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Impact of Plankton Body Size on a Stochastic Plankton System with Lévy Jumps*

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Abstract In consideration of the important impact of plankton body size and Lévy noise on plankton system, a stochastic phytoplankton-zooplankton system with Lévy jump is proposed and investigated in this paper. Firstly, we prove that there is a unique global positive solution to the system by using Lyapunov function and Itǒ formula. Then, some thresholds which depend on plankton body size are given, and they determine the extinction and weak persistence in the mean of plankton populations. In addition, the sufficient conditions for the existence of a stationary distribution of the solution are given. Finally, some numerical simulations are introduced to support the main theoretical results and illustrate the impact of plankton body size and environmental noise on plankton populations.

Keywords Phytoplankton-zooplankton system, Lévy jump, Lyapunov function, Itǒ formula, stationary distribution

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1. Introduction

As the most abundant life form in the aquatic ecosystem, phytoplankton absorbs nutrients from water and generates energy through photosynthesis to support the entire biological community, including fish and heterotrophic bacteria. However, phytoplankton may grow out of control, resulting in toxic or harmful effects on humans, fish, marine mammal under anoxic conditions [1]. These have stimulated many scholars to study the dynamics of phytoplankton blooms in many different ways, so as to explore the possible mechanisms underlying the occurrence or termination of these blooms. Hence, finding some key factors affecting the growth mechanisms of phytoplankton is currently of great interest.

Phytoplankton are a polyphyletic of single-cell primary producers commonly existing in aquatic ecosystems [2]. It is worth noting that the size of phytoplankton cells plays a major part in the metabolism and growth rate of phytoplankton. Further experimental studies showed that the growth rate, metabolic rate and nutrient uptake all depend on the size of phytoplankton cells [3]. In addition, the size of zooplankton is considered to be another prominent element that can significantly affect

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the growth of aquatic plankton, because the size of zooplankton body can change the grazing choices of phytoplankton and maintain a clear state in the aquatic ecosystem [4]. To sum up, the cell size of phytoplankton and the body size of zooplankton are two vital factors affecting the dynamic mechanism of phytoplankton growth in aquatic ecosystems. Because the dynamic mechanism of phytoplankton growth can be qualitatively and quantitatively described by mathematical models, mathematical models, as a powerful tool, have attracted increasing attention from biological mathematicians. However, there are few mathematical ecological models to study the effects of phytoplankton cell size or zooplankton body size or both on the dynamic mechanism of phytoplankton blooms in recent years [5]. For example, Zhao et al. successively studied a phytoplankton-zooplankton model [6–8], and they found that phytoplankton cell size or zooplankton body size has important effects on the spatiotemporal dynamics and growth dynamics of phytoplankton in the complex aquatic environments. This provides a very good idea for future research on phytoplankton-zooplankton models. As we all know, many toxin-producing phytoplankton (TPP) can release toxic chemicals into the aquatic environment, which can inhibit the growth of zooplankton and even kill them. Furthermore, the dynamics of plankton system can be affected by the release toxin.

In fact, in the real aquatic environment, the growth of plankton is inevitably be affected by environmental noise, such as photosynthetic effective radiation, nutrient availability, water temperature, light, acidity, etc., which are usually unpredictable. In this way, it is meaningful to incorporate the unpredictable environmental factors into the aquatic ecosystem, which can help us gain a deeper understanding of the real aquatic ecosystem. Of course, the intrinsic growth rate and mortality of the plankton are always disturbed by environmental noise.

In addition, population system may suffer sudden environmental perturbations, such as, tsunami, earthquakes, volcanoes, floods or hurricanes [9]. Scheffer et al. [10] pointed out that all ecosystems are exposed to gradual changes in climate, nutrient loading, habitat fragmentation or biotic exploitation. However, this smooth change can be interrupted by sudden drastic switches to a contrasting state. In order to model the physical environmental disturbance (occasional catastrophic shocks), it is reasonable to consider another environmental noise, namely the Lévy jump noise, into the underlying population system. Bao et al. [9,11] did some interesting works in this field.

Based on the above research works, in this paper, we will consider the following stochastic phytoplankton-zooplankton model:

$$\begin{cases} \frac{\mathrm{d}P(t)}{\mathrm{d}t} = P(t^-)[r(x)(1-\frac{P(t^-)}{K}) - \alpha C(x,y)Z(t^-)]\mathrm{d}t + \sigma_1 P(t^-)\mathrm{d}B_1(t) \\ \qquad\quad +P(t^-)\int_{\mathbb{Y}}\gamma_1(u)\widetilde{N}(\mathrm{d}t,\mathrm{d}u), \\ \frac{\mathrm{d}Z(t)}{\mathrm{d}t} = Z(t^-)[\beta C(x,y)P(t^-) - \mu - \frac{\theta P^2(t^-)}{m^2+P^2(t^-)}]\mathrm{d}t + \sigma_2 Z(t^-)\mathrm{d}B_2(t) \\ \qquad\quad +Z(t^-)\int_{\mathbb{Y}}\gamma_2(u)\widetilde{N}(\mathrm{d}t,\mathrm{d}u), \end{cases} \tag{1.1}$$

where $P(t)$ is the density of toxin producing phytoplankton population and $Z(t)$ is the density of zooplankton population at time t. All parameters are nonnegative.K is the environmental carrying capacity of TPP population; μ denotes the natural death rate of zooplankton; α is the rate of predation and β is the conversion rate of zooplankton; θ is TPP toxin releasing rate; according to [12] ,the function $\frac{P^2(t)}{m^2+P^2(t)}$ describes the distribution of toxic substance which ultimately contributes to the death of zooplankton populations. In view of [5], $r(x) = \frac{x}{c_1x^2+c_2x+c_3}$ is the maximum specific growth rate of phytoplankton as a function of phytoplankton cell size x, and $c_j (j = 1, 2, 3)$ is a positive empirical constant; $C(x, y) = C_m \exp[-\frac{1}{d}(x-ey)^2]$ is the consumption rate of zooplankton as a function of phytoplankton cell size x (μm^3) and the zooplankton body size y ($\mu g/ind$, dry weight), where the units of the phytoplankton cell size and the zooplankton body size can refer to the references [13, 14], where C_m is the maximum consumption rate, d and e are the consumption rate coefficients. $B_1(t)$ and $B_2(t)$ are mutually independent standard Brownian motions defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions, σ_1 and σ_2 denote the intensities of the white noise. $\widetilde{N}(\mathrm{d}t, \mathrm{d}u) = N(\mathrm{d}t, \mathrm{d}u) - \lambda(\mathrm{d}u)\mathrm{d}t$ is the compensated Poisson random measure, where N is a Poisson counting measure with Lévy measure (jump measure) λ on a measurable subset $\mathbb{Y}$ of $(0, \infty)$ with $\lambda(\mathbb{Y}) < \infty$, and N is independent of $B_i(t)$, the jump intensities $\gamma_i : \mathbb{Y} \times \Omega \to \mathbb{R}$ are bounded and continuous.

The rest of this paper is organized as follows: Section 2 initially presents the basic assumptions and a useful lemma, and we investigate the existence and uniqueness of global positive solutions. Then we apply Itô's formula to obtain sufficient conditions for the extinction and persistence in the mean of the system by constructing a suitable stochastic Lyapunov function. In addition, the sufficient conditions for a stationary distribution are given. In section 3, a series of numerical simulations are then performed to verify the theoretical analysis. We summarize the results and present our conclusions in Section 4.

2. Dynamic behaviors

In this section, we will investigate the existence and uniqueness of global positive solutions of system (1.1), the extinction and persistence in the mean of plankton populations. In addition, the sufficient conditions for the existence of a stationary distribution are given.

2.1. Preliminaries

Consider the following two-dimensional stochastic differential equation with Lévy jumps

$$\mathrm{d}x(t^-) = F(x(t^-), t)\mathrm{d}t + G(x(t^-), t)\mathrm{d}B(t) + \int_{\mathbb{Y}} H(x(t^-), t, u)\widetilde{N}(\mathrm{d}t, \mathrm{d}u),$$

$x(t)$ is the solution of the stochastic differential equation. If $V \in C^{2,1}(\mathbb{R}^2 \times [t_0, +\infty);\ \mathbb{R}_+)$, the random differentiation of V is

$$\begin{aligned} \mathrm{d}V(x,t) =& LV(x,t)\mathrm{d}t + V_x(x^-,t)G(x^-,t)\mathrm{d}B(t) \\ &+ \int_{\mathbb{Y}} \left[V(x^- + H(x^-,t,u),t) - V(x^-,t)\right] \widetilde{N}(\mathrm{d}t,\mathrm{d}u), \end{aligned}$$

and

$$\begin{aligned} LV(x,t) =& V_t(x^-,t) + V_x(x^-,t)F(x^-,t) + \frac{1}{2} trace\left(G^T(x^-,t)V_{xx}G(x^-,t)\right) \\ &+ \int_{\mathbb{Y}} \left[V(x^- + H(x^-,t,u),t) - V(x^-,t) - V_x(x^-,t)H(x^-,t,u)\right] \lambda(\mathrm{d}u), \end{aligned}$$

where $x^- = x(t^-)$.
Assumption 2.1. Throughout this paper, we assume that $1+\gamma_i(u) > 0$, $u \in \mathbb{Y}$, $i = 1, 2$, and there is a constant $c > 0$ such that

$$\int_{\mathbb{Y}} \ln^2(1+\gamma_i(u))\lambda(du) \leq c, \int_{\mathbb{Y}} \gamma_i^2(u)\lambda(du) \leq c.$$

That is to say, the intensity of the Lévy jumps is not too large.

Lemma 2.1. *[15] Suppose that $X(t) \in C(\Omega \times [0, +\infty); \mathbb{R}_+)$, and let Assumption 2.1. hold.*

(i) If there exist two positive constants T and ρ_0 such that

$$\ln X(t) \leq \rho t - \rho_0 \int_0^t X(s)\mathrm{d}s + \sigma B(t) + \sum_{i=1}^{2} \rho_i \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\widetilde{N}(\mathrm{d}s, \mathrm{d}u),$$

for all $t \geq T$, where both σ and ρ_i are constants, then

$$\begin{cases} \langle X(t)\rangle^* \leq \frac{\rho}{\rho_0} \ a.s. & \rho \geq 0; \\ \lim_{t\to+\infty} X(t) = 0 \ a.s. & \rho < 0. \end{cases}$$

(ii) If there exist three positive constants T, ρ and ρ_0 such that

$$\ln X(t) \geq \rho t - \rho_0 \int_0^t X(s)\mathrm{d}s + \sigma B(t) + \sum_{i=1}^{2} \rho_i \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\widetilde{N}(\mathrm{d}s, \mathrm{d}u) \ a.s.,$$

then

$$\langle X(t)\rangle^* \geq \frac{\rho}{\rho_0} \ a.s.$$

Lemma 2.2. *Let Assumption 2.1. hold. Then, for any initial value $(P(0), Z(0)) \in \mathbb{R}_+^2$, the solution $(P(t), Z(t))$ of system (1.1.1) has the property*

$$\lim_{t\to\infty} \sup \frac{\ln P(t)}{t} \leq 0, \ \lim_{t\to\infty} \sup \frac{\ln Z(t)}{t} \leq 0.$$

Proof. For system (1.1), applying Itǒ-Lévy formula we deduce

$$\begin{aligned} e^t \ln P(t) =& \ln P_0 + \int_0^t e^s[\ln P(s) + r(s)(1 - \frac{P(s)}{K}) - \alpha Z(s) - \frac{\sigma_1^2}{2} \\ &- \int_{\mathbb{Y}} (\gamma_1(u) - \ln(1+\gamma_1(u)))\lambda(\mathrm{d}u)]\mathrm{d}s + \int_0^t e^s \sigma_1 \mathrm{d}B_1(s) \\ &+ \int_0^t \int_{\mathbb{Y}} e^s \ln(1+\gamma_1(u))\widetilde{N}(\mathrm{d}t, \mathrm{d}u). \end{aligned}$$

It follows from the inequality $\ln x \leq x - 1$ that

$$e^t \ln P(t) \leq \ln P_0 + \int_0^t e^s[\ln P(s) + r(s) - \frac{r(s)}{K}P(s) - \frac{\sigma_1^2}{2}]\mathrm{d}s$$

$$+\int_0^t e^s\sigma_1 \mathrm{d}B_1(s)+\int_0^t\int_{\mathbb{Y}} e^s\ln(1+\gamma_1(u))\widetilde{N}(\mathrm{d}t,\mathrm{d}u).$$

Due to the property of the function $\ln v-\frac{rv}{K}(\frac{r}{K},v>0)$ which has a maximum value $-1-\ln\frac{r}{K}$ at $v=\frac{K}{r}$, we deduce that

$$\begin{aligned}e^t\ln P(t)\leq{}&\ln P_0+\textstyle\int_0^t e^s[-1+r(s)-\ln\frac{r(s)}{K}-\frac{\sigma_1^2}{2}]\mathrm{d}s\\&+\textstyle\int_0^t e^s\sigma_1\mathrm{d}B_1(s)+\int_0^t\int_{\mathbb{Y}}e^s\ln(1+\gamma_1(u))\widetilde{N}(\mathrm{d}t,\mathrm{d}u).\end{aligned}\tag{2.1}$$

Let $M_1(t)=\int_0^t e^s\sigma_1\mathrm{d}B_1(s), M_2(t)=\int_0^t\int_{\mathbb{Y}}e^s\ln(1+\gamma_1(u))\widetilde{N}(\mathrm{d}t,\mathrm{d}u)$. Then the quadratic variation of $M_1(t)$ and $M_2(t)$ is

$$\langle M_1(t)\rangle(t)=\int_0^t e^{2s}\sigma_1^2\mathrm{d}s\leq\frac{\sigma_1^2}{2}(e^{2t}-1)<\infty,$$

and

$$\langle M_2(t)\rangle(t)=\int_0^t\int_{\mathbb{Y}}e^{2s}|\ln(1+\gamma_1(u))|^2\lambda(\mathrm{d}u)\leq\frac{c(e^{2t}-1)}{2}<\infty.$$

In view of Lemma 4.3 in [9], for any positive numbers a,b,T, we have

$$\begin{aligned}&\mathbb{P}\left\{\sup_{0\leq t\leq T}\Big[M_1(t)-\frac{a}{2}\langle M_1(t)\rangle+M_2(t)\right.\\&\left.-\frac{1}{a}\int_0^t\int_{\mathbb{Y}}[e^{ae^s\ln(1+\gamma_1(u))}-1-ae^s\ln(1+\gamma_1(u)]\lambda(\mathrm{d}u)\mathrm{d}s\Big]>b\right\}\\&\leq e^{-ab}.\end{aligned}$$

Choose $T=n\eta, a=\varepsilon e^{-n\eta}, b=\frac{\xi e^{n\eta\ln n}}{\varepsilon}$, where $n\in\mathbb{N}, 0<\varepsilon<1,\eta>0,\xi>1$. By the Borel-Cantelli lemma, we see that there exists an $\Omega_i\subseteq\Omega$ with $\mathbb{P}(\Omega_i)=1$ such that, for any $\omega\in\Omega_i$, there is an integer $n_0=n_0(\omega)$ such that

$$\begin{aligned}M_1(t)+M_2(t)\leq{}&\frac{\xi e^{n\eta\ln n}}{\varepsilon}+\frac{\varepsilon e^{-n\eta}}{2}\int_0^t e^{2s}\sigma_1^2\mathrm{d}s+\frac{e^{n\eta}}{\varepsilon}\int_0^t\int_{\mathbb{Y}}[(1+\gamma_1(u))^{\varepsilon e^{s-n\eta}}\\&-1-\varepsilon e^{s-n\eta}\ln(1+\gamma_1(u))]\lambda(\mathrm{d}u)\mathrm{d}s,\end{aligned}$$

where $n>n_0, 0\leq t\leq n\eta$. Furthermore, from the inequality $x^p\leq 1+p(x-1)(x\geq 1,0\leq p\leq 1)$ we get

$$\begin{aligned}&\frac{e^{n\eta}}{\varepsilon}\int_0^t\int_{\mathbb{Y}}[(1+\gamma_1(u))^{\varepsilon e^{s-n\eta}}-1-\varepsilon e^{s-n\eta}\ln(1+\gamma_1(u))]\lambda(\mathrm{d}u)\mathrm{d}s\\&\leq\int_0^t\int_{\mathbb{Y}}e^s(\gamma_1(u)-\ln(1+\gamma_1(u)))\lambda(\mathrm{d}u)\mathrm{d}s.\end{aligned}$$

Substituting the above inequality into (2.1) yields

$$\begin{aligned}e^t\ln P(t)\leq{}&\ln P_0+\frac{\xi e^{n\eta}\ln n}{\varepsilon}+\int_0^t e^s(-1-\ln\frac{r(x)}{K}+r(x)-\frac{\sigma_1^2}{2})\mathrm{d}s\\&+\frac{\varepsilon e^{-n\eta}}{2}\int_0^t e^{2s}\sigma_1^2\mathrm{d}s+\int_0^t\int_{\mathbb{Y}}e^s(|\gamma_1(u)|+|\ln(1+\gamma_1(u))|)\lambda(\mathrm{d}u)\mathrm{d}s\end{aligned}$$

$$\le \ln P_0 + \frac{\xi e^{n\eta}\ln n}{\varepsilon} + \int_0^t e^s(-1 - \ln\frac{r(x)}{K} + r(x) - \frac{(1-\varepsilon e^{s-n\eta})\sigma_1^2}{2})\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{Y}} e^s(|\gamma_1(u)| + |\ln(1+\gamma_1(u))|)\lambda(\mathrm{d}u)\mathrm{d}s.$$

Then, for any $\omega \in \Omega_i$ and $(n-1)\eta \le t \le n\eta$ with $n > n_0 + 1$, we have

$$rl \ln P(t) \le \frac{\ln P_0}{e^t} + \frac{\xi e^{\eta}\ln n}{\varepsilon} + \int_0^t e^{s-t}(-1 - \ln\frac{r(x)}{K} + r(x) - \frac{(1-\varepsilon e^{s-n\eta})\sigma_1^2}{2})\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{Y}} e^{s-t}(|\gamma_1(u)| + |\ln(1+\gamma_1(u))|)\lambda(\mathrm{d}u)\mathrm{d}s.$$

By Assumption 2.1, it is readily seen that, for any $0 \le t \le n\eta$, there exists a constant N which is independent of n such that

$$\frac{\ln P(t)}{t} \le \frac{\ln P_0}{te^t} + \frac{\xi e^{\eta}\ln n}{t\varepsilon} + N(\frac{1}{t} - \frac{1}{te^t}).$$

Setting $t \to \infty$ leads to

$$\lim_{t\to\infty}\sup \frac{\ln P(t)}{t} \le 0.$$

On the other hand, the result for $Z(t)$ can be proved in the same way and so we omit it.

□

2.2. Existence and uniqueness of global positive solutions

Before investigating the dynamics of system (1.1), we should first guarantee the existence of global positive solutions. Based on the biological interpretation, we just take the nonnegative solutions into account for system (1.1). The following result can be presented.

Theorem 2.1. *For any given initial value* $(P(0), Z(0)) \in \mathbb{R}_+^2$, *system* (1.1) *has a unique solution* $(P(t), Z(t)) \in \mathbb{R}_+^2$ *for all* $t > 0$ *with probability 1.*

Proof. Since the coefficients of system (1.1) are locally Lipschitz continuous, for any given initial value $(P(0), Z(0)) \in \mathbb{R}_+^2$, there is a unique local solution $(P(t), Z(t)) \in \mathbb{R}_+^2$, for $t \in [0, \tau_{\mathrm{e}})$, where τ_{e} is the explosion time. To show this solution is global, we only need to show that $\tau_{\mathrm{e}} = +\infty$ a.s. Let $k_0 > 0$ be sufficiently large such that $(P(0), Z(0))$ lies within the interval $(\frac{1}{k_0}, k_0)$. For each integer $k \ge k_0$, define the stopping time

$$\tau_k = \inf\left\{t \in [0, \tau_{\mathrm{e}}) : P(t) \notin (\frac{1}{k}, k) \text{ } or \text{ } Z(t) \notin (\frac{1}{k}, k)\right\}.$$

Throughout this paper, we set $\inf \emptyset = \infty$ (as usual $\emptyset$ is the empty set). Obviously, τ_k is increasing as $k \to +\infty$. Let $\tau_\infty = \lim_{k\to+\infty} \tau_k$. Thus, $\tau_\infty \le \tau_{\mathrm{e}}$ a.s. If we can verify $\tau_\infty = +\infty$, then $\tau_{\mathrm{e}} = +\infty$ and $(P(t), Z(t)) \in \mathbb{R}_+^2$ a.s. That is to say, to complete the proof we only need to show that $\tau_\infty = +\infty$ a.s. If $\tau_\infty \ne +\infty$, there exists a pair of constants $T > 0$ and $\varepsilon \in (0,1)$ such that

$$\mathbb{P}\{\tau_\infty \le T\} > \varepsilon.$$

Hence there is an integer $k_1 \geq k_0$, such that for all $k \geq k_1$

$$\mathbb{P}\{\tau_k \leq T\} \geq \varepsilon. \tag{2.2}$$

Define a C^2- function $V : \mathbb{R}^2_+ \to \mathbb{R}_+$ by

$$V(P(t), Z(t)) = \beta[P(t) - M - M\ln(\frac{P(t)}{M})] + \alpha(Z(t) - \ln Z(t) - 1),$$

where M is a positive constant to be determined suitably later.

Applying Itŏ formula to V, we have

$$\begin{aligned} \mathrm{d}V =& LV dt + \beta\sigma_1(P(t) - M)\mathrm{d}B_1(t) + \alpha\sigma_2(Z(t) - 1)\mathrm{d}B_2(t) + \beta\int_{\mathbb{Y}}[\gamma_1(u)P(t-) \\ & - M\ln(1 + \gamma_1(u)]\widetilde{N}(\mathrm{d}t, \mathrm{d}u) + \alpha\int_{\mathbb{Y}}[\gamma_2(u)Z(t-) - \ln(1 + \gamma_2(u)]\widetilde{N}(\mathrm{d}t, \mathrm{d}u), \end{aligned}$$

where

$$\begin{aligned} LV =& \beta r(x)P(t) - \frac{\beta r(x)}{K}P^2(t) - \alpha\beta C(x,y)P(t)Z(t) - \beta M r(x) + \frac{\beta M r(x)P(t)}{K} \\ & + \alpha\beta M C(x,y)Z(t) + \alpha\beta C(x,y)P(t)Z(t) - \alpha\mu Z(t) - \frac{\alpha\theta P^2(t)Z(t)}{m^2 + P^2(t)} \\ & - \alpha\beta C(x,y)P(t) + \alpha\mu + \frac{\alpha\theta P^2(t)}{m^2 + P^2(t)} + \frac{1}{2}\beta M\sigma_1^2 + \frac{1}{2}\alpha\sigma_2^2 \\ & + \beta\int_{\mathbb{Y}}[M\gamma_1(u) - M\ln(1 + \gamma_1(u))]\lambda(\mathrm{d}u) + \alpha\int_{\mathbb{Y}}[\gamma_2(u) - \ln(1 + \gamma_2(u))]\lambda(\mathrm{d}u) \\ \leq& -\frac{\beta r(x)}{K}P^2(t) + (\beta r(x) + \frac{\beta M r(x)}{K} + \frac{\alpha\theta}{2m})P(t) + (\alpha\beta M C(x,y) - \alpha\mu)Z(t) \\ & + \alpha\mu + \frac{1}{2}\beta M\sigma_1^2 + \frac{1}{2}\alpha\sigma_2^2 + \beta\int_{\mathbb{Y}}[M\gamma_1(u) - M\ln(1 + \gamma_1(u))]\lambda(\mathrm{d}u) \\ & + \alpha\int_{\mathbb{Y}}[\gamma_2(u) - \ln(1 + \gamma_2(u))]\lambda(\mathrm{d}u). \end{aligned}$$

Choosing $M = \frac{\mu}{\beta C(x,y)}$ and keeping in mind the fact that

$$0 \leq \int_{\mathbb{Y}}[\gamma_i(u) - \ln(1 + \gamma_i(u))]\lambda(\mathrm{d}u) \leq K_0, i = 1, 2,$$

where K_0 is a constant, we have

$$\begin{aligned} LV \leq& -\frac{\beta r(x)}{K}P^2(t) + P(t)(\beta r(x) + \frac{\beta M r(x)}{K} + \frac{\alpha\theta}{2m}) + \alpha\mu + \frac{1}{2}\beta M\sigma_1^2 + \frac{1}{2}\alpha\sigma_2^2 \\ & + \beta\int_{\mathbb{Y}}[M\gamma_1(u) - M\ln(1 + \gamma_1(u))]\lambda(\mathrm{d}u) + \alpha\int_{\mathbb{Y}}[\gamma_2(u) - \ln(1 + \gamma_2(u))]\lambda(\mathrm{d}u) \\ \leq& K_1, \end{aligned}$$

and K_1 here is a positive constant. Thus we have

$$\mathrm{d}V(P(t), Z(t)) \leq K_1\mathrm{d}t + \beta\sigma_1(P(t) - M)\mathrm{d}B_1(t) + \alpha\sigma_2(Z(t) - 1)\mathrm{d}B_2(t)$$

$$
\begin{aligned}
&+\beta\int_{\mathbb{Y}}(\gamma_1(u)P(t-)-M\ln(1+\gamma_1(u))\widetilde{N}(\mathrm{d}t,\mathrm{d}u)\\
&+\alpha\int_{\mathbb{Y}}(\gamma_2(u)Z(t-)-\ln(1+\gamma_2(u))\widetilde{N}(\mathrm{d}t,\mathrm{d}u). \qquad (2.3)
\end{aligned}
$$

Integrating both sides of (2.3) from 0 to $\tau_k \wedge T$ and taking expectation, we can obtain that,

$$
\begin{aligned}
\mathbb{E}V\left(P(\tau_k\wedge T),Z(\tau_k\wedge T)\right) &\le V\left(P(0),Z(0)\right)+K_1\mathbb{E}(\tau_k\wedge T)\\
&\le V\left(P(0),Z(0)\right)+K_1T. \qquad (2.4)
\end{aligned}
$$

Set $\Omega_k=\{\tau_k\le T\}$, and by (2.2) we get $\mathbb{P}(\Omega_k)\ge\varepsilon$. For each $\omega\in\Omega_k$, it exists that $P(\tau_k,\omega)$, $Z(\tau_k,\omega)$ equals either k or $\frac{1}{k}$. Thereby, we have

$$
\begin{aligned}
&V\left(P(\tau_k,\omega),Z(\tau_k,\omega)\right)\\
&\ge\min\Bigg\{\beta[k-M-M\ln(\frac{k}{M})],\\
&\alpha(k-1-\ln k),\beta[\frac{1}{k}-M+M\ln kM],\alpha(\frac{1}{k}-1+\ln k)\Bigg\}:=H(k).
\end{aligned}
$$

It follows from (2.4) that it is not difficult to see

$$
\begin{aligned}
V(P(0),Z(0))+K_1T &\ge \mathbb{E}\left[I_{\Omega_k}(\omega)V\left(P(\tau_k,\omega),Z(\tau_k,\omega)\right)\right]\\
&\ge\mathbb{P}\left(\Omega_k\right)H(k)\\
&\ge\varepsilon H(k),
\end{aligned}
$$

where I_{Ω_k} is the indicator function of Ω_k. Letting $k\to+\infty$ leads to the contradiction

$$
+\infty>V\left(P(0),Z(0)\right)+K_1T\ge+\infty.
$$

Therefore we show that $\tau_\infty=+\infty$ a.s., so $\tau_e=+\infty$. Then $(P(t),Z(t))$ is the unique global positive solution of system (1.1). This completes the proof of the theorem.

□

Lemma 2.3. *[16] For any initial value $(P(0),Z(0))\in\mathbb{R}^2_+$, there is a unique global positive solution $(P(t),Z(t))$ to model (1.1) a.s., Moreover, there exists a positive constant ζ such that*

$$
\lim_{t\to+\infty}\sup\mathbb{E}(P(t))\le\zeta,\quad \lim_{t\to+\infty}\sup\mathbb{E}(Z(t))\le\zeta.
$$

2.3. Extinction and persistence in the mean

Based on Theorem 2.1 and from the perspective of population dynamics, it is necessary and important to predict and control the development of population. Therefore, we will discuss the properties of extinction and persistence in the mean of system (1.1) in this subsection, and derive some sufficient conditions for them.

For simplicity, we introduce the following notations:

$$
\langle f(t)\rangle=\frac{1}{t}\int_0^t f(s)\mathrm{d}s,\quad f^*=\lim_{t\to\infty}\sup f(t),\quad f_*=\lim_{t\to\infty}\inf f(t).
$$

$$\mu_i = \frac{1}{2}\sigma_i^2 - \int_{\mathbb{Y}}[\ln(1+\gamma_i(u)) - \gamma_i(u)]\lambda(\mathrm{d}u), i = 1, 2.$$

$$N_i(t) = \int_0^t \int_{\mathbb{Y}} \ln(1+\gamma_i(u))\widetilde{N}(\mathrm{d}s, \mathrm{d}u), i = 1, 2.$$

Theorem 2.2. *The phytoplankton $P(t)$ of system* (1.1) *is*

(i) extinct if $r(x) - \mu_1 < 0$;

(ii) non-persistent in mean if $r(x) - \mu_1 = 0$;

(iii) weakly persistent in mean if $r(x) - \mu_1 > 0$.

Proof. Applying the Itǒ formula to the equation of system (1.1), we have

$$\begin{aligned}\mathrm{d}\ln P(t) =&[r(x)(1-\frac{P(t)}{K}) - \alpha Z(t)C(x,y) - \frac{\sigma_1^2}{2} \\ &+ \int_{\mathbb{Y}}[\ln(1+\gamma_1(u)) - \gamma_1(u)]\lambda(\mathrm{d}u)]\mathrm{d}t \\ &+ \sigma_1\mathrm{d}B_1(t) + \int_{\mathbb{Y}}[\ln(1+\gamma_1(u))]\widetilde{N}(\mathrm{d}t, \mathrm{d}u),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\mathrm{d}\ln Z(t) =&[\beta C(x,y)P(t) - \mu - \frac{\theta P^2(t)}{m^2+P^2(t)} - \frac{\sigma_2^2}{2} \\ &+ \int_{\mathbb{Y}}[\ln(1+\gamma_2(u)) - \gamma_2(u)]\lambda(\mathrm{d}u)]\mathrm{d}t \\ &+ \sigma_2\mathrm{d}B_2(t) + \int_{\mathbb{Y}}[\ln(1+\gamma_2(u))]\widetilde{N}(\mathrm{d}t, \mathrm{d}u).\end{aligned} \tag{2.6}$$

Integrating both sides of the above equalities on the interval $[0, t]$ yields

$$\frac{1}{t}\ln\frac{P(t)}{P(0)} = r(x) - \mu_1 - \frac{r(x)}{K}\langle P(t)\rangle - \alpha C(x,y)\langle Z(t)\rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}, \tag{2.7}$$

$$\frac{1}{t}\ln\frac{Z(t)}{Z(0)} = \beta C(x,y)\langle P(t)\rangle - \mu - \mu_2 - \theta\left\langle\frac{P^2(t)}{m^2+P^2(t)}\right\rangle + \frac{\sigma_2 B_2(t)}{t} + \frac{N_2(t)}{t}. \tag{2.8}$$

From (2.7), we get

$$\frac{1}{t}\ln\frac{P(t)}{P(0)} \le r(x) - \mu_1 + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}. \tag{2.9}$$

Based on Theorem 3.4 in [17], taking upper limits on both sides of (2.9) and using condition (i) leads to

$$\left[\frac{1}{t}\ln\frac{P(t)}{P(0)}\right]^* \le r(x) - \mu_1 < 0,$$

which implies $\lim_{t\to\infty} P(t) = 0$, a.s. This completes the proof of (i).

We are now to prove (ii). Given $\epsilon > 0$ small,

$$\frac{1}{t}\ln\frac{P(t)}{P(0)} \le r(x) - \mu_1 + \epsilon - \frac{r(x)}{K}\langle P(t)\rangle + \frac{\sigma_1 B_1(t)}{t} + \frac{N_1(t)}{t}.$$

By Lemma 2.1, if $r(x) - \mu_1 + \epsilon \ge 0$, then

$$\langle P(t)\rangle^* \le \frac{K(r(x) - \mu_1 + \epsilon)}{r(x)}, a.s.$$

Particularly, if $r(x) - \mu_1 = 0$, we have $\langle P(t)\rangle^* \le \frac{\epsilon K}{r(x)}, a.s.$ Then the desired result $\langle P(t)\rangle^* = 0$ follows from the arbitrariness of ϵ.

Now we prove (iii). Similarly, by taking upper limits on both sides of (2.7) we can have

$$[\frac{1}{t}\ln\frac{P(t)}{P(0)}]^* \ge r(x) - \mu_1 - \frac{r(x)}{K}\langle P(t)\rangle^* - \alpha C(x,y)\langle Z(t)\rangle^*. \tag{2.10}$$

Because the left of inequality (2.10) is non-positive from Lemma 2.2, it follows from condition (iii) that

$$\frac{r(x)}{K}\langle P(t)\rangle^* + \alpha C(x,y)\langle Z(t)\rangle^* \ge r(x) - \mu_1 > 0. \tag{2.11}$$

We then claim that $\langle P(t)\rangle^* > 0$ $a.s.$ Otherwise, for $\forall\omega \in \left\{\langle P(t,\omega)\rangle^* = 0\right\}$, we know $\langle Z(t,\omega)\rangle^* > 0$ by (2.11). But from (2.8), we obtain

$$\left[\frac{\ln Z(t,\omega)}{t}\right]^* \le \beta C(x,y)\langle P(t)\rangle^* = 0,$$

which implies $\langle Z(t,\omega)\rangle^* = 0$. This is a contradiction, so we have $\langle P(t)\rangle^* > 0$ $a.s.$ That is, the phytoplankton $P(t)$ is weakly persistent in mean.

Clearly, the zooplankton $Z(t)$ goes to extinction when the phytoplankton $P(t)$ is extinct. For this reason, we always assume that $r(x) - \mu_1 > 0$ in the following discussions. The following theorem gives the survival analysis of zooplankton.

□

Remark 2.1. The extinction and weakly persistent in the mean are in the sense of Wang [18]. Specifically, population $P(t)$ is called extinction if $\lim_{t\to\infty} P(t) = 0$ almost surely; population $P(t)$ is called non-persistence in the mean if $\langle P(t)\rangle^* = 0$ almost surely; population $P(t)$ is called weak persistence in the mean if $\langle P(t)\rangle^* > 0$ almost surely. Again Theorem 2.2 implies that $r(x) - \mu_1$ is the threshold of phytoplankton between extinction and weak persistence in the mean.

Theorem 2.3. *The zooplankton* $Z(t)$ *of system* (1.1) *is*

(i) *extinct if* $\beta\frac{K}{r(x)}(r(x) - \mu_1)C(x,y) < \mu + \mu_2$;

(ii) *weakly persistent in mean if* $\beta\frac{K}{r(x)}(r(x) - \mu_1)C(x,y) > \mu + \mu_2 + \theta$.

Proof. From (2.7), $r(x) - \mu_1 > 0$ and Lemma 2.2, we have

$$\langle P(t)\rangle^* \le \frac{K}{r(x)}(r(x) - \mu_1). \tag{2.12}$$

Moreover, (2.8) indicates

$$\frac{1}{t}\ln\frac{Z(t)}{Z(0)} < \beta C(x,y)\left\langle P(t)\right\rangle - \mu - \mu_2 + \frac{\sigma_2 B_2(t)}{t} + \frac{N_2(t)}{t}.$$

Together with (2.12) and Theorem 3.4 in [17], it follows from condition (i) that

$$\begin{aligned}\left[\frac{1}{t}\ln\frac{Z(t)}{Z(0)}\right]^* &\le \beta C(x,y)\left\langle P(t)\right\rangle^* - \mu - \mu_2\\ &\le \frac{\beta K C(x,y)}{r(x)}(r(x)-\mu_1) - \mu - \mu_2 < 0 \ a.s.\end{aligned}$$

Thus, $\lim_{t\to\infty} Z(t) = 0$, i.e., the zooplankton $Z(t)$ is extinct. This completes the proof of (i).

From (2.7) and (2.8), we obtain

$$\begin{aligned}&\beta C(x,y)\frac{1}{t}\ln\frac{P(t)}{P(0)} + \frac{r}{K}\frac{1}{t}\ln\frac{Z(t)}{Z(0)}\\ =&\beta C(x,y)(r(x)-\mu_1) - \frac{r}{K}(\mu+\mu_2) - \frac{\theta r}{K}\left\langle\frac{P^2(t)}{m^2+P^2(t)}\right\rangle - \alpha\beta C^2(x,y)\left\langle Z(t)\right\rangle\\ &+\frac{\beta C(x,y)\sigma_1 B_1(t)}{t} + \frac{\beta C(x,y)N_1(t)}{t} + \frac{r}{K}\frac{\sigma_2 B_2(t)}{t} + \frac{r}{K}\frac{N_2(t)}{t}\\ \ge&\beta C(x,y)(r(x)-\mu_1) - \frac{r}{K}(\mu+\mu_2) - \frac{\theta r}{K} - \alpha\beta C^2(x,y)\left\langle Z(t)\right\rangle\\ &+\frac{\beta C(x,y)\sigma_1 B_1(t)}{t} + \frac{\beta C(x,y)N_1(t)}{t} + \frac{r}{K}\frac{\sigma_2 B_2(t)}{t} + \frac{r}{K}\frac{N_2(t)}{t}.\end{aligned} \tag{2.13}$$

Taking upper limits on both sides of (2.13) leads to

$$\begin{aligned}&\left[\beta C(x,y)\frac{1}{t}\ln\frac{P(t)}{P(0)} + \frac{r}{K}\frac{1}{t}\ln\frac{Z(t)}{Z(0)}\right]^*\\ &\ge \beta C(x,y)(r(x)-\mu_1) - \frac{r}{K}(\mu+\mu_2+\theta) - \alpha\beta C^2(x,y)\left\langle Z(t)\right\rangle^*.\end{aligned} \tag{2.14}$$

In view of Lemma 2.2, the left of inequality (2.14) is non-positive, then it can be seen from condition (ii) that we have

$$\left\langle Z(t)\right\rangle^* \ge \frac{\beta C(x,y)(r(x)-\mu_1) - \frac{r}{K}(\mu+\mu_2+\theta)}{\alpha\beta C^2(x,y)} > 0,$$

which implies the zooplankton is weakly persistent. The proof is now completed. □

According to Theorems 2.2 and Theorems 2.3, the following Corollary 2.1 is obvious.

Corollary 2.1. *For system* (1.1)*, the following results hold.*

- *(i) If $r(x)-\mu_1<0$, then both the phytoplankton $P(t)$ and the zooplankton $Z(t)$ are extinct;*
- *(ii) If $r(x)-\mu_1>0$ and $\beta\frac{K}{r(x)}(r(x)-\mu_1)C(x,y)<\mu+\mu_2$, then the phytoplankton $P(t)$ is weakly persistent in mean and the zooplankton $Z(t)$ is extinct;*
- *(iii) If $\beta\frac{K}{r(x)}(r(x)-\mu_1)C(x,y)>\mu+\mu_2+\theta$, then both the phytoplankton $P(t)$ and the zooplankton $Z(t)$ are weakly persistent in mean.*

2.4. Stationary distribution

It is known if the noise intensity is sufficiently large then the population may become extinct with probability one. The interesting question is: what happens if the noise is relatively small? Is there a stationary distribution? Our aim in this section is to study the existence of a unique stationary distribution of the solution.

Theorem 2.4. *Suppose that Assumption 2.1 holds. Denote by δ_i the cofactor of the $i-$th diagonal element of the Kirchhoff matrix D_H , where*

$$D_H = \begin{pmatrix} \alpha C(x,y) & -\alpha C(x,y) \\ -\beta C(x,y) & \beta C(x,y) \end{pmatrix}, H = \begin{pmatrix} \frac{r(x)}{K} & \alpha C(x,y) \\ \beta C(x,y) & 0 \end{pmatrix}$$

and $\delta_1 = \beta C(x,y) > 0, \delta_2 = \alpha C(x,y) > 0$. If $\delta_1 \frac{r(x)}{K} - \delta_2 \beta C(x,y) > 0$, then for any given intial value $(P_0, Z_0) \in \mathbb{R}^2_+$, system (1.1) admits a unique stationary distribution $\pi(\cdot)$, that is to say, all the species do not die out almost surely.

Proof. We shall divide the proof into two parts.

Part 1. Define

$$U(t) = \delta_1 \left| \ln P(P_0;t) - \ln P(\tilde{P}_0;t) \right| + \delta_2 \left| \ln Z(Z_0;t) - \ln Z(\tilde{Z}_0;t) \right|,$$

where $P(P_0;t), P(\tilde{P}_0;t), Z(Z_0;t), Z(\tilde{Z}_0;t)$ stand for arbitrary solutions of model (1.1) with initial values $(P_0, Z_0) \in \mathbb{R}^2_+$ and $(\tilde{P}_0, \tilde{Z}_0) \in \mathbb{R}^2_+$, respectively. Making use of Itŏ formula yields

$$\begin{aligned} \mathrm{d}U(t) =& \delta_1 \mathrm{sgn} \left(P(P_0;t) - P(\tilde{P}_0;t) \right) \left[-\frac{r(x)}{K} \left(P(P_0;t) - P(\tilde{P}_0;t) \right) \right. \\ & \left. -\alpha C(x,y) \left(Z(Z_0;t) - Z(\tilde{Z}_0;t) \right) \right] \mathrm{d}t \\ & + \delta_2 \mathrm{sgn} \left(Z(Z_0;t) - Z(\tilde{Z}_0;t) \right) \left[\beta C(x,y) \left(P(P_0;t) - P(\tilde{P}_0;t) \right) \right. \\ & \left. -\theta \left(\frac{P^2(P_0;t)}{m^2 + P^2(P_0;t)} - \frac{P^2(\tilde{P}_0;t)}{m^2 + P^2(\tilde{P}_0;t)} \right) \right] \mathrm{d}t \\ \leq & -\delta_1 \frac{r(x)}{K} \left| P(P_0;t) - P(\tilde{P}_0;t) \right| \mathrm{d}t - \delta_1 \alpha C(x,y) \left| Z(Z_0;t) - Z(\tilde{Z}_0;t) \right| \mathrm{d}t \\ & + \delta_2 \beta C(x,y) \left| P(P_0;t) - P(\tilde{P}_0;t) \right| \mathrm{d}t. \end{aligned}$$

Consequently,

$$\begin{aligned} U(t) \leq & U(0) - (\delta_1 \frac{r(x)}{K} - \delta_2 \beta C(x,y)) \int_0^t \left| P(P_0;t) - P(\tilde{P}_0;t) \right| \mathrm{d}s \\ & - \delta_1 \alpha C(x,y) \int_0^t \left| Z(Z_0;t) - Z(\tilde{Z}_0;t) \right| \mathrm{d}s. \end{aligned}$$

It then follows from $U(t) \geq 0$ that

$$\left(\delta_1 \frac{r(x)}{K} - \delta_2 \beta C(x,y) \right) \int_0^t \left| P(P_0;t) - P(\tilde{P}_0;t) \right| \mathrm{d}s \leq U(0) < \infty,$$

$$\alpha C(x,y)\int_0^t \left|Z(Z_0;t)-Z(\tilde{Z}_0;t)\right| \mathrm{d}s \le U(0) < \infty.$$

Therefore,

$$\left|P(P_0;t)-P(\tilde{P}_0;t)\right| \in L^1[0,\infty), \quad \left|Z(Z_0;t)-Z(\tilde{Z}_0;t)\right| \in L^1[0,\infty). \tag{2.15}$$

Now by (1.1),

$$\mathbb{E}(P(t)) = P(0) + \int_0^t \left[r(x)\mathbb{E}(P(t)) - \frac{r(x)}{K}\mathbb{E}(P(t))^2 - \alpha C(x,y)\mathbb{E}(P(t)Z(t))\right] \mathrm{d}t.$$

That is to say, $\mathbb{E}(P(t))$ is differentiable. Thanks to Lemma 2.3,

$$\begin{aligned}\frac{\mathrm{d}\mathbb{E}(P(t))}{\mathrm{d}t} &= r(x)\mathbb{E}(P(t)) - \frac{r(x)}{K}\mathbb{E}(P(t))^2 - \alpha C(x,y)\mathbb{E}(P(t)Z(t))\\ &\le r(x)\mathbb{E}(P(t)) \le r(x)\zeta,\end{aligned}$$

where $\zeta > 0$ is a constant. Thus $\mathbb{E}(P(t))$ is uniformly continuous. Similarly, $\mathbb{E}(Z(t))$ is uniformly continuous. It then follows from (2.15) and Barbalat's result [20] that

$$\lim_{t\to+\infty}\mathbb{E}\left|P(P_0;t)-P(\tilde{P}_0;t)\right| + \lim_{t\to+\infty}\mathbb{E}\left|Z(Z_0;t)-Z(\tilde{Z}_0;t)\right| = 0. \tag{2.16}$$

Part 2. Denote by $\mathscr{L}(\bar{\mathbb{R}}^2_+)$ all the probability measures defined on $\bar{\mathbb{R}}^2_+$. For arbitrary $Q_1, Q_2 \in \mathscr{L}$, define

$$\mathrm{d}_N(Q_1,Q_2) = \sup_{\varphi\in N}\left|\int_{\bar{\mathbb{R}}^2_+}\varphi(z)Q_1(\mathrm{d}z) - \int_{\bar{\mathbb{R}}^2_+}\varphi(z)Q_2(\mathrm{d}z)\right|,$$

where

$$N = \left\{\varphi : \bar{\mathbb{R}}^2_+ \to \mathbb{R}\Big| |\varphi(z_1)-\varphi(z_2)| \le ||z_1-z_2||, |\varphi(\cdot)| \le 1\right\}.$$

For any bounded Borel measurable function $\varphi \in N$ and $t \ge s > 0$, denoting $Y = (P(t), Z(t))$, for the above $Y_0 = (P_0, Z_0)$, we have

$$\begin{aligned}\left|\mathbb{E}\varphi(Y(Y_0;t+s)) - \mathbb{E}\varphi(Y(Y_0;t))\right| &= \left|\mathbb{E}\left[\mathbb{E}(\varphi(Y(Y_0;t+s))\Big|\mathscr{F}_s)\right] - \mathbb{E}(\varphi(Y(Y_0;t))\right|\\ &= \left|\int_{\bar{\mathbb{R}}^2_+}\mathbb{E}\varphi(Y(\tilde{Y_0};t))\mathbb{P}(s,Y_0,\mathrm{d}\tilde{Y_0}) - \mathbb{E}\varphi(Y(Y_0;t))\right|\\ &\le \int_{\bar{\mathbb{R}}^2_+}\left|\mathbb{E}\varphi(Y(\tilde{Y_0};t)) - \mathbb{E}\varphi(Y(Y_0;t))\right|\mathbb{P}(s,Y_0,\mathrm{d}\tilde{Y_0}).\end{aligned}$$

From (2.16), there is a $T > 0$ such that for $t \ge T$,

$$\sup_{\varphi\in N}\left|\mathbb{E}(\varphi(Y(\tilde{Y_0};t)) - \mathbb{E}(\varphi(Y(Y_0;t))\right| \le \varepsilon.$$

That is to say,

$$\left|\mathbb{E}\varphi(Y(Y_0;t+s)) - \mathbb{E}\varphi(Y(Y_0;t))\right| \le \varepsilon.$$

It then follows from the arbitrariness of φ that

$$\sup_{\varphi\in N}\Big|\mathbb{E}\varphi(Y(Y_0;t+s))-\mathbb{E}\varphi(Y(Y_0;t))\Big|\le\varepsilon.$$

Hence for $t\ge T$ and $s>0$,

$$\mathrm{d}_N(\mathbb{P}(t+s,Y_0,\cdot),\mathbb{P}(t,Y_0,\cdot))\le\varepsilon.$$

In other words, $\{\mathbb{P}(t,Y_0,\cdot):t\ge 0\}$ is Cauchy in $\mathscr{L}(\bar{\mathbb{R}}^2_+)$. It follows that there exists a unique probability measure $\pi(\cdot)\in\mathscr{L}(\bar{\mathbb{R}}^2_+)$ such that for any given initial value $Y_0=(P_0,Z_0)\in\mathbb{R}^2_+$

$$\lim_{t\to+\infty}\mathbb{P}(t,Y_0,\cdot)=\pi(\cdot).$$

Hence, system (1.1) admits a unique stationary distribution $\pi(\cdot)$, all the species do not die out almost surely. The proof is complete.

□

3. Numerical simulations

In this section, we will verify the correctness of the theoretical results obtained through numerical simulation. We choose the same initial value $P(0)=5,\ Z(0)=5$, and the other parameters are: $K=6.5,c_1=0.5,c_2=0.2,c_3=0.8,\beta=1,\mu=0.05,\alpha=1.5,\theta=0.005,C_m=1.8,d=0.5,e=0.5,m=1,\mathbb{Y}=(0,+\infty),\lambda(\mathbb{Y})=1$.

First, when the plankton body size remains unchanged(such as $x=1,y=5$), we will reveal the impact of white noise and Lévy noise on the system (1.1) by considering the following examples.

Example 3.1. On the one hand, let $\sigma_1^2=4.9,\sigma_2^2=2.2,\gamma_1=2.1,\gamma_2=1.1$. It can be calculated that $r(x)-\mu_1=-0.8147<0,\beta\frac{K}{r(x)}(r(x)-\mu_1)C(x,y)-\mu-\mu_2=-0.6717<0$, thus the conditions (i) in Theorem 2.2 and Theorem 2.3 are satisfied, so that all the plankton populations in the system (1.1) become extinct (see Figure 1). On the other hand, let $\sigma_1^2=0.1,\sigma_2^2=0.001,\gamma_1=0.51,\gamma_2=0.001$. It can be calculated that $r(x)-\mu_1=0.7145>0,\beta\frac{K}{r(x)}(r(x)-\mu_1)C(x,y)-\mu-\mu_2-\theta=0.0647>0$, thus the condition (iii) in Theorem 2.2 and the condition (ii) in Theorem 2.3 are satisfied, so that all the plankton populations in the system (1.1) are weakly persistent in mean(see Figure 2).

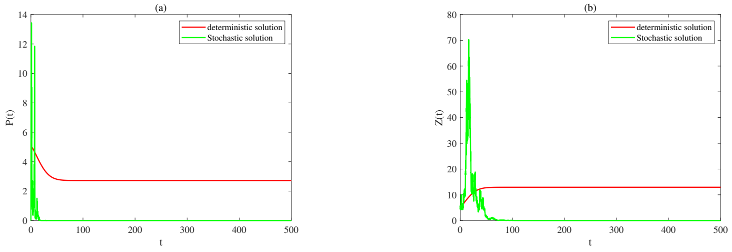

Figure 1. (a) represents the sample paths of phytoplankton $P(t)$; (b) represents the sample paths of zooplankton $Z(t)$.

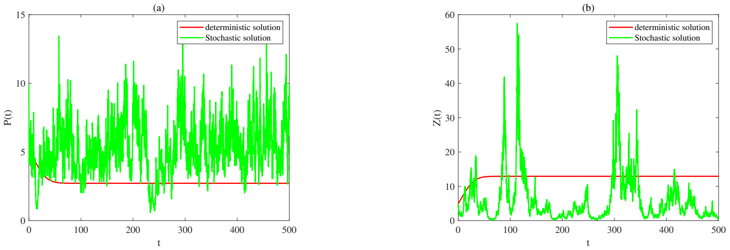

Figure 2. (a) represents the sample paths of phytoplankton $P(t)$; (b) represents the sample paths of zooplankton $Z(t)$.

Example 3.2. Let $\sigma_1^2 = 0.1, \sigma_2^2 = 1.1, \gamma_1 = 0.51, \gamma_2 = 0.001$. It can be calculated that $r(x) - \mu_1 = 0.7145 > 0, \beta\frac{K}{r(x)}(r(x) - \mu_1)C(x, y) - \mu - \mu_2 = -0.4798 < 0$, thus the condition (iii) in Theorem 2.2 and the condition (i) in Theorem 2.3 are satisfied, so that the phytoplankton in the system (1.1) is weakly persistent in mean, and the zooplankton becomes extinct (see Figure 3).

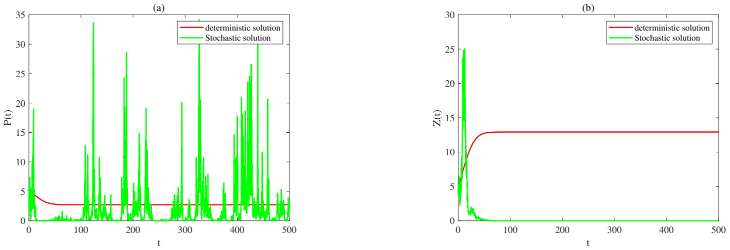

Figure 3. (a) represents the sample paths of phytoplankton $P(t)$; (b) represents the sample paths of zooplankton $Z(t)$.

Next, we will explore the impacts of plankton body size on the dynamics of model (1.1) when the stochastic environmental noise and Lévy noise remains lower intensity to ensure the survival of plankton.(such as $\sigma_1^2 = 0.1, \sigma_2^2 = 0.001, \gamma_1 = 0.01, \gamma_2 = 0.001,$).

Example 3.3. If $x < 0.0404$ or $x > 39.5994$, then it can be calculated that $r(x) - \mu_1 < 0$, so that the phytoplankton population in the system (1.1) becomes extinct (see Figure 4). Furthermore, the zooplankton population becomes extinct.

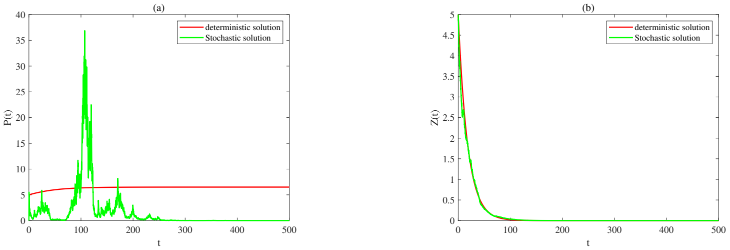

Figure 4. (a) represents the sample paths of phytoplankton $P(t)$ with $x = 0.02$; (b) represents the sample paths of phytoplankton $P(t)$ with $x = 42$.

Example 3.4. Let $x = 1.5, y = 5 < 6.2767$. It can be calculated that $r(x) - \mu_1 = 0.6242 > 0, \beta\frac{K}{r(x)}(r(x) - \mu_1)C(x, y) - \mu - \mu_2 - \theta = 1.4105 > 0$, thus the condition (iii) in Theorem 2.2 and the condition (ii) in Theorem 2.3 are satisfied, so that all the plankton populations in the system (1.1) are weakly persistent in mean(see Figure 5).

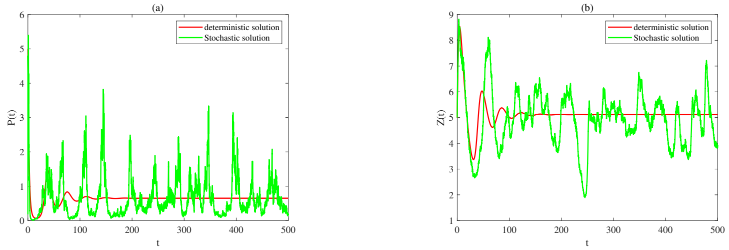

Figure 5. (a) represents the sample paths of phytoplankton $P(t)$ with $x = 1.5$, $y = 5$; (b) represents the sample paths of zooplankton $Z(t)$ with $x = 1.5$, $y = 5$.

Example 3.5. Let $x = 1.5, y = 7 > 6.2767$. It can be calculated that $r(x) - \mu_1 = 0.6242 > 0, \beta\frac{K}{r(x)}(r(x) - \mu_1)C(x, y) - \mu - \mu_2 = -0.0469 < 0$, thus the condition (iii) in Theorem 2.2 and the condition (i) in Theorem 2.3 are satisfied, so that the phytoplankton in the system (1.1) is weakly persistent in mean, and the zooplankton becomes extinct (see Figure 6).

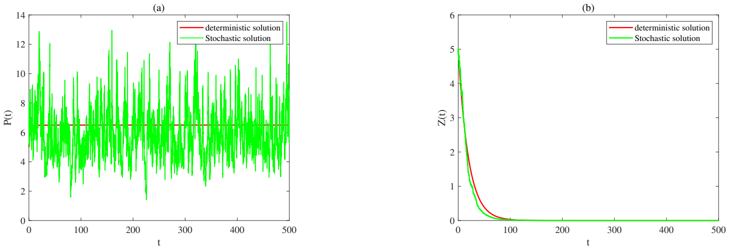

Figure 6. (a) represents the sample paths of phytoplankton $P(t)$ with $x = 1.5$, $y = 7$; (b) represents the sample paths of zooplankton $Z(t)$ with $x = 1.5$, $y = 7$.

Example 3.6. Let $x = 20, y = 37 \in (36.9250, 43.0750)$. It can be calculated that $r(x) - \mu_1 = 0.0477 > 0, \beta\frac{K}{r(x)}(r(x) - \mu_1)C(x, y) - \mu - \mu_2 - \theta = 0.0079 > 0$, thus the condition (iii) in Theorem 2.2 and the condition (ii) in Theorem 2.3 are satisfied, so that all the plankton populations in the system (1.1) are weakly persistent in mean (see Figure 7).

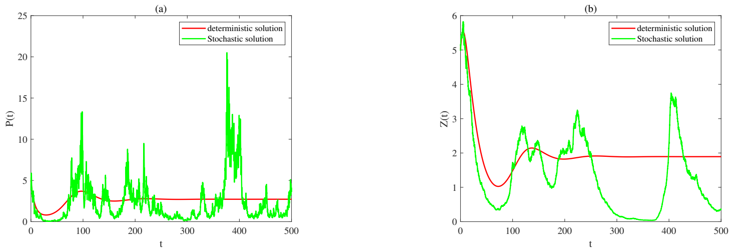

Figure 7. (a) represents the sample paths of phytoplankton $P(t)$ with $x = 20$, $y = 37$; (b) represents the sample paths of zooplankton $Z(t)$ with $x = 20$, $y = 37$.

Example 3.7. Let $x = 20, y = 36$. It can be calculated that $r(x) - \mu_1 = 0.0477 > 0, \beta\frac{K}{r(x)}(r(x) - \mu_1)C(x, y) - \mu - \mu_2 = -0.0486 < 0$. Or let $x = 20, y = 45$. It can be calculated that $r(x)-\mu_1 = 0.0477 > 0, \beta\frac{K}{r(x)}(r(x)-\mu_1)C(x, y)-\mu-\mu_2 = -0.0505 < 0$, thus the condition (iii) in Theorem 2.2 and the condition (i) in Theorem 2.3 are satisfied, so that the phytoplankton in the system (1.1) is weakly persistent in mean, and the zooplankton becomes extinct (see Figure 8 and 9).

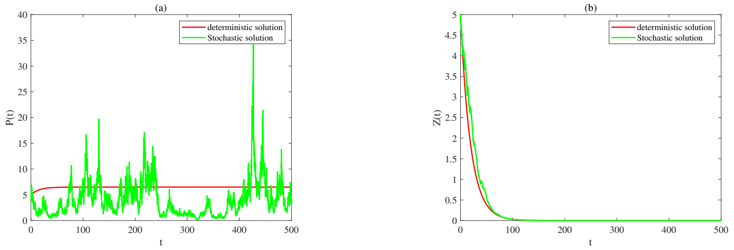

Figure 8. (a) represents the sample paths of phytoplankton $P(t)$ with $x = 20$, $y = 36$; (b) represents the sample paths of zooplankton $Z(t)$ with $x = 20$, $y = 36$.

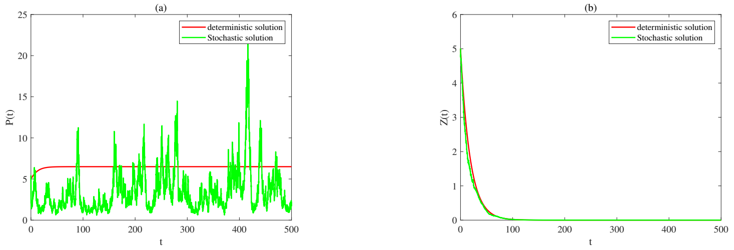

Figure 9. (a) represents the sample paths of phytoplankton $P(t)$ with $x = 20$, $y = 45$; (b) represents the sample paths of zooplankton $Z(t)$ with $x = 20$, $y = 45$.

Example 3.8. Let $x = 1.5, y = 5, K = 1.5$, and other parameter values remain unchanged. It can be calculated that $\delta_1\frac{r(x)}{K} - \delta_2\beta C(x, y) > 0$, thus the condition in Theorem 2.4 is satisfied, so that all the plankton populations in the system (1.1) are persistent (see Figure 10).

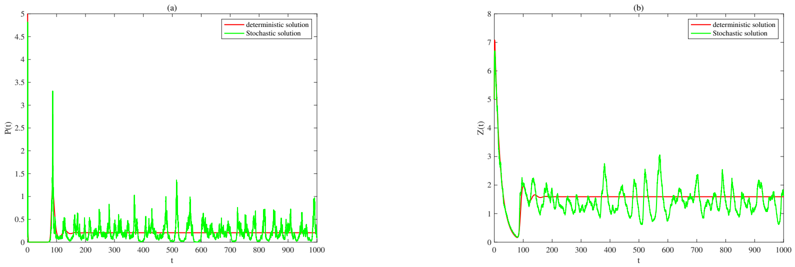

Figure 10. (a) represents the sample paths of phytoplankton $P(t)$ with $x = 1.5$, $y = 5$, $K = 1.5$; (b) represents the sample paths of zooplankton $Z(t)$ with $x = 1.5$, $y = 5$, $K = 1.5$.

4. Conclusion

Due to the unpredictable changes in the natural environment and the adaptive evolution of species size, the stochastic disturbances effect and body sizes should be considered and incorporated into the biological models, which could be represented by the standard Brownian motion, Lévy noise and cell size. In this paper, taking into account the effects of plankton body size, we have studied the dynamic system of the phytoplankton-zooplankton model with white noise and Lévy noise.

In some random environment effects, we find that the small stochastic fluctuations and Lévy noise can remain the long-term survival of plankton (as shown in Figure 2). When the cells of phytoplankton are extremely small or extremely large, plankton will become extinct (as shown in Figure 4). This means that intermediate sized phytoplankton cells are more beneficial to the survival of plankton. When phytoplankton cells are small, increasing the size of zooplankton body can lead to the extinction of zooplankton (as shown in Figures 5 and 6). When the size of phytoplankton cells is large, the size of larger or smaller zooplankton body can lead to their extinction (as shown in Figures 8 and 9), and only the middle sized zooplankton cell can sustain their survival (as shown in Figure 7). In summary, a moderate plankton body size is necessary to maintain the long-term survival of plankton. In addition, we found that under the large environmental fluctuations and Lévy noise, no matter how phytoplankton cells and zooplankton body size change, the plankton will go extinct (as shown in Figure 1).

In the real aquatic environments, however, the growth of plankton is affected by many factors, such as oxygen [35], diffusion and chlorophyll [36], water temperature [37] etc.. So it is necessary to consider the impact of other factors on the growth of plankton. For example, the plankton model with cell size subject to impulsive perturbation could be examined.

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