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Convergence Analysis for the SM-Iteration in Banach Spaces

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Abstract In this article, the fixed point of the SM iterative approach is approximated via Suzuki mapping. In the context of Banach spaces, we provide both weak and strong convergence for the SM iteration. Then, by comparing it with certain well-known iterations, we give some numerical examples to show the effectiveness of SM iteration for the Suzuki-type mapping.

Keywords Suzuki mapping, Opial's condition, SM-iteration

MSC(2010) 47H09, 47H10.

1. Introduction

Fixed point theory is a valuable tool for resolving a variety of practical mathematics difficulties. The Banach contraction principle [5] was the first step toward the fixed point theory on metric spaces. Since many nonlinear analysis problems cannot be solved analytically, iterative approaches for approximating fixed points of various kinds of mappings become essential. In this sense, the development of other processes benefited greatly from the foundation provided by Picard iteration [14]. Though it was successful for contraction mappings, a wider class of mappings constructed on Banach spaces (BS), namely non-expansive mappings, may not necessarily converge to the fixed point, as demonstrated in 1955 by Krasnoselskii [10], where a mapping $\mathfrak{S} : V \to V$, for V being a non-empty closed and convex subset of BS E, is said to be non-expansive if it satisfies the inequality $\|\mathfrak{S}\varkappa - \mathfrak{S}y\| \leq \|\varkappa - y\|$, for all $\varkappa, y \in V$. In addition, if $Fix(\mathfrak{S}) \neq \phi$, where $Fix(\mathfrak{S}) = \{\varkappa \in V : \mathfrak{S}\varkappa = \varkappa\}$ and $\|\mathfrak{S}\varkappa - q\| \leq \|\varkappa - q\|$, for every $\varkappa \in V$, set of fixed points (FP), and $q \in Fix(\mathfrak{S})$, then $\mathfrak{S}$ is called quasi-non-expansive. The main reason for this behaviour is that successive iterations of non-expansive mappings do not need to converge to a fixed point, unlike contraction mappings. Since then, numerous additional iterative procedures have been developed for numerically calculating the fixed points of non-expansive mappings. For instance, one of the first iteration was proposed by Mann [11], which is described as follows: for an arbitrary chosen $\varkappa_0 \in V$, the iteration is defined as:

$$\varkappa_{n+1} = \alpha_n \varkappa_n + (1 - \alpha_n)\mathfrak{S}\varkappa_n, \quad n \geq 0,$$

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where, $\{\alpha_n\}$ is a real sequence in the interval $(0,1)$. After that two-step iterative method, which is Ishikawa [9] iteration, mostly used for finding fixed point of non-expansive mappings: for initial point $\varkappa_0 \in V$, this iteration is defined as

$$\begin{aligned}\varkappa_{n+1} &=(1-\alpha_n)\varkappa_n + \alpha_n \mathfrak{S} y_n,\\ y_n &=(1-\beta_n)\varkappa_n + \beta_n \mathfrak{S}\varkappa_n, \quad n \geq 0,\end{aligned}$$

where $\{\alpha_n\},\{\beta_n\} \in (0,1)$. In the similar manner, Agarwal et al. [2], Noor [12], Abbas and Nazir [1] worked in the same direction. The subsequent three-step iterations, as described by Thakur et al. [19] in 2016, Rathee and Swami [15] in 2020, Ahmad et al. [4] in 2021, will be examined in this follow-up for finding fixed points of non-expansive mapping, which are defined, for self map $\mathfrak{S}$ on V, the initial point $\varkappa_0 \in V$ with real number sequences $\{\alpha_n\},\{\beta_n\},\{\gamma_n\} \in (0,1)$.

In 2016, Thakur [19] defined the following iteration:

$$\begin{aligned}\varkappa_{n+1} &= (1-\alpha_n)\mathfrak{S} z_n + \alpha_n \mathfrak{S} y_n,\\ y_n &= (1-\beta_n) z_n + \beta_n \mathfrak{S} z_n,\\ z_n &= (1-\gamma_n)\varkappa_n + \gamma_n \mathfrak{S}\varkappa_n. \end{aligned} \tag{TH}$$

In 2020, Rathee and Swami [15] proposed an iteration as follows:

$$\begin{aligned}\varkappa_{n+1} &= \mathfrak{S}((1-\alpha_n)\mathfrak{S} z_n + \alpha_n \mathfrak{S} y_n),\\ y_n &= \mathfrak{S}((1-\beta_n)\varkappa_n + \beta_n z_n),\\ z_n &= \mathfrak{S}\varkappa_n. \end{aligned} \tag{SM}$$

In 2021, the following iteration is defined by Ahmad et al. [4]

$$\begin{aligned}\varkappa_{n+1} &= \mathfrak{S}((1-\alpha_n)\mathfrak{S} z_n + \alpha_n \mathfrak{S} y_n),\\ y_n &= \mathfrak{S} z_n,\\ z_n &= (1-\beta_n)\varkappa_n + \beta_n \mathfrak{S}\varkappa_n. \end{aligned} \tag{JK}$$

On the other hand, in 2008, Suzuki [18] made significant progress by defining an intriguing expansion of non-expansive mappings, which is called Suzuki mapping (condition (C)). Suzuki mapping is defined for a self map $\mathfrak{S} : V \to V$ if the following condition holds:

$$\begin{aligned}\frac{1}{2}\|\varkappa - \mathfrak{S}\varkappa\| &\leq \|\varkappa - y\|\\ \Longrightarrow \|\mathfrak{S}\varkappa - \mathfrak{S} y\| &\leq \|\varkappa - y\|, \quad \text{for all } \varkappa, y \in V. \end{aligned} \tag{1.1}$$

It is evident that for certain domain elements, the Suzuki mappings meet the non-expansive criteria. Suzuki [18] used a simple example to illustrate his point that the newly introduced class is larger than the class of non-expansiveness. For this, define a mapping T on $[0,3]$ by

$$T\varkappa = \begin{cases} 0 \text{ if } \varkappa \neq 3\\ 2 \text{ if } \varkappa = 3.\end{cases}$$

This condition (C) caught the attention of many researchers who worked in the direction of finding different fixed point theorems. Here, in the current research, motivated by the results of [3,4,8,20,21], our goal is to broaden the scope of the SM iterative process research to include the larger category of non-expansive mappings, which is Suzuki mapping and compare with the (TH) and (JK) iterations. In this regard, we establish convergence results for Suzuki mappings to fixed points within the framework of uniformly convex Banach spaces. Additionally, illustrative examples are provided to support and validate our results.

2. Preliminaries

In order to understand our new results, let's first go over some theoretical conclusions.

Definition 2.1. [7] A normed vector space X is called uniformly convex (UC) if for each $\epsilon \in [0, 2]$ there is $\delta > 0$ such that for $\varkappa, y \in X, \|\varkappa\| \leq 1, \|y\| \leq 1$ and $\|\varkappa - y\| \geq \epsilon$ imply $\|\frac{\varkappa+y}{2}\| \leq 1 - \delta$.

We consider V to be any non-empty closed and convex subset of BS E, and a bounded sequence $\{\varkappa_n\}$ in E. Assume that b is any fixed element from E and define

- $\mathfrak{R}(b, \{\varkappa_n\})$, the asymptotic radius of $\{\varkappa_n\}$ at b where $\mathfrak{R}(b, \{\varkappa_n\}) = \limsup_{n\to\infty} \|b - \varkappa_n\|$.
- $\mathfrak{R}(V, \{\varkappa_n\})$, the asymptotic radius of $\{\varkappa_n\}$ associated with V where $\mathfrak{R}(V, \{\varkappa_n\}) = \inf\{\mathfrak{R}(b, \{\varkappa_n\}) : b \in V\}$.
- $\mathfrak{F}(V, \{\varkappa_n\})$, the asymptotic center of $\{\varkappa_n\}$ associated with V where $\mathfrak{F}(V, \{\varkappa_k\}) = \{b \in V : \mathfrak{R}(b, \{\varkappa_n\}) = \mathfrak{R}(V, \{\varkappa_n\})\}$.

The set $\mathfrak{F}(V, \{x_n\})$ is widely recognized for being a singleton whenever E is uniformly convex Banach space (UCBS) [6].

Lemma 2.1. *[18] Let V be a non-empty subset of a Banach space E, and if $\mathfrak{S} : V \to V$ is a Suzuki mapping, then*

1. *$\forall\ u \in V$ and $q \in Fix(\mathfrak{S})$, the fact $\|\mathfrak{S}u - \mathfrak{S}q\| \leq \|u - q\|$ holds.*
2. *$\forall\ u, v \in V$, we have*

$$\|u - \mathfrak{S}v\| \leq 3\|u - \mathfrak{S}u\| + \|u - v\|. \tag{2.1}$$

Definition 2.2. [13] A Banach space E is said to satisfy the Opial property if for each weakly convergent sequence $\{\varkappa_n\}$ in E with a weak limit u,

$$\lim_{n\to\infty} \inf \|\varkappa_n - u\| < \lim_{n\to\infty} \inf \|\varkappa_n - v\|$$

holds for all $v \in E$ with $u \neq v$.

The demiclosed principle refers to the following outcome, given by Suzuki [18].

Lemma 2.2. *[18] Let V be any non-empty subset of a BS E having the Opial property, and if $\mathfrak{S} : V \to V$ is a Suzuki mapping, then:*

$$\{u_k\} \subseteq V, u_k \rightharpoonup w, \|u_k - \mathfrak{S}u_k\| \to 0 \implies \mathfrak{S}w = w. \tag{2.2}$$

Here, $u_k \rightharpoonup w$, denotes weak convergence, which means that for any continuous linear functional $\mathfrak{S}$ on Banach space, $\mathfrak{S}(u_k) \to \mathfrak{S}(w)$ as $k \to \infty$.

Senter and Doston [17] introduced the following condition (I):

Definition 2.3. [17] A mapping $\mathfrak{F} : V \to V$ is said to satisfy condition (I), if there exists a non-decreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0, h(a) > 0$ for every $a > 0$, and $\|b - \mathfrak{F}b\| \geq h(d(b, Fix(\mathfrak{F}))$ for all $b \in V$.

Lemma 2.3. *[18] Let V be a weakly compact convex subset of UCBS E and $\mathfrak{F}$ be a self-mapping on V. Assume that $\mathfrak{F}$ satisfies the condition (C). Then $\mathfrak{F}$ has a fixed point.*

Lemma 2.4. *[16] Let $0 < a \leq \gamma_k \leq b < 1$; $\forall\ k \in \mathbb{N}$, $\eta \geq 0$ and $\{y_k\}$, and $\{z_k\}$ be sequences in UCBS, E with $\limsup_{k\to\infty}\|y_k\| \leq \eta, \limsup_{k\to\infty}\|z_k\| \leq \eta$, and $\lim_{k\to\infty}\|\gamma_k y_k + (1-\gamma_k)z_k\| = \eta$. Then, $lim_{k\to\infty}\|y_k - z_k\| = 0$.*

3. Main results

In this section, we prove the weak and strong convergence for the SM iteration.

Lemma 3.1. *Let V be any non-empty closed and convex subset of a BS E, and $\mathfrak{F} : V \to V$ be a Suzuki mapping with $Fix(\mathfrak{F}) \neq \phi$. Suppose $\{\varkappa_n\}$ is the sequence given in* (SM). *Then, $\lim_{n\to\infty}\|\varkappa_n - q\|$ exists for every $q \in Fix(F)$.*

Proof. Take $q \in Fix(\mathfrak{F})$. By equation (SM) and Lemma (2.1), one obtain

$$\|z_n - q\| = \|\mathfrak{F}\varkappa_n - q\| \leq \|\varkappa_n - q\|, \tag{3.1}$$

$$\begin{aligned}\|y_n - q\| &= \|\mathfrak{F}((1-\beta_n)\varkappa_n + \beta_n z_n) - \mathfrak{F}q\| \\ &\leq \|(1-\beta_n)(\varkappa_n - q) + \beta_n(z_n - q)\| \\ &\leq (1-\beta_n)\|\varkappa_n - q\| + \beta_n\|z_n - q\| \\ &\leq \|\varkappa_n - q\|.\end{aligned} \tag{3.2}$$

Thus,

$$\begin{aligned}\|\varkappa_{n+1} - q\| &= \|\mathfrak{F}((1-\alpha_n)\mathfrak{F}z_n + \alpha_n\mathfrak{F}y_n) - q\| \\ &\leq \|(1-\alpha_n)\mathfrak{F}z_n + \alpha_n\mathfrak{F}y_n - q\| \\ &\leq \|(1-\alpha_n)(\mathfrak{F}z_n - q) + \alpha_n(\mathfrak{F}y_n - q)\| \\ &\leq (1-\alpha_n)\|z_n - q\| + \alpha_n\|y_n - q\| \\ &\leq \|\varkappa_n - q\|.\end{aligned} \tag{3.3}$$

Thus, we conclude from equation (3.3) that $\{\|\varkappa_n - q\|\}$ is bounded and non-increasing for all $q \in Fix(\mathfrak{F})$. Hence $\lim_{n\to\infty}\|\varkappa_n - q\|$ exists. □

Theorem 3.1. *Consider E to be UCBS and V be its non-empty closed and convex subset with $\mathfrak{F} : V \to V$ be a Suzuki mapping. Also, consider a sequence $\{\varkappa_n\}$ as in* (SM). *Then, $Fix(\mathfrak{F}) \neq \phi$ if and only if $\{\varkappa_n\}$ is bounded, and $\lim_{n\to\infty}\|\mathfrak{F}\varkappa_n - \varkappa_n\| =$* 0.

Proof. First we assume that $\{\varkappa_n\}$ is bounded and $\lim_{n\to\infty}\|\mathfrak{F}\varkappa_n - \varkappa_n\| = 0$. We shall prove that $Fix(\mathfrak{F}) \neq \phi$. Now let $q \in \mathfrak{F}(V, \{\varkappa_n\})$. By Lemma (2.1), we have

$$\begin{aligned}\mathfrak{R}(\mathfrak{F}q, \{\varkappa_n\}) &= \lim\sup_{n\to\infty}\|\varkappa_n - \mathfrak{F}q\| \\ &\leq 3\lim\sup_{n\to\infty}\|\varkappa_n - \mathfrak{F}\varkappa_n\| + \lim\sup_{n\to\infty}\|\varkappa_n - q\|\end{aligned}$$

$$
\begin{aligned}
&= \limsup_{n\to\infty} \|\varkappa_n - q\| \\
&= \Re(q, \{\varkappa_n\}). \qquad (3.4)
\end{aligned}
$$

It follows that $\mathfrak{S}q \in \mathfrak{F}(V, \{\varkappa_n\})$. Since in UCBS, asymptotic centers are singleton, we have $\mathfrak{S}q = q$. Hence, $Fix(\mathfrak{S}) \neq \phi$.

Conversely, we consider that $Fix(\mathfrak{S}) \neq \phi$. From Lemma (3.1), $\{\varkappa_n\}$ is bounded and $\lim_{n\to\infty}\|\varkappa_n - q\|$ exists. Now, if

$$\lim_{n\to\infty} \|\varkappa_n - q\| = \eta. \qquad (3.5)$$

From equation (3.1), it is known that $\|z_n - q\| \leq \|\varkappa_n - q\|$. Taking lim sup on both sides of equation (3.1) and using (3.5), one obtains

$$\limsup_{n\to\infty} \|z_n - q\| \leq \limsup_{n\to\infty} \|\varkappa_n - q\| = \eta. \qquad (3.6)$$

By Lemma (2.1), we find

$$\limsup_{n\to\infty} \|\mathfrak{S}\varkappa_n - q\| \leq \limsup_{n\to\infty} \|\varkappa_n - q\| = \eta. \qquad (3.7)$$

Now the following inequality holds true:

$$
\begin{aligned}
\|\varkappa_{n+1} - q\| =& \|\mathfrak{S}((1-\alpha_n)\mathfrak{S}z_n + \alpha_n \mathfrak{S}y_n) - q\| \\
\leq& (1-\alpha_n)\|z_n - q\| + \alpha_n\|y_n - q\|,
\end{aligned}
$$

After combining with equation (3.2), we find

$$
\begin{aligned}
\|\varkappa_{n+1} - q\| - \|\varkappa_n - q\| \leq& (1-\alpha_n)\|z_n - q\| + (\alpha_n - 1)\|\varkappa_n - q\|, \\
\Rightarrow \frac{\|\varkappa_{n+1} - q\| - \|\varkappa_n - q\|}{(1-\alpha_n)} \leq& \|z_n - q\| - \|\varkappa_n - q\|,
\end{aligned}
$$

which follows that

$$
\begin{aligned}
\|\varkappa_{n+1} - q\| - \|\varkappa_n - q\| \leq& \frac{\|\varkappa_{n+1} - q\| - \|\varkappa_n - q\|}{(1-\alpha_n)} \\
\leq& \|z_n - q\| - \|\varkappa_n - q\|. \qquad (3.8)
\end{aligned}
$$

Taking limit supremum in equation (3.8) with equations (3.3) and (3.5), one obtains

$$\eta = \limsup_{n\to\infty} \|\varkappa_{n+1} - q\| \leq \limsup_{n\to\infty} \|z_n - q\| \leq \eta,$$

which implies that

$$\limsup_{n\to\infty} \|z_n - q\| = \eta. \qquad (3.9)$$

From equation (3.2), we get

$$
\begin{aligned}
\|y_n - q\| \leq& (1-\beta_n)\|\varkappa_n - q\| + \beta_n\|z_n - q\|, \\
i.e. \limsup_{n\to\infty} \|y_n - q\| \leq& \limsup_{n\to\infty} \|(1-\beta_n)(\varkappa_n - q) + \beta_n(\mathfrak{S}\varkappa_n - q)\| = \eta. \qquad (3.10)
\end{aligned}
$$

From equations (3.5), (3.6), (3.10) and Lemma (2.4), one finds

$$\limsup_{n\to\infty} \|\mathfrak{S}\varkappa_n - \varkappa_n\| = 0.$$

Hence, the proof is complete. □

Theorem 3.2. *Let V, $\mathfrak{S}$ and the sequence $\{\varkappa_n\}$ be the same as defined in Theorem* (3.1)*. Then $\{\varkappa_n\}$ weakly converges to the fixed point of $\mathfrak{S}$.*

Proof. The proof uses a different iterative procedure and is identical to the proof of Theorem 2 in [4]. For the sole purpose of making the document stand alone, we decide to show the proof.

From Theorem (3.1), we proved that $\{\varkappa_n\}$ is bounded and $\lim_{n\to\infty}\|\mathfrak{S}\varkappa_n - \varkappa_n\| = 0$. Since E is uniformly convex, E is reflexive. As a result, it is simple to identify a subsequence, $\{\varkappa_{n_k}\}$ of $\{\varkappa_n\}$ such that $\varkappa_{n_k} \rightharpoonup u$ to some $u \in V$. By Lemma (2.2), $u \in Fix(\mathfrak{S})$. We will demonstrate that u is the weak limit of $\{\varkappa_n\}$. Suppose, if possible, u is not the weak limit of $\{\varkappa_n\}$. Then we get another subsequence, specifically, $\{\varkappa_{n_l}\}$ of $\{\varkappa_n\}$ in such a way that $\varkappa_{n_l} \rightharpoonup v$ and $u \neq v$. Again, from Lemma (2.2), $v \in Fix(\mathfrak{S})$. From Lemma (3.1) and Opial's property, we get

$$\begin{aligned}\lim_{n\to\infty}\|\varkappa_n - u\| &= \lim_{k\to\infty}\|\varkappa_{n_k} - u\| \\ &< \lim_{k\to\infty}\|\varkappa_{n_k} - v\| \\ &= \lim_{n\to\infty}\|\varkappa_n - v\| \\ &= \lim_{l\to\infty}\|\varkappa_{n_l} - v\| \\ &< \lim_{l\to\infty}\|\varkappa_{n_l} - u\| \\ &< \lim_{n\to\infty}\|\varkappa_n - u\|. \end{aligned} \tag{3.11}$$

Therefore, $\lim_{n\to\infty}\|\varkappa_n - u\| \leq \lim_{n\to\infty}\|\varkappa_n - v\|$, obviously contradicting itself. Hence, the proof is complete. □

We now demonstrate the following results on strong convergence.

Theorem 3.3. *Consider V, $\mathfrak{S}$ and the sequence $\{\varkappa_n\}$ be the same as defined in Theorem* (3.1)*. Then sequence $\{\varkappa_n\}$ converges strongly to the fixed point of $\mathfrak{S}$.*

Proof. Once more, the proof is identical to the proof of Theorem 3 in [4]. We can write $Fix(\mathfrak{S}) \neq \phi$ from Lemma (2.3). By Theorem (3.1), $\lim_{n\to\infty}\|\mathfrak{S}\varkappa_n - \varkappa_n\| = 0$. We can quickly identify a strongly convergent subsequence, since the domain M is compact, namely, $\{\varkappa_{k_j}\}$ of $\{\varkappa_n\}$ with a limit, say, x. From Lemma (2.1), we have

$$\|\varkappa_{n_j} - \mathfrak{S}\varkappa\| \leq 3\|\varkappa_{n_j} - \mathfrak{S}\varkappa_{n_j}\| + \|\varkappa_{n_j} - \varkappa\|.$$

Hence, $\varkappa_{n_j} \to \mathfrak{S}\varkappa$ whenever $j \to \infty$, so the uniqueness of limits follows $\mathfrak{S}\varkappa = \varkappa$. By Lemma (3.1), $\lim_{n\to\infty}\|\varkappa_n - \varkappa\|$ exists. Hence, $\{\varkappa_n\}$ strongly converges to limit x. □

Theorem 3.4. *Assume V, $\mathfrak{S}$ and the sequence $\{\varkappa_n\}$ be the same as they are described in Theorem* (3.1)*. Additionally, if $\mathfrak{S}$ satisfies the condition* (I)*, then $\{\varkappa_n\}$ strongly converges to a fixed point of $\mathfrak{S}$.*

Proof. From Lemma (3.1), the existence of $\lim_{n\to\infty}\|\varkappa_n - q\|$ for any $q \in Fix(F)$, implies the existence of $\lim_{n\to\infty} d(\varkappa_n, Fix(\mathfrak{S}))$. Suppose $\lim_{n\to\infty}\|\varkappa_n - q\| = s$, for some $s \geq 0$. If $s = 0$, thereafter we get the desired result. Assume $s \neq 0$, then from condition(I), we find

$$f(d(\varkappa_n, Fix(\mathfrak{S})) \leq d(\varkappa_n, \mathfrak{S}\varkappa_n) = \|\mathfrak{S}\varkappa_n - \varkappa_n\|.$$

From the assumption, $Fix(\mathfrak{F}) \neq \phi$ and using Theorem (3.1), $\lim_{n\to\infty}\|F\varkappa_n - \varkappa_n\| = 0$, which implies

$$\lim_{n\to\infty} d(\varkappa_n, Fix(\mathfrak{F})) = 0.$$

From the properties of the function $\mathfrak{F}$, we find

$$\lim_{n\to\infty} d(\varkappa_n, Fix(\mathfrak{F})) = 0.$$

Let $\{\varkappa_{n_k}\}$ be the subsequence of $\{\varkappa_n\}$ and $\{y_k\} \in Fix(\mathfrak{F})$ such that

$$\|\varkappa_{n_k} - y_k\| < \frac{1}{2^k} \forall k \in \mathbb{N}. \tag{3.12}$$

Using equation (3.3), we obtain

$$\|\varkappa_{n_{k+1}} - y_k\| \leq \|\varkappa_{n_k} - y_k\| \leq \frac{1}{2^k}.$$

For $k \to \infty$, it follows

$$\begin{aligned}\|y_{k+1} - y_k\| \leq & \|y_{k+1} - \varkappa_{n_{k+1}}\| + \|\varkappa_{n_{k+1}} - y_k\| \\ \leq & \frac{1}{2^{k+1}} + \frac{1}{2^k} \\ < & \frac{1}{2^{k-1}} \to 0.\end{aligned}$$

Thus, $\{y_k\} \in Fix(\mathfrak{F})$ is a Cauchy sequence and due to closedness of set $Fix(\mathfrak{F})$, we find $\{y_k\}$ converges to the fixed point q. Assuming $k \to \infty$ in (3.12), we get $\{\varkappa_{n_k}\} \to q \in Fix(\mathfrak{F})$. As $\lim_{n\to\infty}\|\varkappa_n - q\|$ exists, it follows that $\varkappa_n \to q$. Hence, the proof is complete. □

4. Numerical example

In this section, we provide examples to support our findings.

Example 4.1. Consider $V = [0, 2]$, a closed and convex subset of a Banach space $\mathbb{R}$ with a self map $\mathfrak{F} : V \to V$, defined as:

$$\mathfrak{F}\varkappa = \begin{cases} 2 - \varkappa, & if \varkappa < \frac{1}{8} \\ \frac{\varkappa+6}{7}, & if \varkappa \geq \frac{1}{8}. \end{cases} \tag{4.1}$$

By subsequent calculations, we find that $\mathfrak{F}$ is a Suzuki mapping, but it is not non-expansive. For this, choose $\varkappa = 7/69$ and $y = 7/55$, and we notice that

$$\begin{aligned}\|\mathfrak{F}\varkappa - \mathfrak{F}y\| = & |\mathfrak{F}\varkappa - \mathfrak{F}y| = \left|1 - \frac{7}{69} - \frac{337}{385}\right|, \\ = & \frac{27,182}{26,565} > \frac{98}{3795} = \|\varkappa - y\|,\end{aligned}$$

which implies that mapping $\mathfrak{F}$ is not a non-expansive on V.

Next, we divide the solution of the Suzuki property of $\mathfrak{J}$ on V in two cases, given as below.

Case 1: Choose $\varkappa \in [0, 1/8)$; then $(1/2)\|\varkappa - \mathfrak{J}\varkappa\| = \|\varkappa - 1\| \leq \|\varkappa - y\|$ gives $y \geq 1$; i.e. $y \in [1, 2]$. So, one has

$$\begin{aligned}\|\mathfrak{J}\varkappa - \mathfrak{J}y\| &= \left|\frac{y+6}{7} - (2-\varkappa)\right| = \left|\frac{7\varkappa + y - 8}{7}\right| < \frac{6}{7},\\ \|\varkappa - y\| &= |\varkappa - y| > \left|\frac{1}{8} - 1\right| = \frac{7}{8}.\end{aligned}$$

Hence, $(1/2)\|\varkappa - \mathfrak{J}\varkappa\| \leq \|\varkappa - y\| \implies \|\mathfrak{J}\varkappa - \mathfrak{J}y\| \leq \|\varkappa - y\|$.

Case 2: Select $\varkappa \in [(1/8), 2]$; then $(1/2)\|\varkappa - \mathfrak{J}\varkappa\| = (1/2)|((\varkappa+6)/7) - \varkappa| = ((6 - 6\varkappa)/14) \in [0, 3/4]$. For $(1/2)\|\varkappa - \mathfrak{J}\varkappa\| \leq \|\varkappa - y\|$, one has $((6 - 6\varkappa)/14) \leq |y - \varkappa|$, so the following possible situations occur:

(a) Whenever $\varkappa < y, ((6 - 6\varkappa)/14) \leq y - \varkappa \implies y \geq ((4\varkappa + 3)/7) \implies y \in [(1/2), 1] \subset [(1/8), 2]$. So,

$$\|\mathfrak{J}\varkappa - \mathfrak{J}y\| = \left|\frac{\varkappa+6}{7} - \frac{y+6}{7}\right| = \frac{1}{7}\|\varkappa - y\| \leq \|\varkappa - y\|.$$

Therefore, $(1/2)\|\varkappa - \mathfrak{J}\varkappa\| \leq \|\varkappa - y\| \implies \|\mathfrak{J}\varkappa - \mathfrak{J}y\| \leq \|\varkappa - y\|$.

(b) Whenever $\varkappa > y, ((6 - 6\varkappa)/14) \leq \varkappa - y \implies y \leq \varkappa - ((6 - 6\varkappa)/14) = ((20\varkappa - 6)/14) \implies y \in [-2, 2]$. Since $y \in [0, 2], y \leq ((10\varkappa - 3)/7) \implies \varkappa \in [(3/10), (17/10)]$. So, the case is $\varkappa \in [(3/10), (17/10)]$ and $y \in [0, 2]$.

When $\varkappa \in [(3/10), (17/10)]$ and $y \in [(1/8), 2]$ are already included in part (a), we assume $\varkappa \in [(3/10), 1]$ and $y \in [0, (1/8))$; then

$$\begin{aligned}\|\mathfrak{J}\varkappa - \mathfrak{J}y\| &= \left|\frac{\varkappa+6}{7} - (2-y)\right|,\\ &= \left|\frac{\varkappa + 7y - 1}{7}\right|\end{aligned}$$

To make things easier, first we assume $\varkappa \in [(3/10), 1]$ and $y \in [0, (1/8))$. Then $\|\mathfrak{J}\varkappa - \mathfrak{J}y\| \leq (1/8)$ and $\|\varkappa - y\| > (7/40)$. Hence, $\|\mathfrak{J}\varkappa - \mathfrak{J}y\| \leq \|\varkappa - y\|$.

Next, for $\varkappa \in [1, (17/10)]$ and $y \in [0, (1/8))$, we find $\|\mathfrak{J}\varkappa - \mathfrak{J}y\| \leq (9/40)$ and $\|\varkappa - y\| > (7/8)$. Hence, $\|\mathfrak{J}\varkappa - \mathfrak{J}y\| \leq \|\varkappa - y\|$. So, $(1/2)\|\mathfrak{J}\varkappa - \mathfrak{J}y\| \leq \|\varkappa - y\| \implies \|\mathfrak{J}\varkappa - \mathfrak{J}y\| \leq \|\varkappa - y\|$.

Hence, $\mathfrak{J}$ is Suzuki mapping on V. Let $\alpha_n = 2n/(7n+9)$ and $\beta_n = (1/3n + 7))^{(1/2)}, n \in \mathbb{N}$. Table (1) and Figure (1) represents the strong convergence of iterations (SM), leading (JK) and then (TH) to the fixed point $q = 1$. Clearly, (SM) converges faster to $q = 1$.

Table 1. Values of iteration (SM), (JK) and (TH) for the mapping F defined in Example (4.1)

n	TH	JK	SM
1	0.9	0.9	0.9
2	0.998056709530883	0.999277985546009	0.999722387075840
3	0.999962765675244	0.999994380533018	0.999999240115821
4	0.999999290481345	0.999999953941357	0.999999997931432
5	0.999999986511409	0.999999999607583	0.999999999994382
6	0.999999999743813	0.999999999996553	0.999999999999985
7	0.999999999995135	0.999999999999969	1.000000000000000
8	0.999999999999908	1.000000000000000	1.000000000000000
9	0.999999999999998	1	1
10	1	1	1
11	1	1	1

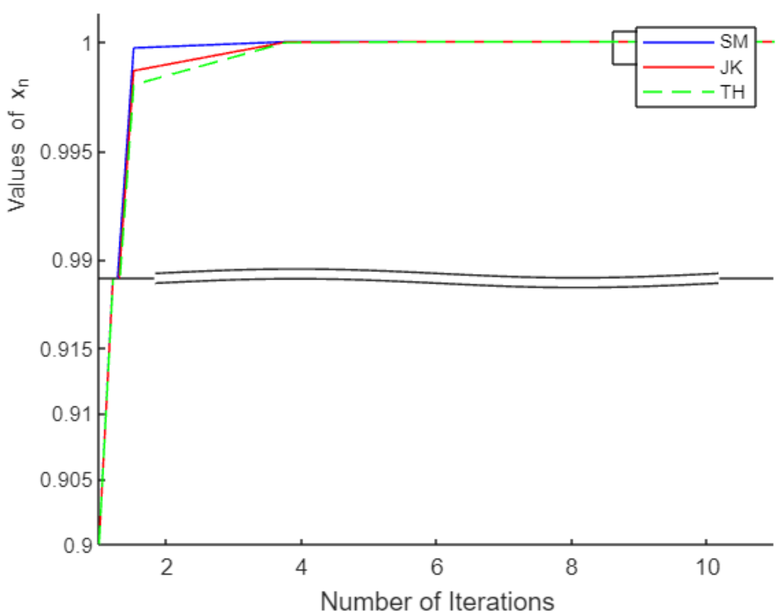

Figure 1. Convergence behaviour of (SM), (JK) and (TH) iterates to $q = 1$, where, q is the unique fixed point of the self map F in Example (4.1) with $\varkappa_1 = 0.9$

Table 2. $\alpha_n = 3n/(5n+7)$ and $\beta_n = (1/(n+7))^{1/2}$

	Iterates number converges to 1		
x_1	TH	JK	SM
0.9	10	9	7
0.5	10	9	7
0.1	10	9	8
0.05	10	9	8

Table 3. $\alpha_n = 1/\sqrt{9n+1}$ and $\beta_n = ((2n+1)^{1/3})/(10n+11)$

	Iterates number converges to 1		
x_1	TH	JK	SM
0.9	10	10	7
0.5	10	10	7
0.1	11	10	8
0.05	11	10	8

Table 4. $\alpha_n = n/(n+5)^{1/7}$ and $\beta_n = 1/\sqrt{(7n+3)}$

	Iterates number converges to 1		
x_1	TH	JK	SM
0.9	9	11	7
0.5	9	11	7
0.1	9	11	7
0.05	9	11	7

5. Conclusion

One can find from Tables 1-4 that (SM) iterative method converges faster than (TH) and (JK) for the Suzuki type mappings.

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