Journal of Nonlinear Modeling and Analysis
Volume 7, Number 3, June 2025, 1037–1053
https://journal.global-sci.org/jnma
DOI:10.12150/jnma.2025.1037

Decay of Solutions to the Three-Dimensional Generalized Navier-Stokes Equation with Nonlinear Damping Term

Yiting Cai[1,†]

Abstract In this paper, we consider the three-dimensional generalized Navier-Stokes equation with a nonlinear damping term $|u|^{\beta-1}(\beta \geq 1)$. Firstly, utilizing the Fourier splitting method, we derive decay estimates for weak solutions to the equations when $\alpha = 0$ and $\beta = 1$, as well as when $0 < \alpha < \frac{3}{4}$ for any $\beta = 2$. Secondly, for $0 < \alpha < \frac{5}{4}$ and any $\beta > \max\{\frac{4\alpha}{3} + 1, 2\}$, we obtain the same result.

Keywords Generalized Navier-Stokes equations, damping term, Fourier splitting method

MSC(2010) 37L05, 35Q35, 35B40.

1. Introduction

In this paper, we investigate the decay of solutions to the following Cauchy problem for the incompressible generalized Navier-Stokes equations with a damping term $|u|^{\beta-1}u(\beta \geq 1)$:

$$\begin{cases} u_t + (u \cdot \nabla)u + \Lambda^{2\alpha}u + \nabla P + |u|^{\beta-1}u = 0, \\ \text{div } u = 0, \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $u = u(x,t) \in \mathbb{R}^3$ and $P = P(x,t) \in \mathbb{R}$ represent the unknown velocity field and the pressure, respectively. u_0 denotes the prescribed initial data satisfying $\text{div} u_0 = 0$. $\alpha \geq 0$, $\beta \geq 1$, are real parameters. $\Lambda^{2\alpha}$ is defined through Fourier transform (see [7])

$$\widehat{\Lambda^{2\alpha} f}(\xi) = |\xi|^{2\alpha} \widehat{f}(\xi), \widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx.$$

In recent studies concerning the well-posedness of Equation (1.1) with $\alpha = 1$, Cai et al. [1] employed the Galerkin approximation method to investigate properties of the system. Their findings revealed the existence of a weak solution for any $\beta > 1$. Furthermore, they determined that for $\beta \geq \frac{7}{2}$, the solution becomes a global strong solution, and it is unique for $\frac{7}{2} \leq \beta \leq 5$. Subsequently, Zhou [2] improved these results, establishing that the strong solution exists globally for $\beta \geq 3$, and it is

†The corresponding author.
Email address: caiyiting@zjnu.edu.cn (Yiting Cai)
[1]Zhejiang Normal University, Jinhua, 321004, P. R. China

unique for all $\beta \geq 1$. More recently, Cai et al. [8] have established the global existence and uniqueness of strong solutions for Equation (1.1) when $\alpha \geq \frac{5}{4}$ for $\beta \geq 1$ and when $\frac{1}{2} + \frac{2}{\beta} \leq \alpha \leq \frac{5}{4}$ for $\frac{8}{3} \leq \beta < +\infty$.

Recently, our attention has been drawn to the asymptotic behavior of the weak solutions of (1.1) with $\alpha = 1$. Through the refinement of the traditional Fourier splitting method, Jia et al. [3] provided the L^2 decay rate of the weak solutions, which holds for $\beta \geq \frac{10}{3}$. Additionally, Jiang and Zhu et al. [4,5] demonstrated that, if the initial condition satisfies $\|e^{\triangle t} u_0\|_{L^2} \leq C(1+t)^{-\mu}$ with $\mu > 0$, the weak solutions of (1.1) with $\beta \geq 3$ obey the bound $\|u(t)\|_{L^2} \leq C(1+t)^{\min\{-\mu, \frac{3}{4}\}}$. Yang et al. [6] further strengthened this result, showing that for $\beta \geq \frac{7}{3}$, the weak solutions satisfy $\|u(t)\|_{L^2} \leq C(1+t)^{-\alpha_0}$, where α_0 is defined as $\alpha_0 = \begin{cases} \min\{\mu, \frac{5}{4}\}, & \beta \in (\frac{7}{3}, 9] \\ \min\{\mu, \frac{3\beta-7}{4(\beta-5)}\}, & \beta \in (9, \infty) \end{cases}$.

Recently, Jiu et al. [10] derived decay estimates for weak solutions of the three-dimensional generalized Navier-Stokes equations. Motivated by [10], we aim to enhance the derived decay estimates of the solution of (1.1) through an iterative approach.

This paper focuses on the long-term behavior of the weak solutions to system (1.1) specifically in the scenario where $\alpha < \frac{5}{4}$. Our aim is to assess the influence of the damping term by utilizing techniques detailed in references [4,6,9,10]. To establish our primary findings, we shall employ the Fourier splitting technique introduced by Schonbek [9]. Our main results are given by the following theorems.

Theorem 1.1. *Let $\alpha = 0$ or $\beta = 1$. For $u_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, the system admits a weak solution such that*

$$\|u\|_2^2 \leq Ce^{-2t},$$

where the constant C only depends on $\|u_0\|_{L^2(\mathbb{R}^3)}$.

Theorem 1.2. *Let $0 < \alpha < \frac{3}{4}$, $\beta = 2$. For $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, the system admits a weak solution such that*

$$\|u\|_2^2 \leq C(1+t)^{-\frac{3}{2\alpha}},$$

where the constant C depends on α and $\|u_0\|_{L^2(\mathbb{R}^3)}$.

Theorem 1.3. *Let $0 < \alpha < \frac{5}{4}$, $\beta > \max\{\frac{4\alpha}{3} + 1, 2\}$. For $u_0 \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$ with $\operatorname{div} u_0 = 0$, the system admits a weak solution such that*

$$\|u\|_2^2 \leq C(1+t)^{-\frac{3}{2\alpha}},$$

where the constant C depends on α, β and $\|u_0\|_{L^2(\mathbb{R}^3)}$.

This paper is organized as follows. In Section 2, we will give some notations and lemmas which will be used in the proof of our main Theorems. We will give the proof Theorem 1.1, 1.2 and 1.3 in Section 3.

2. Preliminaries

The following Gagliardo-Nirenberg inequality plays a very important role in our estimation.

Lemma 2.1 (Gagliardo-Nirenberg inequality [11,12])**.** *Let u belong to L^q in $\mathbb{R}^n$ and its derivatives of order m, $D^m u$, belong to L^r, $1 \le q,\ r \le \infty$. For the derivatives $D^m u$, $0 \le j < m$, the following inequalities hold*

$$\|D^j u\|_{L^p} \le C\|D^m u\|_{L^r}^{\alpha}\|u\|_{L^q}^{1-\alpha}$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha(\frac{1}{r} - \frac{m}{n}) + (1-\alpha)\frac{1}{q}$$

for all α in the interval

$$\frac{j}{m} \le \alpha \le 1.$$

Lemma 2.2 (Plancherel's theorem)**.** *Assume $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{f}, \check{f} \in L^2(\mathbb{R}^n)$ and*

$$\|\hat{f}\|_{L^2(\mathbb{R}^n)} = \|\check{f}\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}.$$

Lemma 2.3 (Young inequality)**.** *Assume $a > 0, b > 0, p > 1, q > 1$, the following inequalities hold*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Lemma 2.4 (Hölder inequality)**.** *Assume $p > 1, q > 1, f \in L^p(\mathbb{R}^3), g \in L^p(\mathbb{R}^3)$. Then $fg \in L^1(\mathbb{R}^3)$ and*

$$\int_{\mathbb{R}^3} fg dx \le \|f\|_{L^p}\|g\|_{L^q}$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

3. Proof of main theorems

In this section, we give the proof of our main theorems.

3.1. Proof of Theorem 1.1 and Theorem 1.2

Proof. For $\alpha = 0$ or $\beta = 1$, taking the inner product (1.1) with u, after integration by parts, we obtain

$$\frac{d}{dt}\|u\|_2^2 + 2\|u\|_2^2 \le 0, \tag{3.1}$$

which implies

$$\|u\|_2^2 \le e^{-2t}\|u_0\|_2^2.$$

The proof of Theorem 1.1 is now complete. □

Proof. For $\beta = 2$, taking the L^2-inner product of the first equation of (1.1) with $u(x,t)$ and integrating over $\mathbb{R}^3$, we derive that

$$\frac{d}{dt}\|u\|_2^2 + 2\|u\|_3^3 = -2\|\Lambda^\alpha u\|_2^2, \tag{3.2}$$

which follows

$$\|u\|_2^2 + 2\int_0^t \|u\|_3^3(\tau)d\tau + 2\int_0^t \|\Lambda^\alpha u\|_2^2(\tau)d\tau = \|u_0\|_2^2.$$

Applying Plancherel's theorem, this yields

$$2\|\Lambda^\alpha u\|_2^2 = 2\int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widehat{u}(\xi)|^2 d\xi.$$

Let

$$S(t) = \{\xi \in \mathbb{R}^3 : |\xi| \le g(t)\},\ g(t) = (\frac{\gamma}{2(1+t)})^{\frac{1}{2\alpha}}, \tag{3.3}$$

where γ is a constant to be determined. Direct calculation we have

$$\begin{aligned} 2\int_{\mathbb{R}^3} |\xi|^{2\alpha}|\widehat{u}(\xi)|^2 d\xi &\ge 2g^{2\alpha}(t)\int_{|\xi|\ge g(t)} |\widehat{u}(\xi)|^2 d\xi + 2\int_{|\xi|\le g(t)} |\xi|^{2\alpha}|\widehat{u}(\xi)|^2 d\xi \\ &\ge 2g^{2\alpha}(t)\int_{\mathbb{R}^3} |\widehat{u}(\xi)|^2 d\xi - 2g^{2\alpha}(t)\int_{|\xi|\le g(t)} |\widehat{u}(\xi)|^2 d\xi. \end{aligned} \tag{3.4}$$

Consequently, from (3.2), we obtain

$$\frac{d}{dt}\|u\|_2^2 + 2\|u\|_3^3 + 2g^{2\alpha}(t)\|u\|_2^2 \le 2g^{2\alpha}(t)\int_{|\xi|\le g(t)} |\widehat{u}(\xi)|^2 d\xi. \tag{3.5}$$

Taking the divergence of (1.1) with $\beta = 2$, we obtain

$$\nabla\cdot[(u\cdot\nabla)u + |u|u] + \Delta P = 0,$$

since u is divergence free. It follows P is a solution of the equation, then we derive

$$\widehat{\Delta P} = -|\xi|^2\widehat{P} = -i\xi\cdot[\widehat{(u\cdot\nabla)u}(\xi,s) + \widehat{|u|u}(\xi,s)].$$

By utilizing the Duhamel principle, one can express $\widehat{u}$ in the following integral form,

$$\widehat{u}(\xi,t) = e^{-|\xi|^{2\alpha}t}\widehat{u_0}(\xi) - \int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})[\widehat{(u\cdot\nabla)u}(\xi,s) + \widehat{|u|u}(\xi,s)]ds.$$

By utilizing the inequalities

$$|\widehat{(u\cdot\nabla)u}(\xi,t)| \le \sum_i^3 |\xi||\widehat{u^i u}(\xi,t)| \le C|\xi|\|u(t)\|_2^2, \tag{3.6}$$

and

$$|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|u}(\xi,s)ds| \le \int_0^t \|u\|_2^2(s)ds \le C(1+t), \tag{3.7}$$

where C is a constant, we can multiply equation (3.5) by $G(t)=e^{2\int_0^t g^{2\alpha}(s)ds}$ and note that the volume $|S(t)|=Cg^3(t)$, we derive

$$\begin{aligned}&\frac{d}{dt}(G(t)\|u\|_2^2)+2G(t)\|u\|_3^3\\ &\le Cg^{2\alpha}(t)G(t)[(1+t)^{-\frac{3}{2\alpha}}+(1+t)^{-(\frac{5}{2\alpha}-2)}+(1+t)^{-(\frac{3}{2\alpha}-2)}].\end{aligned}$$

After integrating with respect to t, it becomes

$$\begin{aligned}&(1+t)^\gamma\|u(t)\|_2^2+2\int_0^t(1+s)^\gamma\|u(s)\|_3^3ds\\ &\le \|u_0\|_2^2+C\int_0^t(1+s)^{\gamma-1}[(1+s)^{-\frac{3}{2\alpha}}+(1+s)^{-(\frac{5}{2\alpha}-2)}+(1+s)^{-(\frac{3}{2\alpha}-2)}]ds\\ &\le \|u_0\|_2^2+C(1+t)^\gamma[(1+t)^{-\frac{3}{2\alpha}}+(1+t)^{-(\frac{5}{2\alpha}-2)}+(1+t)^{-(\frac{3}{2\alpha}-2)}].\end{aligned}$$

Finally, we have

$$\begin{aligned}&\|u(t)\|_2^2+2(1+t)^{-\gamma}\int_0^t(1+s)^\gamma\|u(s)\|_3^3ds\\ &\le C[(1+t)^{-\gamma}+(1+t)^{-\frac{3}{2\alpha}}+(1+t)^{-(\frac{5}{2\alpha}-2)}+(1+t)^{-(\frac{3}{2\alpha}-2)}].\end{aligned}$$

By choosing γ suitably large, we obtain

$$\|u(t)\|_2^2 \le C(1+t)^{-(\frac{3}{2\alpha}-2)}. \tag{3.8}$$

In the subsequent sections, we will utilize this initial preliminary decay to bootstrap and attempt to obtain more precise estimates for $\|u\|_2^2$. There are three cases that we need to consider.

Case 1: When $0<\alpha<\frac{1}{2}$, it is important to note that $\frac{5}{2\alpha}-2>\frac{3}{2\alpha}$, indicating that the decay obtained from $\widehat{(u\cdot\nabla)}u$ is sharper than the decay $(1+t)^{-\frac{3}{2\alpha}}$. We can now improve the decay rate in (3.8). By utilizing (3.7) and (3.8), we can achieve a better decay rate as follows:

$$\int_0^t\|u\|_2^2(s)ds \le C\int_0^t(1+s)^{-(\frac{3}{2\alpha}-2)}ds \le C\frac{2\alpha}{3-6\alpha}[1-(1+t)^{-(\frac{3}{2\alpha}-3)}] \le C.$$

Substituting the above estimate into (3.5) and choosing γ suitably large, we obtain

$$\|u(t)\|_2^2+2(1+t)^{-\gamma}\int_0^t(1+s)^\gamma\|u(s)\|_3^3ds \le C(1+t)^{-\frac{3}{2\alpha}}.$$

Case 2: When $\alpha=\frac{1}{2}$, by using (3.8), we have

$$\begin{aligned}|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)}u(\xi,s)ds| &\le C|\xi|\int_0^t(1+s)^{-1}ds\\ &\le C|\xi|\ln(1+t),\end{aligned} \tag{3.9}$$

and

$$
\begin{aligned}
|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|u}(\xi,s)ds| &\le \int_0^t \|u\|_2^2(s)ds \\
&\le C\int_0^t (1+s)^{-1}ds \\
&\le C\ln(1+t).
\end{aligned}
\tag{3.10}
$$

Therefore, combined with (3.9) and (3.10), we obtain

$$
\begin{aligned}
&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_3^3 ds \\
&\le C[(1+t)^{-\gamma} + (1+t)^{-3} + (1+t)^{-5}\ln^2(1+t) + (1+t)^{-3}\ln^2(1+t)].
\end{aligned}
$$

By choosing γ suitably large gives

$$
\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_3^3 ds \le C(1+t)^{-2},
$$

which implies

$$
\int_0^t \|u\|_2^2(s)ds \le C.
$$

Similar to Case 1, we finally obtain

$$
\|u\|_2^2 \le C(1+t)^{-3}.
$$

Case 3: When $\frac{1}{2} < \alpha < \frac{3}{4}$, by using (3.6) and (3.8), we have

$$
\begin{aligned}
|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)u}(\xi,s)ds| &\le C|\xi|\int_0^t (1+s)^{-(\frac{3}{2\alpha}-2)}ds \\
&\le C|\xi|(1+t)^{-(\frac{3}{2\alpha}-3)},
\end{aligned}
\tag{3.11}
$$

and

$$
\begin{aligned}
|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|u}(\xi,s)ds| &\le \int_0^t \|u\|_2^2(s)ds \\
&\le C\int_0^t (1+s)^{-(\frac{3}{2\alpha}-2)}ds \\
&\le C(1+t)^{-(\frac{3}{2\alpha}-3)}.
\end{aligned}
\tag{3.12}
$$

Combined with (3.11) and (3.12), we have

$$
\begin{aligned}
&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_3^3 ds \\
&\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{9}{2\alpha}-6)}].
\end{aligned}
$$

Since $\alpha < \frac{3}{4}$, we have $\frac{3}{2\alpha} - 2 < \frac{9}{2\alpha} - 6 < \frac{3}{2\alpha}$. Repeating the steps above for n times, we obtain

$$
\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_3^3 ds
$$

$$\leq C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-[\frac{3}{2\alpha}+\frac{3}{2\alpha}(2^n-2)-2(2^n-1)]}].$$

By choosing γ, n suitably large, one has $\frac{3}{2\alpha} + \frac{3}{2\alpha}(2^n-2) - 2(2^n-1) > 1$. Finally we obtain

$$\|u\|_2^2 \leq C(1+t)^{-\frac{3}{2\alpha}}.$$

The proof of Theorem 1.2 is finished. □

3.2. Proof of Theorem 1.3

Proof. By taking L^2-inner product on both sides of the first equation of (1.1) with $u(x,t)$, we have

$$\frac{d}{dt}\|u\|_2^2 + 2\|u\|_{\beta+1}^{\beta+1} = -2\|\Lambda^\alpha u\|_2^2, \tag{3.13}$$

and

$$\|u\|_2^2 + 2\int_0^t \|u\|_{\beta+1}^{\beta+1}(\tau)d\tau + 2\int_0^t \|\Lambda^\alpha u\|_2^2(\tau)d\tau = \|u_0\|_2^2. \tag{3.14}$$

Similar to the proof of Theorem 1.2, from (3.13), it follows that

$$\frac{d}{dt}\|u\|_2^2 + 2\|u\|_{\beta+1}^{\beta+1} + 2g^{2\alpha}(t)\|u\|_2^2 \leq 2g^{2\alpha}(t)\int_{|\xi|\leq g(t)} |\widehat{u}(\xi)|^2 d\xi. \tag{3.15}$$

Taking the divergence of (1.1), we obtain

$$\nabla \cdot [(u\cdot\nabla)u + |u|^{\beta-1}u] + \Delta P = 0,$$

since u is divergence free. It follows that P is a solution of the equation, then we derive

$$\widehat{\Delta P} = -|\xi|^2\widehat{P} = -i\xi\cdot[\widehat{(u\cdot\nabla)}u(\xi,s) + \widehat{|u|^{\beta-1}}u(\xi,s)].$$

In order to estimate the term on the right side, utilizing the Duhamel principle, we obtain

$$\widehat{u}(\xi,t) = e^{-|\xi|^{2\alpha}t}\widehat{u_0}(\xi) - \int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})[\widehat{(u\cdot\nabla)}u(\xi,s) + \widehat{|u|^{\beta-1}}u(\xi,s)]ds.$$

For $\beta > 2$, we have

$$\|u(t)\|_\beta \leq \|u(t)\|_{\beta+1}^{\frac{(\beta-2)(\beta+1)}{\beta(\beta-1)}} \|u(t)\|_2^{\frac{2}{\beta(\beta-1)}}.$$

Consequently, it follows that

$$\begin{aligned} &|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}}u(\xi,s)ds| \\ &\qquad \leq \int_0^t e^{-|\xi|^{2\alpha}(t-s)}\|u\|_\beta^\beta(s)ds \\ &\qquad \leq \int_0^t \|u(s)\|_{\beta+1}^{\frac{(\beta-2)(\beta+1)}{\beta-1}} \|u(s)\|_2^{\frac{2}{\beta-1}} ds \\ &\qquad \leq (\int_0^t \|u(s)\|_2^2 ds)^{\frac{1}{\beta-1}} (\int_0^t \|u(s)\|_{\beta+1}^{\beta+1} ds)^{\frac{\beta-2}{\beta-1}}. \end{aligned} \tag{3.16}$$

Then, we obtian

$$\int_0^t \|u(s)\|_\beta^\beta ds \le C(1+t)^{\frac{1}{\beta-1}}.$$

Mulitiplying (3.15) by $G(t) = e^{2\int_0^t g^{2\alpha}(s)ds}$ yields that

$$\begin{aligned}&\frac{d}{dt}(G(t)\|u\|_2^2) + 2G(t)\|u\|_{\beta+1}^{\beta+1}\\&\le Cg^{2\alpha}(t)G(t)[(1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{5}{2\alpha}-2)} + (1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}].\end{aligned}$$

Integrating the above inequality with respect to t leads to

$$\begin{aligned}&(1+t)^\gamma\|u(t)\|_2^2 + 2\int_0^t (1+s)^\gamma \|u(s)\|_{\beta+1}^{\beta+1} ds\\&\le \|u_0\|_2^2 + C\int_0^t (1+s)^{\gamma-1}[(1+s)^{-\frac{3}{2\alpha}} + (1+s)^{2-\frac{5}{2\alpha}} + (1+s)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}]ds\\&\le \|u_0\|_2^2 + C(1+t)^\gamma[(1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{5}{2\alpha}-2)} + (1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}].\end{aligned}$$

We finally obtain

$$\begin{aligned}&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^\gamma \|u(s)\|_{\beta+1}^{\beta+1} ds\\&\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{5}{2\alpha}-2)} + (1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}].\end{aligned} \tag{3.17}$$

Next, we use an iterative method to complete the proof. Two cases will be considered respectively in the rest of the proof.

3.2.1. $\alpha \in (0, \frac{4}{3}]$

In this subsection, for any $\alpha \in (0, \frac{3}{4}]$ we prove the result for $\beta \in (2, \infty)$. In the case of $0 < \alpha \le \frac{3}{4}, 2 < \beta$ and $\frac{1}{2} < \alpha \le \frac{3}{4}, 2 < \beta \le \frac{4\alpha-1}{2\alpha-1}$, from (3.17), by choosing γ suitably large, we obtain

$$\|u(t)\|_2^2 \le C(1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}. \tag{3.18}$$

In the case of $\frac{1}{2} < \alpha \le \frac{3}{4}, \beta > \frac{4\alpha-1}{2\alpha-1}$, one has $\frac{5}{2\alpha} - 2 < \frac{3}{2\alpha} - \frac{2}{\beta-1}$. Consequently, we have

$$\|u(t)\|_2^2 \le C(1+t)^{-(\frac{5}{2\alpha}-2)}. \tag{3.19}$$

Now we improve the decay rate in (3.18). Five cases will be considered respectively.

(i) $0 < \alpha \le \frac{1}{2}, 2 < \beta$. A bootstrap-type argument will lead to a better decay rate. Using (3.18), inequality (3.16) becomes

$$\begin{aligned}&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds|\\&\le (\int_0^t \|u(s)\|_2^2 ds)^{\frac{1}{\beta-1}}(\int_0^t \|u(s)\|_{\beta+1}^{\beta+1}ds)^{\frac{\beta-2}{\beta-1}}\\&\le C(\int_0^t (1+s)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}ds)^{\frac{1}{\beta-1}}\\&\le C(1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1}-1)\frac{1}{\beta-1}}\\&\le C.\end{aligned} \tag{3.20}$$

Then we obtain

$$\frac{d}{dt}(G(t)\|u\|_2^2)+2G(t)\|u\|_{\beta+1}^{\beta+1}\le Cg^{2\alpha}(t)G(t)(1+t)^{-\frac{3}{2\alpha}}.$$

Integrating with respect to time and choosing γ suitably yields

$$\begin{aligned}\|u(t)\|_2^2+2(1+t)^{-\gamma}\int_0^t(1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds &\le C[(1+t)^{-\gamma}+(1+t)^{-\frac{3}{2\alpha}}]\\ &\le C(1+t)^{-\frac{3}{2\alpha}}.\end{aligned}$$

$(ii)\frac{1}{2}<\alpha\le\frac{3}{4}, 2<\beta<\frac{3+2\alpha}{3-2\alpha}$. Taking (3.18), inequality (3.6) becomes

$$\begin{aligned}&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)u}(\xi,s)ds|\\ &\le C|\xi|\int_0^t(1+s)^{\frac{2}{\beta-1}-\frac{3}{2\alpha}}ds\\ &\le C\frac{1}{1+\frac{2}{\beta-1}-\frac{3}{2\alpha}}|\xi|[(1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1}-1)}-1]\\ &\le C|\xi|(1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1}-1)}.\end{aligned}\tag{3.21}$$

Using (3.18), inequality (3.16) becomes

$$\begin{aligned}&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds|\\ &\le(\int_0^t\|u(s)\|_2^2ds)^{\frac{1}{\beta-1}}(\int_0^t\|u(s)\|_{\beta+1}^{\beta+1}ds)^{\frac{\beta-2}{\beta-1}}\\ &\le C[\int_0^t(1+s)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}ds]^{\frac{1}{\beta-1}}\\ &\le C(1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1}-1)\frac{1}{\beta-1}}.\end{aligned}\tag{3.22}$$

Combined with (3.21) and (3.22), by choosing γ suitably large, we obtain

$$\begin{aligned}&\|u(t)\|_2^2+2(1+t)^{-\gamma}\int_0^t(1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds\\ &\le C[(1+t)^{-\gamma}+(1+t)^{-\frac{3}{2\alpha}}+(1+t)^{-(\frac{11}{2\alpha}-\frac{4}{\beta-1}-2)}+(1+t)^{-[\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{3}{2\alpha}-\frac{2}{\beta-1})\frac{2}{\beta-1}]}]\\ &\le C(1+t)^{-[\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{3}{2\alpha}-\frac{2}{\beta-1})\frac{2}{\beta-1}]}.\end{aligned}$$

When $0<\frac{3}{2\alpha}-\frac{2}{\beta-1}<1$, we obtain

$$\frac{3}{2\alpha}-\frac{2}{\beta-1}<\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{3}{2\alpha}-\frac{2}{\beta-1})\frac{2}{\beta-1}<\frac{3}{2\alpha},$$

and

$$\frac{1}{\alpha}>2-\frac{2}{\beta-1}>(2-\frac{2}{\beta-1})(1-\frac{3}{2\alpha}+\frac{2}{\beta-1}).$$

Repeating the steps above to find a estimate sharper than the decay $(1+t)^{-\frac{3}{2\alpha}}$, for $\beta > 3$, we have

$$\begin{aligned}&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds\\&\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-\frac{3(\beta-1)-4\alpha}{2\alpha(\beta-3)}}].\end{aligned} \tag{3.23}$$

For $\beta = 3$, we derive that

$$\begin{aligned}&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds\\&\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{3n}{2\alpha}-n)}].\end{aligned} \tag{3.24}$$

For $2 < \beta < 3$, we derive that

$$\begin{aligned}&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds\\&\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})(\frac{q^n-1}{q-1})}],\end{aligned} \tag{3.25}$$

where $q = \frac{2}{\beta-1}$. By choosing γ, n suitably large, where n is the number of iterations, and noting $\frac{3(\beta-1)-4\alpha}{2\alpha(\beta-3)} \ge \frac{3}{2\alpha}$, $\frac{3n}{2\alpha} - n \ge \frac{3}{2\alpha}$, $(\frac{3}{2\alpha} - \frac{2}{\beta-1})(\frac{q^n-1}{q-1}) \ge \frac{3}{2\alpha}$, we have

$$\|u(t)\|_2^2 \le C(1+t)^{-\frac{3}{2\alpha}}.$$

$(iii)\frac{1}{2} < \alpha \le \frac{3}{4}, \beta = \frac{3+2\alpha}{3-2\alpha}$. Similar to the proof of Case 2, one has

$$\|u(t)\|_2^2 \le C(1+t)^{-\frac{3}{2\alpha}}.$$

$(iv)\frac{1}{2} < \alpha \le \frac{3}{4}, \frac{3+2\alpha}{3-2\alpha} < \beta \le \frac{4\alpha-1}{2\alpha-1}$. Using (3.18), for $1 + \frac{2}{\beta-1} - \frac{3}{2\alpha} < 0$, inequality (3.6) becomes

$$\begin{aligned}&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)u}(\xi,s)ds|\\&\le C|\xi|\int_0^t (1+s)^{\frac{2}{\beta-1}-\frac{3}{2\alpha}}ds\\&\le C\frac{1}{1+\frac{2}{\beta-1}-\frac{3}{2\alpha}}|\xi|[(1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1}-1)} - 1]\\&\le C|\xi|.\end{aligned} \tag{3.26}$$

And inequality (3.16) becomes

$$\begin{aligned}&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds|\\&\le (\int_0^t \|u(s)\|_2^2 ds)^{\frac{1}{\beta-1}}(\int_0^t \|u(s)\|_{\beta+1}^{\beta+1}ds)^{\frac{\beta-2}{\beta-1}}\\&\le C(\int_0^t (1+s)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}ds)^{\frac{1}{\beta-1}}\\&\le C(1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1}-1)\frac{1}{\beta-1}}\\&\le C.\end{aligned} \tag{3.27}$$

This, combined with (3.26) and (3.27), yields

$$\frac{d}{dt}(G(t)\|u\|_2^2) + 2G(t)\|u\|_{\beta+1}^{\beta+1} \le Cg^{2\alpha}(t)G(t)[(1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-\frac{5}{2\alpha}}].$$

Integrating with respect to time and choosing γ suitably yields

$$\begin{aligned}&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^\gamma \|u(s)\|_{\beta+1}^{\beta+1} ds \\ &\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-\frac{5}{2\alpha}}] \\ &\le C(1+t)^{-\frac{3}{2\alpha}}.\end{aligned}$$

Then we improve the decay rate in (3.19).
$(v)\frac{1}{2} < \alpha \le \frac{3}{4}, \beta > \frac{4\alpha-1}{2\alpha-1}$. By using (3.19), it yields that

$$\begin{aligned}&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)}u(\xi,s)ds| \\ &\le C|\xi|\int_0^t (1+s)^{-(\frac{5}{2\alpha}-2)}ds \\ &\le C\frac{2\alpha}{6\alpha-5}|\xi|[(1+t)^{-(\frac{5}{2\alpha}-3)} - 1] \\ &\le C|\xi|.\end{aligned} \tag{3.28}$$

Next, we consider the estimate from $\widehat{|u|^{\beta-1}u}$. For $\frac{1}{2} < \alpha < \frac{3}{4}$, we obtain

$$\begin{aligned}&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds| \\ &\le (\int_0^t \|u(s)\|_2^2 ds)^{\frac{1}{\beta-1}} (\int_0^t \|u(s)\|_{\beta+1}^{\beta+1} ds)^{\frac{\beta-2}{\beta-1}} \\ &\le C[\int_0^t (1+s)^{-(\frac{5}{2\alpha}-2)}ds]^{\frac{1}{\beta-1}} \\ &\le C.\end{aligned} \tag{3.29}$$

Hence by (3.28) and (3.29)

$$|\widehat{u}(\xi,t)| \le |\widehat{u_0}(\xi)| + C(1+|\xi|),\ \ for\ \xi \in S(t).$$

Combining with (3.15) and integrating with respect to time yields

$$\begin{aligned}&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^\gamma \|u(s)\|_{\beta+1}^{\beta+1} ds \\ &\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-\frac{5}{2\alpha}}].\end{aligned}$$

By choosing γ suitably large, we have

$$\|u(t)\|_2^2 \le C(1+t)^{-\frac{3}{2\alpha}}.$$

3.2.2. $\alpha \in (\frac{3}{4}, \frac{5}{4})$

In this subsection, for any $\alpha \in (\frac{3}{4}, \frac{5}{4})$ we prove the same result for $\beta \in (\frac{4\alpha}{3}+1, \infty)$. In the case of $\frac{4\alpha}{3}+1 < \beta < \frac{4\alpha-1}{2\alpha-1}$, one has $\frac{5}{2\alpha} - 2 \geq \frac{3}{2\alpha} - \frac{2}{\beta-1}$. Hence, by choosing γ suitably large, the estimate takes the form

$$\|u(t)\|_2^2 \leq C(1+t)^{-(\frac{3}{2\alpha}-\frac{2}{\beta-1})}. \tag{3.30}$$

In the case of $\beta \geq \frac{4\alpha-1}{2\alpha-1}$, one has $\frac{5}{2\alpha} - 2 \leq \frac{3}{2\alpha} - \frac{2}{\beta-1}$. Consequently, by choosing γ suitably large, we have

$$\|u(t)\|_2^2 \leq C(1+t)^{-(\frac{5}{2\alpha}-2)}. \tag{3.31}$$

Now we improve the decay rate in (3.30). Two cases will be considered respectively.

(vi) $\frac{3}{4} < \alpha \leq \frac{5}{6}$, $\frac{4\alpha}{3}+1 < \beta < \frac{4\alpha-1}{2\alpha-1}$. In the case of $\frac{4\alpha}{3}+1 < \beta < \frac{3+2\alpha}{3-2\alpha}$, similar to the proof of *(ii)*. In the case of $\beta = \frac{3+2\alpha}{3-2\alpha}$, similar to the proof of Case 2. In the case of $\frac{3+2\alpha}{3-2\alpha} < \beta < \frac{4\alpha-1}{2\alpha-1}$, similar to the proof of *(iv)*.

(vii) $\frac{5}{6} < \alpha < \frac{5}{4}$, $\frac{4\alpha}{3}+1 < \beta < \frac{4\alpha-1}{2\alpha-1}$. In this case, similar to the proof of *(ii)*.

In the rest of this subsection, we improve the decay rate in (3.31). Four cases will be considered respectively.

(viii) $\frac{3}{4} < \alpha < \frac{5}{6}$, $\beta \geq \frac{4\alpha-1}{2\alpha-1}$. By using (3.31), it yields that

$$\begin{aligned} &|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)}u(\xi,s)ds| \\ &\qquad \leq C|\xi|\int_0^t (1+s)^{-(\frac{5}{2\alpha}-2)}ds \\ &\qquad \leq C\frac{2\alpha}{6\alpha-5}|\xi|[(1+t)^{-(\frac{5}{2\alpha}-3)}-1] \\ &\qquad \leq C|\xi|. \end{aligned} \tag{3.32}$$

Next, we consider the estimate from $\widehat{|u|^{\beta-1}u}$. For $\frac{3}{4} < \alpha < \frac{5}{6}$, we obtain

$$\begin{aligned} &|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds| \\ &\qquad \leq (\int_0^t \|u(s)\|_2^2 ds)^{\frac{1}{\beta-1}}(\int_0^t \|u(s)\|_{\beta+1}^{\beta+1}ds)^{\frac{\beta-2}{\beta-1}} \\ &\qquad \leq C[\int_0^t (1+s)^{-(\frac{5}{2\alpha}-2)}ds]^{\frac{1}{\beta-1}} \\ &\qquad \leq C. \end{aligned} \tag{3.33}$$

Hence by (3.32) and (3.33)

$$|\widehat{u}(\xi,t)| \leq |\widehat{u_0}(\xi)| + C(1+|\xi|),\ for\ \xi \in S(t).$$

Combining with (3.15) and integrating with respect to time yields

$$\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^\gamma \|u(s)\|_{\beta+1}^{\beta+1}ds$$

$$\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}}].$$

By choosing γ suitably large, we have

$$\|u(t)\|_2^2 \le C(1+t)^{-\frac{3}{2\alpha}}.$$

(ix)When $\alpha = \frac{5}{6}$ and $\beta \ge \frac{4\alpha-1}{2\alpha-1}$, similar to the proof of Case 2, it goes back to the case $\frac{1}{2} < \alpha < \frac{5}{6}$.

(x) $\frac{5}{6} < \alpha \le 1$, $\beta \ge \frac{4\alpha-1}{2\alpha-1}$. By using (3.31), we obtain that

$$\begin{aligned} &|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)}u(\xi,s)ds| \\ &\le C|\xi|\int_0^t (1+s)^{-(\frac{5}{2\alpha}-2)}ds \\ &\le C\frac{2\alpha}{6\alpha-5}|\xi|[(1+t)^{-(\frac{5}{2\alpha}-3)} - 1] \\ &\le C|\xi|(1+t)^{-(\frac{5}{2\alpha}-3)}. \end{aligned} \tag{3.34}$$

It follows from (3.16) and (3.31) that

$$\begin{aligned} &|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I - \frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds| \\ &\le (\int_0^t \|u(s)\|_2^2 ds)^{\frac{1}{\beta-1}}(\int_0^t \|u(s)\|_{\beta+1}^{\beta+1}ds)^{\frac{\beta-2}{\beta-1}} \\ &\le C[\int_0^t (1+s)^{-(\frac{5}{2\alpha}-2)}ds]^{\frac{1}{\beta-1}} \\ &\le C(1+t)^{-(\frac{5}{2\alpha}-3)\frac{1}{\beta-1}}. \end{aligned} \tag{3.35}$$

Combining (3.34) and (3.35) together and choosing γ suitably easily yields

$$\begin{aligned} &\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds \\ &\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{3}{2\alpha}-\frac{6\alpha-6}{\alpha})} + (1+t)^{-[\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{5}{2\alpha}-2)\frac{2}{\beta-1}]}] \\ &\le C(1+t)^{-[\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{5}{2\alpha}-2)\frac{2}{\beta-1}]}. \end{aligned}$$

When $\frac{5}{2\alpha} - 2 > 0$, one has $\frac{3}{2\alpha} - \frac{2}{\beta-1} < \frac{3}{2\alpha} - \frac{2}{\beta-1} + (\frac{5}{2\alpha} - 2)\frac{2}{\beta-1} < \frac{3}{2\alpha} < \frac{3}{2\alpha} - \frac{6\alpha-6}{\alpha}$. Repeating the steps as for (3.23) and choosing γ suitably gives that

$$\begin{aligned} &\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds \\ &\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{3}{2\alpha}-\frac{6\alpha-6}{\alpha})} + (1+t)^{-\frac{3(\beta-1)-4\alpha}{2\alpha(\beta-3)}}]. \end{aligned} \tag{3.36}$$

Note that $\frac{3(\beta-1)-4\alpha}{2\alpha(\beta-3)} > 1$. It implies that we can also find a decay estimate $(1+t)^{-\frac{3}{2\alpha}}$ in this case.

(xi) $1<\alpha<\frac{5}{4}$,$\beta\geq\frac{4\alpha-1}{2\alpha-1}$. Using (3.6), we obtain

$$\begin{aligned}
&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)u}(\xi,s)ds| \\
&\leq C|\xi|\int_0^t(1+s)^{-(\frac{5}{2\alpha}-2)}ds \\
&\leq C\frac{2\alpha}{6\alpha-5}|\xi|[(1+t)^{-(\frac{5}{2\alpha}-3)}-1] \\
&\leq C|\xi|(1+t)^{-(\frac{5}{2\alpha}-3)}.
\end{aligned} \tag{3.37}$$

And by (3.16), it follows that

$$\begin{aligned}
&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds| \\
&\leq(\int_0^t\|u(s)\|_2^2ds)^{\frac{1}{\beta-1}}(\int_0^t\|u(s)\|_{\beta+1}^{\beta+1}ds)^{\frac{\beta-2}{\beta-1}} \\
&\leq C[\int_0^t(1+s)^{-(\frac{5}{2\alpha}-2)}ds]^{\frac{1}{\beta-1}} \\
&\leq C(1+t)^{-(\frac{5}{2\alpha}-3)\frac{1}{\beta-1}}.
\end{aligned} \tag{3.38}$$

Hence, combining (3.37) and (3.38) together and integrating with respect to time yields

$$\begin{aligned}
&\|u(t)\|_2^2+2(1+t)^{-\gamma}\int_0^t(1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds \\
&\leq C[(1+t)^{-\gamma}+(1+t)^{-\frac{3}{2\alpha}}+(1+t)^{-(\frac{15}{2\alpha}-6)}+(1+t)^{-[\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{5}{2\alpha}-2)\frac{2}{\beta-1}]}].
\end{aligned}$$

Similarly, we consider two cases.In the case of $2+\frac{1}{2\alpha-1}\leq\beta<2+\frac{1}{6\alpha-6}$, one has $\frac{15}{2\alpha}-6>\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{5}{2\alpha}-2)\frac{2}{\beta-1}$, which means

$$\|u(t)\|_2^2\leq C(1+t)^{-[\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{5}{2\alpha}-2)\frac{2}{\beta-1}]}.$$

According to case (ii), the decay estimates from $\widehat{|u|^{\beta-1}u}$ are used for ltaer iteration, and similar to the result of (3.36), we obtain

$$\|u(t)\|_2^2\leq C(1+t)^{-\frac{3}{2\alpha}}.$$

For this reason, we now only discuss the case where the better decay estimates are derived from $\widehat{(u\cdot\nabla)u}$ to simplify the proof. In the case of $\beta\geq 2+\frac{1}{6\alpha-6}$, one has $\frac{15}{2\alpha}-6\leq\frac{3}{2\alpha}-\frac{2}{\beta-1}+(\frac{5}{2\alpha}-2)\frac{2}{\beta-1}$, and we get

$$\|u(t)\|_2^2\leq C(1+t)^{-(\frac{15}{2\alpha}-6)}. \tag{3.39}$$

Using (3.39) and (3.6), we have

$$\begin{aligned}
&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)u}(\xi,s)ds| \\
&\leq C|\xi|\int_0^t(1+s)^{-(\frac{15}{2\alpha}-6)}ds \\
&\leq C\frac{2\alpha}{14\alpha-15}|\xi|[(1+t)^{-(\frac{15}{2\alpha}-7)}-1].
\end{aligned} \tag{3.40}$$

Taking (3.39), inequality (3.16) is equivalent to

$$\begin{aligned}
&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds| \\
&\le (\int_0^t \|u(s)\|_2^2 ds)^{\frac{1}{\beta-1}}(\int_0^t \|u(s)\|_{\beta+1}^{\beta+1}ds)^{\frac{\beta-2}{\beta-1}} \\
&\le C[\int_0^t (1+s)^{-(\frac{15}{2\alpha}-6)}ds]^{\frac{1}{\beta-1}} \\
&\le C(1+t)^{-(\frac{15}{2\alpha}-7)\frac{1}{\beta-1}}.
\end{aligned} \tag{3.41}$$

Hence, combining (3.15), (3.40) and (3.41) together and integrating with respect to time yields

$$\begin{aligned}
&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds \\
&\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{3}{2\alpha}+\frac{32}{2\alpha}-14)} + (1+t)^{-[\frac{3}{2\alpha}+(\frac{15}{2\alpha}-7)\frac{2}{\beta-1}]}].
\end{aligned}$$

In the case of $1<\alpha\le\frac{15}{14}$, one has $\frac{15}{2\alpha}-6>1$. Similar to the proof of $(viii)$ and (ix), by choosing γ suitably, we have

$$\begin{aligned}
&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds \\
&\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{3}{2\alpha}+\frac{32}{2\alpha}-14)} + (1+t)^{-[\frac{3}{2\alpha}+(\frac{15}{2\alpha}-7)\frac{2}{\beta-1}]}] \\
&\le C(1+t)^{-\frac{3}{2\alpha}}.
\end{aligned}$$

In the case of $\frac{15}{14}<\alpha\le\frac{8}{7}$, similar to the proof of (x), by choosing γ suitably, we obtain

$$\begin{aligned}
&\|u(t)\|_2^2 + 2(1+t)^{-\gamma}\int_0^t (1+s)^{\gamma}\|u(s)\|_{\beta+1}^{\beta+1}ds \\
&\le C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{3}{2\alpha}+\frac{16}{\alpha}-14)} + (1+t)^{-(\frac{3}{2\alpha}+(\frac{15}{2\alpha}-7)\frac{2}{\beta-1})}] \\
&\le C(1+t)^{-(\frac{3}{2\alpha}+(\frac{15}{2\alpha}-7)\frac{2}{\beta-1})}.
\end{aligned}$$

Repeating the steps as (x), we obtain the same result.

In tha case of $\frac{8}{7}<\alpha<\frac{5}{4}$ and $\beta\ge 2+\frac{1}{14\alpha-16}$, at this time, $\frac{8}{\alpha}-7<(\frac{15}{2\alpha}-7)\frac{1}{\beta-1}$ and we obtain a sharper decay

$$\|u(t)\|_2^2 \le C(1+t)^{-(\frac{35}{2\alpha}-14)}.$$

Similary, repeat the steps above, after the nth iterative, inequalities become

$$\begin{aligned}
&|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{(u\cdot\nabla)}u(\xi,s)ds| \\
&\le C\frac{2\alpha}{2\alpha b_n-2c_n+1}|\xi|[(1+t)^{-(\frac{2c_n-1}{2\alpha}-b_n)}-1],
\end{aligned} \tag{3.42}$$

and

$$|\int_0^t e^{-|\xi|^{2\alpha}(t-s)}(I-\frac{\xi\otimes\xi}{|\xi|^2})\widehat{|u|^{\beta-1}u}(\xi,s)ds| \le C[(1+t)^{-(\frac{2c_n-1}{2\alpha}-b_n)\frac{1}{\beta-1}}-1], \tag{3.43}$$

where $b_n = 2^{n+1} - 1, c_n = 5 \cdot 2^{n-1} - 2$.

Based on thr discovered patterns, it is easy to see that if $\frac{c_{n-1}}{b_{n-1}} < \alpha \leq \frac{2c_n - 1}{2b_n}$, one has $\frac{2c_n - 1}{2\alpha} - b_n > 0$, similar to the proof of $(viii)$ and (ix). If $\frac{2c_n - 1}{2b_n} < \alpha \leq \frac{c_n}{b_n}$, one has $\frac{3}{2\alpha} + \frac{2c_n}{\alpha} - 2b_n > \frac{3}{2\alpha}$, similar to the proof of (x). If $\frac{c_n}{b_n} < \alpha < \frac{5}{4}, \beta \geq 2 + \frac{1}{2b_n\alpha - 2c_n}$, one has $\frac{c_n}{\alpha} - b_n < (\frac{2c_n - 1}{2\alpha} - b_n)\frac{1}{\beta - 1}$. Combining (3.36), (3.42) and (3.43), we have

$$\begin{aligned} &\|u(t)\|_2^2 + 2(1+t)^{-\gamma} \int_0^t (1+s)^\gamma \|u(s)\|_{\beta+1}^{\beta+1} ds \\ &\leq C[(1+t)^{-\gamma} + (1+t)^{-\frac{3}{2\alpha}} + (1+t)^{-(\frac{3}{2\alpha} - 2b_n + \frac{2c_n}{\alpha})}]. \end{aligned}$$

By choosing n, γ suitably large, where n is the number of iterations, note that $\frac{3}{2\alpha} - 2b_n + \frac{2c_n}{\alpha} > 1$, which means we can still find a decay in case (ix) to obatin

$$\|u(t)\|_2^2 \leq C(1+t)^{-\frac{3}{2\alpha}}.$$

The proof of Theorem 1.3 is finished. □

References

[1] X. Cai, Q. Jiu, *Weak and strong solutions for the incompressible Navier-Stokes equations with damping*, Journal of Mathematical Analysis and Applications, 2008, 343(2), 799-809.

[2] Y. Zhou, *Regularity and uniqueness for the 3D incompressible Navier-Stokes equations with damping*, Applied Mathematics Letters. An International Journal of Rapid Publication, 2012, 25(11), 1822-1825.

[3] Y. Jia, X. Zhang, B. Dong, *The asymptotic behavior of solutions to three-dimensional Navier-Stokes equations with nonlinear damping*, Nonlinear Analysis. Real World Applications. An International Multidisciplinary Journal, 2011, 12(3), 1736-1747.

[4] Z. Jiang, M. Zhu, *The large time behavior of solutions to 3D Navier-Stokes equations with nonlinear damping*, Mathematical Methods in the Applied Sciences, 2012, 35(1), 97-102.

[5] Z. Jiang, *Asymptotic behavior of strong solutions to the 3D Navier-Stokes equations with a nonlinear damping term*, Nonlinear Analysis. Theory, Methods & Applications. An International Multidisciplinary Journal, 2012, 75(13), 5002-5009.

[6] T. Yang, Z. Zhang, *Large time behaviour of solutions to the 3D Navier-Stokes equations with damping*, Zeitschrift für Angewandte Mathematik und Physik. ZAMP. Journal of Applied Mathematics and Physics. Journal de Mathématiques et de Physique Appliquées, 2020, 71(5), Paper No.172, 12.

[7] E. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, USA, 1970.

[8] X. Cai, Y. Zhou, *Global existence of strong solutions for the generalized Navier-Stokes equations with damping*, Acta Mathematicae Applicatae Sinica. English Series, 2022, 38(3), 627-634.

[9] M. Schonbek, *L^2 decay for weak solutions of the Navier-Stokes equations*, Archive for Rational Mechanics and Analysis, 1985, 88(3), 209-222.

[10] Q. Jiu, H. Yu, *Decay of solutions to the three-dimensional generalized Navier-Stokes equations*, Asymptotic Analysis, 2015, 94(1-2), 105-124.

[11] D. Mccormick, J. Robinson, J. Rodrigo, *Generalised Gagliardo-Nirenberg inequalities using weak Lebesgue spaces and BMO*, Milan Journal of Mathematics, 2015, 81(2), 265-289.

[12] L. Nirenberg, *On elliptic partial differential equations*, Annali della Scuola Normale Superiore di Pisa. Classe di Scienze. Serie III, 1959, 13, 115-162.