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Nonlinear SEIS Epidemic Dynamics with Fractional-Order Time: Analytical and Numerical Results

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Abstract This study investigates a novel SEIS epidemic model that incorporates fractional-order derivatives to account for the memory effects of the disease spread. The Caputo derivative is specifically employed. Furthermore, the model considers the influence of behavioral changes in susceptible individuals by incorporating a general non-linear function that depends on their population size. Leveraging recent advancements in fractional differential equations theory, we establish the existence of solutions and analyze the critical conditions for the system's steady states to achieve global asymptotic stability. Finally, the validity and applicability of the theoretical model are corroborated through numerical simulations using real-world data on the prevalence of Pneumococcus.

Keywords Non-linear epidemic model, fractional system, stability of equilibria

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1. Introduction

This work deals with the long-term behavior of a nonlinear fractional SEIS epidemic model with recruitment and varying total population size. Indeed, we divide the host population into three interactive compartments denoted by (S), (E), and (I), where $S(t)$ represents the number of susceptible individuals at time t, $E(t)$ is the number of individuals exposed to the infection but not yet infectious, and $I(t)$ is the number of infected individuals. We assume that the susceptible population has a constant recruitment rate A. Furthermore, we model the number of new cases per unit of time by $\beta\varphi\left(S(t)\right)I(t)$ where β is the transmission rate and φ is an increasing function defined on $[0,\infty[$ such that $\varphi(0)=0$. The function φ can be used to describe how different factors affect the rate of infection. For instance, if $\varphi(s)=s$, it means that the contact rate is fixed and does not depend on the number of susceptible individuals. However, $\varphi(s)=\frac{s}{1+ks}$, means that the contact rate declines as more people become aware of the disease, where k is a parameter

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that measures the effect of awareness [1]. The function φ includes in addition other forms of infection rates, such as the square root factor $\sqrt{s}$ [31], which is a special case of the power form S^p that was studied in [22]. We are interested in the long-term effects of the disease outbreak, so we include both the natural death rate μ and the disease-induced death rate ϵ. We also assume that an exposed person becomes infected at the rate α and that an infected person recovers without any disease-acquired immunity, thus becoming susceptible again at the rate λ. Based on these assumptions, the following integer-order SEIS epidemic model is derived:

$$\begin{cases} \frac{dS}{dt}(t) = A - \mu S(t) - \beta\varphi(S(t))\, I(t) + \lambda I(t), \\ \frac{dE}{dt}(t) = -(\mu+\alpha)\, E(t) + \beta\varphi(S(t))\, I(t), \\ \frac{dI}{dt}(t) = -(\mu+\epsilon+\lambda)\, I(t) + \alpha E(t), \end{cases} \tag{1.1}$$

under positive initial conditions $S(0) = S_0$, $I(0) = I_0$, $E(0) = E_0$. In the paper [10], using the geometrical approach of Li and Muldowney, the authors found the threshold for system model (1.1) in the special case $\varphi(s) = s$ which determines whether the disease dies out or persists in an endemic level. The same results for SEIS with vertical and horizontal transmission are established by Korobeinikov using Lyapunov functions [18]. Recently, Naim et al. [25] presented a detailed analysis of a SEIS model with nonlinear force infection. It is shown that if the basic reproduction number is less than one, then the disease-free equilibrium is globally asymptotically stable. Otherwise, a unique positive equilibrium appears and it is locally asymptotically stable. Furthermore, by using the Lyapunov function approach, global asymptotic stability is obtained under additional conditions. Many types of SEIS epidemic models are studied in the literature [2, 33, 35]. However, to the best of our knowledge, the global stability of system (1.1) is not yet investigated.

On the other hand, many researchers used fractional differential equations to model the evolution of transmissible diseases [20, 29, 30, 36, 37]. These equations involve non-integer order derivatives defined by integrals, making them non-local operators. This feature allows them to capture the memory effect seen in various phenomena. This includes modeling viscoelasticity, polymers, and anomalous diffusion. It also extends to medical applications, such as studying hyperthermia in cancer treatment, and other fields where a non-Markovian approach is more appropriate [5, 23, 32]. Therefore, fractional derivatives have attracted considerable attention in recent years, as they can be applied to formulate simple and unified models for complex materials and processes. The field of fractional calculus offers various fractional-order derivatives, including Riemann-Liouville, Grünwald-Letnikov, and Caputo derivatives [27]. Among these, the Caputo derivative offers distinct advantages for modeling real-world phenomena. Notably, unlike Riemann-Liouville derivatives, the Caputo derivative of a constant is zero. This simplifies the analysis of fractional differential equations involving the Caputo derivative. Additionally, the Caputo derivative allows for straightforward formulation of initial conditions, similar to classical integer-order differential equations. This eases the process of incorporating real-world data into the model. Finally, the Caputo derivative boasts a well-developed mathematical framework with established results on existence, uniqueness, and stability [27]. This robust foundation allows for confident analysis and reliable conclusions when using the Caputo derivative in models. Due to these advantages, we will employ the Caputo derivative in this work. A formal definition is provided in Appendix A. By integrating system (1.1) and changing its uniform

kernel (see [19]), one can transform it into the following Caputo's fractional system

$$\begin{cases} {}_0^C D_t^q S(t) = A - \mu S(t) - \beta\varphi(S(t)) I(t) + \lambda I(t), \\ {}_0^C D_t^q E(t) = -(\mu+\alpha) E(t) + \beta\varphi(S(t)) I(t), \\ {}_0^C D_t^q I(t) = -(\mu+\epsilon+\lambda) I(t) + \alpha E(t), \end{cases} \tag{1.2}$$

with the same initial conditions in the integer system (1.1). The system (1.2) gives rise to several fundamental inquiries. For instance, does the disease described by (1.2) converge to a constant level, either positive or null? Does it exhibit stability or fluctuate periodically or in an unpredictable manner? To address these questions, it is crucial to comprehend the asymptotic behavior of the system (1.2), which will be thoroughly examined in section 3. However, before delving into the analysis of the system's asymptotics, we establish in section 2 that system (1.2) is both mathematically and biologically well-posed. Additionally, in section 4, we present a series of numerical simulations that depict solutions to the system (1.2), utilizing actual data on Pneumococcus prevalence among children under 2 years old.

2. Basic model properties

The sub-population sizes $S(t)$, $E(t)$, and $I(t)$ can never be negative. So, first, we show the existence, positivity, and boundedness of solutions of model (1.2) which are useful in studying its asymptotic properties.

Lemma 2.1. *Let (S,E,I) be a continuous solution to system (1.2) with positive initial condition (S_0,E_0,I_0). If (S,E,I) is defined on $[0,T]$ for some $T\in(0,\infty)$, then, we have*

$$S(t)>0, E(t)>0, I(t)>0, \qquad \forall t\in[0,T], \tag{2.1}$$

$$N(t)\le \min\left(\frac{A}{\mu}, N(0)\right), \qquad \forall t\in[0,T], \tag{2.2}$$

where $N(t)=S(t)+E(t)+I(t)$.

Proof. Define τ_+ as follows $\tau_+=\inf\{t\ge 0,\ S(t)E(t)I(t)=0\}$. Note that $\tau_+>0$. This results from the positivity of S_0,E_0,I_0 and the continuity of the functions solution $S(t)$, $E(t)$ and $I(t)$. We claim that $\tau_+=T$. For the sake of contradiction, suppose that $\tau_+<T$. Then, for all $t\in[0,\tau_+]$, we have

$$\begin{aligned} {}_0^C D_t^q E(t) &\ge -(\mu+\alpha) E(t), \\ {}_0^C D_t^q I(t) &\ge -(\mu+\epsilon+\alpha) I(t). \end{aligned}$$

Therefore, by Lemma A.2, we infer that

$$\begin{aligned} E(t) &\ge E_0\, \mathbb{E}_q\left(-(\mu+\alpha) t^q\right), \\ I(t) &\ge I_0\, E_q\left(-(\mu+\epsilon+\alpha) t^q\right). \end{aligned}$$

In particular, we deduce that $E(\tau_+), I(\tau_+)>0$. Hence, $S(\tau_+)=0$. Furthermore, $\varphi(0)=0$, thereby

$${}_0^C D_t^q S(\tau_+) = A+\lambda I(\tau_+) > 0. \tag{2.3}$$

On the other hand, by Lemma A.1, we have

$$\begin{aligned} {}_0^C D_t^q S(\tau_+) &= \frac{S(\tau_+) - S_0}{\Gamma(1-q)\tau_+^q} + \frac{q}{\Gamma(1-q)} \int_0^{\tau_+} (\tau_+ - s)^{-q-1} \left[S(\tau_+) - S(s)\right] ds, \\ &= \frac{-S_0}{\Gamma(1-q)\tau_+^q} - \frac{q}{\Gamma(1-q)} \int_0^{\tau_+} (\tau_+ - s)^{-q-1} \left[S(s)\right] ds, \\ &< 0, \end{aligned} \tag{2.4}$$

which contradicts (2.3). Then, our claim is true. That is, the assertion (2.1) holds. Now, summing the three equations of (1.2) yields that

$$\begin{aligned} {}^C D_t^q N(t) &= A - \mu N(t) - \varepsilon I(t), \\ &\leq A - \mu N(t). \end{aligned} \tag{2.5}$$

Using Lemma A.2 again, we get

$$\begin{aligned} N(t) &\leq \frac{A}{\mu} + \left(N(0) - \frac{A}{\mu}\right) \mathbb{E}_q\left(-\mu t^q\right) \qquad \forall t \in [0, T] \\ &\leq \max\left(\frac{A}{\mu}, N(0)\right), \end{aligned} \tag{2.6}$$

because $\mathbb{E}_q\left(-\mu t^q\right) \leq 1$. This completes the proof of Lemma 2.1. □

Theorem 2.1. *Assume that φ is a locally Lipschitz continuous function on $[0, \infty)$. For any positive initial condition (S_0, E_0, I_0), system (1.2) possesses a unique solution for all $t \in [0, \infty)$.*

Proof. Set $X(t) = (S(t), E(t), I(t))$. The system (1.2) can be written concisely as

$${}_0^C D_t^q X(t) = f(X(t)),$$

where $f = (f_1, f_2, f_3) : \mathbb{R}^3 \to \mathbb{R}$ such that

$$\begin{cases} f_1(x_1, x_2, x_3) = A - \mu x_1 - \beta\varphi(x_1)x_3 + \lambda x_3, \\ f_2(x_1, x_2, x_3) = -(\mu + \alpha)x_2 + \beta\varphi(x_1)x_3, \\ f_3(x_1, x_2, x_3) = -(\mu + \varepsilon + \lambda)x_3 + \alpha x_2. \end{cases} \tag{2.7}$$

Since φ is supposed locally Lipschitz continuous, it is the same for the field f. Therefore, system (1.2) has a unique local solution [6]. Furthermore, owing the prior estimates (2.1) and (2.2) of Lemma 2.1, and the continuation theorem in [34], the solution $X(t)$ is defined on $[0, \infty)$. □

3. Mains results

First, denote φ^{-1} the inverse mapping of the continuous increasing function φ. The equilibria of model (1.2) are the solutions of the system equations $f(S, E, I) = 0$. Direct calculation shows that system (1.2) has two equilibrium states: the disease-free equilibrium $X^0\left(\frac{A}{\mu}, 0, 0\right)$, and a unique positive equilibrium state $X^*(S^*, E^*, I^*)$

determined by

$$S^* = \varphi^{-1}\left(\frac{(\mu+\alpha)(\mu+\varepsilon+\lambda)}{\alpha\beta}\right),\ I^* = \frac{\alpha}{\mu(\mu+\varepsilon+\lambda)+\alpha(\mu+\varepsilon)}(A-\mu S^*),$$

$E^* = \frac{\mu+\varepsilon+\lambda}{\alpha} I^*$, provided that $A-\mu S^* > 0$ which is equivalent to

$$\mathcal{R}_0 \triangleq \beta\varphi\left(\frac{A}{\mu}\right) \times \frac{\alpha}{\mu+\alpha} \times \frac{1}{\mu+\varepsilon+\lambda} > 1. \tag{3.1}$$

Note that $\beta\varphi\left(\frac{A}{\mu}\right)$ is the number of new cases per unit time caused by a single infectious person in a population of $\frac{A}{\mu}$ susceptible individuals, $\frac{\alpha}{\mu+\alpha}$ is the fraction of individuals progressing from exposed to infectious and $\frac{1}{\mu+\varepsilon+\lambda}$ is the average infectious time taking death into account. Thus, $\mathcal{R}_0$ represents the average number of secondary infections from a single infectious host in a totally susceptible population of size $\frac{A}{\mu}$. It is the basic reproduction number for a disease modeled by system (1.2). In the following results, we prove that the dynamics of (1.2) are completely determined by the threshold parameter $\mathcal{R}_0$.

Theorem 3.1. *Suppose that φ is a differentiable function on $(0,\infty)$. If $\mathcal{R}_0 > 1$, the endemic equilibrium state X^* is uniformly stable and globally Mittag-Leffler attractive. That is*

$$\exists C_1, C_2 > 0,\ \forall t \geqslant 0, \qquad (S(t)-S^*)^2+(E(t)-E^*)^2+(I(t)-I^*)^2 \leqslant C_1\mathbb{E}_q(-C_2 t^q).$$

Proof. See Appendix C.1. □

Theorem 3.2. *Suppose that φ is a differentiable function on $(0,\infty)$. The disease-free equilibrium state X_0 is uniformly stable and globally attractive if and only if $\mathcal{R}_0 \leq 1$.*

Proof. See Appendix C.2. □

4. Numerical simulations

In the previous section, we discussed theoretical findings related to the solutions of the fractional-order model (1.1). To illustrate the practicality of these findings, following the research by Chikhaoui et al. [3], we utilize the dataset presented in their work to simulate the spread of the pneumococcal infectious disease among children under 2 years old. Their study, based on data from the Pediatric Hospital of Ibn Rochd in Casablanca, suggests a significant decrease in patient numbers following a vaccination program implementation. The observed reduction corresponds to a rate of 13.5% per 100,000 children. Therefore, we will simulate a sample of $N = 5,000$ for this purpose, and consider that the infection rate is $\beta = 0.000135$, which is the same transmission rate observed in [4]. To account for mortality, we incorporate a daily death rate μ of 0.000015753. This value is derived from the average annual death rate of 18.9% reported in the data bank [24] conducted during the same timeframe as the clinical investigation by Chikhaoui et al. [3]. To streamline the model, particularly the birth rate parameter is assumed to be $A = \mu N = 6.868$. This simplification is due to the short simulation period in the fractional order scale

and also due to the young age of the population. To ensure comprehensive analysis, additional parameters were retrieved from another source (details in Table 1).

Parameter symbol	Description	Estimate or range	Source
μ	natural death rate of the population	1.5753×10^{-5} day^{-1}	[24]
ϵ	death rate due to the disease	0.05 day^{-1}	[28]
β	disease transmission rate	1.35×10^{-4} person^{-1} day^{-1}	[3,4]
$1/\alpha$	mean duration of incubation	$1-3$ day	[28]
$1/\lambda$	mean duration of infectious period	49.7 day	[14]
q	fractional order	$0.5-1$	assumed

Table 1. Estimates and ranges of input parameters of the system model (1.2).

To implement the numerical simulations, we used Julia which is a high-level, general-purpose dynamic programming language. Since the main objective is to solve numerically our fractional system (1.1), the principle package used is FDE-Solver, which tackles fractional differential equations and focuses on the Caputo definition of fractional derivatives, and offers numerical solutions for these equations. The method used in this code is based on the predictor-corrector approach of Adams-Bashforth-Moulton described in [7]. This approach's convergence and accuracy properties are analyzed in [8]. The stability issues of this method are discussed in [13]. The idea of using multiple iterations for the corrector step when dealing with multiterm FDEs comes from [9]. This code also uses the FFT algorithm from [15] to compute the discrete convolutions efficiently, reducing the computational cost from $\mathcal{N}^2$ to $\mathcal{N}\log^2\mathcal{N}$, where $\mathcal{N}$ is the number of time points for the solution. For more detailed information about the code, we refer to [11,16,26]. In the following, we propose the functional response $\varphi(s) = \exp(1-a)\sqrt{s}$ where the parameter a represents the control measures used to reduce infectious contacts. Figure 1 represents the simulated results from system model (1.2) during a period of 360 days with the parameter values of Table 1 and fractional orders $q = 0.8$. In the first row of this figure, $a = 0$ and thus endemic dynamics are predicted since $\mathcal{R}_0 = 1.1503 > 1$. Whereas in the second row, $\mathcal{R}_0 = 0.4232 < 1$ because $a = 1$, the disease is predicted to die out i.e $\lim_{t\to\infty} I(t) = 0$. However, as we can see from both cases in Figure 1, it is not clear for this period time when the time series $S(t)$, $I(t)$ or $E(t)$ starts to stabilize either around endemic or disease-free equilibrium states. That is why we show figure 2 to depict the long time behavior of the solutions as the final time is 3600 days. More specifically, as we see in the right plot in Figure 2, since the basic reproduction number is $\mathcal{R}_0 > 1$, then, according to Theorem 3.1, the Pneumonia disease will persist at an endemic level. Besides, in virtue of Theorem 3.2, to control the propagation of the disease in the population it is imperative to reduce the value of $\mathcal{R}_0$. One way to achieve this goal is to introduce a control parameter a. The effect of this parameter a on the dynamics of the infectious population will be shown in Figure 3 depicting the curves of the infectious population for different values of a. A specific form of the term multiplied by the contact term $\sqrt{s}$ was chosen, which is $\exp(1-a)$. This modeling approach allows us to express the influence of measures taken to prevent contact between susceptible and infectious individuals. If the measures taken are nonexistent, or rather, if the effort to prevent transmission is negligible, the consequences are exponentially proportional, meaning that 0% effort causes an increase of $\exp(1)$ in the contact rate. If the

efforts made are almost exemplary, then the value of a is equal to 1 or 100%. This can be interpreted as a significant reduction in contacts while only maintaining the "natural" rate intrinsic to the nature of the disease itself. Thus, as the simulations show, the decrease in the number of infectious individuals is significantly important when the value of a tends towards 1. This control parameter acts as a proxy for the vaccination program mentioned in [3]. It indirectly influences the prevalence of new invasive pneumococcal disease (IPD) cases in neonates, similar to how a vaccine reduces a specific infection. Finally, the influence of fractional order on the dynamics of different population classes remains an intriguing question. In fact, the fractional order does not alter the long-term behavior (asymptotic behavior) of the population class sizes. Although only the infective population is explicitly shown, this observation holds true for the other two classes as well. However, smaller values of q can dampen the solution, hindering it from reaching its equilibrium state as quickly. This is clearly depicted in Figure 4 where the solutions $I(t)$ corresponding to each fractional order (see color code) take the longest time to converge toward $I^* \approx 689$ when $q = 0.65$ for example.

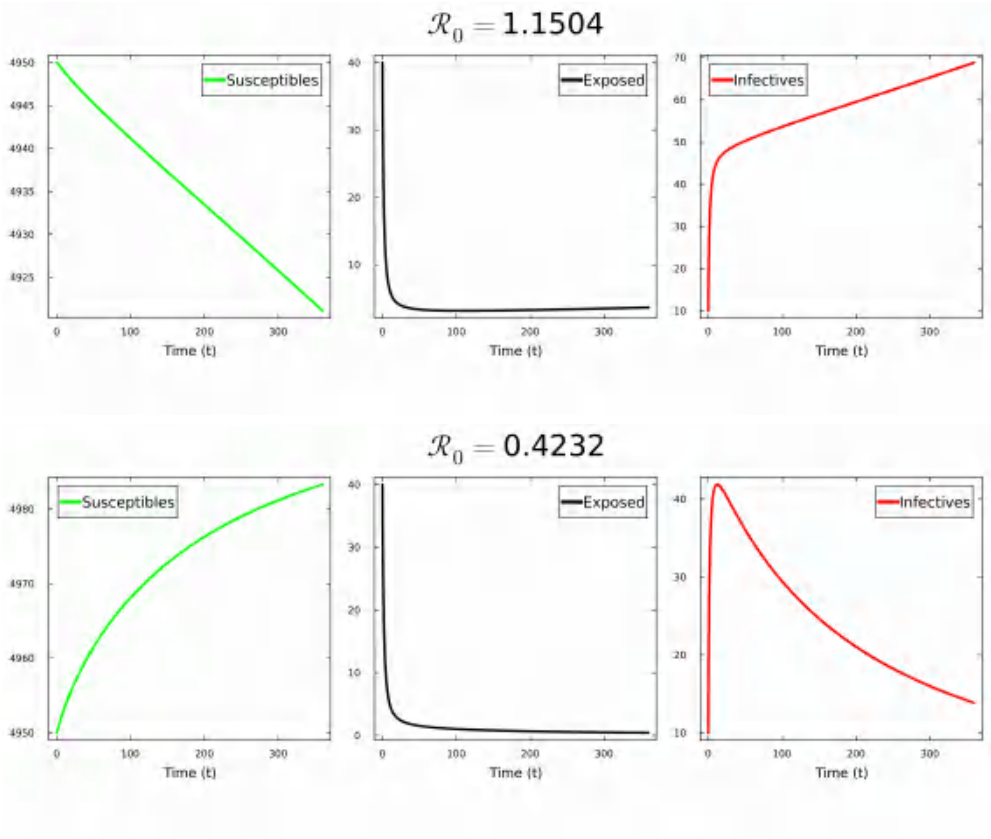

Figure 1. Time series of model (1.2) with the parameter values of Table 1 and $q = 0.8$. In the first row $a = 0$. In the second row $a = 1$.

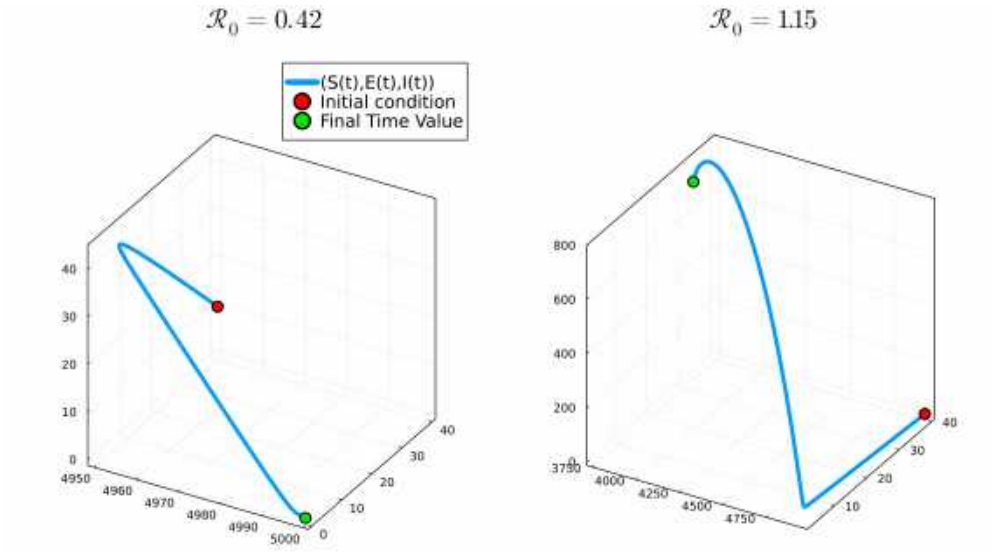

Figure 2. Phase portraits $(S(t), E(t), I(t)$ with $q = 0.95$ and $T = 3600$ days.

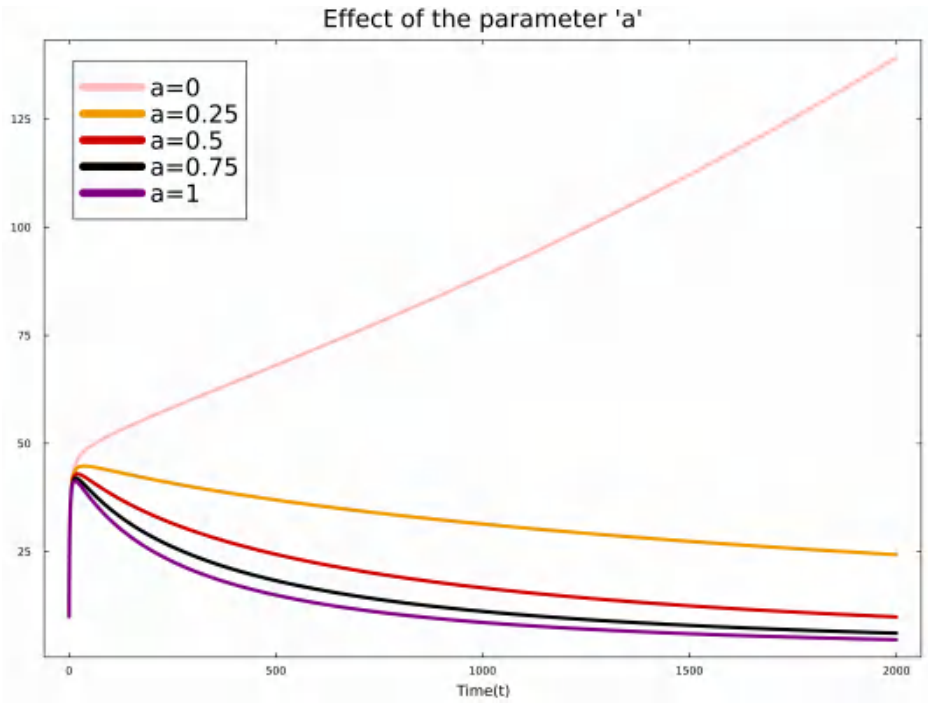

Figure 3. The increase in the value of a causes a decrease in the number of infected individuals.

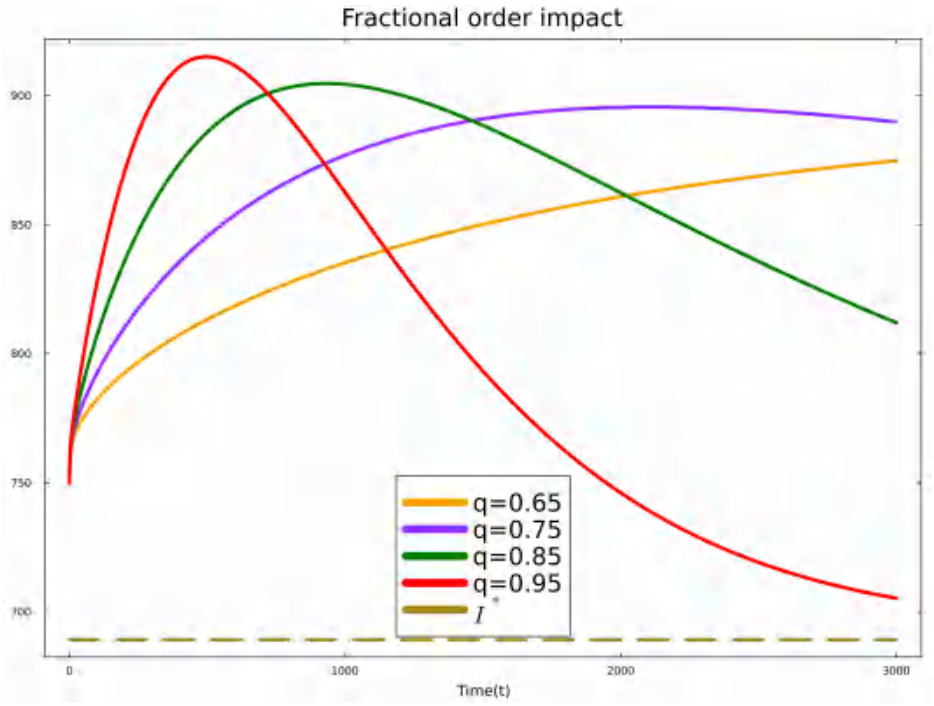

Figure 4. Time series of $I(t)$ with multiple values of q from 0.65 to 0.95. The initial values are exceptionally here $S(0) = 4200, E(0) = 50$ and $I(0) = 750$.

5. Conclusion

In conclusion, this study introduces novel modifications to the classical SEIS epidemic model. Caputo derivatives are incorporated to capture memory effects in disease dynamics, while a generalized incidence rate allows for the modeling of diverse transmission scenarios. The model's mathematical and biological validity is established by demonstrating the existence, positivity, and boundedness of solutions. Furthermore, we derive global asymptotic stability conditions for the system's equilibria based on the basic reproduction number. Notably, the convergence rate for the endemic equilibrium exhibits a Mittag-Leffler-type behavior, analogous to the exponential convergence observed in integer-order models.

Future work focuses on two key areas: expanding model generality and incorporating additional disease stages. Firstly, extending the model to encompass more general incidence functions warrants investigation. This would enable the representation of a broader spectrum of transmission dynamics observed in real-world outbreaks. Secondly, including a recovered compartment with waning immunity is a promising direction for further research. Such a modification would enhance the model's realism by capturing the temporary immunity acquired by recovered individuals, leading to a more accurate representation of disease dynamics. Addressing these areas has the potential to significantly improve our ability to model and predict the complexities of infectious disease outbreaks.

A. Appendix A

A.1. Basic definitions

We start by introducing some key definitions and results from fractional calculus that we will use in the proofs of the main results. For further details on this subject, we refer the reader to the references [6, 17, 27].

Definition A.1. The Riemann-Liouville fractional integral operator of order $q > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$$I^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the gamma function.

Definition A.2. The Caputo fractional derivative of order q of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is defined by

$${}_0^C D_t^q f(t) = I^{n-q} D^n f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-\tau)^{n-q-1} f^{(n)}(s) ds,$$

where $n-1 < q < n$, $n \in \mathbb{N}$) and f has absolutely continuous derivatives up to order $(n-1)$. In particular, when $0 < q \leq 1$, we have

$${}_0^C D_t^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^\alpha} ds.$$

Definition A.3. The one and two parameter Mittag-Leffler functions are defined as

$$\mathbb{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \quad \mathbb{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha, \beta > 0$ and $z \in \mathbb{C}$.

A.2. Useful lemmas

Lemma A.1. *For $q \in (0,1)$ and $X \in \mathcal{C}([0,T])$ with ${}^C D_t^q X \in \mathcal{C}([0,T])$, it holds*

$${}_0^C D_t^q X(t) = \frac{X(t) - X(0)}{\Gamma(1-q) t^q} + \frac{q}{\Gamma(1-q)} \int_0^t (t-s)^{-q-1} [X(t) - X(s)] ds.$$

Lemma A.2. *Let X be a continuous function such that*

$${}_0^C D_t^q X(t) \leq a - bX(t), \qquad \forall t \in [0,T],$$

where $q \in (0,1), a, b \in \mathbb{R}^2$ and $b \neq 0$. Then

$$X(t) \leq \frac{a}{b} - \left(X(0) - \frac{a}{b} \right) \mathbb{E}_q(-bt).$$

Theorem A.1. *Let $V : \Omega \to \boldsymbol{R}$ and $X : [t_0, \infty) \to \Omega$ be two continuous and differential functions, where $\Omega \subset \mathbb{R}^d$ is a convex set. Suppose that V is convex over Ω. Then, for any time instant $t \geq t_0$*

$${}_{t_0}^C D_t^\alpha V(X(t)) \leq (\nabla V(X(t)))^\intercal {}_{t_0}^C D_t^\alpha X(t), \quad \forall \alpha \in (0,1).$$

B. Appendix B

B.1. Useful estimates

A continuous function $\psi : [0,\infty) \to [0,\infty)$ is said to belong to class $\mathcal{K}$ if it is increasing and $\psi(0) = 0$. We have the following double inequality for such types of functions.

Lemma B.1. *Let ψ be a function of class $\mathcal{K}$ differentiable on $(0,\infty)$, and $a, b > 0$. Then, if $c \in (a,b)$ there exists positive constants M_1, M_2 and M_3 such that*

$$\frac{(x-c)\left(\psi(x)-\psi(c)\right)}{\psi(x)} \geq M_1(x-c)^2, \qquad \forall x \in [a,b], \tag{B.1}$$

$$M_2\left(x-c\right)^2 \leqslant \int_c^x \frac{\psi(s)-\psi(c)}{\psi(s)} ds \leqslant M_3\left(x-c\right)^2, \qquad \forall x \in [a,b]. \tag{B.2}$$

Proof. Define the function F on $(0,\infty)$ by

$$F(x) = \begin{cases} \dfrac{\psi(x)-\psi(c)}{(x-c)\psi(x)} & \text{if } x \neq c, \\ \\ \dfrac{\psi'(x)}{2\psi(x)} & \text{if } x = c. \end{cases} \tag{B.3}$$

Since ψ is a differentiable function of class $\mathcal{K}$, the function F is a positive continuous on $(0,\infty)$. Then

$$M_1 = \inf_{x\in[a,b]} F(x) > 0, \qquad M_2 = \sup_{x\in[a,b]} F(x) > 0.$$

Thereby, $(x-c)^2 F(x) \geq M_1(x-c)^2$ on $[a,b]$, which proves (B.1). For the inequality (B.2), we take $M_2 = \frac{1}{2}M_1$ and $M_3 = \frac{1}{2}M_2$. Using the assumption $c \in (a,b)$, (B.1) can be easily obtained by studying, on the interval $[a,b]$, the real functions

$$x \mapsto \int_c^x \frac{\psi(s)-\psi(c)}{\psi(s)} ds - M_i\left(x-c\right)^2, \quad i = 2,3.$$

□

C. Appendix C

C.1. Proof of Theorem 3.1

Define the non-negative functions

$$J_S(t) = \int_{S^*}^{S(t)} \frac{\varphi(x)-\varphi(S^*)}{\varphi(x)} dx,$$

$$J_E(t) = \int_1^{\frac{E(t)}{E^*}} \frac{x-E^*}{x} dx,$$

$$J_I(t) = \int_1^{\frac{I(t)}{I^*}} \frac{x-I^*}{x} dx.$$

Using the equations verified by the components of the positive equilibrium state X^*, one can write the system (1.2) as follows.

$$\begin{cases} {}_0^C D_t^q S(t) = -\mu(S(t)-S^*) - \beta\varphi(S^*)I^*\left(\dfrac{\varphi(S(t))I(t)}{\varphi(S^*)I^*} - 1\right) + \lambda(I(t)-I^*), \\ {}_0^C D_t^q E(t) = \beta\varphi(S^*)I^*E(t)\left(-1 + \dfrac{\varphi(S(t))I(t)E^*}{\varphi(S^*)I^*E(t)}\right), \\ {}_0^C D_t^q I(t) = \alpha E^* I(t)\left(-1 + \dfrac{I^*E(t)}{I(t)E^*}\right). \end{cases}$$

Therefore, using Theorem A.1, we get

$$\begin{aligned} {}_0^C D_t^q J_S(t) \leq & slant \frac{\varphi(S(t)) - \varphi(S^*)}{\varphi(S(t))} {}_0^C D_t^q S(t) \\ = & -\frac{\mu(S(t)-S^*)(\varphi(S(t)) - \varphi(S^*))}{\varphi S(t)} - \beta\varphi(S^*)I^*\left(\frac{\varphi(S(t))I(t)}{\varphi(S^*)I^*} - 1\right) \\ & \times \left(1 - \frac{\varphi(S^*)}{\varphi(S(t))}\right) + \frac{\lambda(\varphi(S(t)) - \varphi(S^*))(I(t)-I^*)}{\varphi(S(t))} \\ = & -\frac{\mu(S(t)-S^*)(\varphi(S(t)) - \varphi(S^*))}{\varphi(S(t))} - \beta\varphi(S(t)I(t)) + \beta\varphi(S^*)I^*\frac{I(t)}{I^*} \\ & + \beta\varphi(S^*)I^* - \beta\varphi(S^*)I^*\frac{\varphi(S^*)}{\varphi(S(t))} + \frac{\lambda(\varphi(S(t)) - \varphi(S^*))(I(t)-I^*)}{\varphi(S(t))}. \end{aligned} \tag{C.1}$$

Using again Theorem A.1, we obtain

$$\begin{aligned} {}_0^C D_t^q J_E(t) \leqslant & \left(\frac{E(t)}{E^*} - 1\right) {}_0^C D_t^q E(t) \\ = & \beta\varphi(S^*)I^*\left(\frac{E(t)}{E^*} - 1\right)\left(-1 + \frac{\varphi(S(t))I(t)E^*}{\varphi(S^*)I^*E(t)}\right) \\ = & \beta\varphi(S^*)I^*\left(-\frac{E(t)}{E^*} + \frac{\varphi(S(t))I(t)}{\varphi(S^*)I^*} + 1 - \frac{\varphi(S(t))I(t)E^*}{\varphi(S^*)I^*E(t)}\right) \\ = & \beta\varphi(S(t))I(t) + \beta\varphi(S^*)I^*\left(1 - \frac{E(t)}{E^*} - \frac{\varphi(S(t))I(t)E^*}{\varphi(S^*)I^*E(t)}\right). \end{aligned} \tag{C.2}$$

Similarly, one can estimate the fractional derivative of J_I as follows.

$$\begin{aligned} {}_0^C D_t^q J_I(t) \leqslant & \left(\frac{I(t)}{I^*} - 1\right) {}_0^C D_t^q I(t) \\ = & \alpha E^*\left(\frac{I(t)}{I^*} - 1\right)\left(-1 + \frac{I^*E(t)}{E^*I(t)}\right) \\ = & \alpha E^*\left(-\frac{I(t)}{I^*} + \frac{E(t)}{E^*} + 1 - \frac{I^*E(t)}{E^*I(t)}\right). \end{aligned} \tag{C.3}$$

Now, consider the following combination of J_S, J_I and J_E defined as follows.

$$J_1(t) = J_S(t) + J_E(t) + \frac{\beta\varphi(S^*)I^*}{\alpha E^*}J_I(t).$$

It follows from the inequalities (C.13), (C.14), (C.3) and the linearity of Caputo's derivative that

$$\begin{aligned}{}_0^C D_t^q J_1(t) \leqslant & \frac{-\mu(S(t)-S^*)(\varphi(S(t))-\varphi(S^*))}{\varphi(S(t))} + \frac{\lambda(I(t)-I^*)(\varphi(S(t))-\varphi(S^*))}{\varphi(S(t))} \\ & + \beta\varphi(S^*)I^*\left(3 - \frac{\varphi(S^*)}{\varphi(S(t))} - \frac{\varphi(S^*)I(t)E^*}{\varphi(S^*)I^*E(t)} - \frac{I^*E(t)}{E^*I(t)}\right).\end{aligned}$$

Using the classical comparison between arithmetic and geometric means, we can see that

$$3 - \frac{\varphi(S^*)}{\varphi(S(t))} - \frac{\varphi(S(t))I(t)E^*}{\varphi(S^*)I^*E(t)} - \frac{I^*E(t)}{I(t)E^*} \leqslant 0.$$

Hence,

$$\begin{aligned}{}_0^C D_t^q J_1(t) \leqslant & \frac{-\mu(S(t)-S^*)(\varphi(S(t))-\varphi(S^*))}{\varphi(S(t))} \\ & + \frac{\lambda(I(t)-I^*)(\varphi(S(t))-\varphi(S^*))}{\varphi(S(t))}.\end{aligned} \tag{C.4}$$

Now, we rewrite the S-equation as follows

$${}_0^C D_t^q S(t) = -\mu(S-S^*) - \beta I(t)(\varphi(S)-\varphi(S^*)) - (\beta\varphi(S^*)-\lambda)(I(t)-I^*).$$

In this case, using the monotonicity of φ, we obtain

$$\begin{aligned}{}_0^C D_t^q J_S(t) \leqslant & \frac{-\mu(S(t)-S^*)(\varphi(S(t))-\varphi(S^*))}{\varphi(S(t))} - \frac{\beta I(t)(\varphi(S(t))-\varphi(S^*))^2}{\varphi(S(t))} \\ & - (\beta\varphi(S^*)-\lambda)\frac{(I(t)-I^*)(\varphi(S(t))-\varphi(S^*))}{\varphi(S(t))} \\ \leqslant & -(\beta\varphi(S^*)-\lambda)\frac{(I(t)-I^*)(\varphi(S(t))-\varphi(S^*))}{\varphi(S(t))}.\end{aligned} \tag{C.5}$$

Note that $\beta\varphi(S^*)-\lambda = \frac{A-\mu S^*}{I^*} > 0$. Hence, if we consider the non-negative function

$$J_2(t) = \frac{\lambda}{\beta\varphi(S^*)-\lambda}J_S(t) + J_1(t),$$

the inequalities (C.4) and (C.5) gives

$${}_0^C D_t^q J_2(t) \leqslant \frac{-\mu(S(t)-S^*)(\varphi(S(t))-\varphi(S^*))}{\varphi(S(t))}. \tag{C.6}$$

In particular, we have

$${}_0^C D_t^q J_2(t) \leqslant 0, \tag{C.7}$$

and therefore the equilibrium endemic equilibrium state X^* is uniformly stable (see, Theorem 3 in [21]). However, in the absence of a similar fractional version of Lassale's principal [12], the inequality (C.5) is not sufficient to show the asymptotic stability of X^*. To reach our goal, we shall show that $\inf_{t\geq 0} E(t) > 0$. Indeed, if not, there will exist a sequence of time (t_n) such that $E(t_n) \underset{n\to\infty}{\longrightarrow} 0$. On the other hand, integrating (C.7), yields that

$$J_E(t_n) \leqslant J_2(t_n) \leqslant J_2(0) \qquad \forall n,$$

which gives by substituting $J_E(t_n)$ by its expression

$$\frac{E(t_n)}{E^*} - 1 - \log \frac{E(t_n)}{E^*} \leqslant J_2(0) \qquad \forall n, \tag{C.8}$$

and this is clearly a contradiction because the left hand of (C.8) is unbounded and tends to ∞ when $n \to \infty$. Similarly, one can show that $\inf_{t\geq 0} I(t) > 0$ and $\inf_{t\geq 0} S(t) > 0$. Let $\eta > 0$ be a sufficiently small real number such that $\eta < \inf_{t\geqslant 0} S(t)$. Choose in (B.1) of lemma B.1 $a = \eta$, $b = \max(\frac{A}{\mu}, N(0))$ and $c = S^*$. So, $c \in (a, b)$. Moreover, in view of the inequality (2.6), $S(t) \in (a, b)$ for all $t \geqslant 0$. Then, for the function φ, there exists $M_1 > 0$ such that

$$\frac{(S(t) - S^*)(\varphi(S(t)) - \varphi(S^*))}{\varphi(S(t))} \geqslant M_1(S(t) - S^*)^2 \quad \forall t \geqslant 0.$$

Thereby, from the inequality (C.6), we deduce that

$${}_0^C D_t^q J_2(t) \leqslant -\mu M_1(S(t) - S^*)^2. \tag{C.9}$$

We continue our proof by considering the non-negative function

$$J_3(t) = \int_{N^*}^{N(t)} (x - N^*)dx + \frac{2\mu + \varepsilon}{\lambda} \int_{S^*+E^*}^{S(t)+E(t)} (x - S^* - E^*)dx,$$

where $N = S^* + E^* + I^*$. The application of Theorem (A.1) and straightforward computation, leads to

$${}_0^C D_t^q J_3(t) \leqslant -\mu(S(t)-S^*)^2 - \mu(E-E^*)^2 - (\mu+\varepsilon)(I(t)-I^*)^2 + J_4\left(S(t) - S^*, E(t) - E^*\right), \tag{C.10}$$

where J_4 is the quadratic given by

$$J_4(x, y) = \frac{-\mu(2\mu + \varepsilon)}{\lambda} x^2 - \left(2\mu + \frac{(2\mu + \varepsilon)(2\mu + \alpha)}{\lambda}\right) xy - (\mu + \alpha)y^2.$$

Let $M_2 > 0$ be a sufficiently large number such that the quadratic $J_4(x, y) - M_2x^2$ is negative definite. Then, we have

$$J_4\left(S(t) - S^*, E(t) - E^*\right) \leqslant M_2(S(t) - S^*)^2. \tag{C.11}$$

Finally, we define the function

$$J_5(t) = \frac{M_2}{\mu M_1} J_2(t) + J_3(t).$$

Combining (C.9), (C.10) and (C.11) yields that

$${}_0^c D_t^q J_5(t) \leqslant -\mu(S(t) - S^*)^2 - \mu(E(t) - E^*)^2 - (\mu + \varepsilon)(I(t) - I^*)^2.$$

Furthermore, in view of (B.2) of lemma B.1, and the fact that $\inf_{t\geqslant 0} S(t)$, $\inf_{t\geqslant 0} E(t)$, $\inf_{t\geqslant 0} I(t) > 0$, there exists $M_3 > 0$ such that

$$-\mu(S(t) - S^*)^2 - \mu(E(t) - E^*)^2 - (\mu + \varepsilon)(I(t) - I^*)^2 \geqslant M_3 J_5(t).$$

Hence, ${}_0^cD_t^q J_5(t) \leqslant -M_5 J_5(t)$, which gives by comparison

$$J_5(t) \leqslant J_5(0)\mathbb{E}_q\left(-M_3 t^q\right).$$

Using again (B.2), there exists also $M_4 > 0$ such that

$$\begin{aligned}(S(t)-S^*)^2+(E(t)-E^*)^2+(I(t)-I^*)^2 &\leqslant M_4 J_5(t)\\ &\leqslant M_4 J_5(0)\mathbb{E}_q(-M_3 t^q).\end{aligned}$$

This, implies the attractivity of the positive equilibrium state (S^*, E^*, I^*) of any solution $(S(t), E(t), I(t))$ starting from any positive initial condition (S_0, I_0, R_0).

C.2. Proof of Theorem 3.2

Suppose that $R_0 \leq 1$. In the following, we discuss two cases.
Case 1. If $\varphi(\frac{A}{\mu}) < \lambda$. So, we choose the variables S, I, N instead of S, E, I and we write their equations as follows

$$\begin{cases} {}_0^cD_t^q S(t) = -\mu\left(S(t)-\dfrac{A}{\mu}\right) - \beta I(t)\left(\varphi(S(t)) - \varphi\left(\dfrac{A}{\mu}\right)\right) + \left(\lambda - \beta\varphi(\dfrac{A}{\mu})\right) I(t),\\ {}_0^cD_t^q I(t) = -(\mu+\varepsilon+\lambda+\alpha)I(t) + \alpha\left(N(t)-\dfrac{A}{\mu}\right) - \alpha\left(S(t)-\dfrac{A}{\mu}\right),\\ {}_0^cD_t^q N(t) = -\mu\left(N(t)-\dfrac{A}{\mu}\right) - \varepsilon I(t).\end{cases}$$

Then, define the non-negative function

$$L_1(t) = \frac{\alpha}{\lambda - \beta\varphi\left(\frac{A}{\mu}\right)}\int_{\frac{A}{\mu}}^{S(t)}\left(x-\frac{A}{\mu}\right)dx + \int_0^{I(t)} x dx + \frac{\alpha}{\varepsilon}\int_0^{N(t)}\left(x-\frac{A}{\mu}\right)dx.$$

Using Theorem (A.1), direct computations leads to

$$\begin{aligned}{}_0^cD_t^q L_1(t) &\leqslant \frac{-\alpha\mu}{\lambda-\beta\varphi(\frac{A}{\mu})}\left(S(t)-\frac{A}{\mu}\right)^2 - (\mu+\varepsilon+\lambda+\alpha)I^2(t) - \frac{\mu\alpha}{\varepsilon}\left(N(t)-\frac{A}{\mu}\right)^2\\ &\leqslant -2(\mu+\varepsilon+\lambda+\alpha)L_1(t).\end{aligned}$$

Thus, $L_1(t) \leqslant L_1(0)\mathbb{E}_q\left(-2(\mu+\varepsilon+\lambda+\alpha)t^q\right)$, which guarantee the global asymptotic stability of E_0.
Case 2. If $\beta\varphi\left(\frac{A}{\mu}\right) \geqslant \lambda$. In this case, we return to the original variables S, E and I, but we rewrite their equations as follows

$$\begin{cases} {}_0^cD_t^q S(t) = -\mu\left(S(t)-\dfrac{A}{\mu}\right) + \beta I(t)\left(\varphi(S(t)) - \varphi(\dfrac{A}{\mu})\right) - \left(\lambda - \beta\varphi(\dfrac{A}{\mu})\right) I(t),\\ {}_0^cD_t^q E(t) = -(\mu+\alpha)E(t) + \beta\left(\varphi(S(t)) - \varphi(\dfrac{A}{\mu})\right) I(t) + \beta\varphi(\dfrac{A}{\mu})I(t),\\ {}_0^cD_t^q I(t) = -(\mu+\varepsilon+\lambda)I(t) + \alpha E(t).\end{cases}$$

Then, we consider the function

$$L_2(t) = \int_{\frac{A}{\mu}}^{S(t)}\left(\varphi(x) - \varphi\left(\frac{A}{\mu}\right)\right)dx + \frac{\beta\varphi\left(\frac{A}{\mu}\right)-\lambda}{\beta}E(t) + \frac{(\mu+\alpha)(\beta\varphi\left(\frac{A}{\mu}\right)-\lambda)}{\beta\alpha}I(t).$$

Therefore

$$\begin{aligned}
{}_0^cD_t^q L_2(t) \leqslant & -\mu(S(t)-\frac{A}{\mu})(\varphi(S)-\varphi\left(\frac{A}{\mu}\right)) - \beta I(t)\left(\varphi(S(t))-\varphi\left(\frac{A}{\mu}\right)\right)^2 \\
& + \left(\beta\varphi\left(\frac{A}{\mu}\right)-\lambda\right)\left(\varphi\left(\frac{A}{\mu}\right)-\frac{(\mu+\alpha)(\mu+\varepsilon+\lambda)}{\beta\alpha}\right) I(t) \\
& \leqslant -\mu\left(S(t)-\frac{A}{\mu}\right)\left(\varphi(S(t))-\varphi\frac{A}{\mu}\right) \\
& \leqslant C_1\left(S(t)-\frac{A}{\mu}\right)^2,
\end{aligned} \tag{C.12}$$

where C_1 is a non-negative constant. So, according to Theorem 3 in the reference [21], the disease-free equilibrium E_0 is uniformly asymptotically stable. To complete its attractivity, we proceed as in the proof of Theorem 3.1. So, we need the additional function

$$L_3(t) = \int_{\frac{A}{\mu}}^{N(t)} (x-\frac{A}{\mu})dx + \frac{2\mu+\varepsilon}{\lambda}\int_{\frac{A}{\mu}}^{S(t)+E(t)} (x-\frac{A}{\mu})dx,$$

where its fractional derivative is estimated by

$${}_0^cD_t^q L_3(t) \leqslant -\mu\left(S(t)-\frac{A}{\mu}\right)^2 - \mu E^2(t) - (\mu+\varepsilon)I^2(t) + C_2\left(S(t)-\frac{A}{\mu}\right)^2, \tag{C.13}$$

such that $C_3 > 0$. Hence, if we put $L_4(t) = \frac{C_2}{\mu C_1}L_2(t) + L_3(t)$, we get from (C.12) and (C.13) that

$${}_0^cD_t^q L_4(t) \leqslant -\mu\left(\left(S(t)-\frac{A}{\mu}\right)^2 + E^2(t) + I^2\right). \tag{C.14}$$

On the other hand, it is easy to see that there exists C_4, $C_5 > 0$ such that

$$L_4(t) \leqslant C_4\left(\left(S(t)-\frac{A}{\mu}\right)^2 + E^2(t) + I^2\right) + C_5(E+I).$$

Let $\eta > 0$ sufficiently small, from the elementary inequality $ab \leqslant \eta a^2 + \frac{1}{4\eta}b^2$, one can write

$$\begin{aligned}
L_4(t) & \leqslant C_4\left(\left(S(t)-\frac{A}{\mu}\right)^2 + E^2(t) + I^2\right) + \frac{C_5}{4\eta}(E^2+I^2) + 2C_5\eta \\
& \leqslant \frac{C_5}{2\eta}\left(\left(S(t)-\frac{A}{\mu}\right)^2 + E^2 + I^2\right) + 2C_5\eta.
\end{aligned}$$

Hence, $\left(S(t)-\frac{A}{\mu}\right)^2 + E^2 + I^2 \geqslant -4\eta^2 + \frac{2\eta}{C_5}L_4(t)$, which gives with (C.14) that

$${}_0^cD_t^q L_4(t) \leqslant 4\mu\eta^2 - \frac{2\mu\eta}{C_5}L_4(t).$$

Therefore, by Lemma (A.2), we get

$$L_4(t) \leqslant 2C_5\eta - (L_4(0) - 2C_5\eta)\mathbb{E}_q\left(\frac{-2\mu\eta}{C_5}t^q\right).$$

Since $\lim_{t\to+\infty} \mathbb{E}_q\left(\frac{-2\mu\eta}{C_5}t^q\right) = 0$, we have $\limsup_{t\to+\infty} L_4(t) \leqslant 2C_5\eta$. Then, letting $\eta \to 0$, gives

$$\limsup_{t\to+\infty} L_4(t) = 0 = \lim_{t\to+\infty} L_4(t).$$

Thus, the attractivity of E_0 is obtained since it is easy to see that there exists a positive constant C_6 such that for all $t \geq 0$, we have $I(t) + E(t) + \left(N(t) - \frac{A}{\mu}\right)^2 \leq CL_4(t)$.

References

[1] R. M. Anderson and R. M. May, *Regulation and stability of host-parasite population interactions*, Journal of animal ecology, 1978, vol. 47(1), 219–247.

[2] X. Chen, J. Cao, J. H. Park and J. Qiu, *Stability analysis and estimation of domain of attraction for the endemic equilibrium of an SEIQ epidemic model*, Nonlinear Dynamics, 2017, 87(2), 975–985.

[3] A. Chikhaoui, N. Nzoyikorera N, I. Diawara, Z. Jouhadi and K. Zerouali, *Burden of invasive pneumococcal disease in children in Casablanca, Morocco four years after the introduction of pneumococcal vaccination*, 2022, Pan African Medical Journal, 41(2).
DOI: 10.11604/pamj.2022.41.2.29449

[4] I. Diawara, K. Zerouali, K. Katfy, B. Zaki, H. Belabbes, J. Najib and N. Elmdaghri, *Invasive pneumococcal disease among children younger than 5 years of age before and after introduction of pneumococcal conjugate vaccine in Casablanca*, Morocco. International Journal of Infectious Diseases, 2015, 40, 95–101.

[5] M. Di Paola, G. Alotta, A. Burlon, and G. Failla, *A novel approach to nonlinear variable-order fractional viscoelasticity*, Philosophical Transactions of the Royal Society A, 2020, 378(2172), 20190296.

[6] K. Diethelm, *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*, Springer Science & Business Media, 2010.

[7] K. Diethelm ans A. D. Freed, *(1998). The FracPECE subroutine for the numerical solution of differential equations of fractional order*, Forschung und wissenschaftliches Rechnen, 1999, 57–71.

[8] K. Diethelm, N. J. Ford and A. D. Freed, *Detailed error analysis for a fractional Adams method*, Numerical algorithms, 2004, 36, 31–52.

[9] K. Diethelm, *Efficient solution of multi-term fractional differential equations using P (EC) m E methods*, 2003, Computing, 71, 305–319.

[10] M. Fan, M. Y. Li and K. Wang, *Global stability of an SEIS epidemic model with recruitment and a varying total population size*, Mathematical biosciences, 2001, vol. 170(2),199–208.

[11] *FdeSolver.jl: Solving fractional differential equations.* https://github.com/JuliaTurkuDataScience/FdeSolver.jl

[12] J. A. Gallegos and M. A. Duarte-Mermoud, *On the Lyapunov theory for fractional order systems*, Applied Mathematics and Computation, 2016, 287, 161–170.

[13] R. Garrappa, *On linear stability of predictor–corrector algorithms for fractional differential equations*, International Journal of Computer Mathematics, 2010, 87(10), 2281–2290.

[14] A. Gray, D. Greenhalgh, L. Hu, X. Mao, and J. Pan, *A stochastic differential equation SIS epidemic model*, SIAM Journal on Applied Mathematics, 2011, vol. 71(3), 876–902.

[15] E. Hairer, C. Lubich and M. Schlichte, *Fast numerical solution of nonlinear Volterra convolution equations*, SIAM journal on scientific and statistical computing, 1985, 6(3), 532–541.

[16] M. Khalighi, G. Benedetti and L. Lahti, *(2022). Fdesolver: A julia package for solving fractional differential equations*, arXiv preprint arXiv: 2022, 2212.12550.

[17] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, elsevier, 2006, vol. 204.

[18] A. Korobeinikov, *Lyapunov functions and global properties for SEIR and SEIS epidemic models*, Mathematical medicine and biology: a journal of the IMA, 2004, vol 21(2), 75–83.

[19] A. Lahrouz, R. Hajjami, M. El Jarroudi and A. Settati, *Mittag-Leffler stability and bifurcation of a nonlinear fractional model with relapse*, Journal of Computational and Applied Mathematics, 2021, 386, 113247.

[20] C. H. Li and A. M. Yousef, *Bifurcation analysis of a network-based sir epidemic model with saturated treatment function*, Chaos: An Interdisciplinary Journal of Nonlinear Science, 2019, 29(3), 033129.

[21] Y. Li, D. Zhao, Y. Chen, I. Podlubny and C. Zhang, *Finite energy Lyapunov function candidate for fractional order general nonlinear systems*, Communications in Nonlinear Science and Numerical Simulation, 2019, 78, 104886.

[22] W. M. Liu, S. A. Levin, and Y. Iwasa, *Influence of nonlinear incidence rates upon the behavior of sirs epidemiological models*, Journal of mathematical biology, 1986, vol. 23, 187–204.

[23] F. Mainardi, *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*, World Scientific, 2022.

[24] *Morocco: Infant mortality rate from 2011 to 2021.* https://www.statista.com/statistics/807047/infant-mortality-in -morocco/.

[25] M. Naim, F. Lahmidi and A. Namir, *Threshold Dynamics of an SEIS Epidemic Model with Nonlinear Incidence Rates*, Differ Equ Dyn Syst, 2021. Doi: 10.1007/s12591-021-00581-9.

[26] F. Ndaïrou, M. Khalighi and L. Lahti, *Ebola epidemic model with dynamic population and memory*, Chaos, Solitons & Fractals, 2023, 170, 113361. DOI: 10.5281/zenodo.7646063

[27] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*, Elsevier, 1998.

[28] Pneumonia, *https://www.cdc.gov/nchs/fastats/pneumonia.htm.*

[29] D. Rostamy and E. Mottaghi, E. (2016).*Stability analysis of a fractional-order epidemics model with multiple equilibriums*, Advances in Difference Equations, 2016, 2016(1), 1–11.

[30] M. R. Sidi Ammi, M. Tahiri and D. F. Torres, *Global stability of a Caputo fractional SIRS model with general incidence rate*, Mathematics in Computer Science, 2021, 15(1), 91–105.

[31] M. Turkyilmazoglu, *A restricted epidemic SIR model with elementary solutions*, Physica A: Statistical Mechanics and its Applications, 2022, vol. 600.

[32] M. Turkyilmazoglu, *Hyperthermia therapy of cancerous tumor sitting in breast via analytical fractional model*, Computers in Biology and Medicine, 2023, 164, 107271.

[33] L. D. Wang and J. Q. Li, *Qualitative analysis of an SEIS epidemic model with nonlinear incidence rate*, Applied Mathematics and Mechanics, 2006, 27(5), 667–672.

[34] C. Wu and X. Liu, *The continuation of solutions to systems of Caputo fractional order differential equations*, Fractional Calculus and Applied Analysis, 2020, 23(2), 591–599.

[35] R. Xu, *Global dynamics of an SEIS epidemic model with saturation incidence and latent period*, Applied Mathematics and Computation, 2012, 218(15), 7927–7938.

[36] Y. Yang and L. Xu, *Stability of a fractional order SEIR model with general incidence* , Applied Mathematics Letters, 2020, 105, 106303.

[37] X. Zhou and M. Wang, *Dynamic analysis of a fractional-order SIRS model with time delay*, Nonlinear Analysis: Modelling and Control, 2022, 27(2), 368–384.