Journal of Nonlinear Modeling and Analysis https://journal.global-sci.org/jnma
Volume 7, Number 2, April 2025, 439–452 DOI:10.12150/jnma.2025.439

On a Class of Discrete Problems with the $p(k)$-Laplacian-like Operators

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Abstract In this paper, we consider a nonlinear discrete problem originating from a capillary phenomena, involving the $p(k)$-Laplacian-like operators with mixed boundary condition. Under appropriate assumptions on the function f and its primitive F near zero and infinity, we investigate the existence and multiplicity of nontrivial solutions by using variational methods and critical point theory.

Keywords Critical point theory, discrete problems, variational methods, $p(k)$-Laplacian-like operators

MSC(2010) 39A10, 35J15, 39A12, 34B15.

1. Introduction

The aim of this article is to establish the existence and multiplicity of solutions for a general discrete problem originating from capillary phenomena, involving the $p(k)$-Laplacian-like operators with mixed boundary condition of the following form:

$$(P)\ \begin{cases} -\Delta\big(m(r-1)a(r,\Delta u(r-1))\big)+\beta(r)|u(r)|^{p(r)-2}u(r)=f(r,u(r)),\ r\in[1,N]_{\mathbb{N}},\\ u(0)=\Delta u(N)=0, \end{cases}$$

where $a(.,.)$ is defined as follows:

$$a(r,s)=\Big(1+\phi_c\left(|s|^{p(r-1)}\right)\Big)|s|^{p(r-1)-2}s,\ \text{for all } r\in[1,N]_{\mathbb{N}} \text{ and } s\in\mathbb{R},$$

and ϕ_c is the so-called mean curvature operator defined as ([23])

$$\phi_c(s):=\frac{s}{\sqrt{1+s^2}},\ s\in\mathbb{R}.$$

Let $[1,N]_{\mathbb{N}}$ be the discrete interval given by $[1,N]_{\mathbb{N}}:=\{1,2,\dots,N\}$, where $N\geq 2$ is a positive integer and Δ denotes the forward difference operator $\Delta u(r):=u(r+1)-u(r)$. In addition, $m:[0,N+1]_{\mathbb{N}}\longrightarrow[1,+\infty)$, $\beta:[0,N+1]_{\mathbb{N}}\longrightarrow[1,+\infty)$

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and $p : [0, N+1]_{\mathbb{N}} \to [2, +\infty)$ are given functions, and for every fixed $r \in [0, N]_{\mathbb{N}}$, $f(r, .) : \mathbb{R} \to \mathbb{R}$ is a continuous function that checks some conditions mentioned below. We say that a function $u : [0, N+1]_{\mathbb{N}} \to \mathbb{R}$ is a solution of problem (P) if it satisfies both equations of (P).

For convenience, for any bounded function $h : [0, N+1]_{\mathbb{N}} \longrightarrow \mathbb{R}$, we will use the following symbols

$$h^+ := \max_{r\in[0,N+1]_{\mathbb{N}}} h(r), \quad h^- := \min_{r\in[0,N+1]_{\mathbb{N}}} h(r).$$

Let

$$\begin{aligned} F(t,s) &:= \int_0^s f(t,\zeta)d\zeta, \\ A(t,s) &:= \int_0^s a(t,\zeta)d\zeta \\ &= \frac{1}{p(t-1)}\left(|s|^{p(t-1)} + \sqrt{1+|s|^{2p(t-1)}} - 1\right), \end{aligned}$$

for all $(t, s) \in [0, N]_{\mathbb{N}} \times \mathbb{R}$, and we put

$$C_{m,\beta} = (m^+ 2^{p^-} + \beta^+) \geq 1, \quad K_0 = \left\{(2N+2)\max\left\{m^+, \beta^+\right\}\right\}^{\frac{p^- - p^+}{p^+ p^-}} \leq 1. \quad (1.1)$$

Now, we introduce the following assumptions on the nonlinear term f and its primitive F at zero and infinity:

(H_1) There exists $\eta < \frac{1}{p^+ C_{m,\beta}}$ such that for all $r \in [1, N]_{\mathbb{N}}$,

$$\limsup_{|x|\to\infty} \frac{F(r,x)}{|x|^{p^-}} \leq \eta.$$

(H_2) There exists $\delta > 0$ such that for all $r \in [1, N]_{\mathbb{N}}$,

$$B_0(r) := \liminf_{x\to 0} \frac{F(r,x)}{|x|^{p^-}} \geq \delta.$$

(H_3) $f(r, -x) = -f(r, x)$ for all $(r, x) \in [1, N]_{\mathbb{N}} \times \mathbb{R}$, i.e., $f(r, x)$ is odd in x.

(H_4) $\lim_{x\to 0} \frac{F(r,x)}{|x|^{p^+}} = 0$, for all $r \in [1, N]_{\mathbb{N}}$.

(H_5) There exists κ such that $\kappa > \frac{2}{p^-} N^{\frac{p^+ - p^-}{p^-}} K_0^{p^+} C_{m,\beta}^{\frac{p^+}{P^-}}$ and

$$\liminf_{|x|\to\infty} \frac{F(r,x)}{|x|^{p^+}} \geq \kappa, \text{ for all } r \in [1, N]_{\mathbb{N}}.$$

In recent years, there have been more and more papers on the topic of difference equations with a variable exponent, which have contributed to the development of research and studies related to the problems of differential equations since it has been frequently used as mathematical models in several domains [9, 14]. Known means from the critical point theory are applied to prove the existence of solutions.

Problems like (P) take the interest from the close connection with the study of the capillary phenomena. These phenomena, which can be briefly explained by considering the effects of two opposing forces: cohesion, i.e. the attractive force between the molecules of the liquid; and adhesion, i.e. the attractive (or repulsive) force between the molecules of the liquid and those of the container. The study of this kind of problems possesses a solid background in physics and other areas of research such as combustible gas dynamics [22], image restoration [5], the analysis of capillary surfaces [3,11] and economic systems processing [21]. For other applications, the reader is referred to [6,7,13] and references therein.

The $p(k)$-Laplacian-like operators have been widely used in areas such as nonlinear partial differential equations, variational calculus and geometric analysis (see [4, 10, 18–20]). In [18], M. Rodrigues studies the existence and multiplicity of solutions for the following problem involving the $p(x)$-Laplacian-like operators originated from a capillary phenomena:

$$(P.1): \quad \begin{cases} -\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u), \quad x\in\Omega, \\ u=0, \quad x\in\partial\Omega, \end{cases}$$

where $\lambda>0$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $p\in C(\overline{\Omega})$ and $p(x)>2$, $\forall x\in\Omega$. Based on the mountain pass theorem, the author showed that the problem has at least one nontrivial solution. The author also proved the existence of a sequence of solutions by using the Fountain theorem. In [4], the authors consider the problem $(P.1)$ where $p:\overline{\Omega}\to\mathbb{R}$ is a Lipschitz continuous function with $1<p^-:=ess\inf_{x\in\Omega}p(x)\leq p(x)\leq p^+:=ess\sup_{x\in\Omega}p(x)<N$ and $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is superlinear but does not satisfy the usual Ambrosetti-Rabinowitz type condition. Under some conditions on f at infinity, they proved that the problem $(P.1)$ admits at least one nontrivial solution. Moreover, the existence of infinite solutions is proved for odd nonlinearity and with some new trickes. In the case $m\equiv 1$ and $\beta\equiv 0$, the problem (P) can be regarded as the variant discrete of $(P.1)$.

Concerning the investigation of discrete boundary value problems we mention, far from being exhaustive, the following recent papers that used critical point theory ([17,23]). In [17], the authors established the existence and uniqueness of positive solution for a discrete boundary value problem of the following form:

$$(P_\lambda) \quad \begin{cases} -\Delta\left(w(r-1)\phi_{p(r-1)}(\Delta u(r-1))\right)+q(r)\phi_{p(r)}(u(r))=\lambda f(r,u(r)), \\ \qquad\qquad\qquad \text{for all } r\in[1,T]_{\mathbb{N}}, \\ u(0)=u(T+1)=0, \end{cases}$$

where $\phi_{p(k)}(s)=|s|^{p(k)-2}s$ is the discrete $p(k)$-Laplacian operator and λ is a positive real parameter. In particular, the authors have studied the existence of at least one, and at least two positive solutions, as well as the uniqueness of a positive solution for the problem (P_λ) which involves the discrete $p(k)$-Laplacian operator. They are based on a local minimum theorem and on a two critical points theorem, which they established the existence of solutions. For uniqueness part, they are based on a Lipschitzian continuous condition on the function f.

We note that papers on the study of discrete problems involving $p(k)$-Laplacian-like operators are relatively rare. So, after reading paper [17], a question was posed: can we obtain some qualitative results such as the existence and multiplicity of solutions for the problem (P_λ) if we replace the $p(k)$-Laplacian operator with the $p(k)$-Laplacian-like operator?

In this article, we will try to answer this question. More precisely, we will establish existence and multiplicity results by applying variational methods and similar techniques to those used in [8].

We organize the remainder of this paper as follows. In Section 2, we introduce some necessary preliminary results. Next, we set the variational setting associated with (P), and finally, in the last section, we will state and prove our main results, and give some examples to illustrate our results.

2. Preliminaries

Consider E as the N-dimensional Banach space [1] defined as

$$E = \Big\{ u : [0, N+1]_{\mathbb{N}} \to \mathbb{R} \mid \ u(0) = \Delta u(N) = 0 \Big\},$$

equipped with the norm

$$\|u\|_{m,\beta} = \left\{ \sum_{r=1}^{N+1} m(r-1)|\Delta u(r-1)|^{p^-} + \beta(r)|u(r)|^{p^-} \right\}^{1/p^-}. \tag{2.1}$$

Let us define

$$\|u\|_{+} = \left\{ \sum_{r=1}^{N+1} m(r-1)|\Delta u(r-1)|^{p^+} + \beta(r)|u(r)|^{p^+} \right\}^{1/p^+}. \tag{2.2}$$

$\|.\|_+$ is a norm on E equivalent with $\|.\|_{m,\beta}$ and by the weighted Hölder inequality, we can conclude that (see [15])

$$K_0\|u\|_{m,\beta} \le \|u\|_+ \le 2^{\frac{p^+ - p^-}{p^+ p^-}} K_0 \|u\|_{m,\beta}, \tag{2.3}$$

where K_0 is defined in (1.1).

Also, we define the norms

$$\|u\|_\infty := \max_{r\in[1,N]_{\mathbb{N}}} |u(r)| \le (2N+2)^{\frac{p^- - 1}{p^-}} \|u\|_{m,\beta}, \tag{2.4}$$

and

$$|u|_q = \left(\sum_{t=1}^{N} |u(t)|^q \right)^{\frac{1}{q}}, \quad \forall u \in E,\ q \ge 2. \tag{2.5}$$

Lemma 2.1. *For every $u \in E$, we have:*

$$|u|_q^q \le N(N+1)^{q-1} \sum_{r=1}^{N+1} |\Delta u(r-1)|^q, \quad \forall q \ge 2. \tag{2.6}$$

$$\sum_{r=1}^{N+1} |\Delta u(r-1)|^q \le 2^q |u|_q^q, \ \forall q \ge 2. \tag{2.7}$$

$$|u|_{p^-}^{p^-} \leq \|u\|_{m,\beta}^{p^-} \leq C_{m,\beta}|u|_{p^-}^{p^-}, \tag{2.8}$$

where $\|.\|_{m,\beta}$ *is defined in* (2.1).

For the first two inequalities, see [12]. The third inequality is immediately obtained from (2.7).

Now, let $\varphi : E \to \mathbb{R}$ be given by the formula

$$\varphi(u) := \sum_{r=1}^{N+1} \left[m(r-1)|\Delta u(r-1)|^{p(r-1)} + \beta(r)|u(r)|^{p(r)} \right].$$

Then, we obtain the following result.

Lemma 2.2. *[16, Lemma 2.1-2.2] For any* $u \in E$*,*

1. *If* $\|u\|_{m,\beta} < 1$*, then*
$$\|u\|_{+}^{p^+} \leq \varphi(u) \leq \|u\|_{m,\beta}^{p^-}.$$
2. *If* $\|u\|_{m,\beta} \geq 1$*, then*
$$\|u\|_{m,\beta}^{p^-} - (N+1)\left(m^+ + \beta^+\right) \leq \varphi(u) \leq \|u\|_{+}^{p^+} + (N+1)\left(m^+ + \beta^+\right).$$

To investigate the problem (P), we consider $L : E \to \mathbb{R}$ the functional associated with problem (P) defined in the following way

$$L(u) = \Lambda(u) + I(u) - \Psi(u), \quad \forall u \in E, \tag{2.9}$$

where

$$\begin{aligned} \Lambda(u) &= \sum_{r=1}^{N+1} m(r-1) A\left(r, \Delta u(r-1)\right) \\ &= \sum_{r=1}^{N+1} \frac{m(r-1)}{p(r-1)} \left(|\Delta u(r-1)|^{p(r-1)} + \sqrt{1 + |\Delta u(r-1)|^{2p(r-1)}} - 1 \right), \end{aligned}$$

$$I(u) = \sum_{r=1}^{N} \frac{\beta(r)}{p(r)} |u(r)|^{p(r)} \quad \text{and} \quad \Psi(u) = \sum_{t=r}^{N} F(r, u(r)).$$

The functional L is well-defined on E and is of class $C^1(E, \mathbb{R})$ with the Gâteaux derivative given by (see [17])

$$\begin{aligned} \langle L'(u), v\rangle = &\sum_{r=1}^{N+1} m(r-1) a(r, \Delta u(r-1)) \Delta v(r-1) + \sum_{r=1}^{N} \beta(r)|u(r)|^{p(r)-2} u(r) v(r) \\ &- \sum_{k=1}^{N} f(r, u(r)) v(r), \end{aligned} \tag{2.10}$$

for all $u, v \in E$.

Since $\max(0, s-1) \leq \sqrt{1+s^2} - 1 \leq s$, $s \geq 0$, we can see easily that:

Lemma 2.3.

1. *For all $u \in E$, we have*

$$\frac{1}{p^+}\sum_{r=1}^{N+1} m(r-1)|\Delta u(r-1)|^{p(r-1)} \leq \Lambda(u) \leq \frac{2}{p^-}\sum_{r=1}^{N+1} m(r-1)|\Delta u(r-1)|^{p(r-1)}. \tag{2.11}$$

2. *For all $u \in E$ such that $|\Delta u(r-1)| \geq 1$ for all $r \in [1, N+1]_{\mathbb{N}}$, we have*

$$\frac{2}{p^+}\sum_{r=1}^{N+1} m(r-1)|\Delta u(r-1)|^{p(r-1)} - \frac{Nm^+}{p^+} \leq \Lambda(u). \tag{2.12}$$

In our approach, we rely on the following critical point theorems:

Theorem 2.1. *(Mountain Pass Theorem, see [2]) Let $(X, \|.\|)$ be a real Banach space. $\Psi \in C^1(X, \mathbb{R})$ satisfies (PS)-condition, i.e., any sequence $(u_n) \subset X$ such that $(\Psi(u_n))$ is bounded and $(\Psi'(u_n)) \longrightarrow 0$ as $n \to +\infty$ has a subsequence which converges in X. Moreover, if $\Psi(0) = 0$ and the following conditions hold:*

1. *There exist positive constants ρ and α such that $\Psi(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$.*
2. *There exists a function $e \in X$ such that $\|e\| > \rho$ and $\Psi(e) \leq 0$.*

Then, the functional Ψ possesses a critical value $c \geq \alpha$. Moreover, the critical value c is characterised by

$$c = \inf_{h\in\Gamma}\max_{t\in[0,1]} \Psi(h(t)),$$

where

$$\Gamma = \{h \in C([0,1], X) / h(0) = 0, h(1) = e\}.$$

Remark 2.1. Since E is a finite dimensional space and the functional L is continuous. Then, to show that L satisfies the (PS)-condition, it suffices to show that it is coercive or anti-coercive in E.

Theorem 2.2. *(see [2]) Let X be a real Banach space and S^{N-1} be the $N-1$ dimensional unit sphere in X. Let $\Psi \in C^1(X, \mathbb{R})$ be even, bounded from below, and satisfying the (PS)-condition. Suppose that $\Psi(0) = 0$ and there is a set $\Omega \subset X$ such that Ω is homeomorphic to S^{N-1} by an odd map and $\sup\{\Psi(u);\ u \in \Omega\} < 0$. Then, Ψ has at least N disjoint pairs of nontrivial critical points in X.*

3. Main results and proofs

Theorem 3.1. *Assume that the hypotheses (H_1)-(H_2) hold, then the problem (P) has at least one non trivial solution in E.*

Furthermore, if the hypothesis (H_3) holds, then problem (P) admits at least N disjoint pairs of nontrivial solutions.

Proof. Let $u \in E$ such that $\|u\|_{m,\beta}$ is large enough so that $|\Delta u(r-1)| \geq 1$ and $|u(r)| \geq 1$. On the one hand, according to Lemma 2.2 and (2.12) we have

$$\Lambda(u) + I(u) = \sum_{r=1}^{N+1} m(r-1)A(r, \Delta u(r-1)) + \sum_{r=1}^{N} \frac{\beta(r)}{p(r)}|u(r)|^{p(r)}$$

$$\geq \frac{2}{p^+}\sum_{r=1}^{N+1} m(r-1)|\Delta u(r-1)|^{p(r-1)} + \sum_{r=1}^{N}\frac{\beta(r)}{p(r)}|u(k)|^{p(r)} - \frac{Nm^+}{p^+}$$
$$\geq \frac{2}{p^+}\left(\sum_{r=1}^{N+1} m(r-1)|\Delta u(r-1)|^{p(r-1)} + \beta(r)|u(r)|^{p(r)}\right) - \frac{Nm^+}{p^+}$$
$$\geq \frac{2}{p^+}\|u\|_{m,\beta}^{p^-} - \frac{2(N+1)(m^+ + \beta^+)}{p^+} - \frac{Nm^+}{p^+}. \tag{3.1}$$

On the other hand, the assumption (H_1) implies that for $\epsilon > 0$, there exists $K > 0$ such that

$$F(r,s) \leq (\eta+\epsilon)|s|^{p^-}, \quad \forall (r,|s|) \in [1,N]_{\mathbb{N}} \times [K,+\infty),$$

and by continuity of the function F, it follows that there exists $c > 0$ such that

$$F(r,s) \leq (\eta+\epsilon)|s|^{p^-} + c, \quad \forall (r,s) \in [1,N]_{\mathbb{N}} \times \mathbb{R}. \tag{3.2}$$

So, for any $u \in E$ we have

$$F(r,u(r)) \leq (\eta+\epsilon)|u(r)|^{p^-} + c, \quad \forall r \in [1,N]_{\mathbb{N}}.$$

Hence, by (2.8) we have

$$\Psi(u) = \sum_{r=1}^{N} F(r,u(r)) \leq (\eta+\varepsilon)|u|_{p^-}^{p^-} + cN$$
$$\leq (\eta+\epsilon)\|u\|_{m,\beta}^{p^-} + cN. \tag{3.3}$$

Using the above relations, we get

$$L(u) = \Lambda(u) + I(u) - \Psi(u)$$
$$\geq \frac{2}{p^+}\|u\|_{m,\beta}^{p^-} - (\eta+\epsilon)\|u\|_{m,\beta}^{p^-} - cN - \frac{2(N+1)(m^+ + \beta^+)}{p^+} - \frac{Nm^+}{p^+}$$
$$\geq \left(\frac{1}{p^+} - (\eta+\epsilon)\right)\|u\|_{m,\beta}^{p^-} - cN - \frac{2(N+1)(m^+ + \beta^+)}{p^+} - \frac{Nm^+}{p^+}. \tag{3.4}$$

Therefore, if we choose $0 < \epsilon < \frac{1}{p^+ C_{m,\beta}} - \eta$, then the functional L is coercive i.e., $L(u) \longrightarrow +\infty$ as $\|u\|_{m,\beta} \longrightarrow \infty$, and L is bounded from below.

Since L is lower semi-continuous, then according to the global minimum principle, it has a global minimum $\tilde{u}$, which is a solution of (P).

It still remains to show that the solution $\tilde{u}$ is nontrivial. By the assumption (H_2), there exists $R > 0$ such that

$$F(r,s) \geq (\delta - \varepsilon)\,|s|^{p^-}, (r,|s|) \in [1,N]_{\mathbb{N}} \times [0,R],$$

where $\varepsilon > 0$.

Let $u \in E, \|u\|_{m,\beta} \leq \xi$ with $\xi = \min\left\{1, R(2N+2)^{\frac{1-p^-}{p^-}}\right\}$.

By (2.4) it follows that

$$|u(r)| \leq \max_{k\in[1,N]_{\mathbb{N}}} |u(k)| \leq R, \quad \forall r \in [1,N]_{\mathbb{N}}.$$

So, by (2.8) we have

$$\sum_{t=1}^{N} F(t,u(t)) \geq (\delta-\varepsilon)\, C_{m,\beta}^{-1}\|u\|_{m,\beta}^{p^-}.$$

On the other hand, for $\|u\|_{m,\beta}<1$, we have $|\Delta u(r-1)|<1$ and $|u(r)|<1$ for all $r\in[1,N+1]_{\mathbb{N}}$. Then,

$$\begin{aligned}\Lambda(u)+I(u) &\leq \frac{1}{p^-}\left(\sum_{r=1}^{N+1} 2m(r-1)|\Delta u(r-1)|^{p^-}+\beta(r)|u(r)|^{p^-}\right)\\ &\leq \frac{2}{p^-}\|u\|_{m,\beta}^{p^-}. \qquad (3.5)\end{aligned}$$

Thus,

$$\begin{aligned}L(u) &= \Lambda(u)+I(u)-\Psi(u)\\ &\leq \frac{2}{p^-}\|u\|_{m,\beta}^{p^-}-(\delta-\varepsilon)\, C_{m,\beta}^{-1}\|u\|_{m,\beta}^{p^-}\\ &\leq \xi^{p^-}\left(\frac{2}{p^-}-C_{m,\beta}^{-1}(\delta-\varepsilon)\right). \qquad (3.6)\end{aligned}$$

Hence, for $\varepsilon>0$ such that

$$\varepsilon<\ \frac{2C_{m,\beta}}{p^-},$$

we get $L(u)<0$, for all $u\in B_\xi$, so

$$\sup_{u\in B_\xi} L(u)<0. \qquad (3.7)$$

Therefore,

$$L(\tilde{u})=\inf_{u\in E} L(u)\leq \inf_{u\in B_\xi} L(u)\leq \sup_{u\in B_\xi} L(u)<0.$$

Since $L(0)=0$, we conclude that $\tilde{u}\neq 0$.

Now, we prove the second result of Theorem 3.1. By condition (H_3), we infer that L is an even function. Moreover, the functional L is coervive, bounded from below in the finite dimensional space E, and thus the (PS)-condition is also satisfied.

Let

$$\Omega=\{u\in E;\ \|u\|_{m,\beta}=\frac{\xi}{N}\}$$

and define $\phi:\Omega\to S^{N-1}$ by

$$\phi(u)=\frac{N}{\xi}u,$$

where S^{N-1} is the unit sphere in $\mathbb{R}^N$. Obviously, ϕ is an odd homeomorphism between S^{N-1} and Ω.

For $u\in\Omega$, we have $\|u\|_{m,\beta}\leq\xi$, then by (3.7) we have $\sup_{u\in B_\xi} L(u)<0$ and thus $\sup_{u\in\Omega} L(u)<0$.

Hence, in view of Lemma 2.2, L admits at least N pairs of nontrivial critical points, which are exactly nontrivial solutions of the (P). Then, the proof of Theorem 3.1 is completed. □

Example 3.1. Let us take $N = 10$, $p(r) = \frac{2}{11}r + 3$, $m(r) = \beta(r) = \frac{1}{2} + \cos^2(\frac{r\pi}{2})$ for $r \in [0, 11]_{\mathbb{N}}$. By simple calculations, we get

$$p^+ = 5, \quad p^- = 3, \quad m^+ = \frac{3}{2}, \quad \beta^+ = \frac{3}{2},$$
$$C_{m,\beta} = (m^+ 2^{p^-} + \beta^+) = \frac{27}{2} \text{ and } K_0 = \left(2N+2\right)^{\frac{p^- - p^+}{p^+ p^-}} \simeq 0.66223.$$

Let f be the function defined as :

$$f(r,x) = \frac{1}{15(2+r)}|x|x, \quad x \in \mathbb{R},$$

and

$$F(r,x) = \frac{1}{45(2+r)}|x|^3, \quad x \in \mathbb{R}.$$

We have

$$\limsup_{|x|\to\infty} \frac{F(r,x)}{|x|^{p^-}} = \frac{1}{45(2+r)} \leq \frac{1}{90} \leq \eta$$

where η is a constant such that $\eta < \frac{1}{p^+ C_{m,\beta}} = \frac{2}{135}$, and

$$B_0(r) := \liminf_{x\to 0} \frac{F(r,x)}{|x|^{p^-}} \geq \frac{1}{45(2+N)} = \frac{1}{540} = \delta.$$

We see easily that assumptions (H_1), (H_2) and (H_3) hold. Then, in view of Theorem 3.1, the problem

$$\begin{cases} -\Delta\left(m(r-1)\left(1 + \frac{|\Delta u(r-1)|^{\frac{2}{11}r+3}}{\sqrt{1+|\Delta u(r-1)|^{\frac{4}{11}r+6}}}\right)\right)|\Delta u(r-1)|^{\frac{2}{11}r+1} + \beta(r)|u(r)|^{\frac{2}{11}r+1}u(r) \\ \quad = f(r,u(r)), \text{ for } r \in [1,10]_{\mathbb{N}}, \\ u(0) = \Delta u(10) = 0, \end{cases}$$

for $r \in [1, 10]_{\mathbb{N}}$, has at least 10 pairs of nontrivial solutions.

Theorem 3.2. *Suppose that the assumptions (H_4)-(H_5) are fulfilled. Then, the problem (P) has at least two nontrivial solutions in E.*

Proof. From the condition (H_4), for $\epsilon > 0$, there exists $K > 0$ such that

$$|F(r,s)| \leq \epsilon |s|^{p^+}, \forall (r,|s|) \in [1,N]_{\mathbb{N}} \times [0,K].$$

Let $u \in E$, such that $\|u\|_{m,\beta} \leq \xi$ with $\xi = \min\left\{1, K(2N+2)^{\frac{1-p^-}{p^-}}\right\}$.

By inequality (2.4) it follows that

$$|u(r)| \leq \max_{t\in[1,N]_{\mathbb{N}}} |u(t)| \leq K, \forall r \in [1,N]_{\mathbb{N}}.$$

So, we conclude that

$$|F(r,u(r))| \leq \epsilon |u(r)|^{p^+}, \quad \text{for all } r \in [1,N]_{\mathbb{N}}.$$

Also by (2.4), we get

$$\Psi(u)=\sum_{r=1}^{N}F(r,u(r))\leq \epsilon N(2N+2)^{\frac{p^{-}-1}{p^{-}}p^{+}}\|u\|_{m,\beta}^{p^{+}}.$$

On the other hand, for $u\in E$ such that $\|u\|_{m,\beta}\leq 1$, by Lemma 2.2 we have

$$\begin{aligned}
\Lambda(u)+I(u) &= \sum_{r=1}^{N+1}m(r-1)A(r-1,\Delta u(r-1))+\sum_{r=1}^{N}\frac{\beta(r)}{p(r)}|u(r)|^{p(r)}\\
&\geq \frac{1}{p^{+}}\sum_{r=1}^{N+1}m(r-1)|\Delta u(r-1)|^{p(r-1)}+\frac{1}{p^{+}}\sum_{r=1}^{N}\beta(r)|u(r)|^{p(r)}\\
&\geq \frac{1}{p^{+}}\varphi(u)\\
&\geq \frac{1}{p^{+}}\|u\|_{+}^{p^{+}}\\
&\geq \frac{1}{p^{+}}K_0^{p^{+}}\|u\|_{m,\beta}^{p^{+}}. \qquad (3.8)
\end{aligned}$$

The combination of the above inequalities, leads us to the following

$$\begin{aligned}
L(u) &\geq \frac{1}{p^{+}}K_0^{p^{+}}\|u\|_{m,\beta}^{p^{+}}-\epsilon N(2N+2)^{\frac{p^{-}-1}{p^{-}}p^{+}}\|u\|_{m,\beta}^{p^{+}}\\
&\geq \left[\frac{1}{p^{+}}K_0^{p^{+}}-\epsilon N(2N+2)^{\frac{p^{-}-1}{p^{-}}p^{+}}\right]\|u\|_{m,\beta}^{p^{+}}\\
&\geq \frac{K_0^{p^{+}}}{2p^{+}}\|u\|_{m,\beta}^{p^{+}}
\end{aligned}$$

for $\epsilon>0$ such that $\epsilon<\frac{K_0^{p^{+}}(2N+2)^{\frac{1-p^{-}}{p^{-}}p^{+}}}{2p^{+}N}$. Then, if we take $\alpha=\frac{K_0^{p^{+}}}{2p^{+}}\rho^{p^{+}}>0$, it follows that

$$L(u)\geq \alpha>0, \forall u\in E \text{ with } \|u\|_{m,\beta}=\rho. \qquad (3.9)$$

According to the hypothesis (H_5), for any $\varepsilon>0$ there exists $R_0>0$ such that

$$\frac{F(r,x)}{|x|^{p^{+}}}\geq \kappa-\varepsilon \quad \text{for} \quad (r,|x|)\in[1,N]_{\mathbb{N}}\times]R_0,+\infty[,$$

so

$$F(r,x)\geq(\kappa-\varepsilon)|x|^{p^{+}} \quad \text{for } (r,|x|)\in[1,N]_{\mathbb{N}}\times]R_0,+\infty[.$$

By the continuity of the function $s\to F(t,s)$, there exists $d>0$ such that

$$F(r,s)\geq(\kappa-\varepsilon)|s|^{p^{+}}-d, \quad \forall(r,s)\in[1,N]_{\mathbb{N}}\times\mathbb{R}.$$

Thus,

$$\Psi(u)=\sum_{r=1}^{N}F(r,u(r))\geq(\kappa-\varepsilon)\sum_{r=1}^{N}|u(r)|^{p^{+}}-dN$$

$$\begin{aligned}&\geq (\kappa-\varepsilon)|u|_{p^+}^{p^+} - dN\\&\geq (\kappa-\varepsilon)N^{\frac{p^- - p^+}{p^-}}|u|_{p^-}^{p^+} - dN\\&\geq (\kappa-\varepsilon)N^{\frac{p^- - p^+}{p^-}}C_{m,\beta}^{\frac{-p^+}{p^-}}\|u\|_{m,\beta}^{p^+}.\end{aligned}$$

On the other hand, for $u \in E$ such that $|u(r)| > 1$ and $|\Delta u(r-1)| > 1$ for all $r \in [1, N]_{\mathbb{N}}$, by (2.11), we have

$$\begin{aligned}\Lambda(u) + I(u) &\leq \frac{2}{p^-}\sum_{r=1}^{N+1}\Big(m(r-1)|\Delta u(r-1)|^{p^+} + \beta(r)|u(r)|^{p^+}\Big)\\&\leq \frac{2}{p^-}\|u\|_{+}^{p^+}\\&\leq \frac{2}{p^-}K_0^{p^+}\|u\|_{m,\beta}^{p^+}.\end{aligned}$$

It follows that,

$$\begin{aligned}L(u) &\leq \frac{2}{p^-}K_0^{p^+}\|u\|_{m,\beta}^{p^+} - N^{\frac{p^- - p^+}{p^-}}C_{m,\beta}^{\frac{-p^+}{p^-}}(\kappa-\varepsilon)\|u\|_{m,\beta}^{p^+} + dN\\&\leq \left[\frac{2}{p^-}K_0^{p^+} - N^{\frac{p^- - p^+}{p^-}}C_{m,\beta}^{\frac{-p^+}{p^-}}(\kappa-\varepsilon)\right]\|u\|_{m,\beta}^{p^+} + dN.\end{aligned}$$

Consequently, for $\varepsilon < \kappa - \frac{2}{p^-}N^{\frac{p^+ - p^-}{p^-}}K_0^{p^+}C_{m,\beta}^{\frac{p^+}{p^-}}$, we have

$$L(u) \to -\infty, \text{ as } \|u\|_{m,\beta} \to \infty.$$

Hence L is anti-coercive on E.

Therefore, we can find a function $\psi \in E \setminus \{0\}$ with a norm large enough and satisfies $L(\psi) < 0$. Furthermore, any (PS) sequence (u_n) associated with L will be bounded in E which is a finite dimensional space. So, the functional L satisfies the (PS)-condition.

Since $L(0) = 0$, by the Mountain Pass Theorem (Theorem 2.1) we conclude that L has a critical value $c \geq \alpha = \frac{K_0^{p^+}}{p^+}\rho^{p^+} > 0$ defined as

$$c = \inf_{g\in\Gamma}\max_{t\in[0,1]} L(g(t)),$$

where

$$\Gamma = \{h \in C([0,1], E)/h(0) = 0, h(1) = e\}.$$

Let $\tilde{u} \in E$ be a critical point corresponding to c, i.e., $L(\tilde{u}) = c$ and $L'(\tilde{u}) = 0$. Clearly, $\tilde{u}$ is a nontrivial solution of (P).

In addition, the functional L is continuous, anti-coercive and bounded from above, then it has a maximum point $\tilde{v} \in E$, i.e., $L(\tilde{v}) = \sup_{u\in E} L(u)$.

Bearing in the mind (3.9), we obtain $L(\tilde{v}) = \sup_{u\in E} L(u) \geq \sup_{u\in\partial B_\rho} L(u) > 0$. Hence, $\tilde{v}$ is a nontrivial solution of (P).

If $\tilde{u} \neq \tilde{v}$, then we have two nontrivial solutions $\tilde{u}$ and $\tilde{v}$. Otherwise, similar to the proof of [8, Theorem 3.1], since $\tilde{u} = \tilde{v}$, we infer that

$$L(\tilde{u}) = \max_{t\in[0,1]} L(g(t)) = L(\tilde{v}).$$

By the continuity of the function $t \longmapsto L(g(t))$ and the fact that $L(0)=0$ and $L(\bar{u})<0$, we conclude that there exists $t_1 \in]0,1[$ such that $L(\tilde{u}) = L\left(g(t_1)\right)$.

Now, we choose $g_2, g_3 \in \Gamma$ such that

$$\Big\{g_2(t) \mid t \in]0,1[\Big\} \cap \Big\{g_3(t) \mid t \in [0,1]\Big\} = \varnothing,$$

and it follows that there are $t_2, t_3 \in]0,1[$ such that

$$L\left(g_2(t_2)\right) = L\left(g_3(t_3)\right) = L\left(\tilde{u}\right) = \max_{t\in[0,1]} L(g(t)).$$

Consequently, we have two different critical points of L.

Finally, the problem (P) has at least two nontrivial solutions. $\square$

Example 3.2. We take, again as in example 3.1, $N=10$, $p(r)=\frac{2}{11}r+3$, $m(r)=\frac{1}{2}+\cos^2(\frac{r\pi}{2})$ and $\beta(r)=\frac{1}{2}+\sin^2(\frac{r\pi}{2})$ for $r\in[0,11]_{\mathbb{N}}$. We have

$$p^+=5, \quad p^-=3, \quad m^+=\beta^+=\frac{3}{2},$$

$$C_{m,\beta}=\frac{27}{2} \text{ and } K_0 \ = \Big(2N+2\Big)^{\frac{p^--p^+}{p^+p^-}} \simeq 0.66223.$$

Let us take the function f as follows:

$$\begin{cases} f(r,x)=30e^{2r}x^5, & r\in[1,10]_{\mathbb{N}},\ |x|\le 1,\\ f(r,x)=30e^{2r}|x|^3x, & r\in[1,10]_{\mathbb{N}},\ |x|>1,\end{cases}$$

and

$$\begin{cases} F(r,x)=5e^{2r}x^6, & r\in[1,10]_{\mathbb{N}},\ |x|\le 1,\\ F(r,x)=6e^{2r}|x|^5-e^{2r}, & r\in[1,10]_{\mathbb{N}},\ |x|>1.\end{cases}$$

We have

$$\lim_{x\to 0}\frac{F(r,x)}{|x|^{p^+}}=0 \ \text{ and } \liminf_{|x|\to\infty}\frac{F(r,x)}{|x|^{p^+}} \ge 6e^2 \ge \kappa,$$

where κ is a constant such that $\kappa > \frac{2}{p^-}N^{\frac{p^+-p^-}{p^-}}K_0^{p^+}C_{m,\beta}^{\frac{p^+}{P^-}} \simeq 30.1659$.

Then, conditions (H_4) and (H_5) hold. So, by virtue of Theorem 3.2, the problem

$$\begin{cases} -\Delta\Bigg(m(r-1)\big(1+\frac{|\Delta u(r-1)|^{\frac{2}{11}r+3}}{\sqrt{1+|\Delta u(r-1)|^{\frac{4}{11}r+6}}}\big)\Bigg)|\Delta u(r-1)|^{\frac{2}{11}r+1}+\beta(r)|u(r)|^{\frac{2}{11}r+1}u(r)\\ \quad = f(r,u(r)), \text{ for } r\in[1,10]_{\mathbb{N}},\\ u(0)=\Delta u(10)=0,\end{cases}$$

has at least two nontrivial solutions.

Acknowledgements

The authors thank the anonymous referees for invaluable comments and insightful suggestions which improved the presentation of this manuscript.

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