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Stability of Phase Locking for Bidirectionally Non-symmetric Coupled Kuramoto Oscillators in a Ring*

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Abstract This paper deals with the stability of phase locking for the identical Kuramoto model, where each oscillator is influenced sinusoidally by two neighboring oscillators. By studying the model with bidirectionally non-symmetric coupling in a ring configuration, all phase-locked solutions are comprehensively delineated, and the basin of attraction for the stable phase-locked state is estimated. The stability of these phase-locked solutions is clearly established, highlighting that only the synchronized state and splay-state are stable equilibria. The crucial tools in this work are the standard linearization technique and the nonlinear analysis arguments.

Keywords Coupled oscillators, stability, phase locking, synchronization

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1. Introduction

Background.- The synchronization (in short, sync) process of large populations of weakly coupled oscillators often appears in natural systems and it has received considerable attention because of its application in diverse areas such as biology, neuroscience, engineering, computer science, economy and sociology [1,2,5,12,19]. Among the many mathematical models of sync, our interest in this paper lies in the Kuramoto model for identical oscillators with a bidirectionally non-symmetric topology. For identical Kuramoto oscillators with all-to-all coupling, a lot of studies have been done for this model, see [3,6,9,17,22]. It is well known that the phase sync is the only stable phase-locked state, which denotes the collapse of all phases into a single phase, see [14]. Hence, almost all initial configurations of phases converge to the phase sync asymptotically. It is reasonable to guess that different asymptotic patterns for Kuramoto oscillators can emerge depending on different network topologies. For example, in [21], Wei et al. consider the periodic sampled-data coupling in the scenario with a generally connected and undirected communication topology, where the connected communication topology means that no isolated oscillator exists in the Kuramoto oscillator network. In [7], Ferguson studies bifurcations in the Kuramoto model on a ring network using a novel vector flow and derives criteria for

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certain bifurcations. As network topologies are varied, different stable phase-locked states will emerge asymptotically, and hence it would be an interesting problem to classify all possible stable asymptotic states and initial configurations converging to a given stable equilibrium. This kind of question comes down to the identification of the basin of attractions in dynamical systems theory. There is some literature addressing this issue for the locally coupled Kuramoto model. The literature [14] studies the stability properties of the Kuramoto model with identical oscillators by linear stability analysis and the authors present a six-node example to point out that a stable non-sync equilibrium arises for oscillators bidirectionally coupled in a ring. In [20], Wiley et al. address the problem of "the size of the sync basin" for the locally coupled Kuramoto model with symmetric forward and backward k-neighbor coupling. They have found that when N is the number of oscillators and $\frac{k}{N}$ is greater than a critical value, then the phase sync is the only stable phase-locked state; as $\frac{k}{N}$ passes below this critical value, other stable phase-locked states are born, which takes the form of the splay-state. In [10, 15], the stability of phase-locking is considered for identical Kuramoto oscillators unidirectionally coupled in a ring. In [23], Zhao et al. consider the Kuramoto model with bidirectionally symmetric coupling network and use Łojasiewicz theory to prove the stability of phase-locked solutions. Dong et al. [4] study the interplay of time-delayed interactions and network structure on the collective behaviors of Kuramoto oscillators. In [11], an adaptive control law for inducing in- and antiphase sync in a pair of relaxation oscillators is proposed. Ito et al. show that the phase dynamics of the oscillators coupled by the control law can be reduced to the dynamics of Kuramoto phase oscillators, and they choose a ring topology to show the time series data of states and controlled thresholds for differential initial conditions.

In this work, we study the dynamical behavior of a finite group of Kuramoto oscillators bidirectionally non-symmetric coupled in a ring by performing nonlinear stability analysis. We consider the oscillators labeled as $1, 2, \ldots, N$, with each i-th oscillator coupled by the $(i+1)$-th and $(i-1)$-th oscillators sinusoidally, in which the first oscillator is coupled by the second and N-th oscillators; the last N-th oscillator is influenced by the 1-th and $(N-1)$-th oscillators. This situation appears in many engineering applications and biological modeling of animal locations [8, 13, 18]. For non-symmetric coupling, the total phase is not conserved quantities which causes considerable mathematical difficulty. Of course this lack of symmetry in the coupling makes the asymptotic dynamics of Kuramoto oscillators richer than that of the mean-field case, because there is room for the emergence of other stable phase-locked states other than sync.

Contributions.- The bidirectionally non-symmetric coupling strength makes it difficult to study the phase-locked states for the Kuramoto oscillators in a ring. The contributions of this paper are threefold. First, we identify the formation of all phase-locked states for system (2.3) (see Theorem 2.1). This will enable us to count and classify the phase-locked states. Second, we prove that the sync and splay-state are the only stable phase-locked state (see Theorem 2.2 and Section 3). Third, we present proper subsets of basins of sync and splay-state using nonlinear analysis arguments asymptotically, which says that a given initial configuration will converge to sync or splay-state (see Theorem 3.1).

Organization of paper.- In Section 2, we present the model system and list the formation of all phase-locked states and their stability properties. In Section 3, based on linearization technique and nonlinear analysis arguments, we strictly prove the stability and instability of all phase-locked states. Finally, Section 4 is devoted to providing a brief summary of this paper.

2. Description of model and main results

In this section, we consider $N(N \geq 3)$ coupled oscillators. First, we discuss the model and its deformation written by phase difference. Second, the definition of phase-locked solutions is given, and then the formation of all phase-locked solutions and their stability are listed.

2.1. Model

In this subsection, we present a sinusoidal coupling system in a ring with a bidirectionally non-symmetric structure. For $i = 1, 2, \ldots, N$, let $\theta_i = \theta_i(t) \in \mathbb{R}$ be the phase of the i-th oscillator. Assume that the i-th oscillator is influenced by the $(i+1)$-th and $(i-1)$-th oscillators only, and $\theta_0 := \theta_N, \theta_{N+1} := \theta_1$. In this circumstance, our governing system with identical oscillators for θ_i reads as

$$\begin{aligned} &\dot{\theta}_i = \Omega + K\sin(\theta_{i+1} - \theta_i) + K^{'}\sin(\theta_{i-1} - \theta_i), \\ &\theta_i(0) = \theta_{i0}, \quad i = 1, 2, \ldots, N, \quad t \geq 0, \end{aligned}$$

where $\dot{\theta}_i := \frac{d\theta_i}{dt}$, Ω represents the intrinsic natural frequency and parameters $K, K^{'}$ are two nonnegative coupling strengths. Throughout this paper, without loss of generality, we may assume that

$$\Omega = 0, \quad t \geq 0.$$

If necessary, we may consider the shifted frame $\theta_i \to \theta_i + \Omega t$. Then the time evolution of $\theta = (\theta_i, \theta_2, \ldots, \theta_N) \in \mathbb{R}^N$ is governed by the following system:

$$\begin{aligned} &\dot{\theta}_i = K\sin(\theta_{i+1} - \theta_i) + K^{'}\sin(\theta_{i-1} - \theta_i), \\ &\theta_i(0) = \theta_{i0}, \quad i = 1, 2, \ldots, N, \quad t \geq 0. \end{aligned} \tag{2.1}$$

Note that if $\theta_i(t), i = 1, 2, \ldots, N$ is the solution to the system (2.1), then its translation $\theta_i(t) + a, a \in \mathbb{R}, i = 1, 2, \ldots, N$, is also a solution, which shows that system (2.1) induces a dynamical system on N-tori $\mathbb{T}^N$.

As a simple case of (2.1), when $K = K^{'} = 0$, then $\theta(t) = \theta(0), t \geq 0$. Most aforementioned works focused on the unidirectionally coupled or the bidirectionally symmetric coupled. Ha et al. [10] and Rogge et al. [15] presented the long-time dynamics of unidirectionally coupled identical Kuramoto oscillators in a ring, when each oscillator is influenced sinusoidally by a single preassigned oscillator, i.e., $K > K^{'} = 0$. In [23], combining a nice theory for Łojasiewicz inequality with the gradient system, Zhao et al. studied the dynamics of bidirectionally coupled with $K = K^{'} > 0$. In the above two cases, the authors identified all the phase-locked states with stability or instability, estimated the basins for stable phase-locked states and showed the convergence rate towards phase-locked states. But these studies did not involve the bidirectionally non-symmetric coupled, i.e.,

$$K > K^{'} > 0 \quad \text{or} \quad K^{'} > K > 0.$$

In this paper, we consider the case of $K > K^{'} > 0$. The analysis for $K^{'} > K > 0$ is similar. In order to highlight that K is larger than $K^{'}$, it is advisable to set

$$K = K_1 + K_2, \quad K^{'} = K_2 \quad \text{and} \quad K_1, K_2 > 0.$$

Then system (2.1) takes the form of

$$\begin{aligned} &\dot{\theta}_i = (K_1 + K_2)\sin(\theta_{i+1} - \theta_i) + K_2 \sin(\theta_{i-1} - \theta_i), \\ &\theta_i(0) = \theta_{i0},\, i = 1, 2, \dots, N. \quad t \geq 0. \end{aligned} \tag{2.2}$$

For any $i = 1, 2, \dots, N$, we denote the phase differences by

$$\phi_i := (\theta_{i+1} - \theta_i) \mod 2\pi,$$

then there exists $k_i \in \mathbb{Z}$ such that $\theta_{i+1} - \theta_i = \phi_i + 2k_i\pi$. Without loss of generality, we may set

$$\phi_i \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right].$$

Hence $\phi_i = \frac{3\pi}{2}$ and $\phi_i = -\frac{\pi}{2}$ are the same, which will be encountered later. From (2.2), the variable $\phi = (\phi_1, \phi_2, \dots, \phi_N)$ satisfies

$$\begin{aligned} &\dot{\phi}_i = (K_1 + K_2)(\sin\phi_{i+1} - \sin\phi_i) - K_2(\sin\phi_i - \sin\phi_{i-1}), \\ &\phi_i(0) = \phi_{i0},\, i = 1, 2, \dots, N. \quad t \geq 0. \end{aligned} \tag{2.3}$$

Next we recall the definitions of phase-locked solutions to system (2.2).

Definition 2.1. Let θ and ϕ be the solutions to systems (2.2) and (2.3), respectively.
(1) θ is a synchronized state to (2.2) if and only if ϕ satisfies

$$\phi = \mathbb{0}_N.$$

(2) θ is the splay-state to (2.2) if and only if ϕ satisfies

$$\phi = \frac{2k\pi}{N}\mathbb{1}_N,\, k = 1, 2, \dots, N-1.$$

Here, $\mathbb{0}_N$ and $\mathbb{1}_N$ denote constant vectors in $\mathbb{T}^N$ defined by

$$\mathbb{0}_N = (0, 0, \dots, 0) \quad \text{and} \quad \mathbb{1}_N = (1, 1, \dots, 1).$$

2.2. Formation and stability of phase-locked states

In this part, we first show the equilibria of (2.3) which also gives the formation of phase-locked states for system (2.2). More specifically, each phase difference of (2.2) is one of the values β or $\pi - \beta$ with the value $\beta \in \mathbb{S}^1$.

Theorem 2.1. *Every equilibrium $\phi^* = (\phi_1^*, \phi_2^*, \dots, \phi_N^*)$ to system (2.3) corresponds to a permutation of the vector*

$$(\underbrace{\beta, \dots, \beta}_{m}, \underbrace{\pi - \beta, \dots, \pi - \beta}_{N-m}), \tag{2.4}$$

where $\beta \in (-\frac{\pi}{2}, \frac{3\pi}{2}]$ and $m \in \{0, 1, \dots, N\}$ satisfy $m\beta + (N-m)(\pi - \beta) = 2\pi k$ for some $k \in \mathbb{Z}$.

Proof. For any equilibrium $\phi^* = (\phi_1^*, \phi_2^*, \ldots, \phi_N^*)$ of the system (2.3), we see that

$$\begin{cases} \sin\phi_3^* - \sin\phi_2^* = \frac{K_2}{K_1+K_2}(\sin\phi_2^* - \sin\phi_1^*), \\ \sin\phi_4^* - \sin\phi_3^* = \frac{K_2}{K_1+K_2}(\sin\phi_3^* - \sin\phi_2^*) = (\frac{K_2}{K_1+K_2})^2(\sin\phi_2^* - \sin\phi_1^*), \\ \sin\phi_5^* - \sin\phi_4^* = \frac{K_2}{K_1+K_2}(\sin\phi_4^* - \sin\phi_3^*) = (\frac{K_2}{K_1+K_2})^3(\sin\phi_2^* - \sin\phi_1^*), \\ \qquad\vdots \\ \sin\phi_1^* - \sin\phi_N^* = \frac{K_2}{K_1+K_2}(\sin\phi_N^* - \sin\phi_{N-1}^*) = (\frac{K_2}{K_1+K_2})^{N-1}(\sin\phi_2^* - \sin\phi_1^*), \\ \sin\phi_2^* - \sin\phi_1^* = \frac{K_2}{K_1+K_2}(\sin\phi_1^* - \sin\phi_N^*) = (\frac{K_2}{K_1+K_2})^N(\sin\phi_2^* - \sin\phi_1^*). \end{cases} \tag{2.5}$$

It follows from the last formula that

$$\left[1 - (\frac{K_2}{K_1+K_2})^N\right](\sin\phi_2^* - \sin\phi_1^*) = 0.$$

From $K_1 > 0, K_2 > 0$, we obtain $\sin\phi_1^* = \sin\phi_2^*$. Plug this relation to (2.5) to find that

$$\sin\phi_1^* = \sin\phi_2^* = \cdots = \sin\phi_N^*.$$

$\phi_i^* \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$ implies that ϕ^* is a permutation of configuration (2.4). As $\sum_{j=1}^N(\theta_{j+1} - \theta_j) = 0$, we see that

$$0 = \sum_{j=1}^N(\phi_j^* + 2k_j\pi) = \sum_{j=1}^N\phi_j^* + 2\pi\sum_{j=1}^N k_j = m\beta + (N-m)(\pi-\beta) + 2\pi\sum_{j=1}^N k_j,\ k_j \in \mathbb{Z}.$$

Therefore, $m\beta + (N-m)(\pi-\beta) = 2\pi k$ for some $k \in \mathbb{Z}$. □

For symbol simplicity, denote

$$\phi^*(\beta, m) := (\underbrace{\beta, \ldots, \beta}_{m}, \underbrace{\pi-\beta, \ldots, \pi-\beta}_{N-m}), \text{ with } \beta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right],\ m \in \{0, 1, \ldots, N\}.$$

It is easy to see that when $\beta = \frac{\pi}{2}$, for any m, $\phi^*(\beta, m)$ can only be

$$(\frac{\pi}{2}, \frac{\pi}{2}, \ldots, \frac{\pi}{2});$$

when $\beta = \frac{3\pi}{2}$, for any m, $\phi^*(\beta, m)$ can only be

$$(\frac{3\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{3\pi}{2}).$$

Next, the main stability result is listed in Theorem 2.2.

Theorem 2.2. *The stability of the equilibrium $\phi^*(\beta, m)$ in system (2.3) is as follows.*
(1) *If $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, then*

- *when $m = N$, $\phi^*(\beta, m)$ is asymptotocally stable;*

- *when* $m \in \{0, 1, \dots, N-1\}$, $\phi^*(\beta, m)$ *is unstable;*

(2) *If* $\beta \in (\frac{\pi}{2}, \frac{3\pi}{2})$, *then*

- *when* $m = 0$, $\phi^*(\beta, m)$ *is asymptotocally stable;*
- *when* $m \in \{1, \dots, N\}$, $\phi^*(\beta, m)$ *is unstable;*

(3) *If* $\beta = \frac{\pi}{2}$ *or* $\frac{3\pi}{2}$, *that is,* $\phi^*(\beta, m)$ *is*

$$(\frac{\pi}{2}, \frac{\pi}{2}, \dots, \frac{\pi}{2}) \quad or \quad (\frac{3\pi}{2}, \frac{3\pi}{2}, \dots, \frac{3\pi}{2}),$$

then $\phi^*(\beta, m)$ *is unstable.*

3. Stability analysis

In this section, we aim at providing a rigorous proof of Theorem 2.2. First of all, we prove Theorem 2.2 (1) and (2) by using standard linearization technique; and then for Theorem 2.2 (3), we will prove that the initial value $\phi(0)$ near $(\frac{\pi}{2}, \frac{\pi}{2}, \dots, \frac{\pi}{2})$ or $(\frac{3\pi}{2}, \frac{3\pi}{2}, \dots, \frac{3\pi}{2})$ does not converge to $(\frac{\pi}{2}, \frac{\pi}{2}, \dots, \frac{\pi}{2})$ or $(\frac{3\pi}{2}, \frac{3\pi}{2}, \dots, \frac{3\pi}{2})$.

3.1. Linearization

The linearization about an equilibrium ϕ^* of system (2.3) is the Jacobian matrix

$$J_{\phi^*} = \begin{pmatrix} -(K_1+2K_2)\cos\phi_1^* & (K_1+K_2)\cos\phi_2^* & \dots & K_2\cos\phi_N^* \\ K_2\cos\phi_1^* & -(K_1+2K_2)\cos\phi_2^* & \dots & 0 \\ 0 & K_2\cos\phi_2^* & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (K_1+K_2)\cos\phi_N^* \\ (K_1+K_2)\cos\phi_1^* & 0 & \dots & -(K_1+2K_2)\cos\phi_N^* \end{pmatrix}.$$

One can observe that J_{ϕ^*} is an non-symmetric real matrix. Proposition 3.1 below indicates that, when $\phi^* = (\beta, \beta, \dots, \beta)$ with $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, J_{ϕ^*} is a specific Toeplitz matrix and ϕ^* is an asymptotically stable equilibrium of system (2.3).

Proposition 3.1. *If* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $m = N$, *then the equilibrium* $\phi^*(\beta, m)$ *is asymptotically stable of system* (2.3).

Proof. For $m = N$, the Jacobian matrix $J_{\phi^*(\beta,N)}$ is

$$\cos\beta \begin{pmatrix} -(K_1+2K_2) & (K_1+K_2) & 0 & 0 & \dots & K_2 \\ K_2 & -(K_1+2K_2) & (K_1+K_2) & 0 & \dots & 0 \\ 0 & K_2 & -(K_1+2K_2) & (K_1+K_2) & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & (K_1+K_2) \\ (K_1+K_2) & 0 & 0 & 0 & \dots & -(K_1+2K_2) \end{pmatrix}.$$

The desired conclusion is proved by calculating eigenvalues of matrix $J_{\phi^*(\beta,N)}$.

Set an $N\times N$ matrix as follows

$$A := \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

It is easy to verify that the characteristic polynomial of A is $|\lambda E - A| = \lambda^N - 1$, so the eigenvalue λ_k and eigenvector x_k are

$$\lambda_k = \cos\frac{2k\pi}{N} + j\sin\frac{2k\pi}{N},\ x_k = (1, \lambda_k, \lambda_k^2, \dots, \lambda_k^{N-1})^{\mathrm{T}},\ k = 0, 1, \dots, N-1.$$

Observing that $J_{\phi^*(\beta,N)}$ is a special Toeplitz matrix and can be represented by

$$J_{\phi^*(\beta,N)} = \cos\beta[-(K_1 + 2K_2)E + (K_1 + K_2)A + K_2A^{N-1}].$$

Hence, we have that

$$\lambda_0(J_{\phi^*(\beta,N)}) = 0,\ \mathrm{Re}\lambda_k(J_{\phi^*(\beta,N)}) = -\cos\beta(K_1+2K_2)(1-\cos\frac{2k\pi}{N}),\ k = 1, \dots, N-1.$$

The eigenvalue 0 is simple and it has an eigenvector $\mathbb{1}_N$, which is due to the global phase shift invariance of system (2.2). $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ implies $\mathrm{Re}\lambda_k(J_{\phi^*(\beta,N)}) < 0$, $k = 1, \dots, N-1$, and hence the asymptotic stability of $\phi^*(\beta, m)$ is obtained. □

Proposition 3.2. *If $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $m \in \{0, 1, \dots, N-1\}$, then the equilibrium $\phi^*(\beta, m)$ is unstable of system* (2.3).

Proof. The proof is divided into three parts. First, we show that the result holds when $m = \frac{N}{2}$. Second, a proof is given for $m \in \{0, 1, \dots, N-2\}$ and $m \neq \frac{N}{2}$. Finally, we consider the result of $m = N-1$.

(1) Owing to $m = \frac{N}{2}$, it is possible to perform a permutation of the phase differences such that $J_{\phi^*(\beta,m)}$ transforms into $-J_{\phi^*(\beta,m)}$. It has been shown that the eigenvalues of linearization are invariant under such a permutation, implying that the set of eigenvalues of $J_{\phi^*(\beta,m)}$ is equal to that of $-J_{\phi^*(\beta,m)}$. Hence, if λ is an eigenvalue of $J_{\phi^*(\beta,m)}$, then so is $-J_{\phi^*(\beta,m)}$.

$J_{\phi^*(\beta,m)}$ has some nonzero eigenvalues, since it is different from the null matrix ($\beta \neq \frac{\pi}{2}$). Gershgorin's theorem [16, Theorem 6.1.1] shows that these nonzero eigenvalues are not located on the imaginary axis. From this it can be concluded that at least one of the eigenvalues of $J_{\phi^*(\beta,m)}$ has a strictly positive real part. This proves the stability of the corresponding phase locking solution.

(2) In order to prove the case of $m \neq \frac{N}{2}$, we consider the characteristic polynomial of matrix $J_{\phi^*(\beta,m)}$:

$$|\lambda E - J_{\phi^*(\beta,m)}|$$

$$= \begin{vmatrix} \lambda + (K_1+2K_2)\cos\phi_1^* & -(K_1+K_2)\cos\phi_2^* & \dots & -K_2\cos\phi_N^* \\ -K_2\cos\phi_1^* & \lambda + (K_1+2K_2)\cos\phi_2^* & \dots & 0 \\ 0 & -K_2\cos\phi_2^* & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -(K_1+K_2)\cos\phi_N^* \\ -(K_1+K_2)\cos\phi_1^* & 0 & \dots & \lambda + (K_1+2K_2)\cos\phi_N^* \end{vmatrix}.$$

By adding all the other rows to the first row, and then extracting the common factor λ from the new first row, we obatin that

$$|\lambda E - J_{\phi^*(\beta,m)}|$$
$$= \lambda \underbrace{\begin{vmatrix} 1 & 1 & \dots & 1 \\ -K_2\cos\phi_1^* & \lambda + (K_1+2K_2)\cos\phi_2^* & \dots & 0 \\ 0 & -K_2\cos\phi_2^* & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -(K_1+K_2)\cos\phi_N^* \\ -(K_1+K_2)\cos\phi_1^* & 0 & \dots & \lambda + (K_1+2K_2)\cos\phi_N^* \end{vmatrix}}_{=:D(\lambda)}. \quad (3.1)$$

From (3.1), 0 is an eigenvalue of $J_{\phi^*(\beta,m)}$. Define other eigenvalues as $\lambda_1, \lambda_2, \dots, \lambda_{N-1}$. Then

$$|\lambda E - J_{\phi^*(\beta,m)}| = \lambda(\lambda-\lambda_1)\cdots(\lambda-\lambda_{N-1}) = \lambda\left[\lambda^{N-1} + \cdots + (-1)^{N-1}\prod_{i=1}^{N-1}\lambda_i\right],$$

which enables us to see that

$$D(\lambda) = \lambda^{N-1} + \cdots + (-1)^{N-1}\prod_{i=1}^{N-1}\lambda_i.$$

In particular, $D(0) = (-1)^{N-1}\prod_{i=1}^{N-1}\lambda_i$.

For $m \in \{0,1,\dots,N-2\}$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, we claim that when $D(0) \neq 0$, $\phi^*(\beta,m)$ is unstable. In fact, If there exist two or more phase differences belonging to $(\frac{\pi}{2}, \frac{3\pi}{2})$, two or more Gershgorin discs lie in the closed right half plane. By [15, Theorem 4], two or more eigenvalues are located in the closed right half plane. Since $D(0) \neq 0$, exactly one eigenvalue is zero, which implies that one or more eigenvalues have a strictly positive real part. So $\phi^*(\beta,m)$ is unstable.

For $m \in \{0,1,\dots,N-2\}$ and $m \neq \frac{N}{2}$, we note that $D(0)$ in (3.1) is

$$\begin{vmatrix} 1 & 1 & \dots & 1 & 1 \\ -K_2\cos\phi_1^* & (K_1+2K_2)\cos\phi_2^* & \dots & 0 & 0 \\ 0 & -K_2\cos\phi_2^* & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (K_1+2K_2)\cos\phi_{N-1}^* & -(K_1+K_2)\cos\phi_N^* \\ -(K_1+K_2)\cos\phi_1^* & 0 & \dots & -K_2\cos\phi_{N-1}^* & (K_1+2K_2)\cos\phi_N^* \end{vmatrix}.$$

For any $i = 1, 2, \dots, N$, extract the common factor $\cos\phi_i^*$ from the i-th column, and bring $\phi^*(\beta, m)$ into it to get

$$D(0) = \prod_{i=1}^{N} \cos\phi_N^* \begin{vmatrix} \frac{1}{\cos\phi_1^*} & \frac{1}{\cos\phi_2^*} & \dots & \frac{1}{\cos\phi_{N-1}^*} & \frac{1}{\cos\phi_N^*} \\ -K_2 & (K_1+2K_2) & \dots & 0 & 0 \\ 0 & -K_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (K_1+2K_2) & -(K_1+K_2) \\ -(K_1+K_2) & 0 & \dots & -K_2 & (K_1+2K_2) \end{vmatrix}$$

$$= (\cos\beta)^m(-\cos\beta)^{N-m}\frac{1}{\cos\beta} \begin{vmatrix} 1 & 1 & \dots & -1 & -1 \\ -K_2 & (K_1+2K_2) & \dots & 0 & 0 \\ 0 & -K_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (K_1+2K_2) & -(K_1+K_2) \\ -(K_1+K_2) & 0 & \dots & -K_2 & (K_1+2K_2) \end{vmatrix}.$$

By adding all the other columns to the first column, and then applying the Laplace expansion theorem (also known as the determinant expansion theorem) to the first column, we obtain that

$$D(0) = (-1)^{N-m}(\cos\beta)^{N-1} \begin{vmatrix} 2m-N & 1 & \dots & -1 & -1 \\ 0 & (K_1+2K_2) & \dots & 0 & 0 \\ 0 & -K_2 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & (K_1+2K_2) & -(K_1+K_2) \\ 0 & 0 & \dots & -K_2 & (K_1+2K_2) \end{vmatrix}$$

$$= (-1)^{N-m}(2m-N)(\cos\beta)^{N-1} \begin{vmatrix} (K_1+2K_2) & -(K_1+K_2) & \dots & 0 \\ -K_2 & (K_1+2K_2) & \dots & 0 \\ 0 & -K_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & -(K_1+K_2) \\ 0 & 0 & \dots & (K_1+2K_2) \end{vmatrix}$$

$$= (-1)^{N-m}(2m-N)(\cos\beta)^{N-1}\frac{x_1^N - x_2^N}{x_1 - x_2}, \tag{3.2}$$

where the final determinant is of order $(N-1)$, and x_1, x_2 $(x_1 > x_2 > 0)$ are the two different roots of $x^2 - (K_1+2K_2)x + K_2(K_1+K_2) = 0$. Thus, $m \neq \frac{N}{2}$ and $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ show $D(0) \neq 0$, leading to the conclusion.

(3) For $m = N-1$, equation (3.2) enables us to see that

$$D(0) = -(N-2)(\cos\beta)^{N-1}\frac{x_1^N - x_2^N}{x_1 - x_2},$$

in which x_1, x_2 are the same as in (3.2). Then we have $D(0) < 0$. Due to $m = N-1$, the phase locking solution possesses exactly one phase difference $\phi_i \in (\frac{\pi}{2}, \frac{3\pi}{2})$, exactly one Gershgorin disc lies in the closed half plane. At most one eigenvalue is positive. Recalling

$$D(0) = (-1)^{N-1}\prod_{i=1}^{N-1}\lambda_i < 0,$$

it follows

$$\mathrm{Re}(\lambda_i) \neq 0,\, i = 1, 2, \ldots, N-1 \quad \text{and} \quad \exists!\lambda_{i_0}, s.t.\, \mathrm{Re}(\lambda_{i_0}) > 0.$$

If not, we have $\mathrm{Re}(\lambda_i) < 0$, $i = 1, 2, \ldots, N-1$, which contradicts $D(0) < 0$. Therefore, $\phi^*(\beta, m)$ is unstable. □

Remark 3.1. It can be seen from Proposition 3.2 that in order to fully consider the possible value of N, a separate proof is provided for $m = \frac{N}{2}$. However, if N is an odd number, then $\frac{N}{2}$ is not an integer. At this point, the proof of Proposition 3.2 (1) is omitted.

We next complete the proof of Theorem 2.2 (1) − (2).

Proof of Theorem 2.2 (1)**.** The result can be obtained directly from Proposition 3.1 and Proposition 3.2.

Proof of Theorem 2.2 (2)**.** The result of $\beta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $m = 0$ can be directly obtained by that of $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $m = N$. Similarly, the result of $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $m \in \{0, 1, \ldots, N-1\}$ enable us to conclude that of $\beta \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $m \in \{1, 2, \ldots, N\}$.

3.2. Global dynamics

In this subsection, we prove Theorem 2.2 (3) and consider the global dynamics of system (2.3). Also, we present a non-trivial subset of the basin for the stable phase-locked state in (2.2). Define $U(\phi) = N - \sum_{i=1}^{N}\cos\phi_i$. The equation (2.3) means

$$\begin{aligned}
\frac{dU(\phi)}{dt} &= \sum_{i=1}^{N}\sin\phi_i\dot{\phi}_i \\
&= \sum_{i=1}^{N}\sin\phi_i\left[(K_1+K_2)(\sin\phi_{i+1} - \sin\phi_i) - K_2(\sin\phi_i - \sin\phi_{i-1})\right] \\
&= (K_1+K_2)\sum_{i=1}^{N}(\sin\phi_i\sin\phi_{i+1} - \sin^2\phi_i) + K_2\sum_{i=1}^{N}(\sin\phi_i\sin\phi_{i-1} - \sin^2\phi_i) \\
&= -\frac{K_1+K_2}{2}\sum_{i=1}^{N}(\sin^2\phi_{i+1} + \sin^2\phi_i - 2\sin\phi_{i+1}\sin\phi_i)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{K_2}{2}\sum_{i=1}^{N}(\sin^2\phi_i + \sin^2\phi_{i-1} - 2\sin\phi_i\sin\phi_{i-1}) \\
&= -\frac{K_1+K_2}{2}\sum_{i=1}^{N}(\sin\phi_{i+1} - \sin\phi_i)^2 - \frac{K_2}{2}\sum_{i=1}^{N}(\sin\phi_i - \sin\phi_{i-1})^2,
\end{aligned}
$$

where we use $\sum_{i=1}^{N}\sin^2\phi_{i+1} = \sum_{i=1}^{N}\sin^2\phi_i = \sum_{i=1}^{N}\sin^2\phi_{i-1}$ and $\phi_{N+1} = \phi_1, \phi_0 = \phi_N$. It can be noted that $\frac{dU(\phi)}{dt} \le 0$ in the torus and every trajectory goes to the set

$$
C = \{\frac{dU(\phi)}{dt} = 0\} = \{\sin\phi_{i+1} = \sin\phi_i, i = 1, 2, \dots, N\}, \tag{3.3}
$$

which contains the phase-locking solutions as its only invariants. Similar to the analysis in [14], we see that almost all the trajectories in the torus converge to one of the stable phase-locking solutions.

Let $\phi(t)$ be the smooth solution of system (2.3). For $\phi = (\phi_1, \phi_2, \dots, \phi_N)$, we introduce the indices

$$
M \in \arg\min_{i=1,2,\dots,N}\phi_i \quad \text{and} \quad m \in \arg\max_{i=1,2,\dots,N}\phi_i,
$$

which indicate $\phi_M = \max\{\phi_1, \phi_2, \dots, \phi_N\}$ and $\phi_m = \min\{\phi_1, \phi_2, \dots, \phi_N\}$. For time-varying configuration $\phi(t)$, the indices M and m depend on t and the extremal phase differences $\phi_M - \phi_m$ are Lipschitz continuous and piecewise differentiable.

Lemma 3.1. *Let ϕ be the smooth solution of system* (2.3) *with initial condition*

$$
\phi(0) \in B_{sp} := \left\{\phi \in \mathbb{R}^N : -\frac{\pi}{2} < \phi_m \le \phi_M < \frac{\pi}{2}\right\}.
$$

Then ϕ_M is noninceasing and ϕ_m is nondecreasing for $t \ge 0$.

Proof. • Step 1: Show that for some $T \in (0, \infty]$, if

$$
-\frac{\pi}{2} < \phi_m(t) \le \phi_M(t) < \frac{\pi}{2}, \quad t \in [0, T),
$$

then ϕ_M is noninceasing and ϕ_m is nondecreasing.

For $t \in [0, T)$, we see

$$
\begin{aligned}
-\frac{\pi}{2} < \frac{\phi_{M+1} - \phi_M}{2} \le 0, \qquad 0 \le \frac{\phi_{m+1} - \phi_m}{2} < \frac{\pi}{2}, \\
0 \le \frac{\phi_M - \phi_{M-1}}{2} < \frac{\pi}{2}, \qquad -\frac{\pi}{2} < \frac{\phi_m - \phi_{m-1}}{2} \le 0,
\end{aligned}
$$

and

$$
-\frac{\pi}{2} < \frac{\phi_{i+1} + \phi_i}{2} < \frac{\pi}{2}, \quad i = 1, 2, \dots, N.
$$

These inequalities indicate

$$
\begin{aligned}
\dot{\phi}_M(t) &= (K_1 + K_2)(\sin\phi_{M+1} - \sin\phi_M) - K_2(\sin\phi_M - \sin\phi_{M-1}) \\
&= 2(K_1 + K_2)\cos\left(\frac{\phi_{M+1} + \phi_M}{2}\right)\sin\left(\frac{\phi_{M+1} - \phi_M}{2}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - 2K_2 \cos\left(\frac{\phi_M + \phi_{M-1}}{2}\right) \sin\left(\frac{\phi_M - \phi_{M-1}}{2}\right) \\
&\le 0, \ \ t \in [0, T),
\end{aligned}
$$

and

$$
\begin{aligned}
\dot{\phi}_m(t) &= (K_1 + K_2)\left(\sin\phi_{m+1} - \sin\phi_m\right) - K_2\left(\sin\phi_m - \sin\phi_{m-1}\right) \\
&= 2(K_1 + K_2)\cos\left(\frac{\phi_{m+1} + \phi_m}{2}\right) \sin\left(\frac{\phi_{m+1} - \phi_m}{2}\right) \\
&\quad - 2K_2 \cos\left(\frac{\phi_m + \phi_{m-1}}{2}\right) \sin\left(\frac{\phi_m - \phi_{m-1}}{2}\right) \\
&\ge 0, \ \ t \in [0, T).
\end{aligned}
$$

Therefore, the function ϕ_M is nonincreasing and ϕ_m is nondecreasing for all $t \in [0, T)$.

• Step 2: Show that the maximal time interval with the aforementioned monotonicity property is in fact $T = \infty$.

Define the following set

$$
\mathcal{T} := \left\{ T > 0 \,\middle|\, -\frac{\pi}{2} < \phi_m(t) \le \phi_M(t) < \frac{\pi}{2}, \ \forall\, t \in [0, T) \right\}.
$$

Since $\phi(0) \in B_{sp}$ and the continuity of $\phi_i(t)$, there exists a sufficiently small $\delta > 0$ such that

$$
-\frac{\pi}{2} < \phi_m(t) \le \phi_M(t) < \frac{\pi}{2}, \quad t \in [0, \delta).
$$

Hence, $\mathcal{T} \ne \emptyset$ and $T_0 := \sup \mathcal{T}$ can be well defined. Next we claim that $T_0 = \infty$. Suppose not, i.e., $T_0 < \infty$. Then

$$
-\frac{\pi}{2} < \phi_m(t) \le \phi_M(t) < \frac{\pi}{2}, \quad t \in [0, T_0).
$$

Step 1 tells us that

$$
\phi_m(0) \le \phi_m(t) \le \phi_M(t) \le \phi_M(0), \quad t \in [0, T_0),
$$

and

$$
\lim_{t \to T_0^-} \phi_m(t), \ \lim_{t \to T_0^-} \phi_M(t) \text{ exist.}
$$

The definition of T_0 implies

$$
\text{either} \quad \lim_{t \to T_0^-} \phi_m(t) = -\frac{\pi}{2} \quad \text{or} \quad \lim_{t \to T_0^-} \phi_M(t) = \frac{\pi}{2},
$$

which is contradictory to

$$
\lim_{t \to T_0^-} \phi_m(t) \ge \phi_m(0) > -\frac{\pi}{2}, \quad \lim_{t \to T_0^-} \phi_M(t) \le \phi_M(0) < \frac{\pi}{2}.
$$

This leads to $T_0 = \infty$, showing the desired result. □

Theorem 3.1. *Let ϕ be the smooth solution of system* (2.3) *with initial condition* $\phi(0) \in B_{sp}$ *which satisfies*

$$\sum_{i=1}^{N} \phi_i(0) = 2k\pi \quad for\ some\ k \in \mathbb{Z}\ with - \frac{N}{4} < k < \frac{N}{4}.$$

Then

$$\lim_{t\to\infty} \phi(t) = \frac{2k\pi}{N} \mathbb{1}_N.$$

Proof. • Step 1: We refine the estimate in Lemma 3.1 and claim that ϕ_m is strictly increasing and ϕ_M is strictly decreasing.

Note that

$$\dot{\phi}_m(t) = (K_1 + K_2)(\sin\phi_{m+1} - \sin\phi_m) - K_2(\sin\phi_m - \sin\phi_{m-1}) \geq 0,\ t \in [0, \infty).$$

If ϕ_m is not strictly increasing, then we can find some open interval I such that

$$(K_1 + K_2)(\sin\phi_{m+1} - \sin\phi_m) - K_2(\sin\phi_m - \sin\phi_{m-1}) = 0,\ t \in I.$$

Then it follows from the graph of sinusoidal function on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ that we should have

$$\text{either} \quad \phi_m = \phi_{m-1} = \phi_{m+1} \quad \text{or} \quad \phi_{m+1} < \phi_m < \phi_{m-1}.$$

The second case certainly contradicts the definition of ϕ_m. So we have

$$\phi_m = \phi_{m-1} = \phi_{m+1},\ t \in I.$$

This implies that for $t \in I$,

$$\begin{aligned} 0 &= \dot{\phi}_m = \dot{\phi}_{m+1} \\ &= (K_1 + K_2)(\sin\phi_{m+2} - \sin\phi_{m+1}) - K_2(\sin\phi_{m+1} - \sin\phi_m) \\ &= (K_1 + K_2)(\sin\phi_{m+2} - \sin\phi_{m+1}), \end{aligned}$$

and $\phi_{m+2} = \phi_{m+1}$. Using the similar argument we can obtain

$$\phi_l = \phi_m, \quad for\ any\ \ t \in I\ \ and\ \ l = 1, 2, \ldots, N,$$

which implies that ϕ is an equilibrium. However, this contradicts the initial condition which is not an equilibrium. Thus, ϕ_m is strictly increasing. The strict decrease of ϕ_M can be proved by the similar argument.

• Step 2: Following Step 1, we have

$$-\frac{\pi}{2} < \phi_m(0) < \phi_m(t) \leq \phi_M(t) < \phi_M(0) < \frac{\pi}{2},\ t \in [0, \infty).$$

This implies that $\phi_m(t)$ and $\phi_M(t)$ converge as $t \to \infty$ and

$$-\frac{\pi}{2} < \phi_m(0) < \lim_{t\to\infty} \phi_m(t) \leq \lim_{t\to\infty} \phi_M(t) < \phi_M(0) < \frac{\pi}{2}. \tag{3.4}$$

Combining (3.3) and Theorem 2.1, we see that the solution of (2.3) converges to some phase-locking with ϕ_i being β or $\pi - \beta$ for some $\beta \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$. Hence,

$$\lim_{t\to\infty} \phi_m(t), \lim_{t\to\infty} \phi_M(t) \in \{\beta, \pi - \beta\}.$$

Due to

$$\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Leftrightarrow \pi - \beta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

we invoke the relation (3.4) and use the Squeeze theorem to find that

$$\lim_{t\to\infty} \phi_i(t) = \beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \, i = 1, 2, \ldots, N.$$

It follows from the conservation law

$$\sum_{i=1}^{N} \phi_i(t) = \sum_{i=1}^{N} \phi_i(0) = 2k\pi, \; t \geq 0, k \in \mathbb{Z}$$

that

$$N\beta = \lim_{t\to\infty} \sum_{i=1}^{N} \phi_i(t) = 2k\pi, k \in \mathbb{Z} \quad \text{i.e.,} \quad \beta = \frac{2k\pi}{N}, k \in \mathbb{Z}.$$

Therefore, the desired result is obtained. □

Proof of Theorem 2.2 (3)**.** From Theorem 3.1, the initial value $\phi(0)$ near $(\frac{\pi}{2}, \frac{\pi}{2}, \ldots, \frac{\pi}{2})$ converges to $\frac{2k\pi}{N}\mathbb{1}_N$, so the equilibrium $(\frac{\pi}{2}, \frac{\pi}{2}, \ldots, \frac{\pi}{2})$ is unstable.

From $\phi_i \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $\phi_i = \frac{3\pi}{2}$ and $\phi_i = -\frac{\pi}{2}$ are the same. By Theorem 3.1, the initial value $\phi(0)$ near $(-\frac{\pi}{2}, -\frac{\pi}{2}, \ldots, -\frac{\pi}{2})$ converges to $\frac{2k\pi}{N}\mathbb{1}_N$, so the equilibrium $(\frac{3\pi}{2}, \frac{3\pi}{2}, \ldots, \frac{3\pi}{2})$ is also unstable.

Remark 3.2. (1) Suppose that the initial configuration $\phi(0)$ satisfies the conditions in Theorem 3.1 with $k \neq 0$. Then the trajectory $\theta(t)$ of system (2.2) converges to a splay-state.
(2) If the initial configuration $\phi(0)$ satisfies

$$\phi_i(0) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), i = 1, 2, \ldots, N \quad \text{and} \quad \sum_{i=1}^{N} \phi_i(0) = 0,$$

then we have $k = 0$ in Theorem 3.1, which indicates

$$N\beta = \lim_{t\to\infty} \sum_{i=1}^{N} \phi_i(t) = 0.$$

Then $\beta = 0$ means that the trajectory $\theta(t)$ of system (2.2) converges to a phase sync state.

Remark 3.3. Let $\theta(t)$ be the smooth solution of system (2.2) with initial condition

$$-\frac{\pi}{2} < \min_{i=1,2,\ldots,N} (\theta_{i+1}(0) - \theta_i(0)) \mod 2\pi \leq \max_{i=1,2,\ldots,N} (\theta_{i+1}(0) - \theta_i(0)) \mod 2\pi < \frac{\pi}{2},$$

then

$$\begin{aligned}
\lim_{t\to\infty} \dot{\theta}_i(t) &= \lim_{t\to\infty} [(K_1 + K_2)\sin(\theta_{i+1} - \theta_i) + K_2 \sin(\theta_{i-1} - \theta_i)] \\
&= \lim_{t\to\infty} [(K_1 + K_2)\sin\phi_i - K_2 \sin\phi_{i-1}] \\
&= K_1 \sin\frac{2k\pi}{N},
\end{aligned}$$

where the constant $k = \sum_{i=1}^{N} k_i$ and $\phi_i = \theta_{i+1} - \theta_i + 2k_i\pi$.

4. Conclusion

This work introduces the dynamical properties of identical Kuramoto oscillators that are bidirectionally non-symmetric coupled in a ring configuration. Through the definition of phase difference, the formation of all phase-locked states is described, and the almost global stability of the stable phase-locking solution is shown. Additionally, estimations for the basins of attraction for these stable phase-locked states are provided. These results enrich the study of the non-symmetric coupled, and in addition, k-nearest-neighbor couplings will be an interesting problem to be invested in future work.

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