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Dynamics of an n-Patch Predator-Prey Model with Allee Effect*

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Abstract A class of n-patch predator-prey diffusion models with the Allee effect is established. The influence of the Allee effect and diffusion of prey on the existence and stability of the equilibrium point are investigated. Firstly, sufficient conditions for the permanence and extinction of the system are analyzed. Secondly, by constructing a new Lyapunov function in terms of graph theory, we obtain a sufficient condition of the global asymptotical stability for the positive equilibrium point. Finally, our results of this paper are verified by Matlab simulation.

Keywords n-patch, stability, Allee effect, predation-prey model

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1. Introduction

Considering the influence of predator and prey diffusion behavior, Levin [1] first established the population dynamics model in a patch environment, and studied the related problems of predator-prey diffusion model [2–9]. Up to now, the dynamics problem of patch predator-prey model has received lots of attention. For example, Kuang and Takeuchi [2] investigated the following autonomous predator-prey systems

$$\begin{aligned} \dot{x}_1 &= x_1 g_1(x_1) - y p_1(x_1) + d(x_2 - x_1), \\ \dot{x}_2 &= x_2 g_2(x_2) - y p_2(x_2) + d(x_1 - x_2), \\ \dot{y} &= y[-s(y) + c_1 p_1(x_1) + c_2 p_2(x_2)]. \end{aligned}$$

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They obtained sufficient conditions for permanence, local stability and global stability of the system. After this, many scholars have extended Kuang and Takeuchi's model to consider the influence of impulse [7], fear effect [8], age structure [9] and other factors on the patch predator-prey model. Since the dynamic analysis in [2–9] is mainly concentrated in the patch environment, the patch predator-prey model for high-dimensional systems is not applicable. Li and Shuai [10] considered the n-patch predator-prey model. In [10], the authors used the results of graph theory to construct the global Lyapunov function of a large-scale coupled system from a single vertex system, and then obtained the stability of the system.

In addition, population density is limited by the environment. Excessive sparseness or overcrowding can inhibit the growth of the population, which means that the species has the range for the population growth. This phenomenon is called the Allee effect [11–14]. In this paper, we assume that the prey growth rate is affected by the Allee effect, so in the absence of predators, the per capita growth rate function becomes $g(x) = r\left(1 - \frac{x}{k}\right)(x + m)$, where m is the Allee effect parameter. Recently, Pal, Samanta [15] and Saha [16] studied the diffusion dynamics of a predator-prey system with a strong Allee effect in a two-patch environment. In [15, 16], they obtained the existence and stability criteria of the positive equilibrium point in the presence and absence of diffusion. Biswas and Pal [17] introduced the strong Allee effect into the three-patch model. Biswas and Pal showed the dynamics and the asymptotic behavior of the system and proved the occurrence of Hopf bifurcation. The Allee effect plays an important role in the dynamic behavior of the predator prey system. However, there are few results to deal with the weak Allee effect of n-path predator-prey.

Motivated by the above discussions, we discuss the weak Allee effect of n-path predator-prey in this paper. By using Li and Shuai's technique of constructing Lyapunov function, the conditions for the stability of the positive equilibrium point of the n-patch predator-prey model are given. We also provide examples to demonstrate the effectiveness of the proposed stability results. It also shows how the weak Allee effect determines the existence and stability of positive equilibrium.

This paper is organized as follows: In Section 2, our mathematical model of n-patch predator-prey with Allee effect is presented and some preliminaries are given. In Section 3, the main results for both permanence and asymptotically stability of n-patch predator-prey model with the weak Allee effect are proposed. An effective numerical simulation is presented in Section 4 to illustrate the main results. Concluding remarks are collected in Section 5.

2. The model and preliminaries

In this paper, we consider the following system

$$\begin{cases} \frac{dx_i}{dt} = r_i(1 - \frac{x_i}{k_i})(x_i + m_i)x_i - b_i {x_i}^2 - q_i x_i y + \sum\limits_{l=1}^{n} d_{il}(x_l - x_i), \\ \frac{dx_n}{dt} = r_n(1 - \frac{x_n}{k_n})x_n + \sum\limits_{l=1}^{n} d_{nl}(x_l - x_n), \\ \frac{dy}{dt} = -sy + y\sum\limits_{i=1}^{n-1} e_i x_i . i = 1, 2, ..., n-1. \end{cases} \tag{2.1}$$

In the system, $n \geq 2$, $x_l(t)$ is the number of prey populations in the $l, l = 1, 2, ..., n$, patch at time t; $y(t)$ is the number of predator populations at time t; r_l, k_l are the intrinsic growth rate and environmental capacity of the prey population in the l patch, respectively. m_i is the Allee effect coefficient of the prey population in the $i, i = 1, 2, ..., n-1$ patch; m_i quantifies the intensity of the Allee effect so that it is weak with $0 < m_i < k_i, i = 1, 2, ..., n-1$. b_i the density restriction factors of the population affected by food and environment; q_i is the proportion coefficient of the amount of prey eaten by predators per unit time proportional to the number of prey population, and e_i represents the predator's prey intake rate. s is the mortality rate of the predator population; $d_{jl}, j, l = 1, 2, ..., n$ is the constant diffusivity of the prey population from patch l to patch j.

Therein, $\frac{q_1}{e_1} = \frac{q_2}{e_2} = ... = \frac{q_i}{e_i}$ indicates that the predator has the same conversion rate for the same prey; $d_{jl} = d_{lj}, j, l = 1, 2, ..., n$ indicates that the migration constant from patch j to patch l is equal to the migration constant from patch l to patch j; $d_{ll} = 0$ indicates that the migration constant from patch l to patch l is 0.

Define vector

$$(\mathbf{X}(t), y(t)) = (x_1(t), x_2(t), ..., x_n(t), y(t))^T \in R_+^{n+1}.$$

$I_0 = (x_1(0), x_2(0), ..., x_n(0), y(0))$, and assume that I_0 satisfies nonnegative initial conditions $x_l(0), y(0) > 0, l = 1, 2, ..., n$. According to the existence and uniqueness theorem of solution of the system of ordinary differential equations in literature [18], there exists a unique solution satisfying the condition.

Definition 2.1. The weight matrix of a weighted digraph G with n vertices is $D = (d_{lj})_{n\times n}$ and satisfies $d_{lj} \geq 0$. A weighted digraph (G, D) is strongly connected if and only if the weighted matrix D is irreducible. Furthermore the Laplace matrix $L = [l_{ij}]$ of a weighted digraph (G, D) with a strongly connected graph is as follows:

$$l_{lj} = \begin{cases} -d_{lj}, & l \neq j, \\ \sum_{k\neq l} d_{lk}, & l = j. \end{cases}$$

Definition 2.2. Let $(\mathbf{X}(t), y(t))$ be a solution to system (2.1) if there are constants $0 < c_i < C_i, i = 1, 2$ for any initial value $I_0 \in intR_+^n$ such that

$$0 < c_1 \leq \liminf_{t\to+\infty} x_l(t, x_l(0)) \leq \limsup_{t\to+\infty} x_l(t, x_l(0)) \leq C_1,$$

and

$$0 < c_2 \leq \liminf_{t\to+\infty} y(t, y(0)) \leq \limsup_{t\to+\infty} y(t, y(0)) \leq C_2,$$

then it is called the permanence of the system (2.1).

Lemma 2.1 (Theorem2.2, [10])**.** *Assume $n \geq 2$. If*

$$\gamma_l = \sum_{\mathcal{T}\in\mathbb{T}_l} w(\mathcal{T}),\ l = 1, 2, \ldots, n, \tag{2.2}$$

then

$$\sum_{l,j=1}^{n} \gamma_l a_{lj} G_l(x_l) = \sum_{l,j=1}^{n} \gamma_l a_{lj} G_j(x_j), \tag{2.3}$$

where $\mathbb{T}_l$ is the set of all spanning trees $\mathcal{T}$ of (G, D) that are rooted at vertex l, and $w(\mathcal{T})$ is the weight of $\mathcal{T}$. In particular, if (G, D) is strongly connected, then $\gamma_l > 0$ for $1 \leq l \leq n$, $G_l(x_l)$ is arbitrary functions.

Lemma 2.2 (Theorem6.3, [19]). *Suppose that permanence holds for the set of difference equations or the set of differential equations. Then there exists an equilibrium point in* $intR_+^n$.

Lemma 2.3. *Every solution* $(\boldsymbol{X}(t), y(t))$ *with initial conditions* $x_l(0), l = 1, 2, ..., n$, $y(0) > 0$, *remains positive for all* $t > 0$.

Proof. System (2.1) is equivalent to system (2.4)

$$\begin{cases} \frac{dX}{dt} = g(t, X, y), \\ \frac{dy}{dt} = g_{n+1}(t, X, y), \end{cases} \tag{2.4}$$

where $X = (x_1, ..., x_{n-1}, x_n)^T, g(t, X, y) = (g_1(t, X, y), ..., g_{n-1}(t, X, y), g_n(t, X, y))^T$, $g_i(t, X, y) = r_i(1 - \frac{x_i}{k_i})(x_i + m_i)x_i - b_i{x_i}^2 - q_i x_i y + \sum\limits_{l=1}^{n} d_{il}(x_l - x_i), i = 1, 2, ..., n-1$,

$g_n(t, X, y) = r_n(1 - \frac{x_n}{k_n})x_n + \sum\limits_{l=1}^{n} d_{nl}(x_l - x_n), g_{n+1}(t, X, y) = -sy + y\sum\limits_{i=1}^{n-1} e_i x_i$.

When $x_i = 0$, and $x_n, y \geq 0$, $g_i(t, X, y) = \sum\limits_{l=2}^{n} d_{il}x_l \geq 0$; when $x_n = 0$, and $x_i, y \geq 0$, $g_n(t, X, y) = \sum\limits_{l=1}^{n-1} d_{nl}x_l \geq 0$; when $y = 0$, and $x_i, x_n \geq 0$, $g_{n+1}(t, X, y) = 0$.

We know from Proposition B.7 [20] that when $x_l(0) > 0, y(0) > 0$, the solution of system (2.1) satisfying positive initial value can remain constant positive. □

3. Main results

3.1. Permanence and extinction

In this section, we discuss the boundedness of solutions, the permanence and extinction of the system. The existence of positive equilibrium point is analyzed.

Theorem 3.1. *Every solution of the system equation (2.1) is uniformly bounded in* R_+^{n+1}.

Proof. We define

$$V_1 = \max\{x_l\}, l = 1, ..., n.$$

Calculating the upper right derivative of V_1 along the positive solution of system (2.1), we have

(1) If $V_1(t) = x_i(t), i = 1, ..., n-1$, then

$$\begin{aligned} D^+V_1(t) &= \dot{x}_i(t) \\ &\leq x_i(t)r_i(1 - \tfrac{x_i(t)}{k_i})(x_i(t) + m_i) \\ &\leq V_1(t)r_i(1 - \tfrac{V}{k_i})(V_1(t) + m_i).\ (i = 1, ..., n-1). \end{aligned}$$

(2) If $V_1(t) = x_n(t)$, then

$$\begin{aligned} D^+V_1(t) &= \dot{x}_n(t) \\ &\leq x_n(t)r_n(1 - \tfrac{x_n(t)}{k_n}) \\ &\leq V_1(t)r_n(1 - \tfrac{V_1(t)}{k_n}). \end{aligned}$$

Therefore,

$$\limsup_{t\to+\infty} V_1(t) \le \max\{k_l\}. \tag{3.1}$$

From(3.1), we derive $C_1 = \max\{k_l\}$, $l = 1, \ldots, n$.

$$\limsup_{t\to+\infty} V_1(t) \le C_1. \tag{3.2}$$

We can obtain

(a) If $V_1(0) = \max\{x_l(0)\} \le C_1$, then $\max\{x_l(t)\} \le C_1, t > 0$.

(b) If $V_1(0) = \max\{x_l(0)\} > C_1$, then let $\alpha > 0$,

$$-\alpha = \max\{C_1 r_i(1 - \frac{C_1}{k_i})(C_1 + m_i), C_1 r_n(1 - \frac{C_1}{k_n})\}.$$

If $V_1(0) = x_l(0) > C_1$ holds, then there exists $\varepsilon > 0$, such that if $t \in [0, \varepsilon)$, $V_1(t) = x_l(t) > C_1$ and we have

$$D^+V_1(x_l(t)) = \dot{x}_l(t) < -\alpha < 0\ , l = 1, ..., n.$$

So there exists $T_1 > 0$ if $t > T_1$, and we have

$$V_1(t) = \max\{x_l(t)\} \le C_1. \tag{3.3}$$

In addition,we define for a solution $(X(t), y(t))$ of system (2.1),

$$U_1(t) = U_1(X, y) \equiv A_1 \sum_{l=1}^{n} x_l(t) + B_1 y(t), \tag{3.4}$$

where $B_1 = \frac{A_1 q_1}{e_1}$, $\theta_n = \frac{k_n(r_n+1)}{2r_n}$, $A_1 = \frac{1}{\sum\limits_{i=1}^{n-1} \theta_i r_i(1-\frac{\theta_i}{k_i})(\theta_i+m_i)-b_i\theta_i{}^2+\theta_i+\theta_n r_n(1-\frac{\theta_n}{k_n})+\theta_n}$, $\theta_i = \frac{r_i k_i - r_i m_i - b_i k_i + \sqrt{r_i{}^2 m_i(m_i+k_i)+r_i k_i(r_i k_i+3)+b_i k_i(k_i-2r_i(k_i-m_i))}}{3r_i}$.
Then we have

$$\begin{aligned}
\dot{U}_1(t) &= A_1 \sum_{l=1}^{n} \dot{x}_l(t) + B_1 \dot{y}(t) \\
&= A_1[\sum_{i=1}^{n-1} x_i r_i(1 - \tfrac{x_i}{k_i})(x_i + m_i) - b_i x_i{}^2 + x_n r_n(1 - \tfrac{x_n}{k_n})] - B_1 s y \\
&\le A_1 F_1 - [A_1 \sum_{l=1}^{n} x_l + B_1 s y].
\end{aligned}$$

Now, if we take $F_1 = \sum\limits_{i=1}^{n-1} x_i r_i(1 - \frac{x_i}{k_i})(x_i + m_i) - b_i x_i{}^2 + x_i + x_n r_n(1 - \frac{x_n}{k_n}) + x_n$, then the value of F_1 is maximum at $(\theta_1, \theta_2, ..., \theta_n)$ and $F_{1\max} = \frac{1}{A_1} > 1$.

Therefore, $\dot{U}_1 \le A_1 F_{1\max} - \alpha_1 U_1$ where $\alpha_1 = \min\{1, s\}$. This implies

$$\dot{U}_1 \le 1 - \alpha_1 U_1$$

and we get

$$\limsup_{t\to+\infty} U_1(t) \le \frac{1}{\alpha_1}.$$

Because of (3.4),

$$\limsup_{t\to+\infty} B_1 y(t) = \limsup_{t\to+\infty}(U_1(t) - A_1 \sum_{l=1}^{n} x_l(t)) \le \lim_{t\to+\infty} \sup U_1(t) \le \frac{1}{\alpha_1}.$$

So there exist a positive constant $C_2 = \frac{1}{B_1\alpha_1}$ such that

$$\lim_{t\to+\infty} \sup y(t) \le C_2. \tag{3.5}$$

From (3.3) and (3.5) we take $M = \max\{C_1, C_2\}$, so

$$x_l(t) \le M,\ l = 1,2,...,n,\ y(t) \le M.$$

Hence every solution of the system equation (2.1) is uniformly bounded in R_+^{n+1}. □

Theorem 3.2. *If positive constants $C_1, C_2, and\ \mu$ exist such that the following conditions are satisfied:*

(i) $r_i m_i - q_i C_2 > 0$*;*

(ii) $\max\{k_l\} \le C_1 \le \min\{k_i(\frac{r_iC_1+r_im_i+1}{r_iC_1+r_im_i+b_ik_i}), k_n(1+\frac{1}{r_n})\}.i = 1,...,n-1$*;*

(iii) $s < \sum_{i=1}^{n-1} e_i\mu$*,*

then the system (2.1) is said to be permanence, that is, the system has a positive equilibrium.

Proof. Suppose that $(X(t), y(t))$ is a solution of system (2.1) which satisfies $I_0 > 0$. We define

$$V_2(t) = \min\{x_l\}.$$

According to Theorem 3.1, there exists C_2 such that

$$\limsup_{t\to+\infty} y(t) \le C_2.$$

In accordance with Lemma 2.3 if $t > T$ and condition (i) holds, then by calculating the lower right derivative of $V_2(t)$ along the positive solution of system (2.1), we obtain

$$D_+V_2(t) = \dot{x}_l(t) \ge \begin{cases} x_i(t)[-\frac{r_i}{k_i}x_i^2(t) + (r_i - \frac{r_im_i}{k_i} - b)x_i(t) + r_im_i - q_iC_2], \\ x_n(t)[r_n - \frac{r_n}{k_n}x_n(t)]. \end{cases}$$
$$= \begin{cases} x_i(t)[(x_i(t) + \bar{\mu}_{i1})(\mu_{i2} - x_i(t))], i = 1,2,...,n-1. \\ x_n(t)[r_n - \frac{r_n}{k_n}x_n(t)], \end{cases}$$

where

$$\Delta_i = (r_i - \frac{r_im_i}{k_i} - b_i)^2 + 4\frac{r_i}{k_i}(r_im_i - q_iC_2) > 0,\ \bar{\mu}_{i1} = |\mu_{i1}|,$$

$$\mu_{i1} = \frac{r_i k_i - r_i m_i - b_i k_i - k_i\sqrt{\Delta_i}}{2r_i},\ \mu_{i2} = \frac{r_i k_i - r_i m_i - b_i k_i + k_i\sqrt{\Delta_i}}{2r_i} > 0.$$

We get

$$\liminf_{t\to+\infty} x_i(t) \geq \mu, \tag{3.6}$$

where $\mu = \min\{\mu_{i2}\}, i = 1, \ldots, n-1$.

Let $c_1 = \min\{\mu, k_n\}$. So

$$\liminf_{t\to+\infty} V_2(t) \geq c_1.$$

We can obtain

(a)If $V_2(0) = \min\{x_l(0)\} \geq c_1$, then $\min\{x_l(t)\} \geq c_1, t > 0$.

(b)If $V_2(0) = \min\{x_l(0)\} < c_1$, then let

$$u = \min\{x_i(0)[(c_1 + \bar{\mu}_{i1})(\mu_{i2} - c_1)], x_n(0)[r_n - \frac{r_n}{k_n} c_1]\}.$$

If $V_2(0) = x_l(0) < c_1$ holds, then there exists $\varepsilon > 0$, such that if $t \in [0, \varepsilon)$,$V_2(t) = x_l(t)$ and we have

$$D_+V_2(x_l(t)) = \dot{x}_l(t) > u > 0,\ l = 1, 2, ..., n.$$

Furthermore, there exists $\bar{T}_1 > 0$ if $t > \bar{T}_1$, and we have $V_2(t) = \min\{x_l(t)\} > c_1$. On the whole

$$\liminf_{t\to+\infty} x_l(t) \geq c_1 > 0, l = 1, 2, ..., n.$$

Hence we let

$$U_2(t) = A_2 \sum_{l=1}^{n} x_l(t) + B_2 y(t).$$

Then

$$\begin{aligned}
\dot{U}_2(t) &= A_2 \sum_{l=1}^{n} \dot{x}_l(t) + \dot{B}_2 y(t) \\
&= A_2[\sum_{i=1}^{n-1} x_i r_i (1 - \tfrac{x_i}{k_i})(x_i + m_i) - b_i x_i^2 + x_n r_n (1 - \tfrac{x_n}{k_n})] - B_2 s y \\
&\geq A_2 F_2 - [A_2 \sum_{l=1}^{n} x_l - B_1 s y] \\
&\geq A_2 F_2 - \alpha_2 U_2,
\end{aligned}$$

where $f_i(x_i) = r_i(1 - \frac{x_i}{k_i})(x_i + m_i) - b_i x_i + 1,\ f_n(x_n) = r_n(1 - \frac{x_n}{k_n}) + 1,\ F_2(x_l) = \sum_{l=1}^{n} x_l f_l(x_l)$. We know $x_l \in (c_1, C_1)$ from the previous proof. And because of the (ii) and $f_l(0) = 0,\ f_l(c_1) > 0,\ f_l(C_1) > 0$. Then

$$F_{2\min} = \min\left\{\sum_{l=1}^{n} c_1 f_l(c_1), \sum_{l=1}^{n} c_1 f_l(C_1)\right\} = \frac{1}{A_2}.$$

We get $\dot{U}_2 \geq 1 - \alpha_2 U_2$. This implies $\liminf_{t\to+\infty} U_2(t) \geq \alpha_2$ and $\liminf_{t\to+\infty} y(t) \leq \frac{1}{B_2\alpha_2}$, where $\alpha_2 = \max\{1, s\}$. We also know

$$\dot{y} \geq y(\sum_{l=1}^{n-1} e_i \mu - s) > 0$$

from condition (iii). We know $t = t_0,\ y(t_0) > 0$ from Lemma 2.3. There exists $t_0 > 0$ if $t' > t_0$ when $t' \to +\infty$, $\liminf_{t' \to +\infty} y(t) > y(t_0) > 0$. So there exists a positive constant $c_2 = y(t_0)$ such that

$$\liminf_{t \to +\infty} y(t) > c_2.$$

Combining Theorem 3.1, we can know that there exist positive constants c_i, C_i, $i = 1, 2$ such that

$$0 < c_1 \le \liminf_{t \to +\infty} x_l(t) \le \limsup_{t \to +\infty} x_l(t) \le C_1,$$

and

$$0 < c_2 < \liminf_{t \to +\infty} y(t) \le \limsup_{t \to +\infty} y(t) \le C_2,$$

then the system (2.1) is permanence. Furthermore, from Lemma 2.2(Hutson and Schmitt's Theorem 6.3 [19]), the system (2.1) has a positive equilibrium. □

Remark 3.1. By Theorems 3.1 and 3.2, when $m < 0$ is a strong Allee effect, the boundedness of the solution of the system is satisfied and Theorem 3.1 holds, but the sufficient conditions for the permanence of the system are not satisfied and Theorem 3.2 does not hold.

Theorem 3.3. *If* $s > \frac{ne^0 r^0}{a^0}$, *then the predator population will go extinct, where* $e^0 = \max\{e_i\}$, $r^0 = \max\{r_i m_i, r_n\}$, $a^0 = \min\{\frac{r_n}{k_n}, a_i\}, i = 1, 2, ..., n-1.$

Proof. Let

$$H_1 = \sum_{l=1}^{n} x_l.$$

Compute the derivative of H_1 along (2.1) to get

$$\begin{aligned} \dot{H}_1 &= \sum_{l=1}^{n} \dot{x}_l \\ &\le \sum_{i=1}^{n-1} x_i r_i (1 - \tfrac{x_i}{k_i})(x_i + m_i) - b_i x_i^2 - x_n r_n (1 - \tfrac{x_n}{k_n}) \\ &\le r^0 H_1 - \tfrac{a^0}{n} H_1^2, \end{aligned}$$

where $r^0 = \max\{r_i m_i, r_n\}$, $a^0 = \min\{\frac{r_n}{k_n}, a_i\}$, $-a_i = r_i - \frac{r_i m_i}{k_i} - b_i$, $a_i > 0$. Hence

$$\limsup_{t \to +\infty} H_1(t) \le \tfrac{nr^0}{a^0}.$$

From the fourth equation in (2.1), consider the constants $e^0 = \max\{e_i\}$ and $l_1 > 0$. Then it satisfies that when t is large enough

$$\dot{y} \le y(-s + e^0 H_1) \le -l_1 y,$$

as $t \to +\infty, y \to 0$. When $s > \frac{ne^0 r^0}{a^0}$, the predator population becomes extinct. □

Remark 3.2. From Theorems 3.2 and 3.3, we can see that the magnitude of the mortality parameter of the predator population is very important for the permanence of the system. When $s < \sum_{i=1}^{n-1} e_i\mu$, the system (2.1) exhibits permanence. When $s > \frac{ne^0r^0}{a^0}$, the predators in system (2.1) tend to go extinct. The positive equilibrium point will not exist.

3.2. Stability of equilibrium

In this section, we study the existence and stability of the equilibria point of the system (2.1). Let's assume that the equilibrium point of the system is $E = (\mathbf{X}, y)$, $\mathbf{X} = (x_1, x_2, ..., x_n)$. It is clear that the extinction equilibrium $E_0 = (\mathbf{0}, 0)$, $\mathbf{0} = (0, 0, ..., 0)$ exists, while the edge equilibrium $E_1 = (\mathbf{0}, y^*)$, $y^* > 0$ does not.

Theorem 3.4. *If the following conditions are satisfied*

(i) $d_{lj} = d, l \neq j, l, j = 1, ..., n;$

(ii) $r_i m_i = r_n = (n-1)d, i = 1, ..., n-1,$

then extinction balance $E_0 = (\boldsymbol{0}, 0)$ *is the saddle point.*

Proof. The Jacobian matrix of system (2.1) at the extinction equilibrium point $E_0 = (\mathbf{0}, 0)$ is

$$J(E_0) = \begin{vmatrix} J_1 & 0 \\ 0 & -s \end{vmatrix}_{(n+1)\times(n+1)},$$

where

$$J_1 = \begin{vmatrix} r_1m_1 - (n-1)d & d & \cdots & d & d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ d & d & \cdots & r_{n-1}m_{n-1} - (n-1)d & d \\ d & d & \cdots & d & r_n - (n-1)d \end{vmatrix}_{n\times n}.$$

The characteristic equation of the variation matrix $J(E_0)$ is given by,

$$|\lambda E - J(E_0)| = |\lambda E - J_1| (\lambda + s).$$

It's obvious that $\lambda = -s$. We just need to discuss $|\lambda E - J_1| = 0$.

Condition (ii) implies that $r_i m_i - (n-1)d = r_n - (n-1)d = 0$.

$$|\lambda E - J_1| = \begin{vmatrix} \lambda & -d & \cdots & -d & -d \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -d & -d & \cdots & \lambda & -d \\ -d & -d & \cdots & -d & \lambda \end{vmatrix}_{n\times n} = (\lambda - (n-1)d)(\lambda + d)^{n-1}.$$

Clearly there are multiple roots $\lambda = -d$ with multiple $(n-1)$ and a positive root $\lambda = (n-1)d$. Therefore, we can say that there exists an eigenvalue greater than 0. In summary, the system (2.1) is unstable at the extinction equilibrium point. □

Through the above description, we obtain the sufficient condition of the existence of the positive equilibrium point E^*. Next, we analyze the stability of the positive equilibrium point E^*.

Theorem 3.5. *If a positive equilibrium E^* exists, when the following conditions are satisfied*

(i) $d_{lj} \neq 0, j \neq l, l, j = 1, 2, ..., n$. *The diffusion coefficient matrix is irreducible;*

(ii) $\frac{(r_i - b_i)k_i}{r_i} \leq m_i < k_i$,

then it is unique and globally asymptotically stable in R_+^{n+1}.

Proof. Uniqueness can be deduced from global asymptotic stability, so only the global asymptotic stability of positive equilibrium points is investigated.

Let $E^* = (\mathbf{X}^*, y^*)$ denote the positive equilibrium. Consider a Lyapunov function

$$V = \delta\left(y - y^* + y^* \ln \frac{y}{y^*}\right) + \sum_{l=1}^{n} \gamma_l \varepsilon_l \left(x_l - x_l^* + x_l^* \ln \frac{x_l}{x_l^*}\right), \tag{3.7}$$

where $\delta = \frac{\gamma_i \varepsilon_i q_i}{\delta e_i}.\gamma_l = \sum_{\mathcal{T} \in \mathbb{T}_l} w(\mathcal{T})$. Differentiating V along (2.1) gives

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(2.1)} =& \sum_{i=1}^{n-1} \gamma_i \varepsilon_i (x_i - x_i^*)(r_i \left(1 - \frac{x_i}{k_{\ i}}\right)(x_i + m_i) - b_i x_i - q_i y + \sum_{l=1}^{n} d_{il}(\frac{x_l}{x_i} - 1)) \\
&+ \gamma_n \varepsilon_n (x_n - x_n^*)(r_n \left(1 - \frac{x_n}{k_n}\right) + \sum_{l=1}^{n} d_{nl}(\frac{x_l}{x_n} - 1)) \\
&+ \delta(y - y^*)(\sum_{i=1}^{n-1} e_i x_i - s) \\
=& \sum_{i=1}^{n-1} \gamma_i \varepsilon_i (x_i - x_i^*)^2 (r_i - \frac{r_i m_i}{k_i} - b_i) - \gamma_i \varepsilon_i (x_i - x_i^*)^2 \frac{r_i}{k_i}(x_i + x_i^*) \\
&- \gamma_i \varepsilon_i q_i (x_i - x_i^*)(y - y^*) + \sum_{i=1}^{n-1} \sum_{l=1}^{n} \gamma_i \varepsilon_i d_{il}(\frac{x_l}{x_l^*} - \frac{x_i}{x_i^*} + 1 - \frac{x_l x_i^*}{x_l^* x_i}) x_l^* \\
&- \gamma_n \varepsilon_n \frac{r_n}{k_n}(x_n - x_n^*)^2 + \sum_{l=1}^{n} \gamma_n \varepsilon_n d_{nl}(\frac{x_l}{x_l^*} - \frac{x_n}{x_n^*} + 1 - \frac{x_l x_n^*}{x_l^* x_n}) x_l^* \\
&+ \sum_{i=1}^{n-1} \delta e_i (x_i - x_i^*)(y - y^*) - s\delta(y - y^*).
\end{aligned}$$

Because of (ii), we have

$$\begin{aligned}
\left.\frac{dV}{dt}\right|_{(2.1)} &\leq \sum_{j,l=1}^{n} \gamma_j \varepsilon_j d_{jl}(\frac{x_l}{x_l^*} - \frac{x_j}{x_j^*} + 1 - \frac{x_l x_j^*}{x_l^* x_j}) x_l^* \\
&= \sum_{j,l=1}^{n} \gamma_j \varepsilon_j d_{jl} x_l^* (G_j(x_j) - G_l(x_l) + 1 - \frac{x_l x_j^*}{x_l^* x_j} + \ln \frac{x_l x_j^*}{x_l^* x_j}),
\end{aligned}$$

of which $a_{jl} = d_{jl}\varepsilon_j x_l^*$, $G_l(x_l) = -\frac{x_l}{x_l^*} + \ln\frac{x_l}{x_l^*}$. Here we use one fact:$1 - c + \ln c \leq 0$ for $a \geq 0$ with equality holding if $c = 1$. We have shown that γ_l and $G_l(x_l)$ satisfy the assumptions of Lemma 2.1. Therefore,

$$\left.\frac{dV}{dt}\right|_{(2.1)} \leq \sum_{j,l=1}^{3} \gamma_j a_{jl}(G_j(x_j) - G_l(x_l)) = 0.$$

When $\left.\frac{dV}{dt}\right|_{(2.1)} = 0$, by $r_i - \frac{r_i m_i}{k_i} - b_i < 0, i = 1, 2, ..., n-1$, we obtain $(r_i - \frac{r_i m_i}{k_i} - b_i)(x_i - x_i^*)^2 = 0$, $r_n(x_n - x_n^*)^2 = 0$. We can solve $x_l = x_l^*, l = 1, 2, ..., n$. $s\delta(y - y^*) = 0$, when $y = y^*$.

If vertices j and l are connected by edges, then $d_{jl} > 0$. Thus $1 - \frac{x_j x_l^*}{x_j^* x_l} + \ln\frac{x_j x_l^*}{x_j^* x_l}$. Due to $1 - c + \ln c \leq 0$ and $1 - c + \ln c = 0$ is equivalent to $c = 1$, it follows that $\frac{x_l}{x_l^*} = \frac{x_j}{x_j^*} = 1$, when $x_l = x_l^*, l = 1, 2, ..., n$.

From the irreducible property of diffusivity, the graph G is strongly connected, which means that any vertex j and l are connected. Thus, $x_l = x_l^*, l = 1, 2, ..., n$.

In summary, $\{(\mathbf{X}, y) | \left.\frac{dV}{dt}\right|_{(2.1)} = 0\}$. The maximal closed invariant subsets are single-point sets E^*. According to LaSalle's Invariance Principle, E^* the equilibrium point is globally asymptotically stable in $R_+^{n+1} = \{(\mathbf{X}, y) | \mathbf{X} = (x_l, x_2, ..., x_n), x_l > 0,\ y > 0,\ l = 1, 2, ...n.\}$. □

Remark 3.3. Set $y = 0$ in system (2.1) to get the following system

$$\begin{cases} \frac{dx_i}{dt} = r_i(1 - \frac{x_i}{k_i})(x_i + m_1)x_i - b_i {x_i}^2 + \sum\limits_{l=1}^{n} d_{il}(x_l - x_i), \\ \frac{dx_n}{dt} = r_n(1 - \frac{x_n}{k_n})x_n + \sum\limits_{l=1}^{n} d_{nl}(x_l - x_n). \end{cases} \tag{3.8}$$

For system (3.8), we can still obtain the existence of the positive equilibrium point $E_* = (x_{1*}, x_{2*}, ..., x_{n*})$ of system (3.8) by Theorems 3.1 and 3.2. All solutions of the system (2.1) $x_l(t)$ still satisfy the inequality

$$0 < c_1 \leq \liminf_{t\to+\infty} x_l(t) \leq \limsup_{t\to+\infty} x_l(t) \leq C_1.$$

For the proof of the stability of the positive equilibrium point E_*, we take a similar method to the proof of Theorem 3.5. Consider the Lyapunov function

$$V = \sum_{l=1}^{n} \gamma_l \varepsilon_l \left(x_l - x_l^* + x_l^* \ln\frac{x_l}{x_l^*} \right). \tag{3.9}$$

When $\delta = 0$ in equation (3.7), the global asymptotic stability of the positive equilibrium point E_* can still be obtained.

4. Numerical simulation

In this paper, we primarily analyze two situations : (1)the permanence and stability of the predator-prey system ; (2)the effect of the parameter value of predator population mortality s on the system. To simulate the model presented in this paper, we consider the following model for $n = 2$.

$$\begin{cases} \frac{dx_1}{dt} = 3x_1(1 - \frac{x_1}{3})(x_1 + 2.9) - 0.1{x_1}^2 - 2x_1 y + (x_2 - x_1), \\ \frac{dx_2}{dt} = 0.5x_2(1 - x_2) + (x_1 - x_2), \\ \frac{dy}{dt} = -y + 0.76699yx_1. \end{cases} \tag{4.1}$$

First of all, we choose the initial values $x_1(0) = 0.6,\ x_2(0) = 0.2,\ y(0) = 0.4$. According to the prearranged model, the required parameters of the system can be calculated. $C_1 = 3,\ C_2 = 3.5,\ \mu = 1.3038,\ c_1 = 1$ and $c_2 = 0.4$. By substituting the parameters into the judgment conditions in the theorem, it can be found that when the obtained parameters satisfy Theorems 3.2 and 3.5, the system (4.1) exhibits uniform persistence and the positive equilibrium point E^* is stable, as shown in Figure 1. Starting from different initial values, it can be seen that the solution of system (4.1) satisfies

$$0 < 1 \le \liminf_{t\to+\infty} x_l(t, x_l(0)) \le \limsup_{t\to+\infty} x_l(t, x_l(0)) \le 3,$$

and

$$0 < 0.4 \le \liminf_{t\to+\infty} y(t, y(0)) \le \limsup_{t\to+\infty} y(t, y(0)) \le 3.5.$$

So the solution curves of system (4.1) are all in the cube with red border, and the arrows will tend to the positive equilibrium point with time, as shown in Figure 1.

Finally, the effect of changes in s on the predator population was observed when the other parameters were fixed. According to Theorems 3.2 and 3.3, when $s < 1.0032$, the predator population exists and tends to a normal number point. When $s > 2.3010$, the predator population tends to 0. Four groups of s values were selected to draw the variation rules of predator and prey population density respectively, which are $s = 0.97$, $s = 1.003$, $s = 2.5$ and $s = 4$, as shown in Figure 2.

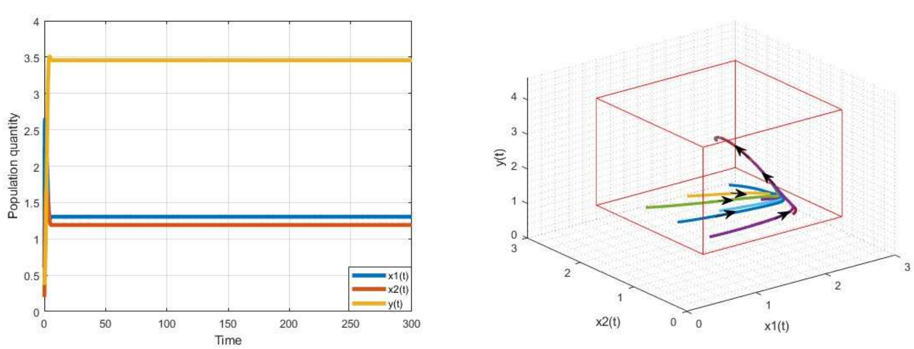

Figure 1. The stability of the positive equilibrium point of the system (4.1). Left: the population density of predators and prey tends to be stable over time; Right: the solution of the system (4.1) is selected from different initial values and tends to be stable in a bounded red cube.

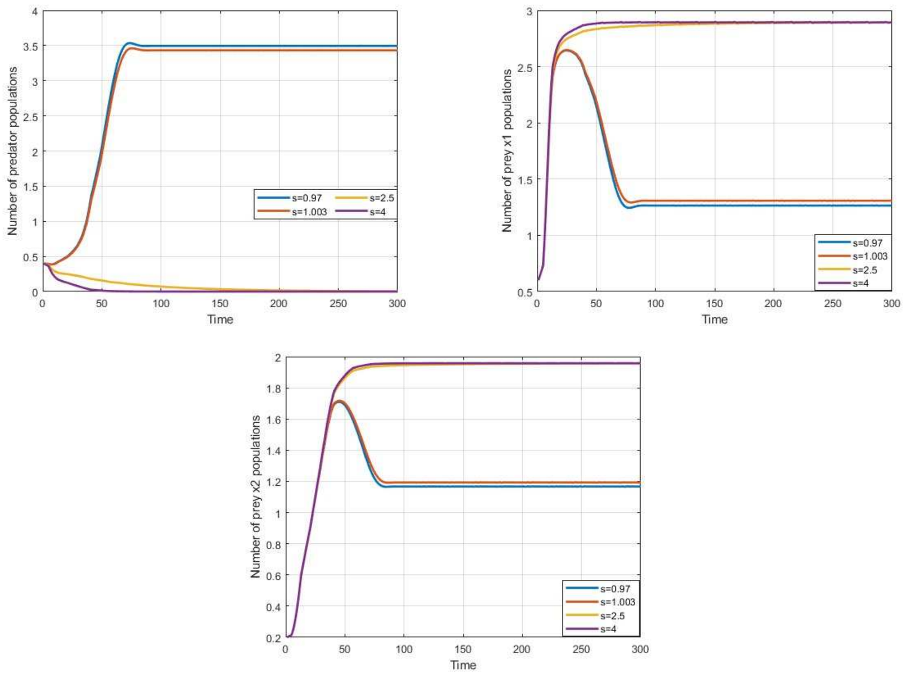

Figure 2. Trend of predator and prey population density with parameter s in system (4.1).

5. Discussion and summary

In this paper, the dynamic behavior of a predator-prey diffusion model with the Allee effect is studied. Firstly, we obtain the boundedness and permanence of the system. We discover that the system cannot achieve the coexistence under the strong Allee effect, while under the weak Allee effect, the system can be the permanence and guarantee the existence of the positive equilibrium point. Secondly, we prove the sufficient conditions for the global stability of the positive equilibrium point by constructing a Lyapunov function. We can see that the population diffusion and weak Allee effect are crucial for the coexistence and stability of the system. Finally, we analyze the influence of the parameter s on the predator population, and calculate the parameter threshold of the parameter s on the permanence and extinction of the predator. From Figure 2, we can understand the influence of parameter s on population density. Due to time constraints, we will continue to consider the strong Allee effect, bifurcation behavior, and even random diffusion factors in the future.

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