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Characterization of Distributions through Stochastic Models under Fuzzy Random Variables

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Abstract This paper is noteworthy because it investigates a novel method for comparing the expectations of stochastic models in fuzzy contexts. Actuarial science and economics both depend on stochastic models. Understanding the novel concepts of stochastic comparison of stochastic models based on the exponential order is the main advantage of this study. We solved the preservation properties and theorem, created a new definition, and put the fuzzy mean inactive time order definition into practice. Stochastic models are handled in a variety of applications.

Keywords Fuzzy set, random variables, stochastic orders

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1. Introduction

Applied probability, statistics, dependability, operations research, economics, and allied domains have demonstrated the value of stochastic ordering. A variety of stochastic ordering and related features have quickly emerged over the years. Let X be a nonnegative random variable that denotes the lifetime of a system with a distribution function F, survival function $\bar{F} = 1 - F$, and density function f. The conditional random variable $X_t = (X - t|x > t)\,, t \geq 0$, is known as the residual life of the system after X_t, given that it has already survived up to X_t. The mean residual life (MRL) function of X is the expectation of X_t, which is given by

$$\mu_x(t) = \begin{cases} \int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right) dt, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

The MRL function is an important characteristic in various fields such as reliability engineering, survival analysis, and actuarial studies. It has been extensively

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studied in the literature, especially for binary systems, that is, when there are only two possible states for the system: either working or failing. The hazard rate HR function of X provides another useful reliability measure.

$$r_x(t) = \frac{f(x)}{\bar{F}(t)}, \quad t \geq 0.$$

The HR function is very helpful in characterising how the probability of witnessing the event varies over time and in identifying the appropriate failure distributions using qualitative data regarding the failure mechanism. The MRL function has been shown to be more effective in replacement and repair procedures, even though the shape of the HR function is still significant. The HR function only accounts for the possibility of a sudden failure at any given time. According to A. Arriaza, M.A. Sordo, and A. Surez Llorenz [1], there is a group of methods called transform stochastic orderings that help compare the remaining lifetimes and inactive periods at specific points. A new stochastic order, called the star order, was introduced. It is positioned between the convex order and the other two transform orderings. I. Arab, PE. Oliveira, and T. Wiklund [2] have developed a novel concept for the simple and thorough characterisation of instances in which one beta distribution is smaller than another based on the convex transform order. They find patterns in how likely it is for a beta-distributed random variable to be greater than its average or most common value. Arevalillo and H. Navarro [3] allow for stochastic comparisons of vectors with a multivariate skew-normal distribution. The new way of comparing is built on a specific change related to the multivariate skew-normal distribution and a common method used for comparing single skewed parts of that change. A. Arriaza, A. Crescenzo, M. A. Sordo, and Suarez-Llorens propose three functional measures of the shape of univariate distributions [4]. These metrics are appropriate with respect to the convex transform order. To close a gap in the literature, F. Belzunce, C. Martinez-Riquelme, and M. Perera [5] concentrate on giving sufficient conditions for a few well-known stochastic orders in dependability while handling their discrete forms. In particular, they discovered ways to compare two discrete random variables in certain stochastic orders by looking at the unimodality of the likelihood ratio. J.H. Cha and F.G. Badia [6] have studied the mean residual life, the bending property of the failure rate, the reversed hazard rate, and the mean inactive duration in mixtures. The idea of relative spacing's was first developed by F. Belzunce, C. Martinez-Riquelme, and M. Perera [7]. They demonstrate the relevance of this idea in several situations, such as economy and reliability, and we offer various results for evaluating relative spacing's among two populations. Belzunce, F., Ruiz, J.M., and Ruiz, M.C. [8] have compared organised structures formed from either a single group of parts or two distinct groups, based on various shifting and proportional stochastic orders. Izadkhah, S., and Kayid, M. [9] have proposed and explored a new type of stochastic order. Izadkhah, S., and Kayid, M. [9] look into important properties of the new stochastic order related to convolution, mixture, and shock model reliability methods. Xie, M., and Lai, C.D. [10] present an extensive summary of the theory and applications of dependence and ageing using mathematical methods for survival and reliability studies. A. Patra and C. Kundu [11] have enhanced the study of older properties of residual lifetime mixture models and stochastic comparisons. They looked at two different mixture models by using methods such as likelihood ratio, hazard rate, mean residual life, and variance residual life orders, along with two kinds of mixing distributions and two kinds of baseline distributions. Recently, A. Patra and C. Kundu [12] discussed the stochastic comparison and ageing properties of RLRT (ITRT) based on variance residual life and discovered a novel aspect of stochastic ageing. Sufficient standards for the residual life and inactive time's log-concavity and log-convexity have been given by Misra, N., Gupta, N., and Dhariyal, I.D. [13]. In addition, we perform stochastic comparisons between the inactivity time and residual life in terms of the typical stochastic order, the mean residual life order, and the failure rate order. A well-known MRL order has been introduced and examined in the literature, based on the MRL function. Nanda.K, Bhattacharjee.S, and N. Balakrishnan [15]. Numerous writers have studied the MRL order's uses in survival and reliability analysis throughout the years (see Shaked and Shanthikumar [17] and Muller and Stoyan [14]). However, the literature suggests that the proportional stochastic order generalises several existing concepts of stochastic comparisons of random variables. Proportional stochastic orders have been explored by numerous researchers as enlarged versions of the prominent stochastic orders prevalent in the literature right now, such as Ramos-Romero and Sordo-Diaz [16]. Nanda et al. [15] conducted a new study on various partial ordering effects related to the MRL order and examined reliability models using the MRL function. Their review was quite effective.

An Introduction to Stochastic Orders discusses this helpful tool, which may be used to assess probabilistic models in a range of domains, including finance, economics, survival analysis, risks associated with stock trading, and reliability. It provides a general foundation on the subject for academics and students who wish to use this data as a tool for their research. It includes thorough explanations of the main findings in various areas, along with examples related to probabilistic models and talks about key features of many stochastic orders in both single-variable and multiple-variable situations. In applied probability, stochastic ordering among random variables has been shown to be an effective method for comparing system reliability. Marketers view stochastic orderings as a crucial tool for making decisions in the face of uncertainty. To create a mathematical or financial model that can find every possible outcome for a particular circumstance or issue, stochastic modelling uses random input variables. The probability distribution of potential outcomes is its primary concern. Examples include Shaked, M. and Shanthikumar, J.G. [17], Markov models, and regression models. The model functions as a realistic case simulation to gain a more profound understanding of the system, investigate unpredictability, and evaluate uncertain scenarios that delineate all possible outcomes and the trajectory of the system's evolution. Thus, to optimise profitability, experts and investors can develop their own business practices and make better management decisions with the aid of this modelling technique.

1.1. Fuzziness

There are two typical scenarios in the real world when an observed variable gets fuzzy. In the first scenario, the response variable cannot be measured exactly due to technical measurement conditions. As a result, data cannot be recorded explicitly with precise (non-fuzzy) numbers; instead, it can only be done in linguistic terms to demonstrate the necessary tolerance to errors in measurement. The second scenario involves the response to the variable being given in linguistic forms, such as a farmer's report about his products or an expert's linguistic report, which are not numeric. To analyse the experiment, the data in both scenarios may be represented as a nation of fuzzy sets. Fuzzy set theory has to be used to model and manage the findings from experiments in many applied fields since the values obtained from experiment outcomes are often fuzzy. Many people have utilised the fuzzy sets theory in a variety of scientific areas since Zadeh (1965) introduced it to the scientific world.

Developing exponential stochastic models of fuzzy random variables is the aim of this paper. A fundamental concept of fuzzy sets and stochastic ordering, as well as a definition of fuzzy random variables and fuzzy random vectors, is provided in Section 2 along with a few definitions and equations. Section 3 explains how to compare randomness using the stop-loss premium of the convex order and describes the characteristics of the convex ordering for fuzzy random variables. The convex ordering of the set of fuzzy random variables was graphically depicted in a clear and understandable manner. 3.5.5 Stochastic model comparisons aid the process of making investment decisions by forecasting results in unpredictable circumstances, particularly those involving the stock market. It is regarded as an insurance company that, for example, based its price list on the exponential principles of premium computation, using a distribution function in the literature called inactivity time. This section covers the exponential inactivity order of a random variable and preservation features under specific dependability operations. With the aid of a theorem and proof, the continuous nonnegative fuzzy random variable with a probability density function is elaborated. Numerous fields, including agriculture, systems biology, production, weather forecasting, and biochemistry, have benefitted from the extensive applications of stochastic models in real life. Finally, we have resolved the question of what practical applications exist for stochastic ordering under fuzzy random variables.

Ordinary stochastic order between system lives has been achieved when components are interconnected, according to Sangita Das and Suchandan Kayal [18]. Sufficient conditions under which the reversed hazard rate order between the second-largest order statistics hold are studied for the independent heterogeneous distributions. Shrahili M [19] examines systems with heterogeneous components and dependent exponential lifetimes, both in parallel and series. We intend to connect the component lifetimes via an Archimedean copula, and we consider the underlying dependence to be Archimedean.

2. Preliminaries

Definition 2.1. Let $\mathcal{X}$ be a set of all values. Next a fuzzy set $\tilde{A} = \{(x, \mu_A(x))/x \in \mathcal{X}\}$ of $\mathcal{X}$ is determined by the role it plays in membership $\mu_A : \mathcal{X} \to [0,1]$.

Definition 2.2. The $\alpha - cut$ of the set of $\tilde{A}$ is indicated by its for every $(0 \leq \alpha \leq 1). \tilde{A}_\alpha = \{x \in \mathcal{X}; \mu_{\tilde{A}}(x) \geq \alpha\}$.

Definition 2.3.

1. For each $\alpha \in (0,1]$, both $[\mathcal{X}_1^u, \mathcal{X}_\beta^u]$ defined as $\mathcal{X}_\alpha^l(\omega)(x) = inf\{x \in \omega; \mathcal{X}_\alpha^l(\omega)(x) \geq \alpha\}$ and $\mathcal{X}_\beta^u = sup\{x \in \omega; \mathcal{X}_\alpha^u(\omega)(x) \geq \beta\}$ are finite real valued random variables defined on $(\Omega, \mathcal{A}, \mathcal{P})$, that the mathematical expectations $E\left(\mathcal{X}_\alpha^l\right)$ and $E\left(\mathcal{X}_\alpha^l\right)$ exist.
2. For each, $\omega \in \Omega$ and $\alpha \in (0,1], \mathcal{X}_\alpha^l(\omega)(x) \geq \alpha$ and $\mathcal{X}_\alpha^u(\omega)(x) \geq \beta$.

Definition 2.4. If $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$ fuzzy random variables with fuzzy cumulative distribution function $\bar{F}$ and $\bar{G}$ respectively; then $\bar{\mathcal{X}} \leq_{\text{st}} \bar{\mathcal{Y}} \Longleftrightarrow \bar{F}(t) \geq \tilde{G}(t) \forall t$.

Definition 2.5. If $\mathcal{X}$ and Y are two fuzzy random variables then

$$\mathcal{X} \leq_{\text{st}} \mathcal{Y} \Longleftrightarrow \left\{P\left(\mathcal{X}_\alpha^l \geq t\right) \bigvee P\left(\mathcal{X}_\alpha^u \geq t\right)\right\} \leq \left\{P\left(\mathcal{Y}_\alpha^l \geq t\right) \bigvee P\left(\mathcal{Y}_\alpha^u \geq t\right)\right\}.$$

Definition 2.6. If $\mathcal{X}$ and $\mathcal{Y}$ are two fuzzy random variables then

$$\mathcal{X} \leq_{\text{st}} \mathcal{Y} \Longleftrightarrow E\left[f\left(\mathcal{X}_\alpha^l\right)\right] \bigvee E\left[f\left(\mathcal{X}_\alpha^u\right)\right] \leq E\left[g\left(\mathcal{Y}_\alpha^l\right)\right] \bigvee E\left[g\left(\mathcal{Y}_\alpha^u\right)\right]$$

, for all increasing functions f.

Example 2.1. If $\mathcal{X}$ and $\mathcal{Y}$ are two exponential fuzzy random variables with mean λ and μ respectively such that $\lambda < \mu$, then $\mathcal{X} \leq_{\text{st}} \mathcal{Y}$.

3. Stochastic comparison of the exponential orders

Definition 3.1. Consider two consecutive sequence set of fuzzy random variables $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ and $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ such that, $e^{t\mathcal{X}} E[\varphi\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}] \leq e^{t\mathcal{Y}} E[\varphi\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}]$, for all convex functions φ, provided expectations exists. Then the sequence $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ is said to be stochastically dominant of $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ in the convex order denoted as $\mathcal{X} \leq_{\text{FCO}} \mathcal{Y}$, where

$$\mathcal{X} = \left[{}^{\vee\&\wedge}_{\alpha \leq \beta \leq 1}\{x_\alpha^l, x_\beta^u\}\right]$$

and

$$\mathcal{Y} = \left[{}^{\vee\&\wedge}_{\alpha \leq \beta \leq 1}\{y_\alpha^l, y_\beta^u\}\right].$$

3.1. Properties of convex ordering of set fuzzy random variables

Let $\mathcal{X} = [\mathcal{X}_\alpha^l, \mathcal{X}_\alpha^u]$ and $\mathcal{Y} = [\mathcal{Y}_\alpha^l, \mathcal{Y}_\alpha^u]$ be two sets of fuzzy random variables. Then the following conditions are satisfied;

- If $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$, is said to be stochastically dominant of $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ in convex order, sense that, if $\mathcal{X} \leq_{\text{FCO}} \mathcal{Y}$,

  $$E\left[\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}\right] \leq_{\text{FCO}} E\left[\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}\right]$$

  and

  $$Var\left[\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}\right] \leq_{\text{FCO}} Var\left[\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}\right].$$

- If $[\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}] \leq_{\text{FCO}} [\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}]$ and $[\{Z_1, Z_2, Z_3, \ldots, Z_n\}]$ is independent of $[\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}]$ and $[\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}]$, $[\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}] + [\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}] \leq_{\text{FCO}} [\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}] + [\{Z_1, Z_2, Z_3, \ldots, Z_n\}]$.

- Let $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ and $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$, be two sets of consecutive fuzzy random variables. Then,

  $$\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\} \leq_{\text{CO}} \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\} \Leftrightarrow -\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\},$$
  $$\leq_{\text{FCO}} -\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}.$$

- Let $\mathcal{X}$ and $\mathcal{Y}$, be two sets of consecutive fuzzy random variables such that $E(\mathcal{X}) = E(\mathcal{Y})$. Then $\mathcal{X} \leq_{\text{FCO}} \mathcal{Y}$, if and only if $|E\mathcal{X} - \delta| \leq_{\text{FCO}} |E\mathcal{Y} - \delta|$ for all $\delta \in \varnothing$
- The convex order closed under mixtures: let $\mathcal{X}$ and $\mathcal{Y}$ and Z be random variables such that $[\mathcal{X}/Z = \varnothing] \leq_{\text{FCO}} [(\mathcal{Y}/Z = \varnothing]$ for all $\varnothing$ in the support of Z. Then $\mathcal{X} \leq_{\text{FCO}} \mathcal{Y}$.
- The convex order closed under convolution: let $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ and $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ be sets of independent fuzzy random variables. If $\mathcal{X}_i \leq_{\text{FCO}} \mathcal{Y}_i$, for $i = 1, 2, 3, \ldots, n$, $\sum_j^m \mathcal{X}_j \leq_{\text{FCO}} \sum_j^m \mathcal{Y}_j$.
- Let $\mathcal{X}$ be a set of consecutive fuzzy random variable with finite mean. Then, $+E[\mathcal{X}] \leq_{\text{FCO}} 2\mathcal{X}$.
- Let $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ and $\mathcal{Y}$ be $(N + 1)$ sets of consecutive independent fuzzy random variables. If $\mathcal{X}_i \leq_{\text{FCO}} \mathcal{Y}$ for $i = 1, 2, 3, \ldots, n$. $\sum_j^m A_i\mathcal{X}_j \leq_{\text{FCO}} \mathcal{Y}$, whenever $A_i \geq 0$, $j = 1, 2, 3, \ldots, n$ and $\sum_{i=1}^n A_i{=}1$.
- Let $\mathcal{X}$ and $\mathcal{Y}$ be two sets of non negative consecutive fuzzy random variables. Then $\sum_j^m \mathcal{X}_j \leq_{FCO} \sum_j^m \mathcal{Y}_j$. If and only if
  $E[\varphi(\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}, \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\})] \leq_{\text{FCO}}$
  $E[\varphi(\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}, \{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\})]$ for all $\varphi \in \varnothing$, where $\varnothing = \{\varphi : \varnothing^2 \to \varnothing : \varphi(\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}, \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}) - f(\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}, \{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}) is\ convex\ for\ all\ \mathcal{X} \in \mathcal{Y}.\}$
- Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be a pair of consecutive independent fuzzy random variables and let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be a pair of consecutive independent fuzzy random variables.
- If $\sum_j^m \mathcal{X}_j \leq_{\text{FCO}} \sum_j^m \mathcal{Y}_j, j = 1, 2$ then $[\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}, \{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_m\}] \leq_{\text{FCO}} [\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}, \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_m\}]$.
  A clear illustration of the properties of convex ordering of set fuzzy random variables by graphical representation can be found in the following figures.

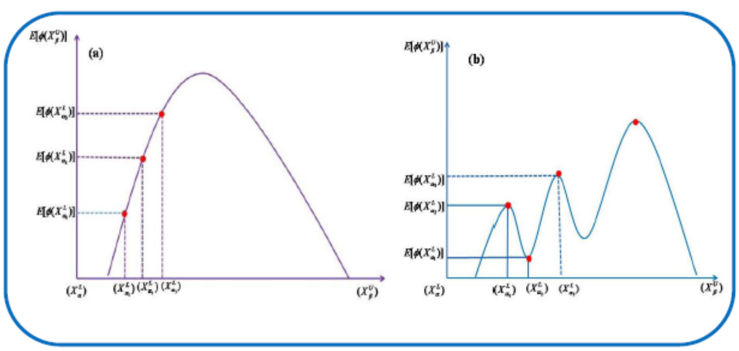

Figure 1. Fig(a)- convex orderings set of fuzzy random variables Fig (b)- non convex orderings set of fuzzy random variables

3.2. Comparisons of stochastic models fuzzy random variables

The integral form has applications in actuarial science, reliability, and economics in numerous stochastic comparison relations. A class F of measurable functions generates a stochastic order relation referred to as an integral stochastic comparison, or $\leq_F$. In particular, given two sets of fuzzy random variables, $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ and $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$, $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ is said to be stochastically dominant over $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$, in the F sense, expressed as $E[\varphi\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}] \leq_F E[\varphi\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}]$ for all the functions $\varphi \in F$.

As long as the presumption in the equation above is met. Marshal and Muller[20] looked at such stochastic evaluations in a fairly broad context. The $f_{\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\ldots,\mathcal{X}_n\}}$ and $f_{\{\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\ldots,\mathcal{Y}_n\}}$ corresponding to $\mathcal{X}$ and $\mathcal{Y}$ are ordered, not the particular configurations of these fuzzy random variables, as should be noted. Here, we revisit the exponential order as one of these analogies.

Let $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ and $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ be two sets of fuzzy random variables with distributions $f_{\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\ldots,\mathcal{X}_n\}}$ and $f_{\{\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\ldots,\mathcal{Y}_n\}}$ and denote their survival functions by

$$\bar{F}_{\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\ldots,\mathcal{X}_n\}} = (1 - f_{\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\ldots,\mathcal{X}_n\}})$$

and

$$\bar{F}_{\{\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\ldots,\mathcal{Y}_n\}} = (1 - f_{\{\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\ldots,\mathcal{Y}_n\}})$$

respectively. Their exponential functions are defined as, for all $S > 0$,

$$\varphi_{\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\ldots,\mathcal{X}_n\}}(S) = E[e^{S\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\ldots,\mathcal{X}_n\}}]$$

and

$$\varphi_{\{\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\ldots,\mathcal{Y}_n\}}(S) = E[e^{S\{\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\ldots,\mathcal{Y}_n\}}].$$

Let us consider $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ and $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ be two sets of fuzzy random variables. $\mathcal{X}$ is said to be smaller than $\mathcal{Y}$ in the exponential order denoted as $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\} \leq_{\text{FEO}} \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ if $E\left(e^{t_0\mathcal{Y}}\right)$ is finite for some $t_0 > 0$, and $\varphi_{\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\ldots,\mathcal{X}_n\}}(S) \leq \varphi_{\{\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\ldots,\mathcal{Y}_n\}}(S)$, for all $S > 0$.

Notice that
$\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\} \leq_{\text{FEO}} \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\} \Leftrightarrow \int_0^\infty e^{SU}\bar{F}_{\{\mathcal{X}_1,\mathcal{X}_2,\mathcal{X}_3,\ldots,\mathcal{X}_n\}}(U)dU$ $\leq_{\text{FEO}} \int_0^\infty e^{SU}\bar{F}_{\{\mathcal{Y}_1,\mathcal{Y}_2,\mathcal{Y}_3,\ldots,\mathcal{Y}_n\}}(U)dU.$

This is more important in the idea of reliability. From a probabilistic standpoint, the exponential order mandates that the moment-generating functions of the non-negative random factors $\mathcal{X}$ and $\mathcal{Y}$ be laid out in chronological order. Additionally, the exponential order communicates the collective preferences of all decision-makers via utility features from $\delta(\mathcal{X}) = 1 - e^{-S\mathcal{X}}$. Exponential orders of fuzzy random variables have numerous meanings in an actuarial framework. For instance, consider a financial institution that uses the quadratic premium calculation principle as its basis for its cost list.

In this instance, the premium amount $\sum \tau_S(\mathcal{X})$ related to the risk $\mathcal{X}$ is provided by

$$\sum \tau_S(\mathcal{X}) = 1/SlnE(e^{t\mathcal{X}}).$$

From the above equation, $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\} \leq_{FEO} \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\} \Leftrightarrow$ $\sum \tau_S(\mathcal{X}) = 1/S \;\; lnE(e^{t\mathcal{X}}) \leq_{FEO} \sum \tau_S(\mathcal{Y}) = 1/SlnE(e^{t\mathcal{X}})$.

Here are other interpretations, features, and uses of the exponential order as reported by Klar, Muller, and Denuit[20]. Variables of the type $\mathcal{X}_t = [t - \mathcal{X}/\mathcal{X} \leq t]$ are of significance in many reliability engineering problems for fixed $t\epsilon(0, L_\mathcal{X})$ and $L_\mathcal{X} = \sup\left\{t : F_{\mathcal{X}(t)} < 1\right\}$, with a distribution function $F_t(S) = P[t - \mathcal{X} \leq S/\mathcal{X} \leq t]$ and a known inactivity time in the literature.

Definition 3.2. Both $\mathcal{X}$ and $\mathcal{Y}$ are two continuous nonzero fuzzy random factors with the following attributes: f and g are their probability density functions; F and G are their distribution functions; and $\bar{F}$ and $\bar{G}$ are their survival values. After that, the progression of the

(i) Exponential of the likelihood ratio defined by $\mathcal{X} \leq_{\text{ELRO}} \mathcal{Y}$,
$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\frac{f_{\mathcal{X}}}{g_{\mathcal{X}}}\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right)\right) \leq_{ELR} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{Y}})\left(\frac{f_{\mathcal{Y}}}{g_{\mathcal{Y}}}\left(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u\right)\right)$.

(ii) Exponential of a typical duration order of inactivity as stated by $\mathcal{X} \leq_{\text{EMITO}} \mathcal{Y}$,
$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\frac{\int_0^t \bar{F}_{\mathcal{X}}(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)d\mathcal{X}}{F_t(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)}\right) \leq_{EMITO} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{Y}})\left(\frac{\int_0^t \bar{G}_{\mathcal{Y}}(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u)d\mathcal{X}}{G_t(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u)}\right)$.

(iii) The reversed hazard rate order's exponential stated by $\mathcal{X} \leq_{EMITO} \mathcal{Y}$,
$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E\left(e^{t\mathcal{X}}\right)\left(\frac{\bar{F}_{\mathcal{X}}\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right)}{f_{\mathcal{X}}\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right)}\right) \leq_{ERHRO} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E\left(e^{t\mathcal{Y}}\right)\left(\frac{\bar{G}_{\mathcal{X}}\left(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u\right)}{g_{\mathcal{Y}}\left(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u\right)}\right)$.

(iv) Exponential of the hazard rate order defined by $\mathcal{X} \leq_{EMITO} \mathcal{Y}$,
$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\frac{f_{\mathcal{X}}(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)}{\bar{F}_{\mathcal{X}}(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)}\right) \leq_{EHRO} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{Y}})\left(\frac{g_{\mathcal{Y}}(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u)}{\bar{G}_{\mathcal{Y}}(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u)}\right)$.

(v) The mean residual life order exponential as described by $\mathcal{X} \leq_{EMRO} \mathcal{Y}$,
$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\frac{\int_0^t \bar{F}_{\mathcal{X}}(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)d\mathcal{X}}{\bar{F}_t(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)}\right) \leq_{EMRLO} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{Y}})\left(\frac{\int_0^t \bar{G}_{\mathcal{Y}}(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u)d\mathcal{X}}{\bar{G}_t(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u)}\right)$.

(vi) Exponential serves as the decreasing order typical residual life order computed by $\mathcal{X} \leq_{DEMRO} \mathcal{X}_{\nabla'}$,
$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})(\frac{\int_0^t \bar{F}_{\mathcal{X}}(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)d\mathcal{X}}{\bar{F}_t(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)}) \leq_{\text{EDMRLO}} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})(\frac{\int_0^t \bar{G}_{\mathcal{X}}(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)d\mathcal{X}}{\bar{G}_t(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)})$.

Following figure describes the graphical representation of stochastic orders of fuzzy random variables.

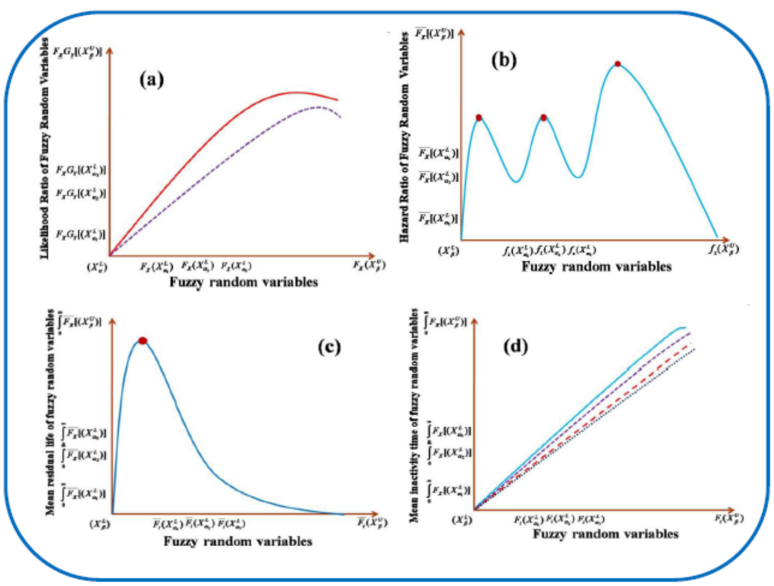

Figure 2. (a-d) Graphical representation of stochastic orders of Fuzzy random variables

These graphical representations serve as visual tools for comparing distributions of random variables and for stochastic ordering. They provide a natural understanding of the strengths, weaknesses, and probabilities associated with the variables being evaluated. In this study, we investigate the exponential order of mean inactive time within a fuzzy context. We then discuss the preservation characteristics of the exponential order in activity time under convolution and combined operations. Subsequently, we present various applications of shock models and highlight a few basic instances of their use to identify situations in which the random variables in this series are similar. Throughout the entire paper, we substituted the terms "increasing" and "decreasing" for monotone non-decreasing and monotone non-increasing, respectively. Furthermore, all FRVs under consideration are assumed to be perfectly continuous, with 0 and 1 serving as the usual left points of their supports, and all expectations are implicitly regarded as limiting whenever they are mentioned. expectancies are implicitly assumed to be limiting whenever they appear.

3.3. Preservation properties

Dependability theory places importance on an order's preservation properties under certain dependability operations. Some features of the exponential of inactivity order of a random variable are covered in this section.

Let $\mathcal{X}$ and $\mathcal{Y}$ be two continuous nonnegative fuzzy random factors that have the following traits distribution functions F and G, survival measures $\bar{F}$ and $\bar{G}$, and probability density functions f and g, respectively.

$$\psi^*_{\mathcal{X}_t}\left(\mathcal{X}^l_\alpha,\mathcal{X}^u_\alpha\right)=\underset{\alpha\le\beta\le 1}{\vee\&\wedge}\,E\left(e^{-t\mathcal{X}}\right)\left(\frac{\int_0^t \bar{F}_{\mathcal{X}}\left(\mathcal{X}^l_\alpha,\mathcal{X}^u_\beta\right)d\mathcal{X}}{\bar{F}_t(\mathcal{X}^l_\alpha,\mathcal{X}^u_\beta)}\right)$$

and

$$\psi^*_{\mathcal{Y}_t}\left(\mathcal{Y}^l_\alpha,\mathcal{Y}^u_\alpha\right)=\underset{\alpha\le\beta\le 1}{\vee\&\wedge}\,E(e^{-t\mathcal{Y}})\,\left(\frac{\int_0^t \bar{F}_{\mathcal{X}}(\mathcal{X}^l_\alpha,\mathcal{X}^u_\beta)d\mathcal{X}}{\bar{F}_t(\mathcal{X}^l_\alpha,\mathcal{X}^u_\beta)}\right).$$

This condition holds true by the previous definition, then
$\mathcal{X}\le_{EMITO}\mathcal{Y}\Leftrightarrow\psi^*_{\mathcal{X}_t}\left(\mathcal{X}^l_\alpha,\mathcal{X}^u_\alpha\right)\le_{EMITO}\psi^*_{\mathcal{Y}_t}\left(\mathcal{Y}^l_\alpha,\mathcal{Y}^u_\alpha\right)$.

Proposition 3.1. *Imagine two continuous nonnegative fuzzy random parameters, $\mathcal{X}$ and $\mathcal{Y}$, with the following traits probability density functions f and g, distribution functions F and G, and inheriting functions $\bar{F}$ and $\bar{G}$, respectively. Then*

$$\mathcal{X}\le_{EMITO}\mathcal{Y}\Leftrightarrow\underset{\alpha\le\beta\le 1}{\vee\&\wedge}\,E(e^{t\mathcal{X}})\left(\frac{\int_0^t \bar{F}_{\mathcal{X}}(\mathcal{X}^l_\alpha,\mathcal{X}^u_\beta)d\mathcal{X}}{\bar{F}_t(\mathcal{X}^l_\alpha,\mathcal{X}^u_\beta)}\right)$$
$$\le_{EMITO}\underset{\alpha\le\beta\le 1}{\vee\&\wedge}\,E(e^{t\mathcal{X}})\,\left(\frac{\int_0^t \bar{G}_{\mathcal{X}}(\mathcal{X}^l_\alpha,\mathcal{X}^u_\beta)d\mathcal{X}}{\bar{G}_t(\mathcal{X}^l_\alpha,\mathcal{X}^u_\beta)}\right)$$

is decreasing in $t\in(0,t_{\mathcal{X}})\cap(0,t_{\mathcal{Y}})$, for all $t>0$.

Proof. Let us observed that

$$\psi^{*}_{\mathcal{X}_t}\left(\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha\right) = \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{-t\mathcal{X}})\left(\frac{\int_0^t F_{\mathcal{X}}(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)d\mathcal{X}}{F_t(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)}\right)$$

$$= \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{-t\mathcal{X}})\left(\frac{\int_0^t F_{\mathcal{X}}(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)d\mathcal{X}}{\frac{\partial}{\partial\mathcal{X}}F_t(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta).}\right)$$

given $t > 0$, by previous equations $\mathcal{X} \leq_{\text{EMITO}} \mathcal{Y} \Leftrightarrow$

$$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{-t\mathcal{X}})\left(\frac{\int_0^t F_{\mathcal{X}}(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)d\mathcal{X}}{\frac{\partial}{\partial\mathcal{X}}F_t(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)}\right) \leq_{\text{EMITO}} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{-t\mathcal{Y}})\left(\frac{\int_0^t G_{\mathcal{Y}}(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta)d\mathcal{Y}}{\frac{\partial}{\partial\mathcal{X}}G_t(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta)}\right)$$

$$\Leftrightarrow$$

$$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{-t\mathcal{X}})\left(\frac{\int_0^t \frac{\partial}{\partial\mathcal{X}}F_t(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta)d\mathcal{Y}}{G_{\mathcal{Y}}(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta)}\right)$$

$$\leq_{\text{EMITO}} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{-t\mathcal{Y}})\left(\frac{\int_0^t \frac{\partial}{\partial\mathcal{X}}F_t\left(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta\right)d\mathcal{X}}{G_{\mathcal{Y}}(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta)}\right)$$

$$\Leftrightarrow$$

$$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\frac{\int_0^t \bar{F}_{\mathcal{X}}(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)d\mathcal{X}}{\bar{F}_t(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)}\right) \leq_{\text{EMITO}} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\frac{\int_0^t \bar{G}_{\mathcal{X}}(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)d\mathcal{X}}{\bar{G}_t(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta)}\right)$$

is decreasing in $t \in (0, t_{\mathcal{X}}) \cap (0, t_{\mathcal{Y}})$, for all $t > 0$. □

Theorem 3.1. *Let us take $\mathcal{X}_1, \mathcal{X}_2$ and Z be three continuous nonnegative fuzzy random variable with probability density function f, g and h, distribution function ; F, G and H, survival functions $\bar{F}$, $\bar{G}$ and $\bar{H}$ respectively, then $\mathcal{X}_1 \leq_{EMITO} \mathcal{X}_2$ and Z is log-concave then $\mathcal{X}_1 + Z \leq_{EMITO} \mathcal{X}_2 + Z$.*

Proof. The previous preposition, it is enough to show that for all $0 \leq t_1 \leq t_2$ *and* $\mathcal{X} > 0$.

$$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\int_0^\infty \int_{-\infty}^{t_1} \frac{P[\mathcal{X}_1 \leq u - \left(\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha\right)]\{f(t_1-u)\}dud\mathcal{X}}{P[\mathcal{X}_2 \leq u - (\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha)]\{g(t_2-u)\}dud\mathcal{X}}\right) \geq_{EMITO}$$

$$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\int_0^\infty \int_{-\infty}^{t_1} \frac{P[\mathcal{X}_1 \leq u - \left(\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha\right)]\{f(t_2-u)\}dud\mathcal{X}}{P[\mathcal{X}_2 \leq u - (\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha)]\{g(t_1-u)\}dud\mathcal{X}}\right).$$

Since Z is nonnegative then $g(t-u) = 0$ when $t < u$, hence the above inequality is equivalent to

$$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\int_0^\infty \int_{-\infty}^{t_1} \frac{[F_{\mathcal{X}_1} \leq u - \left(\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha\right)]\{f(t_1-u)\}dud\mathcal{X}}{[F_{\mathcal{X}_2} \leq u - (\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha)]\{g(t_2-u)\}dud\mathcal{X}}\right) \geq_{EMITO}$$

$$\underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}})\left(\int_0^\infty \int_{-\infty}^{t_1} \frac{[G_{\mathcal{X}_1} \leq u - \left(\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha\right)]\{f(t_2-u)\}dud\mathcal{X}}{[G_{\mathcal{X}_2} \leq u - (\mathcal{X}^l_\alpha, \mathcal{X}^u_\alpha)]\{g(t_2-u)\}dud\mathcal{X}}\right),$$

$0 \le t_1 \le t_2$ or equivalently,

$$\underset{\alpha\le\beta\le1}{\vee\&\wedge}\begin{vmatrix}\int_{\mathbf{0}}^{\infty}\int_{-\infty}^{\mathbf{t_1}}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)[\Phi_1]\{\Phi_3 du d\mathcal{X}\}] & \int_{\mathbf{0}}^{\infty}\int_{-\infty}^{\mathbf{t_1}}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)[\Phi_1]\{\Phi_3 du d\mathcal{X}\}] \\ \int_{\mathbf{0}}^{\infty}\int_{-\infty}^{\mathbf{t_1}}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)[\Phi_1]\{\Phi_4 du d\mathcal{X}\}] & \int_{\mathbf{0}}^{\infty}\int_{-\infty}^{\mathbf{t_1}}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)[\Phi_1]\{\Phi_4 du d\mathcal{X}\}]\end{vmatrix} \ge 0$$

and

$$\underset{\alpha\le\beta\le1}{\vee\&\wedge}\begin{vmatrix}\int_{\mathbf{0}}^{\infty}\int_{-\infty}^{\mathbf{t_1}}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)[\Phi_1]\{\Phi_3 du d\mathcal{X}\}] & \int_{\mathbf{0}}^{\infty}\int_{-\infty}^{\mathbf{t_1}}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)[\Phi_2]\{\Phi_3 du d\mathcal{X}\}] \\ \int_{\mathbf{0}}^{\infty}\int_{-\infty}^{\mathbf{t_1}}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)[\Phi_1]\{\Phi_4 du d\mathcal{X}\}] & \int_{\mathbf{0}}^{\infty}\int_{-\infty}^{\mathbf{t_1}}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)[\Phi_2]\{\Phi_4 du d\mathcal{X}\}]\end{vmatrix} \ge 0,$$

where $\Phi_1 = F_{\mathcal{X}_2}(u - (\mathcal{X}_\alpha^l, \mathcal{X}_\alpha^u))$, $\Phi_2 = F_{\mathcal{X}_1}(u - (\mathcal{X}_\alpha^l, \mathcal{X}_\alpha^u))$, $\Phi_3 = g(t_2 - u)$, $\Phi_4 = g(t_1 - u)$, by the well known basic composition formula

$$\underset{\alpha\le\beta\le1}{\vee\&\wedge}\int_{u_1<u_2}^{\infty}\int_{u_1<u_2}^{\infty}\begin{vmatrix} g(t_2-u_1) & g(t_2-u_2) \\ g(t_1-u_1) & g(t_1-u_2)\end{vmatrix}\times$$

$$\begin{vmatrix}\int_0^{\infty}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)\left[F_{\mathcal{X}_2}(u_1-(\mathcal{X}_\alpha^l,\mathcal{X}_\alpha^u))\right] & \int_0^{\infty}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)\left[F_{\mathcal{X}_1}(u_1-(\mathcal{X}_\alpha^l,\mathcal{X}_\alpha^u))\right] \\ \int_0^{\infty}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)\left[F_{\mathcal{X}_2}(u_2-(\mathcal{X}_\alpha^l,\mathcal{X}_\alpha^u))\right] & \int_0^{\infty}\mathbf{E}\left(\mathbf{e}^{\mathbf{t}\mathcal{X}}\right)\left[F_{\mathcal{X}_1}(u_2-(\mathcal{X}_\alpha^l,\mathcal{X}_\alpha^u))\right]\end{vmatrix} du_1\, du_2$$

Seeing that the first determinate is non-positive because of g's log-concavity and the second determinant being non-positive due to $\mathcal{X}_1 \le_{\text{EMITO}} \mathcal{X}_2$. leads us to the conclusion $\mathcal{X}_1 = \left\{\underset{\alpha\le\beta\le1}{\vee\&\wedge} \mathrm{F}_{\mathcal{X}_1}\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right)\right\}$ and $\mathcal{X}_2 = \{\underset{\alpha\le\beta\le1}{\vee\&\wedge} \mathrm{F}_{\mathcal{X}_2}\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right)\}$ is complete the proof. □

Lemma 3.1. *If $\mathcal{X}_1 \le_{EMITO} \mathcal{Y}_1$ and $\mathcal{X}_2 \le_{EMITO} \mathcal{Y}_2$, where $\mathcal{X}_1$ is independent of $\mathcal{X}_2$ and $\mathcal{Y}_1$ is independent of $\mathcal{Y}_2$ with probability density function f and g, distribution function F and $G$and survival functions $\bar{F}$ and $\bar{G}$ respectively, the following statements hold:*

(i) If $\mathcal{X}_1$ and $\mathcal{Y}_2$ have log-concave densities, then $\mathcal{X}_1 + \mathcal{X}_2 \le_{EMITO} \mathcal{Y}_1 + \mathcal{Y}_2$,

(ii) If $\mathcal{X}_2$ and $\mathcal{Y}_1$ have log-concave densities, then $\mathcal{X}_1 + \mathcal{X}_2 \le_{EMITO} \mathcal{Y}_1 + \mathcal{Y}_2$.

Proof. (i).

$$\underset{\alpha\le\beta\le1}{\vee\&\wedge}E(e^{t\mathcal{X}})\left(\frac{\int_0^t \bar{F}_{\mathcal{X}}(\mathcal{X}_\alpha^l,\mathcal{X}_\beta^u)d\mathcal{X}}{\bar{F}_t(\mathcal{X}_\alpha^l,\mathcal{X}_\beta^u)}\right) \le_{EMITO} \underset{\alpha\le\beta\le1}{\vee\&\wedge}E(e^{t\mathcal{X}})\left(\frac{\int_0^t \bar{G}_{\mathcal{X}}(\mathcal{X}_\alpha^l,\mathcal{X}_\beta^u)d\mathcal{X}}{\bar{G}_t(\mathcal{X}_\alpha^l,\mathcal{X}_\beta^u)}\right).$$

The following chain inequality, which (i), follows by theorem 3.1:

$$\mathcal{X}_1 + \mathcal{X}_2 \le_{\text{EMITO}} \mathcal{X}_1 + \mathcal{Y}_2 \le_{\text{EMITO}} \mathcal{Y}_1 + \mathcal{Y}_2,$$

$$\mathcal{X} = \underset{\alpha\le\beta\le1}{\vee\&\wedge}E(e^{t\mathcal{X}})\left(\frac{\int_0^t F_{\mathcal{X}}(\mathcal{X}_\alpha^l,\mathcal{X}_\beta^u)d\mathcal{X}}{F_t(\mathcal{X}_\alpha^l,\mathcal{X}_\beta^u)}\right)$$

and

$$\mathcal{Y} = {}^{\vee\&\wedge}_{\alpha\leq\beta\leq 1} E(e^{t\mathcal{Y}}) \left(\frac{\int_0^t G_{\mathcal{Y}}(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta) d\mathcal{Y}}{G_t(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta)} \right).$$

(ii).

The evidence for (ii) is analogous. The following outcome can be obtained by repeatedly employing lemma 3.1 and the closure property of log-concaves under convolution.

□

Theorem 3.2. *Let us consider* $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ *and* $\{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ *two sequence sets of random variables* $\mathcal{X}_i$ *is said to be smaller than* $\mathcal{Y}_i$ *in the exponential order denoted as,* $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\} \leq \{\mathcal{Y}_1, \mathcal{Y}_2, \mathcal{Y}_3, \ldots, \mathcal{Y}_n\}$ *and have log-concave densities for all* i*, then* $\sum_{i=1}^n \mathcal{X}_i \leq_{EMITO} \sum_{i=1}^n \mathcal{Y}_i$*,* $i = 1, 2, 3, \ldots, n$*.*

Proof. We shall employ induction to demonstrate the theorem. Certainly, the result true for $n = 1$.

Assume that the result is true for $q = n - 1$, that is

$$\sum_{i=1}^{n-1} \mathcal{X}_i \leq_{\text{EMITO}} \sum_{i=1}^{n-1} \mathcal{Y}_i,$$

$$\mathcal{X} = {}^{\vee\&\wedge}_{\alpha\leq\beta\leq 1} E\left(e^{t\mathcal{X}}\right) \left(\frac{\int_0^t F_{\mathcal{X}}\left(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta\right) d\mathcal{X}}{F_t\left(\mathcal{X}^l_\alpha, \mathcal{X}^u_\beta\right)} \right)$$

and

$$\mathcal{Y} = {}^{\vee\&\wedge}_{\alpha\leq\beta\leq 1} E\left(e^{t\mathcal{Y}}\right) \left(\frac{\int_0^t G_{\mathcal{Y}}\left(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta\right) d\mathcal{Y}}{G_t\left(\mathcal{Y}^l_\alpha, \mathcal{Y}^u_\beta\right)} \right).$$

Note that each of the two sides of above equation has a log-concave density. Applying previous lemma the results follows. The following concepts will be used in the sequel. □

Definition 3.3. A function $F_{\mathcal{X}\mathcal{Y}} : \mathcal{X} \times \mathcal{Y} \to [0, 1]$ is said to be totally positive set of order 2 for all $\alpha_1 < \alpha_2$ and $\beta_1 < \beta_2$ $\{ (\alpha_1, \alpha_2) \in \mathcal{X}, (\beta_1, \beta_2) \in \mathcal{Y}\}$

$$\begin{vmatrix} F_{\mathcal{X}\mathcal{Y}}(\alpha_1, \beta_1) & F_{\mathcal{X}\mathcal{Y}}(\alpha_1, \beta_2) \\ F_{\mathcal{X}\mathcal{Y}}(\alpha_2, \beta_1) & F_{\mathcal{X}\mathcal{Y}}(\alpha_2, \beta_2) \end{vmatrix} \geq 0.$$

Let us take $\mathcal{X}(\delta)$ a distribution function-containing random variable $F_{\mathcal{X}(\delta)}$ and let $\mathcal{Y}(\delta)$, another fuzzily distributed random variable with a distribution function $F_{\mathcal{Y}(\delta)}$, for $i = 1, 2$, and support $\mathcal{R}^+$. The following is a closure of exponential inactivity time order under mixture.

Theorem 3.3. *Let us take* $\mathcal{X}(\delta)$ *set of random variable* $\delta \in \mathcal{R}^+$ *and independent of* $\varnothing_1$ *and* $\varnothing_2$*. If* $\varnothing_1 \leq_{FLR} \varnothing_2$ *and if* $\mathcal{X}(\delta_1) \leq_{EMITO} \mathcal{X}(\delta_2)$ *whenever* $\delta_1 \leq \delta_2$*, then* $\mathcal{Y}(\varnothing_1) \leq_{EMITO} \mathcal{Y}(\varnothing_2)$*.*

Proof. Let $F_{\mathcal{X}}$ be the distribution function of $\mathcal{X}(\delta_i)$ with $i = 1, 2$. We know that

$$F_{\mathcal{X}_i} = \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t F_{\mathcal{X}(\delta_i)}(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) d\mathcal{X}}{F_t(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)} \right),$$

because of previous preposition, we should prove that,
$\psi_{\mathcal{X}_i}^* \left(\mathcal{X}_\alpha^l, \mathcal{X}_\alpha^u\right) = \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t F_{\mathcal{X}(\delta_i)}\left(t-(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)\right) d\mathcal{X}}{F_t\left(t-(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)\right)} \right)$ is totally positive order2 in (i, t). But actually

$$\begin{aligned}
\psi_{\mathcal{X}_i}^* \left(\mathcal{X}_\alpha^l, \mathcal{X}_\alpha^u\right) &= \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \int_0^\infty F_{\mathcal{X}(\delta_i)} \left(t-(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)\right) d\mathcal{X} dt}{F_t\left(t-(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)\right)} \right) \\
&= \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \int_0^\infty F_{\mathcal{X}(\delta_i)} \left(t-(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)\right) dG_{\mathcal{Y}(\varnothing_i)} d\mathcal{X}}{F_t\left(t-(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)\right)} \right) \\
&= \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \int_0^\infty \psi_{\mathcal{X}_i}^*(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) dG_{\mathcal{Y}(\varnothing_i)} d\mathcal{X}}{\psi_{\mathcal{X}_i}^*(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)} \right) \\
&= \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \int_0^\infty \psi_{\mathcal{X}_i}^*(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) \psi_{\mathcal{Y}_i}^*(\mathcal{Y}_\alpha^l, \mathcal{Y}_\alpha^u) d\mathcal{X} d\mathcal{Y}}{\psi_{\mathcal{X}_i}^*(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) \psi_{\mathcal{Y}_i}^*(\mathcal{Y}_\alpha^l, \mathcal{Y}_\alpha^u)} \right),
\end{aligned}$$

by assumption $\mathcal{X}(\delta_1) \leq_{\text{EMITO}} \mathcal{X}(\delta_2)$ whenever$\delta_1 \leq \delta_2$, we have that $\psi_{\mathcal{X}_i}^* \left(\mathcal{X}_\alpha^l, \mathcal{X}_\alpha^u\right)$ is totally positive order2 in (δ, t), while form assumption $(\varnothing_1) \leq_{\text{FLR}} (\varnothing_2)$ follows that $\psi_{\mathcal{Y}_i}^* \left(\mathcal{Y}_\alpha^l, \mathcal{Y}_\alpha^u\right)$ is totally positive order2 in (δ, i). Thus assertion from the basic composition formula.

Let $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ be of random variables with distributions $f_{\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}}$ and denote their survival functions by , $\bar{f}_{\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}} = \left(1 - f_{\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}}\right)$. Let $\underline{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ and $\underline{\beta} = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_n\}$ be two sets of probability vectors. A probability vector $\underline{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ with $\alpha_i > 0$ for $i = 1, 2, 3, \ldots, n$ is said to be smaller than probability vector $\underline{\beta} = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_n\}$ in the sense of discrete likelihood ratio order, denoted as $\alpha_i \leq_{\text{DFLR}} \beta_i$, if

$$\frac{\beta_i}{\alpha_i} \leq_{\text{DFLR}} \frac{\beta_j}{\alpha_j} \text{ for all } 1 \leq i \leq j \leq n.$$

Let us take $\mathcal{X}$ and $\mathcal{Y}$ two continuous random variable with probability density f and g, distribution F and G, and survival functions $\bar{F}$ and $\bar{G}$ respectively.

$$F(\mathcal{X}) = \sum_{i=1}^{n} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \alpha_i F_{\mathcal{X}_i}(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) d\mathcal{X}}{\alpha_i F_t(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u)} \right)$$

and

$$G(\mathcal{Y}) = \sum_{i=1}^{n} \underset{\alpha\leq\beta\leq 1}{\vee\&\wedge} E(e^{t\mathcal{Y}}) \left(\frac{\int_0^t \beta_i G_{\mathcal{Y}_i}(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u) d\mathcal{Y}}{\beta_i G_t(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u)} \right).$$

Conditions under which $\mathcal{X}$ and $\mathcal{Y}$ are analogous with regard to the exponential inactivity time order of random variables are established by the following. □

Theorem 3.4. *Let* $\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}$ *be of random variables with distributions* $f_{\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}}$ *and denote their survival functions by ,* $\bar{F}_{\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}} = \left(1 - F_{\{\mathcal{X}_1, \mathcal{X}_2, \mathcal{X}_3, \ldots, \mathcal{X}_n\}}\right)$*, such that* $\mathcal{X}_1 \leq_{EMITO} \mathcal{X}_2 \leq_{EMITO} \cdots \leq_{EMITO} \mathcal{X}_n$ *and* *let* $\underline{\alpha} = \{\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n\}$ *and* $\underline{\beta} = \{\beta_1, \beta_2, \beta_3, \ldots, \beta_n\}$ *such that* $\alpha_i \leq_{DFLR} \beta_i$. *Let random* $\mathcal{X}$ *and* $\mathcal{Y}$ *have distribution* $f_{\mathcal{X}}$ *and* $g_{\mathcal{Y}}$ *defined by the previous equation. Then* $\mathcal{X} \leq_{EMITO} \mathcal{Y}$.

Proof. Because of the previous preposition, we need to establish that

$$\sum_{\mathbf{i=1}}^{\mathbf{n}} \left(\frac{\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \beta_j F_{\mathcal{X}_i}\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right) d\mathcal{X}}{\beta_j F_t\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right)} \right)}{\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \alpha_i G_{\mathcal{Y}_i}\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right) d\mathcal{X}}{\alpha_j G_t\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right)} \right)} \right)$$

$$\leq \sum_{\mathbf{i=1}}^{\mathbf{n}} \left(\frac{\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{Y}}) \left(\frac{\int_0^t \beta_j F_{\mathcal{X}_i}\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right) d\mathcal{X}}{\beta_j F_t\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right)} \right)}{\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{Y}}) \left(\frac{\int_0^t \alpha_i G_{\mathcal{Y}_i}\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right) d\mathcal{X}}{\alpha_j G_t\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right)} \right)} \right).$$

The aforementioned equation can be demonstrated to be equivalent by multiplying, by the denominators and eliminating equal terms.

$$\sum_{i=1}^{n} \sum_{i=1}^{n} \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \alpha_i \beta_j F_{\mathcal{X}_i}\left(\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right) - t\right) d\mathcal{X}}{\alpha_i \beta_j F_t\left(\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right) - t\right)} \right) \right)$$
$$\times \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \alpha_i \beta_j G_{\mathcal{Y}_i}\left(\left(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u\right) - t\right) d\mathcal{X}}{\alpha_i \beta_j G_t\left(\left(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u\right) - t\right)} \right) \right)$$
$$\leq \sum_{i=1}^{n} \sum_{i=1}^{n} \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \alpha_i \beta_j F_{\mathcal{X}_i}\left(\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right) - t\right) d\mathcal{X}}{\alpha_j \beta_i F_t\left(\left(\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u\right) - t\right)} \right) \right)$$
$$\times \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \alpha_i \beta_j G_{\mathcal{Y}_i}\left(\left(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u\right) - t\right) d\mathcal{X}}{\alpha_j \beta_i G_t\left(\left(\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u\right) - t\right)} \right) \right),$$

where $i \neq j$. Now for each fixed pair (i, j) with $i < j$ we have

$$\left[\beta_i \alpha_j \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \ (\psi_1) \right) \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \ (\psi_2) \right)\right]$$
$$+ \left[\beta_j \alpha_i \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \ (\psi_1) \right) \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \ (\psi_2) \right)\right]$$
$$- \left[\beta_i \alpha_j \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \ (\psi_1) \right) \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \ (\psi_2) \right)\right]$$
$$+ \left[\beta_j \alpha_i \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \ (\psi_1) \right) \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \ (\psi_2) \right)\right]$$
$$= (\beta_i \alpha_j \beta_j \alpha_i)[\left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E\left(e^{t\mathcal{X}}\right) \left(\frac{\int_0^t \alpha_i \beta_j G_{\mathcal{Y}_i}\left((\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u) - t\right) d\mathcal{X}}{G_t\left((\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u) - t\right)} \right) \right)$$
$$\left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E\left(e^{t\mathcal{X}}\right) \left(\frac{\int_0^t \alpha_i \beta_j F_{\mathcal{X}_i}\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right) d\mathcal{X}}{F_t\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right)} \right) \right)$$
$$- \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \alpha_i \beta_j F_{\mathcal{X}_i}\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right) d\mathcal{X}}{F_t\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right)} \right) \right)$$

$$\times \left(\underset{\alpha \leq \beta \leq 1}{\vee \& \wedge} E(e^{t\mathcal{X}}) \left(\frac{\int_0^t \alpha_i \beta_j G_{\mathcal{Y}_i}\left((\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u) - t\right) d\mathcal{X}}{G_t\left((\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u) - t\right)} \right) \right),$$

where $\psi_1 = \frac{\int_0^t \alpha_i \beta_j G_{\mathcal{Y}_i}\left((\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u) - t\right) d\mathcal{X}}{G_t\left((\mathcal{Y}_\alpha^l, \mathcal{Y}_\beta^u) - t\right)}$, $\psi_2 = \frac{\int_0^t \alpha_i \beta_j F_{\mathcal{X}_i}\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right) d\mathcal{X}}{F_t\left((\mathcal{X}_\alpha^l, \mathcal{X}_\beta^u) - t\right)}$.

This is nonnegative because both terms are nonnegative by assumption. The proof is complete. The above holds maximum value of random variable. The same preservation properties and theorems true for models like likelihood ratio order, hazard rate order, mean residual life.

□

3.4. Practical utilization of stochastic models

Stochastic ordering provides a framework for comparing the distributional properties of fuzzy random variables. Fuzzy random variables that indicate imprecision and uncertainty can be used in conjunction with stochastic ordering to model and analyse a number of real-world problems. The following categories of applications for the stochastic ordering of fuzzy random variables are possible:

Risk analysis: It is feasible to evaluate and rank the riskiness of various financial assets or investment portfolios using stochastic mandating of fuzzy random variables. By considering the stochastic dominance relationships between fuzzy random variables, investors can make more informed decisions about risk management and asset allocation.

Quality control: When measurements are ambiguous or imprecise, we can use stochastic orders to gauge the calibre of products or production processes. Assume that a certain approach or product is stochastically superior to another. We can determine this by using fuzzy random variables and stochastic ordering, which may capture the fuzziness and variability in the quality attributes.

Reliability analysis: Stochastic ordering of fuzzy random variables enables reliability engineering to compare and measure the reliability of various systems or individual components. By minding the stochastic dominance relationships, engineers can evaluate the performance and robustness of different designs and make decisions about system maintenance and improvement.

Insurance and actuarial science: Stochastic orders are useful in actuarial science and insurance, especially when fuzzy risk models are being used. They can be used to evaluate the insurance companies' solvency and financial stability, as well as to weigh the risks involved in various insurance options.

Decision-making under uncertainty: Fuzzy random variables with stochastic ordering come in use whenever there is ambiguity and imprecision in the decision factors and objectives. By using stochastic dominance criteria, decision-makers can determine which options or strategies are better based on their distributional properties.

Environmental modeling: It is feasible to employ stochastic ordering of fuzzy random variables in environmental modelling and analysis. They can be used, for example, to assess the impact of confusing and imprecise factors on environmental processes or to compare and rank pollution levels from different emission sources.

The mentioned applications demonstrate how flexible stochastic ordering is when dealing with fuzzy random variables, making it easier to study unclear and uncertain systems in various areas.

4. Conclusion

In actuarial science, one of the most important roles is the exponential order of a stochastic model. In the current study, we propose different preservation features for mixture and convolution reliability-proof fuzzy random variables with exponential stochastic order. Applications such as the hazard rate order, the mean residual life order, and the reverse hazard rate order using stochastic models are outlined. Examples are given to show how the results may be exploited to find the mean inactivity time order of exponential type fuzzy random variables. Our findings also have implications for dependability, risk theory, and statistics. Future studies can take into account the extra features and uses of this novel ordering.

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