3/2}\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\right]\right\}$$
$$\cdot\ \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[t^{2m+1+\frac{1}{2}\delta}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}u(x,\tau)|^2\mathrm{d}x\right]\right\}.$$

The proof of Lemma 2.2 is finished now. □

2.3. The comprehensive analysis

The main purpose is to make use of the representations of the Fourier transformations of the global smooth solutions and the elementary estimates to establish estimates for the following energies

$$t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x,$$

$$t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m[u(x,t)-v(x,t)]|^2\mathrm{d}x,$$

for all positive constants $0<\delta<4$ and $0<\varepsilon<1$, for all real constants $m\geq 0$ and for all $t>0$. The comprehensive analysis will play very important roles when we accomplish the sharp rate decay estimates and the optimal decay estimates for all order derivatives of the global smooth solution.

Lemma 2.3. *There holds the following estimate*

$$\begin{aligned}
& t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x \\
\leq\ & \frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}}|\phi(x)|\mathrm{d}x\right\}^2 \\
+\ & \frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2 \\
+\ & \frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\right\}^2 \\
+\ & \frac{5}{2\pi t}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
\cdot\ & \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[t^{m+1+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\psi(x,\tau)|\mathrm{d}x\right]\right\}^2 \\
+\ & \frac{C(m)}{2\pi t}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
\cdot\ & \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[t^{3/2}\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\right]\right\} \\
\cdot\ & \left\{\sup_{(1-\varepsilon)t\leq\tau\leq t}\left[t^{2m+1+\frac{1}{2}\delta}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}u(x,\tau)|^2\mathrm{d}x\right]\right\}.
\end{aligned}$$

This estimate is true for all positive constants $0<\delta<4$ *and* $0<\varepsilon<1$*, for all real constants* $m\geq 0$ *and for all* $t>0$*. The positive constant* $C(m)>0$ *only depends on* m*.*

Proof. The main idea is to couple together the Fourier transformation, the Parseval's identity, the representation of the Fourier transformation of the global smooth solution, the change of variables $\eta=t^{1/2}\xi$ and the elementary estimates in Lemma 2.1 and Lemma 2.2. We have the following computations and estimates

$$\begin{aligned}
& t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x \\
=\ & \frac{t^{2m+3/2}}{2\pi}\int_{\mathbb{R}}|\xi|^{4m}|\widehat{u}(\xi,t)|^2\mathrm{d}\xi \\
=\ & \frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m}|t^{1/2}\widehat{u}(t^{-1/2}\eta,t)|^2\mathrm{d}\eta \\
=\ & \frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\Big|\exp(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&-\int_0^t \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2\mathrm{d}\eta \\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\Big|\exp(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta) \\
&+\int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&-\int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&+\int_{(1-\varepsilon)t}^t \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \\
&-\int_{(1-\varepsilon)t}^t \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2\mathrm{d}\eta \\
&\le\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\Big|\exp(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\Big|^2\mathrm{d}\eta \\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\Big|\int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \\
&\cdot\ \widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2\mathrm{d}\eta \\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\Big|\int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \\
&\cdot\ \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2\mathrm{d}\eta \\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\Big|\int_{(1-\varepsilon)t}^t \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \\
&\cdot\ \widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2\mathrm{d}\eta \\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\Big|\int_{(1-\varepsilon)t}^t \exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \\
&\cdot\ \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Big|^2\mathrm{d}\eta \\
&\le\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}}|\phi(x)|\mathrm{d}x\right\}^2 \\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2 \\
&+\frac{5}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}|\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\right\}^2 \\
&+\frac{5}{2\pi t}\left\{\int_{\mathbb{R}}\frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}
\end{aligned}
$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{m+1+\frac{1}{4}(1+\delta)} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\psi(x,\tau)|\mathrm{d}x \right] \right\}^2$$
$$+ \frac{C(m)}{2\pi t} \left\{ \int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{3/2} \int_{\mathbb{R}} |u(x,\tau)|^2 \mathrm{d}x \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{2m+1+\frac{1}{2}\delta} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} u(x,\tau)|^2 \mathrm{d}x \right] \right\}.$$

In these estimates, we have applied the elementary estimates in Lemma 2.1 and Lemma 2.2. The proof of Lemma 2.3 is finished now. □

Lemma 2.4. *There holds the following estimate*

$$t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m [u(x,t)-v(x,t)]|^2 \mathrm{d}x$$
$$\le \frac{2}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha\varepsilon|\eta|^2) \mathrm{d}\eta \left\{ \int_0^\infty \int_{\mathbb{R}} |\mathcal{N}(u(x,t))| \mathrm{d}x\mathrm{d}t \right\}^2$$
$$+ \frac{C(m)}{2\pi t} \left\{ \int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}} \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{3/2} \int_{\mathbb{R}} |u(x,\tau)|^2 \mathrm{d}x \right] \right\}$$
$$\cdot \left\{ \sup_{(1-\varepsilon)t\le\tau\le t} \left[t^{2m+1+\frac{1}{2}\delta} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} u(x,\tau)|^2 \mathrm{d}x \right] \right\}.$$

This estimate is true for all positive constants $0<\delta<4$ *and* $0<\varepsilon<1$, *for all real constants* $m\ge 0$ *and for all* $t>0$.

Proof. The idea in the proof of this lemma is the same as the idea in the proof of Lemma 2.3.

We have the following computations and estimates

$$t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m [u(x,t)-v(x,t)|^2 \mathrm{d}x$$
$$= \frac{t^{2m+3/2}}{2\pi} \int_{\mathbb{R}} |\xi|^{4m} |\widehat{u}(\xi,t)-\widehat{v}(\xi,t)|^2 \mathrm{d}\xi$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m} |t^{1/2}[\widehat{u}(t^{-1/2}\eta,t)-\widehat{v}(t^{-1/2}\eta,t)]|^2 \mathrm{d}\eta$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_0^t \exp\left\{ (-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t}) \right\} \right.$$
$$\left. \cdot\ \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta$$
$$= \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_0^{(1-\varepsilon)t} \exp\left\{ (-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t}) \right\} \right.$$
$$\cdot\ \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$

$$
\begin{aligned}
&+ \int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2 \mathrm{d}\eta \\
&\le \frac{2}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\left|\int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\right. \\
&\quad \cdot \left.\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta \\
&+ \frac{2}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\left|\int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\right. \\
&\quad \cdot \left.\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2 \mathrm{d}\eta \\
&\le \frac{2}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}} |\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\right\}^2 \\
&+ \frac{C(m)}{2\pi t}\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
&\quad \cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[t^{3/2}\int_{\mathbb{R}} |u(x,\tau)|^2\mathrm{d}x\right]\right\} \\
&\quad \cdot \left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[t^{2m+1+\frac{1}{2}\delta}\int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}u(x,\tau)|^2\mathrm{d}x\right]\right\}.
\end{aligned}
$$

The proof of Lemma 2.4 is finished. □

2.4. The sharp rate decay estimates

The main purpose is to couple together an elegant iteration technique, the comprehensive analysis in Lemma 2.3 and the elementary decay estimate to establish the sharp rate decay estimates for all order derivatives of the global smooth solution.

Lemma 2.5. *There hold the following sharp rate decay estimates*

$$
\sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\} < \infty,
$$

for all positive constants $m > 0$.

Proof. Let us use a very simple elegant iteration technique to establish the result. First of all, recall that there holds the elementary decay estimate

$$
\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}} |u(x,t)|^2\mathrm{d}x\right\} < \infty.
$$

For any positive constant $m > 0$, there exists a positive integer $k \ge 4$ and a positive constant $0 < \delta_0 < 1$, such that $m = k\frac{1-\delta_0}{4}$. For example, we may let

$$
k = 4 + [4m], \qquad \delta_0 = 1 - \frac{4m}{4+[4m]}.
$$

Let

$$
m_0 = 0,
$$

$$\begin{aligned} m_1 &= \frac{1-\delta_0}{4}, \\ m_2 &= 2\frac{1-\delta_0}{4}, \\ m_3 &= 3\frac{1-\delta_0}{4}, \\ &\cdots\cdots\cdots\cdots \\ m_k &= k\frac{1-\delta_0}{4}. \end{aligned}$$

Note that $m_k = m$ and $m_{i-1} = m_i + \frac{1}{4}(\delta_0 - 1)$, for all $i = 1, 2, 3, \cdots, k$. Now in Lemma 2.3, letting $m = m_1$, we immediately obtain the estimate

$$\sup_{t>0}\left\{t^{2m_1+3/2}\int_{\mathbb{R}}|(-\triangle)^{m_1}u(x,t)|^2\mathrm{d}x\right\} < \infty,$$

where we have applied the decay estimate

$$\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\} < \infty.$$

Now letting $m = m_2$, we obtain the slightly better estimate

$$\sup_{t>0}\left\{t^{2m_2+3/2}\int_{\mathbb{R}}|(-\triangle)^{m_2}u(x,t)|^2\mathrm{d}x\right\} < \infty.$$

Repeating this procedure for finitely many times by letting $m = m_1$, $m = m_2$, $m = m_3$, $\cdots\cdots$, $m = m_k$, eventually, we obtain the sharp rate decay estimate

$$\sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}u(x,t)|^2\mathrm{d}x\right\} < \infty.$$

The proof of Lemma 2.5 is finished now. □

2.5. The existence and uniqueness of the global smooth solution

The main purpose is to couple together the existence of the global weak solution and the sharp rate decay estimates for all order derivatives to demonstrate the existence and uniqueness of the global smooth solution of the Cauchy problem for the general Korteweg-de Vries-Burgers equation (1.1)-(1.2).

Recall that there hold the following uniform energy estimates

$$\sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^{m}u(x,t)|^2\mathrm{d}x\right\} < \infty,$$

for all order derivatives of the global weak solution.

Let us make some changes of variables. Let

$$z = \frac{x}{t^{1/2}}, \qquad U(z,t) = u(x,t)t,$$

for all $(z,t) \in \mathbb{R} \times \mathbb{R}^+$.

Now there hold the following uniform energy estimates

$$\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial z^2}\right)^m U(z,t)\right|^2 \mathrm{d}z = t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x,$$

$$\sup_{t>0}\left\{\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial z^2}\right)^m U(z,t)\right|^2 \mathrm{d}z\right\} = \sup_{t>0}\left\{t^{2m+3/2}\int_{\mathbb{R}}\left|\left(-\frac{\partial^2}{\partial x^2}\right)^m u(x,t)\right|^2 \mathrm{d}x\right\} < \infty,$$

for all positive constants $m > 0$.

Let the function $\phi \in H^1(\mathbb{R})$. Then we have

$$\phi(y) - \phi(x) = \int_x^y \phi'(t)\mathrm{d}t,$$

$$|\phi(y) - \phi(x)|^2 = \left|\int_x^y \phi'(t)\mathrm{d}t\right|^2$$

$$\le |y-x|\int_x^y |\phi'(t)|^2\mathrm{d}t \le |y-x|\int_{\mathbb{R}} |\phi'(x)|^2\mathrm{d}x,$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}$ and $x < y$.

Therefore, ϕ is Lipschitz continuous, as long as $\phi \in H^1(\mathbb{R})$.

By coupling together Sobolev embedding theorem

$$H^{2m}(\mathbb{R}) \hookrightarrow C^{2m-2}(\mathbb{R}),$$

and these uniform estimates, we see that

$$U \in C^\infty(\mathbb{R}\times\mathbb{R}^+).$$

Finally, we find that the global weak solution is a global smooth solution

$$u \in C^\infty(\mathbb{R}\times\mathbb{R}^+).$$

The uniqueness may be established by using standard energy method. The proof is finished now. □

There exist positive constants $C_1(m) > 0$, $C_2(m) > 0$, $C > 0$, such that there hold the following estimates

$$\sup_{(1-\varepsilon)t\le\tau\le t}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\right\} \le C,$$

$$\sup_{(1-\varepsilon)t\le\tau\le t}\left\{t^{2m+1+\frac{1}{2}\delta}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}u(x,\tau)|^2\mathrm{d}x\right\} \le C_1(m),$$

$$\sup_{(1-\varepsilon)t\le\tau\le t}\left\{t^{m+1+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\psi(x,\tau)|\mathrm{d}x\right\} \le C_2(m),$$

for all real constants $m \ge 0$ and for all $t \ge 0$.

Therefore

$$I_4(m,t)$$

$$= \int_{\mathbb{R}}|\eta|^{4m+2}\left|\int_{(1-\varepsilon)t}^t \exp\Big\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\Big\}\widehat{\psi}(t^{-1/2},\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta$$

$$\begin{aligned}
&\le C_4(m)t^{-1}, \\
&I_5(m,t) \\
&= \int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \right. \\
&\cdot \left. \widehat{\mathcal{N}(u)}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
&\le C_5(m)t^{-1},
\end{aligned}$$

for all $m \ge 0$ and for all $t > 0$.

Remark 2.6. From the comprehensive analysis, we may derive the positive constants $\mathcal{A} = \mathcal{A}(\alpha, \delta, \varepsilon, m)$ and $\mathcal{C} = \mathcal{C}(\alpha, \delta, \varepsilon, m)$ very easily. The constants $\mathcal{B} = \mathcal{B}(\alpha, \delta, \varepsilon, m)$ and $\mathcal{D} = \mathcal{D}(\alpha, \delta, \varepsilon, m)$ will be derived later.

2.6. The fundamental limits

The main purpose is to apply the elementary estimates and the sharp rate decay estimates to establish several fundamental limits. These limits are true for all real constants $m \ge 0$.

Recall that the auxiliary functions are defined by

$$\begin{aligned}
&\Gamma_1(\eta,t) = \exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta) \\
&+ \int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \widehat{\psi}(t^{-1/2}\eta, \tau)\mathrm{d}\tau, \\
&\Gamma_2(\eta,t) = \exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta) \\
&+ \int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \widehat{\psi}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \\
&- \int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \widehat{\mathcal{N}(u)}(t^{-1/2}\eta, \tau)\mathrm{d}\tau, \\
&\Gamma_3(\eta,t) = \int_0^{(1-\varepsilon)t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\} \widehat{\mathcal{N}(u)}(t^{-1/2}\eta, \tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\eta, t) \in \mathbb{R} \times \mathbb{R}^+$.

Lemma 2.6. *There hold the fundamental exact limits*

$$\begin{aligned}
&\lim_{t\to\infty} \Gamma_1(\eta,t) = \exp(-\alpha|\eta|^2 + \mathrm{i}\gamma|\eta|\eta)\left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t\right\}, \\
&\lim_{t\to\infty} \Gamma_2(\eta,t) = \exp(-\alpha|\eta|^2 + \mathrm{i}\gamma|\eta|\eta) \\
&\cdot \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}, \\
&\lim_{t\to\infty} \Gamma_3(\eta,t)] = \exp(-\alpha|\eta|^2 + \mathrm{i}\gamma|\eta|\eta)\left\{\int_0^\infty \int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\},
\end{aligned}$$

for all $\eta \in \mathbb{R}$ and for all $0 < \varepsilon < 1$.

Proof. The proof follows from an application of Lebesgue's dominated convergence theorem.

First of all, note that the functions

$$\phi \in L^1(\mathbb{R}), \qquad \psi \in L^1(\mathbb{R}\times\mathbb{R}^+), \qquad \mathcal{N}(u) \in L^1(\mathbb{R}\times\mathbb{R}^+).$$

Therefore, the Fourier transformations

$$\widehat{\phi}(t^{-1/2}\eta), \qquad \widehat{\psi}(t^{-1/2}\eta,\tau), \qquad \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)$$

are continuous functions of η and t, for each fixed $\tau > 0$.

Other details are skipped because they are very easy. □

Lemma 2.7. *There hold the following fundamental exact limits*

$$\begin{aligned}
\lim_{t\to\infty} \Lambda_1(\eta,t) &= \lim_{t\to\infty} \Gamma_1(\eta,t),\\
\lim_{t\to\infty} \Lambda_2(\eta,t) &= \lim_{t\to\infty} \Gamma_2(\eta,t),\\
\lim_{t\to\infty} \Lambda_3(\eta,t) &= \lim_{t\to\infty} \Gamma_3(\eta,t),
\end{aligned}$$

for all $\eta \in \mathbb{R}$.

Proof. The proof is long, but the main point is to establish the following estimates

$$\begin{aligned}
|\Lambda_1(\eta,t) - \Gamma_1(\eta,t)| &\le \frac{C}{t^{1/2}},\\
|\Lambda_2(\eta,t) - \Gamma_2(\eta,t)| &\le \frac{C}{t^{1/2}},\\
|\Lambda_3(\eta,t) - \Gamma_3(\eta,t)| &\le \frac{C}{t^{1/2}},
\end{aligned}$$

for all $(\eta,t) \in \mathbb{R}\times\mathbb{R}^+$, where $C > 0$ is a positive constant, independent of (η,t).

Recall that there hold the following relationships

$$\begin{aligned}
&\Lambda_1(\eta,t) - \Gamma_1(\eta,t)\\
&= \int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Lambda_2(\eta,t) - \Gamma_2(\eta,t)\\
&= \int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&- \int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\Lambda_3(\eta,t) - \Gamma_3(\eta,t)\\
&= \int_{(1-\varepsilon)t}^{t} \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

for all $(\eta,t) \in \mathbb{R}\times\mathbb{R}^+$.

Multiplying these equations by $\alpha|\eta|^2$, we obtain

$$\begin{aligned}
&\alpha|\eta|^2[\Lambda_1(\eta,t) - \Gamma_1(\eta,t)]\\
&= \int_{(1-\varepsilon)t}^{t} \alpha|\eta|^2 \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}$$

$$
\begin{aligned}
&\alpha|\eta|^2[\Lambda_2(\eta,t)-\Gamma_2(\eta,t)]\\
&=\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\\
&-\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,\\
&\alpha|\eta|^2[\Lambda_3(\eta,t)-\Gamma_3(\eta,t)]\\
&=\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,
\end{aligned}
$$

for all $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$.

Now, there hold the following estimates

$$
\begin{aligned}
&\alpha|\eta|^2|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|\\
&\le\left|\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|\\
&\le\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{-\alpha|\eta|^2(1-\frac{\tau}{t})\right\}\left\{\int_{\mathbb{R}}|\psi(x,\tau)|\mathrm{d}x\right\}\mathrm{d}\tau\\
&\le C[1-\exp(-\alpha\varepsilon|\eta|^2)]t^{-1/2},\\
&\alpha|\eta|^2|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|\\
&\le\left|\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|\\
&+\left|\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|\\
&\le\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{-\alpha|\eta|^2(1-\frac{\tau}{t})\right\}\left\{\int_{\mathbb{R}}|\psi(x,\tau)|\mathrm{d}x\right\}\mathrm{d}\tau\\
&+\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{-\alpha|\eta|^2(1-\frac{\tau}{t})\right\}\left\{\int_{\mathbb{R}}|\mathcal{N}(u(x,\tau))|\mathrm{d}x\right\}\mathrm{d}\tau,\\
&\le C[1-\exp(-\alpha\varepsilon|\eta|^2)]t^{-1/2},\\
&\alpha|\eta|^2|\Lambda_3(\eta,t)-\Gamma_3(\eta,t)|\\
&\le\left|\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|\\
&\le\int_{(1-\varepsilon)t}^{t}\alpha|\eta|^2\exp\left\{-\alpha|\eta|^2(1-\frac{\tau}{t})\right\}\left\{\int_{\mathbb{R}}|\mathcal{N}(u(x,\tau))|\mathrm{d}x\right\}\mathrm{d}\tau\\
&\le C[1-\exp(-\alpha\varepsilon|\eta|^2)]t^{-1/2},
\end{aligned}
$$

for all $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$. In the above analysis, we have used the following estimates

$$
\begin{aligned}
&\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}}|\psi(x,t)|\mathrm{d}x\right\}<\infty,\\
&\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty.
\end{aligned}
$$

That is

$$|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)| \le C\varepsilon \left\{\frac{1-\exp(-\alpha\varepsilon|\eta|^2)}{\alpha\varepsilon|\eta|^2}\right\} t^{-1/2} \le \frac{C}{t^{1/2}},$$
$$|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)| \le C\varepsilon \left\{\frac{1-\exp(-\alpha\varepsilon|\eta|^2)}{\alpha\varepsilon|\eta|^2}\right\} t^{-1/2} \le \frac{C}{t^{1/2}},$$
$$|\Lambda_3(\eta,t)-\Gamma_3(\eta,t)| \le C\varepsilon \left\{\frac{1-\exp(-\alpha\varepsilon|\eta|^2)}{\alpha\varepsilon|\eta|^2}\right\} t^{-1/2} \le \frac{C}{t^{1/2}},$$

for all $\eta \in \mathbb{R}$, except for $\eta = 0$. Motivated by the limit

$$\lim_{|\eta|\to 0} \left\{\frac{1-\exp(-\alpha\varepsilon|\eta|^2)}{\alpha\varepsilon|\eta|^2}\right\} = 1,$$

to include the case $\eta = 0$ in the above estimates, we may define

$$\frac{1-\exp(-\alpha\varepsilon|\eta|^2)}{\alpha\varepsilon|\eta|^2} = 1,$$

for $\eta = 0$.

Overall, we have the estimates

$$|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)| \le \frac{C}{t^{1/2}},$$
$$|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)| \le \frac{C}{t^{1/2}},$$
$$|\Lambda_3(\eta,t)-\Gamma_3(\eta,t)| \le \frac{C}{t^{1/2}},$$

for all $(\eta,t) \in \mathbb{R}\times\mathbb{R}^+$.

By using squeeze theorem, it is easy to see that

$$\lim_{t\to\infty} |\Lambda_1(\eta,t)-\Gamma_1(\eta,t)| = 0,$$
$$\lim_{t\to\infty} |\Lambda_2(\eta,t)-\Gamma_2(\eta,t)| = 0,$$
$$\lim_{t\to\infty} |\Lambda_3(\eta,t)-\Gamma_3(\eta,t)| = 0,$$

for all $\eta \in \mathbb{R}$. The proof of Lemma 2.7 is finished now. □

Lemma 2.8. *There hold the following fundamental limits*

$$\lim_{t\to\infty} \left\{\int_{\mathbb{R}} |\eta|^{4m+2} |\Gamma_1(\eta,t)|^2 \mathrm{d}\eta\right\}$$
$$= \left\{\int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\} \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t\right\}^2,$$
$$\lim_{t\to\infty} \left\{\int_{\mathbb{R}} |\eta|^{4m+2} |\Gamma_2(\eta,t)|^2 \mathrm{d}\eta\right\}$$
$$= \left\{\int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}$$
$$\cdot \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,$$

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Gamma_3(\eta,t)|^2\mathrm{d}\eta\right\}$$
$$=\left\{\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\left\{\int_0^{\infty}\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all real constants $m\geq 0$.

Proof. The limits follow from Lebesgue's dominated convergence theorem, the elementary estimates in Lemma 2.1 and the fundamental limits in Lemma 2.6. □

Lemma 2.9. *There hold the following fundamental limits*

$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}=0,$$
$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}=0,$$
$$\lim_{t\to\infty}\left\{\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_3(\eta,t)-\Gamma_3(\eta,t)|^2\mathrm{d}\eta\right\}=0,$$

for all real constants $m\geq 0$.

Proof. The key point is to establish the following estimates

$$\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\leq C_1(m)t^{-1},$$
$$\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\leq C_2(m)t^{-1},$$
$$\int_{\mathbb{R}}|\eta|^{4m+2}|\Lambda_3(\eta,t)-\Gamma_3(\eta,t)|^2\mathrm{d}\eta\leq C_3(m)t^{-1},$$

for all $m\geq 0$ and for all $t>0$, where $C_1(m)>0$, $C_2(m)>0$ and $C_3(m)>0$ are positive constants, independent of $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$.

The proof of these estimates follows from the elementary estimates in Lemma 2.2 and the sharp rate decay estimates in Lemma 2.5.

Recall that there hold

$$\Lambda_1(\eta,t)-\Gamma_1(\eta,t)$$
$$=\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$
$$\Lambda_2(\eta,t)-\Gamma_2(\eta,t)$$
$$=\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$-\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$
$$\Lambda_3(\eta,t)-\Gamma_3(\eta,t)$$
$$=\int_{(1-\varepsilon)t}^{t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau,$$

for all $(\eta,t)\in\mathbb{R}\times\mathbb{R}^+$.

In fact, we have the following estimates

$$
\begin{aligned}
&\int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_1(\eta,t) - \Gamma_1(\eta,t)|^2 \mathrm{d}\eta \\
= &\int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_{(1-\varepsilon)t}^{t} \exp\Big\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\Big\} \widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\le &\frac{1}{t}\left\{\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
\cdot &\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[t^{m+1+\frac14(1+\delta)}\int_{\mathbb{R}} |(-\triangle)^{m-\frac12+\frac14(1+\delta)}\psi(x,\tau)|\mathrm{d}x\right]\right\}^2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_2(\eta,t) - \Gamma_2(\eta,t)|^2 \mathrm{d}\eta \\
= &\int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_{(1-\varepsilon)t}^{t} \exp\Big\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\Big\} \widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right. \\
+ &\left. \int_{(1-\varepsilon)t}^{t} \exp\Big\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\Big\} \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\le &2\int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_{(1-\varepsilon)t}^{t} \exp\Big\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\Big\} \widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
+ &2\int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_{(1-\varepsilon)t}^{t} \exp\Big\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\Big\} \right. \\
\cdot &\left. \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau \right|^2 \mathrm{d}\eta \\
\le &\frac{2}{t}\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta \\
\cdot &\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[t^{m+1+\frac14(1+\delta)}\int_{\mathbb{R}} |(-\triangle)^{m-\frac12+\frac14(1+\delta)}\psi(x,\tau)|\mathrm{d}x\right]\right\}^2 \\
+ &\frac{C(m)}{t}\left\{\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\} \\
\cdot &\left\{\sup_{1-\varepsilon)t\le\tau\le t}\left[t^{3/2}\int_{\mathbb{R}} |u(x,\tau)|^2\mathrm{d}x\right]\right\} \\
\cdot &\left\{\sup_{(1-\varepsilon)t\le\tau\le t}\left[t^{2m+1+\frac12\delta}\int_{\mathbb{R}} |(-\triangle)^{m-\frac12+\frac14(1+\delta)}u(x,\tau)|^2\mathrm{d}x\right]\right\} \\
\le &\frac{C(m)}{t},
\end{aligned}
$$

and

$$
\int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_3(\eta,t) - \Gamma_3(\eta,t)|^2 \mathrm{d}\eta
$$

$$= \int_{\mathbb{R}} |\eta|^{4m+2} \left| \int_{(1-\varepsilon)t}^{t} \exp\Big\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\Big\} \widehat{\mathcal{N}(u)}(t^{-1/2}\eta, \tau)\mathrm{d}\tau \right| \mathrm{d}\eta$$

$$\leq \frac{C(m)}{t} \left\{ \int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}} \mathrm{d}\eta \right\}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t \leq \tau \leq t} \left[t^{3/2} \int_{\mathbb{R}} |u(x,\tau)|^2 \mathrm{d}x \right] \right\}$$

$$\cdot \left\{ \sup_{(1-\varepsilon)t \leq \tau \leq t} \left[t^{2m+1+\frac{1}{2}\delta} \int_{\mathbb{R}} |(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)} u(x,\tau)|^2 \mathrm{d}x \right] \right\}$$

$$\leq \frac{C(m)}{t},$$

for all real constants $m \geq 0$ and for all $t > 0$. Now the limits follow from the application of the squeeze theorem. □

Lemma 2.10. *There hold the following fundamental limits*

$$\lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_1(\eta,t)|^2 \mathrm{d}\eta \right\}$$
$$= \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \left\{ \int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t \right\}^2,$$

$$\lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_2(\eta,t)|^2 \mathrm{d}\eta \right\}$$
$$= \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\}$$
$$\cdot \left\{ \int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t \right\}^2,$$

$$\lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_3(\eta,t)|^2 \mathrm{d}\eta \right\}$$
$$= \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \left\{ \int_0^\infty \int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t \right\}^2,$$

for all real constants $m \geq 0$.

Proof. The key point in the proof is to demonstrate that

$$\lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_1(\eta,t)|^2 \mathrm{d}\eta \right\} = \lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Gamma_1(\eta,t)|^2 \mathrm{d}\eta \right\},$$
$$\lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_2(\eta,t)|^2 \mathrm{d}\eta \right\} = \lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Gamma_2(\eta,t)|^2 \mathrm{d}\eta \right\},$$
$$\lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_3(\eta,t)|^2 \mathrm{d}\eta \right\} = \lim_{t\to\infty} \left\{ \int_{\mathbb{R}} |\eta|^{4m+2} |\Gamma_3(\eta,t)|^2 \mathrm{d}\eta \right\},$$

for all real constants $m \geq 0$.

First of all, by using Cauchy-Schwartz's inequality and the definitions of the auxiliary functions $\Lambda_i(\eta,t)$ and $\Gamma_i(\eta,t)$, we have the following elementary estimates

$$\left| \int_{\mathbb{R}} |\eta|^{4m+2} |\Lambda_1(\eta,t)|^2 \mathrm{d}\eta - \int_{\mathbb{R}} |\eta|^{4m+2} |\Gamma_1(\eta,t)|^2 \mathrm{d}\eta \right|$$

$$\le \int_{\mathbb{R}} |\eta|^{4m+2}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta$$
$$+ 2\left\{\int_{\mathbb{R}} |\eta|^{4m+2}|\Lambda_1(\eta,t)-\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\int_{\mathbb{R}} |\eta|^{4m+2}|\Gamma_1(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2},$$
$$\left|\int_{\mathbb{R}} |\eta|^{4m+2}|\Lambda_2(\eta,t)|^2\mathrm{d}\eta - \int_{\mathbb{R}} |\eta|^{4m+2}|\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right|$$
$$\le \int_{\mathbb{R}} |\eta|^{4m+2}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta$$
$$+ 2\left\{\int_{\mathbb{R}} |\eta|^{4m+2}|\Lambda_2(\eta,t)-\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\int_{\mathbb{R}} |\eta|^{4m+2}|\Gamma_2(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2},$$
$$\left|\int_{\mathbb{R}} |\eta|^{4m+2}|\Lambda_3(\eta,t)|^2\mathrm{d}\eta - \int_{\mathbb{R}} |\eta|^{4m+2}|\Gamma_3(\eta,t)|^2\mathrm{d}\eta\right|$$
$$\le \int_{\mathbb{R}} |\eta|^{4m+2}|\Lambda_3(\eta,t)-\Gamma_3(\eta,t)|^2\mathrm{d}\eta$$
$$+ 2\left\{\int_{\mathbb{R}} |\eta|^{4m+2}|\Lambda_3(\eta,t)-\Gamma_3(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2}\left\{\int_{\mathbb{R}} |\eta|^{4m+2}|\Gamma_3(\eta,t)|^2\mathrm{d}\eta\right\}^{1/2},$$

These estimates are true for all real constants $m \ge 0$. The limits follow from the squeeze theorem and the fundamental limits in Lemma 2.8 and Lemma 2.9. The proof of Lemma 2.10 is finished. □

2.7. The exact limits

The main purpose is to make complete use of the fundamental limits and several traditional ideas to accomplish the exact limits for all order derivatives of the global smooth solution.

The Proof of Theorem 1.3: Let us make the change of variables $\eta = t^{1/2}\xi$, where $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}$ and $t > 0$. Now by coupling together the Fourier transformation, the Parseval's identity and the representation of the Fourier transformation of the global smooth solution, we have the following computations

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\}$$
$$= \lim_{t\to\infty}\left\{\frac{t^{2m+3/2}}{2\pi}\int_{\mathbb{R}} |\xi|^{4m}|\widehat{u}(\xi,t)|^2\mathrm{d}\xi\right\}$$
$$= \lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m}|t^{1/2}\widehat{u}(t^{-1/2}\eta,t)|^2\mathrm{d}\eta\right\}$$
$$= \lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\left|\exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\right.\right.$$
$$+ \int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$\left.\left. - \int_0^t \exp\left\{(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}$$
$$= \lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\left|\exp(-\alpha|\eta|^2 + \mathrm{i}\beta\eta^3 t^{-1/2} + \mathrm{i}\gamma|\eta|\eta)\widehat{\phi}(t^{-1/2}\eta)\right.\right.$$

$$+\int_0^{(1-\varepsilon)t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\psi}(t^{-1/2}\eta,\tau)\mathrm{d}\tau$$
$$-\int_0^{(1-\varepsilon)t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\Bigg|^2\mathrm{d}\eta\Bigg\}$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta$$
$$\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all real constants $m\geq 0$.

Very similarly, we have the following computations

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m[u(x,t)-v(x,t)]|^2\mathrm{d}x\right\}$$
$$=\lim_{t\to\infty}\left\{\frac{t^{2m+3/2}}{2\pi}\int_{\mathbb{R}}|\xi|^{4m}|\widehat{u}(\xi,t)-\widehat{v}(\xi,t)|^2\mathrm{d}\xi\right\}$$
$$=\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m}|t^{1/2}[\widehat{u}(t^{-1/2}\eta,t)-\widehat{v}(t^{-1/2}\eta,t)]|^2\mathrm{d}\eta\right\}$$
$$=\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\left|\int_0^t\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\right.\right.$$
$$\left.\left.\cdot\ \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}$$
$$=\lim_{t\to\infty}\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\left|\int_0^{(1-\varepsilon)t}\exp\left\{(-\alpha|\eta|^2+\mathrm{i}\beta\eta^3t^{-1/2}+\mathrm{i}\gamma|\eta|\eta)(1-\frac{\tau}{t})\right\}\right.\right.$$
$$\left.\left.\cdot\ \widehat{\mathcal{N}(u)}(t^{-1/2}\eta,\tau)\mathrm{d}\tau\right|^2\mathrm{d}\eta\right\}$$
$$=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all real constants $m\geq 0$.

2.8. The linear results

The main purpose is to establish the exact limits for all order derivatives of the global smooth solution of the Cauchy problem for the corresponding linear equation (1.3)-(1.4).

Lemma 2.11. *There hold the following exact limits for the global smooth solution of the corresponding linear equation* (1.3)

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m v(x,t)|^2\mathrm{d}x\right\}$$
$$=\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all real constants $m\geq 0$.

Proof. The idea in the proof is the same as that of Theorem 1.3. The details are skipped. □

Lemma 2.12. *There hold the following results*

$$\int_{\mathbb{R}} |\eta|^{4m+4} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \frac{4m+3}{4\alpha} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta,$$
$$\int_{\mathbb{R}} |\eta|^{4m+6} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta = \frac{(4m+3)(4m+5)}{(4\alpha)^2} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2)\mathrm{d}\eta,$$

for all real constants $m \geq 0$.

Proof. They follow from the integration by parts. The details are skipped. □

Lemma 2.13. *The ratios of the exact limits are given by*

$$\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}} |(-\triangle)^{m+1/2}v(x,t)|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^{m}v(x,t)|^2\mathrm{d}x\right]\right\} = \frac{4m+3}{4\alpha},$$
$$\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}} |(-\triangle)^{m+1}v(x,t)|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^{m}v(x,t)|^2\mathrm{d}x\right]\right\} = \frac{(4m+3)(4m+5)}{(4\alpha)^2},$$

for all real constants $m \geq 0$.

Proof. It follows from the results of Lemma 2.11 and Lemma 2.12. The details are skipped. □

2.9. The ratios of the exact limits

The main purpose is to compute the ratios of the exact limits for each fixed $m \geq 0$.

Lemma 2.14. *There hold the following results*

$$\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}} |(-\triangle)^{m+1/2}u(x,t)|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^{m}u(x,t)|^2\mathrm{d}x\right]\right\} = \frac{4m+3}{4\alpha},$$
$$\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}} |(-\triangle)^{m+1}u(x,t)|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^{m}u(x,t)|^2\mathrm{d}x\right]\right\} = \frac{(4m+3)(4m+5)}{(4\alpha)^2},$$
$$\left\{\lim_{t\to\infty}\left[t^{2m+5/2}\int_{\mathbb{R}} |(-\triangle)^{m+1/2}[u(x,t)-v(x,t)]|^2\mathrm{d}x\right]\right\}$$
$$/\left\{\lim_{t\to\infty}\left[t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^{m}[u(x,t)-v(x,t)]|^2\mathrm{d}x\right]\right\} = \frac{4m+3}{4\alpha},$$
$$\left\{\lim_{t\to\infty}\left[t^{2m+7/2}\int_{\mathbb{R}} |(-\triangle)^{m+1}[u(x,t)-v(x,t)]|^2\mathrm{d}x\right]\right\}$$

$$/ \left\{ \lim_{t\to\infty} \left[t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m[u(x,t)-v(x,t)]|^2 \mathrm{d}x \right] \right\} = \frac{(4m+3)(4m+5)}{(4\alpha)^2},$$

for all real constants $m \geq 0$.

Proof. It follows from Theorem 1.3 and Lemma 2.12. □

2.10. The optimal decay estimates

The main purpose is to couple together the comprehensive analysis and the exact limits for all order derivatives of the global smooth solution to establish the following optimal decay estimates

$$t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \leq \mathcal{A}(\alpha,\delta,\varepsilon,m) + \mathcal{B}(\alpha,\delta,\varepsilon,m)t^{-1},$$
$$t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m [u(x,t)-v(x,t)]|^2 \mathrm{d}x \leq \mathcal{C}(\alpha,\delta,\varepsilon,m) + \mathcal{D}(\alpha,\delta,\varepsilon,m)t^{-1},$$

for all order derivatives of the global smooth solution, for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$ and for all sufficiently large t.

For convenience, let us define $\mathcal{I} = \mathcal{I}(m)$ and $\mathcal{J}$ by

$$\mathcal{I}(m) = \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta,$$
$$\mathcal{J} = \left\{ \int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t \right\}^2.$$

Recall that there hold the exact limits

$$\lim_{t\to\infty} \left\{ t^{m+3/2} \int_{\mathbb{R}} |(-\triangle)^m \psi(x,t)| \mathrm{d}x \right\} = \mathcal{L}(m),$$
$$\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \right\} = \mathcal{I}(m)\mathcal{J},$$

for all real constants $m \geq 0$.

By using the squeeze theorem, we have the exact limits

$$\lim_{t\to\infty} \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{m+3/2} \int_{\mathbb{R}} |(-\triangle)^m \psi(x,\tau)| \mathrm{d}x \right] \right\} = \mathcal{L}(m),$$
$$\lim_{t\to\infty} \left\{ \sup_{(1-\varepsilon)t\leq\tau\leq t} \left[\tau^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,\tau)|^2 \mathrm{d}x \right] \right\} = \mathcal{I}(m)\mathcal{J},$$

for all $m \geq 0$.

Therefore, there exists a sufficiently large positive constant T, such that

$$\sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{m+3/2} \int_{\mathbb{R}} |(-\triangle)^m \psi(x,\tau)| \mathrm{d}x \right\} \leq (1+\varepsilon_1)\mathcal{L}(m),$$
$$\sup_{(1-\varepsilon)t\leq\tau\leq t} \left\{ \tau^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,\tau)|^2 \mathrm{d}x \right\} \leq (1+\varepsilon_2)\mathcal{I}(m)\mathcal{J},$$

for all $t > T$, where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are sufficiently small positive constants. In particular, there hold the following estimates

$$\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{m+1+\frac{1}{4}(1+\delta)}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}\psi(x,\tau)|\mathrm{d}x\right\}$$
$$\leq(1+\varepsilon_1)\mathcal{L}(m-\frac{1}{2}+\frac{1}{4}(1+\delta)),$$

and

$$\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{3/2}\int_{\mathbb{R}}|u(x,\tau)|^2\mathrm{d}x\right\}\leq(1+\varepsilon_2)\mathcal{I}(0)\mathcal{J},$$
$$\sup_{(1-\varepsilon)t\leq\tau\leq t}\left\{\tau^{2m+1+\frac{1}{2}\delta}\int_{\mathbb{R}}|(-\triangle)^{m-\frac{1}{2}+\frac{1}{4}(1+\delta)}u(x,\tau)|^2\mathrm{d}x\right\}$$
$$\leq(1+\varepsilon_2)\mathcal{I}(m-\frac{1}{2}+\frac{1}{4}(1+\delta))\mathcal{J},$$

for all $t > T$.

Remark 2.7. Now the positive constants $\mathcal{B}=\mathcal{B}(\alpha,\delta,\varepsilon,m)$ and $\mathcal{D}=\mathcal{D}(\alpha,\delta,\varepsilon,m)$ may be derived clearly.

2.11. The completion of the proofs of the theorems

The proof of Theorem 1.1 may be finished by using the mathematical analysis in Subsection 2.5. □

The proof of Theorem 1.2 may be finished by using Lemma 2.5. □

The proof of Theorem 1.3 may be finished by the mathematical analysis in Subsection 2.7. □

The proof of Theorem 1.4 may be finished by coupling together the results of Theorem 1.3 and Lemma 2.14. □

The proof of Theorem 1.5 may be finished by coupling together the results of Lemma 2.3, Lemma 2.4, Theorem 1.3 and the mathematical analysis in Subsection 2.10. □

The proof of Theorem 1.6 may be finished by using Lemma 2.11. □

The proof of Theorem 1.7 may be finished by using Lemma 2.13. □

3. Conclusion and remarks

3.1. Summary

Consider the Cauchy problem for the general Korteweg-de Vries-Burgers equation

$$\frac{\partial}{\partial t}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial^3}{\partial x^3}u+\gamma\mathcal{H}\frac{\partial^2}{\partial x^2}u+\frac{\partial}{\partial x}\mathcal{N}(u)=f(x,t),$$
$$u(x,0)=u_0(x).$$

Also, consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v-\alpha\frac{\partial^2}{\partial x^2}v+\beta\frac{\partial^3}{\partial x^3}v+\gamma\mathcal{H}\frac{\partial^2}{\partial x^2}v=f(x,t),$$

$$v(x,0) = u_0(x).$$

This model equation generalizes the nonlinear Korteweg-de Vries-Burgers equation

$$\frac{\partial}{\partial t}u + \frac{\partial^3}{\partial x^3}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}\mathcal{N}(u) = f(x,t),$$

and the nonlinear Benjamin-Ono-Burgers equation

$$\frac{\partial}{\partial t}u + \mathcal{H}\frac{\partial^2}{\partial x^2}u - \alpha\frac{\partial^2}{\partial x^2}u + \frac{\partial}{\partial x}\mathcal{N}(u) = f(x,t).$$

In these equations, $\mathcal{H} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ represents the Hilbert operator, defined by the principal value of the singular integral

$$[\mathcal{H}\phi](x) = \frac{1}{\pi}\text{ P. V. }\int_{\mathbb{R}} \frac{\phi(y)}{x-y}\mathrm{d}y.$$

(A1) Suppose that the initial function and the external force satisfy the following conditions

$$u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}),$$
$$f \in C^\infty(\mathbb{R}\times\mathbb{R}^+) \cap L^1(\mathbb{R}\times\mathbb{R}^+) \cap L^1(\mathbb{R}^+, L^2(\mathbb{R})).$$

(A2) Suppose that there exist real scalar smooth functions

$$\phi \in C^1(\mathbb{R}) \cap L^1(\mathbb{R}),$$
$$\psi \in C^\infty(\mathbb{R}\times\mathbb{R}^+) \cap L^1(\mathbb{R}\times\mathbb{R}^+),$$

such that the initial function and the external force are given by

$$u_0(x) = \frac{\partial}{\partial x}\phi(x), \qquad f(x,t) = \frac{\partial}{\partial x}\psi(x,t),$$

for all $(x,t) \in \mathbb{R}\times\mathbb{R}^+$.

(A3) Suppose that there exist the following limits

$$\lim_{t\to\infty}\left\{t^{m+3/2}\int_{\mathbb{R}} |(-\triangle)^m\psi(x,t)|\mathrm{d}x\right\} \equiv \mathcal{L}(m),$$

for all real constants $m \geq 0$.

(A4) Suppose that the nonlinear function $\mathcal{N} = \mathcal{N}(u) \in C^\infty(\mathbb{R})$. There exists a positive constant $C > 0$, independent of u, such that

$$|\mathcal{N}(u)| \leq C(|u|^2 + |u|^5),$$

for all $u \in \mathbb{R}$.

Suppose that there exists the following limit

$$\lim_{u\to 0}\frac{\mathcal{N}(u)}{u^2} \equiv \mathcal{L} \neq 0,$$

for some real nonzero constant $\mathcal{L} \in \mathbb{R}$.

(A5) Suppose that there exists a global weak solution

$$u \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R})), \qquad \frac{\partial}{\partial x} u \in L^2(\mathbb{R} \times \mathbb{R}^+).$$

(A6) Suppose that there holds the following elementary decay estimate

$$\sup_{t>0} \left\{ t^{3/2} \int_{\mathbb{R}} |u(x,t)|^2 \mathrm{d}x \right\} < \infty.$$

We have accomplished the following results under these mathematical assumptions.

(1) There exists a unique global smooth solution

$$\begin{aligned} &u \in C^\infty(\mathbb{R} \times \mathbb{R}^+), \\ &u \in L^\infty(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \frac{\partial}{\partial x} u \in L^2(\mathbb{R}^+, H^{2m}(\mathbb{R})), \qquad \forall m > 0, \end{aligned}$$

to the Cauchy problem for the general Korteweg-de Vries-Burgers equation (1.1)-(1.2).

(2) There hold the following sharp rate decay estimates

$$\sup_{t>0} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \right\} < \infty,$$

for all positive constants $m > 0$.

(3) There hold the following explicit representations for the exact limits

$$\begin{aligned} &\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \right\} \\ &= \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \\ &\cdot \left\{ \int_{\mathbb{R}} \phi(x) \mathrm{d}x + \int_0^\infty \int_{\mathbb{R}} \psi(x,t) \mathrm{d}x\mathrm{d}t - \int_0^\infty \int_{\mathbb{R}} \mathcal{N}(u(x,t)) \mathrm{d}x\mathrm{d}t \right\}^2, \\ &\lim_{t\to\infty} \left\{ t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m [u(x,t) - v(x,t)]|^2 \mathrm{d}x \right\} \\ &= \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} |\eta|^{4m+2} \exp(-2\alpha|\eta|^2) \mathrm{d}\eta \right\} \left\{ \int_0^\infty \int_{\mathbb{R}} \mathcal{N}(u(x,t)) \mathrm{d}x\mathrm{d}t \right\}^2, \end{aligned}$$

for all order derivatives of the global smooth solution.

(4) There hold the following optimal decay estimates

$$\begin{aligned} &t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x \le \mathcal{A}(\alpha,\delta,\varepsilon,m) + \mathcal{B}(\alpha,\delta,\varepsilon,m) t^{-1}, \\ &t^{2m+3/2} \int_{\mathbb{R}} |(-\triangle)^m [u(x,t) - v(x,t)]|^2 \mathrm{d}x \le \mathcal{C}(\alpha,\delta,\varepsilon,m) + \mathcal{D}(\alpha,\delta,\varepsilon,m) t^{-1}, \end{aligned}$$

for all order derivatives of the global smooth solution, for all positive constants $0 < \delta < 4$ and $0 < \varepsilon < 1$ and for all sufficiently large t.

The four positive constants are given by

$$\begin{aligned}
&\mathcal{A}(\alpha,\delta,\varepsilon,m)\\
&= \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta \left\{\int_{\mathbb{R}} |\phi(x)|\mathrm{d}x\right\}^2\\
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{\int_0^\infty\int_{\mathbb{R}} |\psi(x,t)|\mathrm{d}x\mathrm{d}t\right\}^2\\
&+ \frac{5}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{\int_0^\infty\int_{\mathbb{R}} |\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\mathcal{B}(\alpha,\delta,\varepsilon,m)\\
&= C(m)\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\mathcal{L}\left(m-\frac{1}{2}+\frac{1}{4}(1+\delta)\right)\\
&+ C(m)\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot \left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+1+\delta}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty\int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^4,\\
&\mathcal{C}(\alpha,\delta,\varepsilon,m)\\
&= \frac{2}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+2}\exp(-2\alpha\varepsilon|\eta|^2)\mathrm{d}\eta \left\{\int_0^\infty\int_{\mathbb{R}} |\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\mathcal{D}(\alpha,\delta,\varepsilon,m)\\
&= C(m)\left\{\frac{1}{2\pi}\int_{\mathbb{R}} \frac{[1-\exp(-\alpha\varepsilon|\eta|^2)]^2}{\alpha^2|\eta|^{1+\delta}}\mathrm{d}\eta\right\}\\
&\cdot \left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^2\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot \left\{\frac{1}{2\pi}\int_{\mathbb{R}} |\eta|^{4m+1+\delta}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\
&\cdot \left\{\int_{\mathbb{R}} \phi(x)\mathrm{d}x + \int_0^\infty\int_{\mathbb{R}} \psi(x,t)\mathrm{d}x\mathrm{d}t - \int_0^\infty\int_{\mathbb{R}} \mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^4.
\end{aligned}$$

Let us make the main results clear. Let the nonlinear function $\mathcal{N}(u) = u^p$, where $p = 2, 3, 4, 5$, then all the results in

(1) Theorem 1.1,

(2) Theorem 1.2,

(3) Theorem 1.3,

(4) Theorem 1.4,

(5) Theorem 1.5

are true.

3.2. Remarks

Remark 3.1. (A) The exact limits

$$\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
&\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m[u(x,t)-v(x,t)]|^2\mathrm{d}x\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,
\end{aligned}$$

are increasing functions of m.

(B) The exact limits

$$\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
&\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m[u(x,t)-v(x,t)]|^2\mathrm{d}x\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,
\end{aligned}$$

are decreasing functions of the diffusion coefficient α.

(C) The exact limits

$$\begin{aligned}
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\
&\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,\\
&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m[u(x,t)-v(x,t)]|^2\mathrm{d}x\right\}\\
&=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,
\end{aligned}$$

are independent of

(1) any norm of any order derivative of the function u_0,

(2) any norm of any order derivative of the function f,

(3) any norm of any order derivative of the function $\mathcal{N}(u)$.

The exact limits depend only on

(1) the integral of the auxiliary function ϕ,
(2) the integral of the auxiliary function ψ,
(3) the integral of the nonlinear function $\mathcal{N}(u)$.

Remark 3.2. The exact limits

$$\begin{aligned}&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\}\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\\ &\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,\end{aligned}$$

of the general Korteweg-de Vries-Burgers equation (1.1) reduce to the exact limits

$$\begin{aligned}&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m v(x,t)|^2\mathrm{d}x\right\}\\ &=\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t\right\}^2,\end{aligned}$$

of the corresponding linear equation (1.3), if the nonlinear function $\mathcal{N}(u)$ is dropped.

Remark 3.3. Let us see why there exists the integral

$$\int_0^\infty\int_{\mathbb{R}}|\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t<\infty.$$

First of all, from the elementary energy equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x+2\alpha\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}u(x,t)\right|^2\mathrm{d}x=2\int_{\mathbb{R}}u(x,t)f(x,t)\mathrm{d}x,$$

for all $t>0$, we may easily obtain the uniform energy estimate

$$\begin{aligned}&\left\{\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x+2\alpha\int_0^t\int_{\mathbb{R}}\left|\frac{\partial}{\partial x}u(x,\tau)\right|^2\mathrm{d}x\mathrm{d}\tau\right\}^{1/2}\\ &\le\left\{\int_{\mathbb{R}}|u_0(x)|^2\mathrm{d}x\right\}^{1/2}+\int_0^\infty\left\{\int_{\mathbb{R}}|f(x,t)|^2\mathrm{d}x\right\}^{1/2}\mathrm{d}t.\end{aligned}$$

Secondly, we have the mathematical assumption

$$\sup_{t>0}\left\{t^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty.$$

Overall, we have the following estimates

$$C\equiv\sup_{t>0}\left\{(1+t)^{3/2}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\right\}<\infty,$$

and

$$\sup_{t>0}\left\{(1+t)^{1/2}\|u(\cdot,t)\|_{L^\infty}\right\}<\infty.$$

Recall that

$$|\mathcal{N}(u)|\le C(|u|^2+|u|^5),$$

for all $u\in\mathbb{R}$.

Therefore, the improper integral

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}|\mathcal{N}(u(x,t))|\mathrm{d}x\mathrm{d}t\\ \le{}& C\int_0^\infty\int_{\mathbb{R}}\left\{|u(x,t)|^2+|u(x,t)|^5\right\}\mathrm{d}x\mathrm{d}t\\ \le{}& C\int_0^\infty\left\{1+\|u(\cdot,t)\|_{L^\infty}^3\right\}\int_{\mathbb{R}}|u(x,t)|^2\mathrm{d}x\mathrm{d}t\\ \le{}& C\int_0^\infty\frac{1}{(1+t)^{3/2}}\mathrm{d}t<\infty,\end{aligned}$$

exists.

Remark 3.4. The results in Theorem 1.3 and Theorem 1.5 indicate that we may use the solution of the corresponding linear equation (1.3)-(1.4) to approximate the solution of the general Korteweg-de Vries-Burgers equation (1.1)-(1.2) very well.

Remark 3.5. The exact limits and the optimal decay estimates for all order derivatives of the global smooth solution will have strong influences in long time accurate numerical simulations.

Remark 3.6. Consider the Cauchy problem for the Benjamin-Bona-Mahony-Burgers equation

$$\begin{aligned}&\frac{\partial}{\partial t}u-\frac{\partial^3}{\partial x^2\partial t}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial}{\partial x}u+\frac{\partial}{\partial x}\mathcal{N}(u)=f(x,t),\\ &u(x,0)=u_0(x),\end{aligned}$$

and the Cauchy problem for the corresponding linear equation

$$\begin{aligned}&\frac{\partial}{\partial t}v-\frac{\partial^3}{\partial x^2\partial t}v-\alpha\frac{\partial^2}{\partial x^2}v+\beta\frac{\partial}{\partial x}v=f(x,t),\\ &v(x,0)=u_0(x).\end{aligned}$$

We have reasons to believe that there hold the same limits

$$\begin{aligned}&\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m u(x,t)|^2\mathrm{d}x\right\}\\ ={}&\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+2}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\\ &\cdot\left\{\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t-\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,\end{aligned}$$

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}}|(-\triangle)^m[u(x,t)-v(x,t)]|^2\mathrm{d}x\right\}$$
$$=\left\{\frac{1}{2\pi}\int_{\mathbb{R}}|\eta|^{4m+4}\exp(-2\alpha|\eta|^2)\mathrm{d}\eta\right\}\left\{\int_0^\infty\int_{\mathbb{R}}\mathcal{N}(u(x,t))\mathrm{d}x\mathrm{d}t\right\}^2,$$

for all real constants $m \geq 0$, under the same assumptions as in this paper.

3.3. Open problems

Open Problem 1: Consider the Cauchy problem for the following general Korteweg-de Vries-Burgers equations

$$\frac{\partial}{\partial t}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial^3}{\partial x^3}u+\gamma\mathcal{H}\frac{\partial^2}{\partial x^2}u+\frac{\partial}{\partial x}\mathcal{N}(u)=f(x,t),$$
$$u(x,0)=u_0(x).$$

Also, consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v+\frac{\partial^3}{\partial x^3}v-\alpha\frac{\partial^2}{\partial x^2}v=f(x,t),$$
$$v(x,0)=u_0(x).$$

Suppose that the initial function and the external force satisfy

$$u_0\in L^1(\mathbb{R})\cap L^2(\mathbb{R}),\qquad f\in L^1(\mathbb{R}\times\mathbb{R}^+)\cap L^1(\mathbb{R}^+,L^2(\mathbb{R})),$$
$$\int_{\mathbb{R}}u_0(x)\mathrm{d}x\neq 0,\qquad \int_0^\infty\int_{\mathbb{R}}f(x,t)\mathrm{d}x\mathrm{d}t\neq 0,$$
$$\int_{\mathbb{R}}\phi(x)\mathrm{d}x+\int_0^\infty\int_{\mathbb{R}}\psi(x,t)\mathrm{d}x\mathrm{d}t\neq 0.$$

However, we do not make the following assumption

$$u_0\in H^{2m}(\mathbb{R}),\qquad f\in L^2(\mathbb{R}^+,H^{2m}(\mathbb{R})),$$

where $m > 0$ is some positive constant.

The author has accomplished the main results for the cases

$$\mathcal{N}(u)=u^3,\qquad u^4,\qquad u^5,$$
$$\mathcal{N}(u)=u^3+u^4,\qquad u^3+u^4+u^5.$$

However, the main results are still open for the cases

$$\mathcal{N}(u)=u^2,\qquad u^2+u^3,\qquad u^2+u^3+u^4,$$
$$\mathcal{N}(u)=\sin(u^2),\qquad \arctan(u^2),\qquad \ln(1+u^2).$$

Open Problem 2: Consider the Cauchy problem for the general Benjamin-Bona-Mahony-Burgers equation

$$\frac{\partial}{\partial t}u-\frac{\partial^3}{\partial x^2\partial t}u-\alpha\frac{\partial^2}{\partial x^2}u+\beta\frac{\partial}{\partial x}u+\frac{\partial}{\partial x}\mathcal{N}(u)=f(x,t),$$
$$u(x,0)=u_0(x).$$

Also, consider the Cauchy problem for the corresponding linear equation

$$\frac{\partial}{\partial t}v - \frac{\partial^3}{\partial x^2 \partial t}v - \alpha\frac{\partial^2}{\partial x^2}v + \beta\frac{\partial}{\partial x}v = f(x,t),$$
$$v(x,0) = u_0(x).$$

See [5], [6], [7], [41], [42].

Can we establish the following limits

$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^m u(x,t)|^2 \mathrm{d}x\right\},$$
$$\lim_{t\to\infty}\left\{t^{2m+3/2}\int_{\mathbb{R}} |(-\triangle)^m [u(x,t)-v(x,t)]|^2 \mathrm{d}x\right\},$$

in terms of α, m and the following integrals

$$\int_{\mathbb{R}} u_0(x)\mathrm{d}x, \qquad \int_0^\infty \int_{\mathbb{R}} f(x,t)\mathrm{d}x\mathrm{d}t,$$

for all real constants $m \geq 0$? The nonlinear function

$$\mathcal{N}(u) = u^2, \qquad \alpha_0 u^2 + \beta_0 u^3, \qquad \gamma_0 \sin(u^2), \qquad \delta_0 \arctan(u^2), \qquad \varepsilon_0 \ln(1+u^2),$$

where $\alpha_0 \in \mathbb{R}$, $\beta_0 \in \mathbb{R}$, $\gamma_0 \in \mathbb{R}$, $\delta_0 \in \mathbb{R}$, $\varepsilon_0 \in \mathbb{R}$ are real nonzero constants.

We hope to solve the open problems in the future.

3.4. Some technical lemmas used

Lemma 3.1. *There holds the Parseval's identity*

$$\int_{\mathbb{R}} |\phi(x)|^2 \mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}} |\widehat{\phi}(\xi)|^2 \mathrm{d}\xi,$$

for all functions $\phi \in L^2(\mathbb{R})$.

Lemma 3.2. *For every positive constant* $m > 0$, *there exists a positive constant* $C(m) > 0$, *such that*

$$\left\{\int_{\mathbb{R}} |(-\triangle)^m [\phi(x)\psi(x)]|\mathrm{d}x\right\}^2$$
$$\leq C(m)\left\{\int_{\mathbb{R}} |\phi(x)|^2 \mathrm{d}x\right\}\left\{\int_{\mathbb{R}} |(-\triangle)^m \psi(x)|^2 \mathrm{d}x\right\}$$
$$+ C(m)\left\{\int_{\mathbb{R}} |(-\triangle)^m \phi(x)|^2 \mathrm{d}x\right\}\left\{\int_{\mathbb{R}} |\psi(x)|^2 \mathrm{d}x\right\},$$

for all functions $\phi \in H^{2m}(\mathbb{R})$ *and* $\psi \in H^{2m}(\mathbb{R})$.

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