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Normal Form for the 1:1 Resonance Problems for Delayed Reaction-Diffusion Systems*

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Abstract This article presents a direct method for calculating the normal form coefficients of a 1:1 resonant Hopf bifurcation in reaction-diffusion systems with time delay and Neumann boundary conditions. The formulas obtained in this paper can be easily implemented using a computer algebra system such as Maple or Mathematica.

Keywords Normal form, 1:1 resonance, Hopf bifurcation, delayed reaction-diffusion

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1. Introduction

Hopf and generalized Hopf bifurcations have been extensively studied by many researchers (e.g., see Ref. [1–5], and are associated with a pair of purely imaginary eigenvalues at an equilibrium. If the Jacobian of a system evaluated at a critical point involves two pairs of purely imaginary eigenvalues, the so-called "double-Hopf bifurcation may occur. Such bifurcations may exhibit more complicated and interesting dynamic behavior such as quasi-periodic motions on tori, and chaos (e.g., see Ref. [6–10]). A bifurcation is called non-resonant if the ratio of the two eigenvalues is not a rational number, otherwise it is called resonant. The most important resonance is the 1:1 non-semisimple case, in which the purely imaginary eigenvalues at criticality are assumed to be double and non-semisimple. This bifurcation has been presented as an open problem in Kopell and Howard [11] and in Guckenheimer and Holmes [7] and in Ref. [12–14]. To date there has been little research on the 1:1 resonant Hopf bifurcation.

The 1:1 resonance is important in a number of applications such as wind-induced oscillations of bundled conductors and aircraft longitudinal dynamics when the eigenvalues corresponding to a pair of elastic modes approach each other and the imaginary axis, see Ref. [15].

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Normal form theory is one of the basic methods for the study of non-linear dynamics such as the singularity, Hopf bifurcation and homoclinic and heteroclinic bifurcations. The theory of normal form is concerned with constructing a series of near identity non-linear transformations that simplify the non-linear systems as much as possible. With the aid of normal form theory, we may obtain a set of simpler differential equations, which is topologically equivalent to the original systems. Being "simpler" means that some non-linear terms may be eliminated from the original differential equations. Also the normal form for a 1:1 resonance Hopf bifurcation was expressed by some researchers see Ref. [12, 13, 16], but there are no explicit formulas relating the coefficients of the original system to those of the normal form.

The main attention of the paper is focused on developing a new and efficient computation of the normal forms for 1:1 resonant Hopf bifurcation. This bifurcation has linear codimension-3, and a centre subspace of dimension 4. With the help of the results presented in this paper, one can apply the analysis to any physical problem exhibiting a generalized Hopf bifurcation with non-semisimple 1:1 resonance.

The aim of this paper is two-fold: first, to present an explicit formula for the normal form of a generalized Hopf bifurcation with non-semisimple 1:1 resonance. Second, to use the results with those obtained to the vector field.

2. Decomposition of the phase space

In this section, we explore the decomposition behaviour of abstract reaction diffusion retarded functional differential equation with parameters in the phase space $\mathcal{C} = C([-\tau, 0]; X^m)$, described by

$$\dot{u}(t) = D\Delta u + L(\mu)u_t + F(u_t, \mu), \tag{2.1}$$

where $u_t \in \mathcal{C}$ is defined by $u_t(\theta) = u(t+\theta), -\tau \leq \theta \leq 0$, $\mu \in R^p$ is a parameter vector in a neighborhood V of zero. $L(\mu) : V \to L(\mathcal{C}, X^m)$ is C^k for $k \geq 3$ and $F : \mathcal{C} \times R^p \to X^m$ is $C^k (k \geq 2)$ with $F(0, \mu) = 0, DF(0, \mu) = 0$, $D = diag(d_1, d_2, \cdots d_m)$ and $d_i > 0$ for $i = 1, 2, \cdots m$. $X^m : \{(u_1, u_2, \cdots u_m^T) \in (H^2(0, l\pi))^m : \frac{\partial u_i}{\partial x}(0, t) = \frac{\partial u_i}{\partial x}(l\pi, t) = 0, i = 1, 2, \cdots m\}$ is the real-valued Hilbert space.

For Laplacian operator Δ, we have the following properties (see [17–20]).

(P1) $D\Delta$ generates a C_0 semigroup $\{T(t)\}_{t\geq 0}$ on X^m with $|T(t)| \leq Me^{\omega t}(t \geq 0)$for some $M \geq 1, \omega \in R$ and $T(t)$ is a compact operator for $t > 0$;

(P2) The eigenfunctions $\{\beta_k^j : k \in N_0 = N \cup 0, j = 0, 1, \cdots m\}$ of $D\Delta$, with corresponding eigenvalues $\{\mu_k : k \in N_0\}$, form an orthonormal basis for X^m where $\mu_k = -\frac{k^2}{l^2}, k \in N_0$ and $\beta_k^j = \gamma_m e_j, \gamma_m = \frac{cos\frac{k}{l}}{x}||cos\frac{k}{l}x|| : k \in N_0, j = 0, 1, \cdots m$. Denote

$$< \nu(.), \beta_k > := \begin{pmatrix} < \nu(.), \beta_k^1 > \\ < \nu(.), \beta_k^2 > \\ \cdots \\ < \nu(.), \beta_k^m > \end{pmatrix}$$

for $\nu \in \mathcal{C}$ and $\beta_k = (\beta_k^1, \beta_k^2, \cdots \beta_k^m)$.

Define $L = L(0)$. Then, the linear homogeneous reaction diffusion retard functional differential equation **(2.1)** can be written as

$$\dot{u}(t) = D\Delta u + Lu_t. \tag{2.2}$$

We make the following hypothesis:

(H1) L can be extended to a bounded linear operator from BC to X^m, where

$$BC = \left\{ \varphi : [-\tau, 0] \to X^m : \varphi \ \text{ continuous on } [-\tau, 0), \exists \lim_{\theta \to 0^-} \varphi(\theta) \in X^m \right\}$$

with the sup norm.

Let $\mathcal{A}$ be the infinitesimal generator such that

$$\mathcal{A}\varphi = \dot{\varphi}, Dom(\mathcal{A}) = \{\varphi \in C^1([-\tau, 0], X^m) : \dot{\varphi}(0) = D\Delta\varphi(0) + L\varphi\},$$

and the spectrum of $\mathcal{A}$ coincides with the point spectrum of $\mathcal{A}$, i.e.

$$\sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) = \{\lambda \in C : \Delta(\lambda)y = 0, for y \in Dom(\Delta), y \neq 0\},$$

where

$$\Delta(\lambda)y = \lambda y - D\Delta y - L(e^{\lambda} y), \tag{2.3}$$

is the characteristic equation of (**2.3**). Using the theorem of reaction-diffusion, (**2.3**) is equivalent to the sequence of equations $det\Delta_k(\lambda) = 0$, where

$$\Delta_k(\lambda) := \lambda I - \mu_k D - L_k(e^{\lambda}), k \in N_0,$$

and the linear equation (**2.2**) is equivalent to a sequence of functional differential equations

$$\dot{x}(t) = -\mu_k D x(t) + L_k x_t, \tag{2.4}$$

with the characteristic equation given by (**2.4**), where $x_t =< u_t, \beta_k >\in \mathcal{C}$ and satisfying

$$-\mu_k \varphi(0) + L\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta), \quad \forall \varphi \in \mathcal{C}. \tag{2.5}$$

Here, $\eta(\theta)(\theta \in [-\tau, 0])$ is an $m \times m$ matrix function of bounded variation.

Define the bilinear form between $\mathcal{C}$ and $\mathcal{C}' = C([0, \tau], X^{m^*})$ by

$$(\psi, \varphi)_k = \psi(0)\varphi(0) - \int_{-\tau}^{0} \int_{0}^{\theta} \psi(\xi - \theta) d\eta(\theta)\varphi(\xi) d\xi, \forall \psi \in \mathcal{C}', \forall \varphi \in \mathcal{C}.$$

In the following, we will consider the case where L_0 has a simple purely imaginary eigenvalues $\pm i\omega$ with algebraic multiplicity 2 and geometric 1 and all other eigenvalues have negative real parts when $k = k_0$. $\mathcal{C}$ can be decomposed as

$$\mathcal{C} = P \oplus Q, \quad \text{where } Q = \left\{ \varphi \in \mathcal{C} :< \varphi, \psi >= 0, \forall \psi \in P^* \right\},$$

with dim $p = 3$, where P is the eigenspace generalized by the eigenfunction corresponding to $i\omega$. Choose the bases Φ and Ψ for P and P^* respectively such that

$$< \Psi, \Phi >= I, \quad \dot{\Phi} = \Phi J, \quad \dot{\Psi} = -J\Psi,$$

where $\Phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \bar{\phi_1}(\theta), \bar{\phi_2}(\theta))$ which can be detedminded by

$$\phi_1(\theta) = \phi_1^0 e^{i\omega\theta}, \phi_2(\theta) = (\phi_2^0 + \theta\phi_1^0)e^{i\omega\theta}.$$

$$(\mathcal{A} - i\omega)\phi_1^0 = 0, (\mathcal{A} - i\omega)\phi_2^0 = \phi_1^0.$$

$\Psi(s) = (\psi_1(s), \psi_2(s), \bar{\psi_1}(s), \bar{\psi_2}(s))^T$ which can be detedminded by

$$\psi_2(s) = \psi_2^0 e^{-i\omega s}, \psi_1(s) = (\psi_1^0 - s\psi_2^0)e^{-i\omega s}.$$

$$(\mathcal{A}^* + i\omega)\psi_2^0 = 0, (\mathcal{A}^* + i\omega)\psi_1^0 = \psi_2^0.$$

I is the $m \times m$ identity matrix and

$$J = \begin{pmatrix} i\omega & 1 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega & 1 \\ 0 & 0 & 0 & -i\omega \end{pmatrix}.$$

Following the ideas in [5], we consider the enlarged phase space BC, and

$$BC = \left\{ \varphi : [-\tau, 0] \to \mathbb{X}^m : \varphi \text{ continuous on } [-\tau, 0), \exists \lim_{\theta \to 0^-} \varphi(\theta) \in \mathbb{X}^m \right\}.$$

Then, the continuous projection $\pi : BC \to P$, defined by

$$\pi(\varphi) = \Phi(\Psi, < \varphi, \beta_k >).\beta_k,$$

allows us to decompose the enlarged phase space $BC = P \oplus \mathrm{Ker}\pi$. We decompose $u \in \mathcal{C}_0^1$ as $u(t) = \Phi x(t)\beta_k + y$, where $x(t) = (\Psi, < u(.), \beta_k >)$ and let

$$\tilde{F}(u, \mu) = \tilde{F}(x, y, \mu) = L(\mu) - L(0) + \frac{1}{2!}F_2(x, y, \mu) + \frac{1}{3!}F_3(x, y, \mu) + h.o.t.$$

Then the original system becomes

$$\begin{cases} \dot{x} = Jx + \Psi(0) < \tilde{F}(\Phi x\beta_k, y, \mu), \beta_k > \\ \frac{dy}{dt} = \mathcal{L}_1(y) + H(\tilde{\Phi}x, y, \mu), \end{cases} \tag{2.6}$$

where

$$H(\tilde{\Phi}x, y, \mu) = (I - \pi)X_0 - < \tilde{F}(\Phi x\beta_k, y, \mu), \beta_k > .$$

With the formal Taylor expansions, system (**2.6**) can be rewritten as

$$\begin{cases} \dot{x} = Jx + \sum_{j\geq 2} \frac{1}{j!} f_j^1(x, y, \mu), \\ \frac{dy}{dt} = \mathcal{L}_1(w) + \sum_{j\geq 2} \frac{1}{j!} f_j^2(x, y, \mu). \end{cases} \tag{2.7}$$

Let

$$M_j^1 U_j^1 = D_x U_j^1(x, \mu)Jx - U_j^1(x, \mu),$$

$$M_j^2 U_j^2 = D_x U_j^2(x, \mu)Jx - L_1 U_j^2(z, \mu).$$

Similar to the work of Song [3], (**2.7**) can be transforms into the following normal form:

$$\begin{cases} \dot{x} = Jx + \frac{1}{2!}g_2^1(x,0,\mu) + \frac{1}{3!}g_3^1(x,0,0) + \cdots, \\ \dot{y} = \mathcal{L}_1(y) + \frac{1}{2!}g_2^2(x,0,\mu) + \frac{1}{3!}g_3^2(x,0,0) + \cdots, \end{cases} \tag{2.8}$$

where $g_i = (g_j^1, g_j^2),\ j = 2, 3$, given by

$$g_j^1 = \tilde{f}_j^1 - M_j^1 U_j^1,\ g_j^2 = \tilde{f}_j^2 - M_j^2 U_j^2$$

with

$$\tilde{f}_2^1(x,0,\mu) = f_2^1(x,0,\mu), \quad \tilde{f}_2^2(x,0,\mu) = f_2^2(x,0,\mu)$$

and

$$\begin{aligned} \tilde{f}_3^1(x,0,0) =& f_3^1(x,0,0) + \frac{3}{2}[D_x f_2^1(x,0,0)U_2^1(x,0,0) + D_y f_2^1(x,0,0)U_2^2(x,0,0) \\ & - D_x U_2^1(x,0,0) g_2^1(x,0,0)))]. \end{aligned}$$

3. Explicit formulas of $g_2^1(x,\mu,0)$ with three parameters

We need to compute $g_2^1(x,\mu,0)$ in (**2.8**) with three parameters. For a normed space Z, denote $V_j^6(Z)$ the linear space of homogeneous polynomials of $(x,\mu) = (x_1, x_2, x_3, x_3, \mu_1, \mu_2, \mu_3)$ with degree j and with coefficients in Z and define M_j the operator in V_j^7 with the range in the same space by

$$M_j(p,h) = (M_j^1 p, M_j^2 h),$$

where

$$M_j^1 p = D_x P(x,\mu)Jx - JP(x,\mu), \qquad \forall P(x,\mu) \in H_{4+3}^2,$$

and

$$P(x,\mu) = \begin{pmatrix} P_1(x,\mu) \\ P_2(x,\mu) \\ P_3(x,\mu) \\ P_4(x,\mu) \end{pmatrix}, D_x P(x,\mu) = \begin{pmatrix} \frac{\partial P_1}{\partial x_1} & \frac{\partial P_1}{\partial x_2} & \frac{\partial P_1}{\partial x_3} & \frac{\partial P_1}{\partial x_4} \\ \frac{\partial P_2}{\partial x_1} & \frac{\partial P_2}{\partial x_2} & \frac{\partial P_2}{\partial x_3} & \frac{\partial P_2}{\partial x_4} \\ \frac{\partial P_3}{\partial x_1} & \frac{\partial P_3}{\partial x_2} & \frac{\partial P_3}{\partial x_3} & \frac{\partial P_3}{\partial x_4} \\ \frac{\partial P_4}{\partial x_1} & \frac{\partial P_4}{\partial x_2} & \frac{\partial P_4}{\partial x_3} & \frac{\partial P_4}{\partial x_4} \end{pmatrix}.$$

$$M_j^2 h = M_j^2 h(x,\mu) = D_x h(x,\mu)Jx - A_{Q^1} h(x,\mu).$$

Using M_j^1, we have the following decompositions,

$$V_j^6(R^3) = \mathrm{Im}(M_j^1) \oplus (\mathrm{Im}(M_j^1))^c, V_j^6(R^3) = \mathrm{Ker}(M_j^1) \oplus (\mathrm{Ker}(M_j^1))^c.$$

Then, $g_2^1(z,0,\mu)$ can be expressed as

$$g_2^1(x,0,\mu) = \mathrm{Proj}_{(\mathrm{Im}(M_2^1))^c} f_2^1(x,0,\mu).$$

According to fact that $\int_0^{l\pi} \gamma_k^2 dx = 1$, we obtain the following theorem.

Theorem 3.1. *Let*

$$f_2^1(x,0,\mu) = \Sigma_{1\le i\le j\le 4, 1\le m\le 4} a_{i,j}^m x_i x_j e_m + \Sigma_{1\le i,m\le 4, 1\le k\le 3} b_{i,j}^m x_i \mu_k e_m$$

$$+\Sigma_{1\le k\le l\le 3, 1\le m\le 4} c_{k,l}^m \mu_k \mu_l e_m.$$

Then

$$g_2^1(x,0,\mu) = \begin{pmatrix} 0 \\ (b_{11}^2\mu_1 + b_{12}^2\mu_2 + b_{13}^2\mu_3)x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \left(b_{31}^4\mu_1 + b_{32}^4\mu_2 + b_{33}^4\mu_3\right) x_3 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ \left((b_{21}^2 + b_{11}^1)\mu_1 + (b_{22}^2 + b_{12}^1)\mu_2 + \left(b_{23}^2 + b_{13}^1\right)\mu_3\right) x_2 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ \left((b_{41}^4 + b_{31}^3)\mu_1 + (b_{42}^4 + b_{32}^3)\mu_2 + (b_{43}^4 + b_{33}^3)\mu_3\right) x_4 \end{pmatrix}.$$

4. Explicit formulas of $g_3^1(x,0,0)$

Now let us compute $U_2^1(x,\mu)$ by $M_2^1 U_2^1(x,\mu) = \text{Proj}_{(\text{Im}(M_2^1))} f_2^1(x,0,0)$.

The matrix of M_2^1 is

$$\begin{pmatrix} A_1 & -I_{10} & O & O \\ O & A_1 & O & O \\ O & O & A_2 & -I_{10} \\ O & O & O & A_2 \end{pmatrix},$$

where

$$A_1 = \begin{pmatrix} i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -i\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & i\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -i\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & -i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3i\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -3i\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3i\omega \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 3i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3i\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & i\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -i\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\omega \end{pmatrix}.$$

Denote $f_2^1(x,0,0) = \Sigma_{1\le i\le j\le 4, 1\le m\le 4} a_{i,j}^m x_i x_j e_m$ and let $U_2^1(x) = \Sigma_{1\le i\le j\le 4, 1\le m\le 4} u_{i,j}^m x_i x_j e_m$. Then we have

$$\begin{pmatrix} u_{1,1}^1 \\ u_{1,2}^1 \\ u_{1,3}^1 \\ u_{1,4}^1 \\ u_{2,2}^1 \\ u_{2,3}^1 \\ u_{2,4}^1 \\ u_{3,3}^1 \\ u_{3,4}^1 \\ u_{4,4}^1 \end{pmatrix} = \begin{pmatrix} 3i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3i\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & i\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -i\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\omega \end{pmatrix} \begin{pmatrix} a_{1,1}^1 \\ a_{1,2}^1 \\ a_{1,3}^1 \\ a_{1,4}^1 \\ a_{2,2}^1 \\ a_{2,3}^1 \\ a_{2,4}^1 \\ a_{3,3}^1 \\ a_{3,4}^1 \\ a_{4,4}^1 \end{pmatrix} + \begin{pmatrix} a_{1,1}^2 \\ a_{1,2}^2 \\ a_{1,3}^2 \\ a_{1,4}^2 \\ a_{2,2}^2 \\ a_{2,3}^2 \\ a_{2,4}^2 \\ a_{3,3}^2 \\ a_{3,4}^2 \\ a_{4,4}^2 \end{pmatrix},$$

$$\begin{pmatrix} u_{1,1}^2 \\ u_{1,2}^2 \\ u_{1,3}^2 \\ u_{1,4}^2 \\ u_{2,2}^2 \\ u_{2,3}^2 \\ u_{2,4}^2 \\ u_{3,3}^2 \\ u_{3,4}^2 \\ u_{4,4}^2 \end{pmatrix} = \begin{pmatrix} 3i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & i\omega & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3i\omega & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & i\omega & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & i\omega & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -i\omega & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -i\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -i\omega \end{pmatrix} \begin{pmatrix} a_{1,1}^2 \\ a_{1,2}^2 \\ a_{1,3}^2 \\ a_{1,4}^2 \\ a_{2,2}^2 \\ a_{2,3}^2 \\ a_{2,4}^2 \\ a_{3,3}^2 \\ a_{3,4}^2 \\ a_{4,4}^2 \end{pmatrix},$$

$$\begin{pmatrix} u_{1,1}^3 \\ u_{1,2}^3 \\ u_{1,3}^3 \\ u_{1,4}^3 \\ u_{2,2}^3 \\ u_{2,3}^3 \\ u_{2,4}^3 \\ u_{3,3}^3 \\ u_{3,4}^3 \\ u_{4,4}^3 \end{pmatrix} = \begin{pmatrix} -\frac{i}{3\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{9\omega^2} & -\frac{i}{3\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\omega^2} & -\frac{i}{\omega} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2i}{27\omega^3} & \frac{1}{9\omega^2} & 0 & 0 & -\frac{i}{3\omega} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\omega^2} & 0 & 0 & -\frac{i}{\omega} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2i}{\omega^3} & \frac{1}{\omega^2} & 0 & \frac{1}{\omega^2} & -\frac{i}{\omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\omega^2} & \frac{i}{\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2i}{\omega^3} & \frac{1}{\omega^2} & \frac{i}{\omega} \end{pmatrix} \begin{pmatrix} a_{1,1}^3 \\ a_{1,2}^3 \\ a_{1,3}^3 \\ a_{1,4}^3 \\ a_{2,2}^3 \\ a_{2,3}^3 \\ a_{2,4}^3 \\ a_{3,3}^3 \\ a_{3,4}^3 \\ a_{4,4}^3 \end{pmatrix} + \begin{pmatrix} a_{1,1}^4 \\ a_{1,2}^4 \\ a_{1,3}^4 \\ a_{1,4}^4 \\ a_{2,2}^4 \\ a_{2,3}^4 \\ a_{2,4}^4 \\ a_{3,3}^4 \\ a_{3,4}^4 \\ a_{4,4}^4 \end{pmatrix},$$

$$\begin{pmatrix} u_{1,1}^4 \\ u_{1,2}^4 \\ u_{1,3}^4 \\ u_{1,4}^4 \\ u_{2,2}^4 \\ u_{2,3}^4 \\ u_{2,4}^4 \\ u_{3,3}^4 \\ u_{3,4}^4 \\ u_{4,4}^4 \end{pmatrix} = \begin{pmatrix} -\frac{i}{3\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2}{9\omega^2} & -\frac{i}{3\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\omega} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\omega^2} & -\frac{i}{\omega} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{2i}{27\omega^3} & \frac{1}{9\omega^2} & 0 & 0 & -\frac{i}{3\omega} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\omega^2} & 0 & 0 & -\frac{i}{\omega} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{2i}{\omega^3} & \frac{1}{\omega^2} & 0 & \frac{1}{\omega^2} & -\frac{i}{\omega} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\omega} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{\omega^2} & \frac{i}{\omega} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2i}{\omega^3} & \frac{1}{\omega^2} & \frac{i}{\omega} \end{pmatrix} \begin{pmatrix} a_{1,1}^4 \\ a_{1,2}^4 \\ a_{1,3}^4 \\ a_{1,4}^4 \\ a_{2,2}^4 \\ a_{2,3}^4 \\ a_{2,4}^4 \\ a_{3,3}^4 \\ a_{3,4}^4 \\ a_{4,4}^4 \end{pmatrix}.$$

Computation of $U_2^2(x,\mu)$ by $M_2^1 U_2^1(x,\mu) = \mathrm{Proj}_{(\mathrm{Im}(M_2^1))} f_2^1(x,0,0)$.
Using $M_2^2 U_2^2(x,0,0) = f_2^2(x,0,0)$, and defining $U_2^2(x,0,0) = h(x)$, we have

$$D_x h(x) Jx - \dot{h}(x) + X_0[\dot{h}(x)(0) - L(0)h(x)] = f_2^2(x,0,0),$$

where $\dot{h}$ stands for the derivative of $h(x)$ respective to θ and $h(x)$ can be written as

$$\begin{aligned} h(x)(\theta) &= h_{2000}x_1^2 + h_{0200}x_2^2 + h_{0020}x_3^2 + h_{0002}x_4^2 + h_{1100}x_1x_2 \\ &\quad + h_{1010}x_1x_3 + h_{1001}x_1x_4 + h_{0110}x_2x_3 + h_{0101}x_2x_4 + h_{0011}x_3x_4. \end{aligned}$$

Similar to the work of Ref. [13], we have

$$\begin{aligned} &\dot{h}_{2000} - 2i\omega h_{2000} - h_{1100} = 2\Phi(\theta)\Psi(0)a_{22}, \dot{h}_{0200}(0) - L(h_{0200}) = 2a_{22}, \\ &\dot{h}_{0200} - 2i\omega h_{0200} = 2\Phi(\theta)\Psi(0)a_{11}, \dot{h}_{2000}(0) - L(h_{2000}) = 2a_{11}, \\ &\dot{h}_{1010} = 2\Phi(\theta)\Psi(0)a_{13}, \dot{h}_{1010}(0) = 2a_{13}, \\ &\dot{h}_{0101} = h_{0110} + h_{1001} + 2\Phi(\theta)\Psi(0)a_{24}, \dot{h}_{0101}(0) - L(h_{0101}) = 2a_{24}, \\ &\dot{h}_{1100} - 2i\omega h_{1100} - h_{1100} = 2h_{2000} + 2\Phi(\theta)\Psi(0)a_{12}, \dot{h}_{1100}(0) - L(h_{1100}) = 2a_{12}, \\ &\dot{h}_{1001} = h_{1010} + 2\Phi(\theta)\Psi(0)a_{14}, \dot{h}_{1001}(0) - L(h_{1001}) = 2a_{14}, \\ &\dot{h}_{0110} = h_{1010} + 2\Phi(\theta)\Psi(0)a_{23}, \dot{h}_{1001}(0) - L(h_{1001}) = 2a_{23}, \end{aligned}$$

where $a_{ij} = (a_{ij}^{(1)}, a_{ij}^{(2)}, a_{ij}^{(3)}, a_{ij}^{(4)})^T$, $i, j = 1, 2, 3, 4$.

Theorem 4.1. *Let* $\tilde{f}_3^1(x, 0, 0) = \Sigma_{1\le i\le j\le k\le 4, 1\le m\le 4} a_{i,j,k}^m x_i x_j x_k e_m$. *Then*

$$\begin{aligned} g_3^1(x,0,0) &= \begin{pmatrix} 0 \\ a_{113}^2 x_1^2 x_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (a_{114}^2 + a_{113}^1) x_1^2 x_4 \\ 0 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ (a_{123}^2 + 2a_{113}^1) x_1 x_2 x_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (a_{124}^2 + 2a_{114}^1 - a_{123}^1 - 2a_{223}^2) x_1 x_2 x_4 \\ 0 \\ 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ a_{133}^4 x_1 x_3^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (a_{233}^4 + a_{133}^3) x_2 x_3^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (a_{134}^4 + 2a_{133}^3) x_1 x_3 x_4 \end{pmatrix} \\ &+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ (a_{234}^4 + 2a_{233}^3 - a_{134}^3 - 2a_{144}^4) x_2 x_3 x_4 \end{pmatrix}. \end{aligned} \tag{4.1}$$

According to **Theorem 3.1** and **4.1**, we have the following about normal forms of vector fields.

Theorem 4.2. *Suppose that the Jacobian of vector field (2.1) evaluated at a critical point involves double purely imaginary eigenvalues with geometric multiplicity one. Then, the reduced normal form with unfolding has the following form on the center manifold near $X = 0$:*

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} i\omega & 1 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega & 1 \\ 0 & 0 & 0 & -i\omega \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ (b^2_{11}\mu_1 + b^2_{12}\mu_2 + b^2_{13}\mu_3)x_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (b^4_{31}\mu_1 + b^4_{32}\mu_2 + b^4_{33}\mu_3)x_3 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ ((b^2_{21} + b^1_{11})\mu_1 + (b^2_{22} + b^1_{12})\mu_2 + (b^2_{23} + b^1_{13})\mu_3)x_2 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ ((b^4_{41} + b^3_{31})\mu_1 + (b^4_{42} + b^3_{32})\mu_2 + (b^4_{43} + b^3_{33})\mu_3)x_4 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ a^2_{113}x_1^2x_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (a^2_{114} + a^1_{113})x_1^2x_4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (a^2_{123} + 2a^1_{113})x_1x_2x_3 \\ 0 \\ 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ (a^2_{124} + 2a^1_{114} - a^1_{123} - 2a^2_{223})x_1x_2x_4 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ a^4_{133}x_1x_3^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (a^4_{233} + a^3_{133})x_2x_3^2 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ 0 \\ 0 \\ (a^4_{134} + 2a^3_{133})x_1x_3x_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (a^4_{234} + 2a^3_{233} - a^3_{134} - 2a^4_{144})x_2x_3x_4 \end{pmatrix}. \tag{4.2}$$

It is worth noting that the first 4 vectors in the basis are complex conjugates of the last 4, as expected. Hence, we can confirm that the coefficients in the equation satisfy the following conditions:

$$a^2_{113} = \bar{a}^4_{133}, \quad a^2_{114} = \bar{a}^4_{233}, \quad a^1_{113} = \bar{a}^3_{133}, \quad a^2_{123} = \bar{a}^4_{134},$$

$$a^2_{124} = \bar{a}^4_{234}, \quad a^1_{114} = \bar{a}^3_{233}, \quad a^1_{123} = \bar{a}^3_{134}, \quad a^2_{223} = \bar{a}^4_{144}.$$

Denote $\lambda_1 = (b^2_{11}\mu_1 + b^2_{12}\mu_2 + b^2_{13}\mu_3)$, $\bar{\lambda_1} = (b^4_{31}\mu_1 + b^4_{32}\mu_2 + b^4_{33}\mu_3)$; $\lambda_2 = ((b^2_{21} + b^1_{11})\mu_1 + (b^2_{22} + b^1_{12})\mu_2 + (b^2_{23} + b^1_{13})\mu_3)$, $\bar{\lambda_2} = ((b^4_{41} + b^3_{31})\mu_1 + (b^4_{42} + b^3_{32})\mu_2 + (b^4_{43} + b^3_{33})\mu_3)$; $z_1 = x_1$, $\bar{z_1} = x_3$, $z_2 = x_2$, $\bar{z_2} = x_4$, the normal form up to the third order is

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} i\omega & 1 \\ 0 & i\omega \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \lambda_1 z_1 + \lambda_2 z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ a^2_{113} z_1^2 \bar{z_1} \end{pmatrix} + \begin{pmatrix} 0 \\ (a^2_{114} + a^1_{113}) z_1^2 \bar{z_2} \end{pmatrix}$$

$$+ \begin{pmatrix} 0 \\ (a^2_{123} + 2a^1_{113}) z_1 z_2 \bar{z_1} \end{pmatrix} \varphi + \begin{pmatrix} 0 \\ (a^2_{124} + 2a^1_{114} - a^1_{123} - 2a^2_{223}) z_1 z_2 \bar{z_2} \end{pmatrix}. \quad (4.3)$$

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$, $\varphi = \theta_1 - \theta_2$, $a = a^2_{113}$, $b = a^2_{114} + a^1_{113}$, $c = a^2_{123} + 2a^1_{113}$, and $d = a^2_{124} + 2a^1_{114} - a^1_{123} - 2a^2_{223}$. Therefore, we can obtain the following:

$$\begin{cases} \dot{r_1} = r_2 cos\varphi, \\ \dot{r_2} = r_1[Re(\lambda_1)cos\varphi - Im(\lambda_2)sin\varphi] + Re(\lambda_2)r_2 + r_1^3[Re(a)cos\varphi - Im(a)sin\varphi], \\ \qquad + r_1^2 r_2[Re(b)cos2\varphi - Im(b)sin2\varphi + Re(c)] + r_1 r_2^2[Re(d)cos\varphi - Im(d)sin\varphi, \\ \dot{\varphi} = -\frac{r_2}{r_1}sin\varphi - \frac{r_1}{r_2}[Im(\lambda_1)cos\varphi + Re(\lambda_2)sin\varphi] - \frac{r_1^3}{r_2}[Im(a)cos\varphi + Re(a)sin\varphi] \\ \qquad - r_1^2[Im(b)cos2\varphi + Re(b)sin2\varphi + Im(c)] - r_1 r_2[Im(d)cos\varphi + Re(d)sin\varphi. \end{cases} \quad (4.4)$$

The number of the positive equilibrium points of **(4.4)** corresponds to the number of the periodic solutions of **(2.1)**.

5. Conclusion

In this paper, we have derived the normal form computational formulas for 1:1 resonant Hopf bifurcation based on normal form theory. The core of the research lies in constructing a novel and efficient computational method to solve for the normal form of 1:1 resonant Hopf bifurcation. The bifurcation features a linear codimension-3, and a centre subspace of dimension 4.

6. Appendix the calculation of complementary space

6.1. Proof of Theorem 3.1

Let x_1, x_2, x_3, x_4 be independent variables, μ_1, μ_2, μ_3 be independent parameters, $\mathbf{C}$ be the complex number field, and

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, e_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}; X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}.$$

Let i, j, m be positive integers, and

$$H^2_{4+3}(X,\mu) = \{\Sigma_{1\le i\le j\le 4, 1\le m\le 4} a^m_{i,j} x_i x_j e_m + \Sigma_{1\le i,j,m\le 4} b^m_{i,j} x_i \mu_j e_m +$$

$$\Sigma_{1\le i\le j\le 4, 1\le m\le 4} c^m_{i,j} \mu_i \mu_j e_m | a^m_{i,j}, b^m_{i,j}, c^m_{i,j} \in \mathbf{C}\}.$$

With the general inner product, H^2_{4+3} is an inner product space over $\mathbf{C}$.

Define the linear translation M^1_2 on H^2_{4+3} by J as follows:

Denote E_n the $n\times n$ identity matrix, E_{ij} the matrix that (i,j) element is 1 and other elements 0, and

$$x_5 = 0, \quad e_0 = (\,0,\,0,\,0,\,0\,)^{\mathrm{T}}.$$

Define two maps α, β :

$$\alpha(i) = \begin{cases} 1, \; i = 1, 2, \\ -1, \; i = 3, 4. \end{cases} \qquad \beta(i) = \begin{cases} 0, \; i = 0, 2, 4, \\ 1, \; i = 1, 3. \end{cases}$$

Lemma 6.1. *For any $x_i x_j e_m, x_i \mu_k e_m, \mu_k \mu_l e_m \in H^2_{4+3}$, we have*

$$\begin{aligned} M^1_2(x_i x_j e_m) = {} & [\alpha(i)+\alpha(j)-\alpha(m)] i\omega x_i x_j e_m + [\beta(i) x_{i+1} x_j + \beta(j) x_i x_{j+1}] e_m \\ & -\beta(m-1) x_i x_j e_{m-1}, \\ M^1_2(x_i \mu_k e_m) = {} & [\alpha(i)-\alpha(m)] i\omega x_i \mu_k e_m + \beta(i) x_{i+1}\mu_k e_m - \beta(m-1) x_i \mu_k e_{m-1}, \\ M^1_2(\mu_k \mu_l e_m) = {} & -\alpha(m) i\omega \mu_k \mu_l e_m - \beta(m-1)\mu_k\mu_l e_{m-1}. \end{aligned}$$

Proof.

$$\begin{aligned} & M^1_2(x_i x_j e_m) \\ = {} & (x_i E_{mj} + x_j E_{mi})((x_2 + i\omega x_1)e_1 + i\omega x_2 e_2 + (x_4 - i\omega x_3)e_3 - i\omega x_4 e_4) \\ & -(i\omega E_{11} + E_{12} + i\omega E_{22} - i\omega E_{33} + E_{34} - i\omega E_{44}) x_i x_j e_m \\ = {} & [\alpha(i)+\alpha(j)-\alpha(m)] i\omega x_i x_j e_m + [\beta(i) x_{i+1} x_j + \beta(j) x_i x_{j+1}] e_m \\ & -\beta(m-1) x_i x_j e_{m-1}. \end{aligned}$$

$$M_2^1(x_i\mu_k e_m)$$
$$= \mu_k E_{mi}((i\omega x_1 + x_2)e_1 + i\omega x_2 e_2 - (i\omega x_3 + x_4)e_3 - i\omega x_4 e_4)$$
$$-(i\omega E_{11} + E_{12} + i\omega E_{22} - i\omega E_{33} + E_{34} - i\omega E_{44})x_i\mu_k e_m$$
$$= [\alpha(i) - \alpha(m)]i\omega x_i\mu_k e_m + \beta(i)x_{i+1}\mu_k e_m - \beta(m-1)x_i\mu_k e_{m-1}.$$
$$M_2^1(\mu_k\mu_l e_m)$$
$$= -(i\omega E_{11} + E_{12} + i\omega E_{22} - i\omega E_{33} + E_{34} - i\omega E_{44})\mu_k\mu_l e_m \qquad \square$$
$$= -\alpha(m)i\omega\mu_k\mu_l e_m - \beta(m-1)\mu_k\mu_l e_{m-1}.$$

Proof. [Theorem 3.1] (1) Cleverly choose a standard orthogonal basis of H_{4+3}^2 as follows:

$$f_1 = x_1x_1e_1,\ \ f_2 = x_1x_2e_1,\ \ f_3 = x_2x_2e_1,\ \ f_4 = x_1x_3e_1,\ \ f_5 = x_1x_4e_1,$$
$$f_6 = x_2x_3e_1,\ \ f_7 = x_2x_4e_1,\ \ f_8 = x_3x_3e_1,\ \ f_9 = x_3x_4e_1,\ \ f_{10} = x_4x_4e_1,$$
$$f_{11} = x_1\mu_1e_1,\ f_{12} = x_2\mu_1e_1,\ f_{13} = x_1\mu_2e_1,\ f_{14} = x_2\mu_2e_1,\ f_{15} = x_1\mu_3e_1,$$
$$f_{15} = x_2\mu_3e_1,\ f_{17} = x_3\mu_1e_1,\ f_{18} = x_4\mu_1e_1,\ f_{19} = x_3\mu_2e_1,\ f_{20} = x_4\mu_2e_1,$$
$$f_{21} = x_3\mu_3e_1,\ f_{22} = x_4\mu_3e_1,\ f_{23} = \mu_1\mu_1e_1,\ f_{24} = \mu_1\mu_2e_1,\ f_{25} = \mu_1\mu_3e_1,$$
$$f_{26} = \mu_2\mu_2e_1,\ f_{27} = \mu_2\mu_3e_1,\ f_{28} = \mu_3\mu_3e_1;$$

$$f_{29} = x_1x_1e_2,\ f_{30} = x_1x_2e_2,\ f_{31} = x_2x_2e_2,\ f_{32} = x_1x_3e_2,\ f_{33} = x_1x_4e_2,$$
$$f_{34} = x_2x_3e_2,\ f_{35} = x_2x_4e_2,\ f_{36} = x_3x_3e_2,\ f_{37} = x_3x_4e_2,\ f_{38} = x_4x_4e_2,$$
$$f_{39} = x_1\mu_1e_2,\ f_{40} = x_2\mu_1e_2,\ f_{41} = x_1\mu_2e_2,\ f_{42} = x_2\mu_2e_2,\ f_{43} = x_1\mu_3e_2,$$
$$f_{44} = x_2\mu_3e_2,\ f_{45} = x_3\mu_1e_2,\ f_{46} = x_4\mu_1e_2,\ f_{47} = x_3\mu_2e_2,\ f_{48} = x_4\mu_2e_2,$$
$$f_{49} = x_3\mu_3e_2,\ f_{50} = x_4\mu_3e_2,\ f_{51} = \mu_1\mu_1e_2,\ f_{52} = \mu_1\mu_2e_2,\ f_{53} = \mu_1\mu_3e_2,$$
$$f_{54} = \mu_2\mu_2e_2,\ f_{55} = \mu_2\mu_3e_2,\ f_{56} = \mu_3\mu_3e_2;$$

$$f_{57} = x_1x_1e_3,\ f_{58} = x_1x_2e_3,\ f_{59} = x_2x_2e_3,\ f_{60} = x_1x_3e_3,\ f_{61} = x_1x_4e_3,$$
$$f_{62} = x_2x_3e_3,\ f_{63} = x_2x_4e_3,\ f_{64} = x_3x_3e_3,\ f_{65} = x_3x_4e_3,\ f_{66} = x_4x_4e_3,$$
$$f_{67} = x_1\mu_1e_3,\ f_{68} = x_2\mu_1e_3,\ f_{69} = x_1\mu_2e_3,\ f_{70} = x_2\mu_2e_3,\ f_{71} = x_1\mu_3e_3,$$
$$f_{72} = x_2\mu_3e_2,\ f_{73} = x_3\mu_1e_2,\ f_{74} = x_4\mu_1e_2,\ f_{75} = x_3\mu_2e_2,\ f_{76} = x_4\mu_2e_2,$$
$$f_{77} = x_3\mu_3e_3,\ f_{78} = x_4\mu_3e_3,\ f_{79} = \mu_1\mu_1e_3,\ f_{80} = \mu_1\mu_2e_3,\ f_{81} = \mu_1\mu_3e_3,$$
$$f_{82} = \mu_2\mu_2e_3,\ f_{83} = \mu_2\mu_3e_3,\ f_{84} = \mu_3\mu_3e_3;$$

$f_{85} = x_1x_1e_4,\quad f_{86} = x_1x_2e_4,\quad f_{87} = x_2x_2e_4,\quad f_{88} = x_1x_3e_4,\quad f_{89} = x_1x_4e_4,$

$f_{90} = x_2x_3e_4,\quad f_{91} = x_2x_4e_4,\quad f_{92} = x_3x_3e_4,\quad f_{93} = x_3x_4e_4,\quad f_{94} = x_4x_4e_4,$

$f_{95} = x_1\mu_1e_4,\quad f_{96} = x_2\mu_1e_4,\quad f_{97} = x_1\mu_2e_4,\quad f_{98} = x_2\mu_2e_4,\quad f_{99} = x_1\mu_3e_4,$

$f_{100} = x_2\mu_3e_4,\ f_{101} = x_3\mu_1e_4,\ f_{102} = x_4\mu_1e_4,\ f_{103} = x_3\mu_2e_4,\ f_{104} = x_4\mu_2e_4,$

$f_{105} = x_3\mu_3e_4,\ f_{106} = x_4\mu_3e_4,\ f_{107} = \mu_1\mu_1e_4,\ f_{108} = \mu_1\mu_2e_4,\ f_{109} = \mu_1\mu_3e_4,$

$f_{110} = \mu_2\mu_2e_4,\ f_{111} = \mu_2\mu_3e_4,\ f_{112} = \mu_3\mu_3e_4.$

Using Lemma **6.1**, we get the matrix of $M_2^1(H_{4+3}^2)$ on the basis $f_1, f_2, \cdots, f_{112}$.

$$M = \left(\begin{array}{c|c|c|c} A & -E_{28} & O & O \\ \hline O & A & O & O \\ \hline O & O & B & -E_{28} \\ \hline O & O & O & B \end{array}\right),$$

where

$$A = \left(\begin{array}{c|c|c|c|c|c} i\omega E_3 + C_1 & O & O & O & O & O \\ \hline O & -i\omega E_4 + C_2 & O & O & O & O \\ \hline O & O & -3i\omega E_3 + C_1 & O & O & O \\ \hline O & O & O & C_3 & O & O \\ \hline O & O & O & O & -2i\omega E_6 + C_3 & O \\ \hline O & O & O & O & O & -i\omega E_6 \end{array}\right),$$

$$B = \left(\begin{array}{c|c|c|c|c|c} 3i\omega E_3 + C_1 & O & O & O & O & O \\ \hline O & i\omega E_4 + C_2 & O & O & O & O \\ \hline O & O & -i\omega E_3 + C_1 & O & O & O \\ \hline O & O & O & 2i\omega E_6 + C_3 & O & O \\ \hline O & O & O & O & C_3 & O \\ \hline O & O & O & O & O & i\omega E_6 \end{array}\right),$$

$$C_1 = \begin{pmatrix} 0\,0\,0 \\ 2\,0\,0 \\ 0\,1\,0 \end{pmatrix}, C_2 = \begin{pmatrix} 0\,0\,0\,0 \\ 1\,0\,0\,0 \\ 1\,0\,0\,0 \\ 0\,0\,1\,1 \end{pmatrix}, C_3 = \begin{pmatrix} 0\,0\,0\,0\,0\,0 \\ 1\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,1\,0\,0\,0 \\ 0\,0\,0\,0\,0\,0 \\ 0\,0\,0\,0\,1\,0 \end{pmatrix}.$$

Using elementary column transformation, M can be simplified as follows:

$$M \to \left(\begin{array}{c|c|c|c|c|c|c|c|c} E_{10} & & & & & & & & \\ \hline & E_6 & & & & & & & \\ \hline & & E_{22} & & & & & & \\ \hline & -C_3 & & O_{6\times 6} & & & & & \\ \hline & & & & E_{28} & & & & \\ \hline & & & & & E_6 & & & \\ \hline & & & & & & E_{22} & & \\ \hline & & & & & -C_3 & & O_{6\times 6} & \\ \hline & & & & & & & & E_6 \end{array}\right).$$

Let

$$(g_1, g_2, \cdots, g_{38}, g_{45}, g_{46}, \cdots, g_{100}, g_{107}, g_{108}, \cdots, g_{112})$$

$$= (f_1, f_2, \cdots, f_{112}) \left(\begin{array}{c|c|c|c|c|c|c|c|c} E_{10} & & & & & & & & \\ \hline & E_6 & & & & & & & \\ \hline & & E_{22} & & & & & & \\ \hline & -C_3 & & O_{6\times 6} & & & & & \\ \hline & & & & E_{28} & & & & \\ \hline & & & & & E_6 & & & \\ \hline & & & & & & E_{22} & & \\ \hline & & & & & -C_3 & & O_{6\times 6} & \\ \hline & & & & & & & & E_6 \end{array} \right).$$

Then, $g_1, g_2, \cdots, g_{38}, g_{45}, g_{46}, \cdots, g_{100}, g_{107}, g_{108}, \cdots, g_{112}$ is a basis of $M_2^1(H_{4+3}^2)$.

Because

$$\left(\begin{array}{c|c|c|c|c|c|c|c|c} E_{10} & & & & & & & & \\ \hline & E_6 & & & & & & & \\ \hline & & E_{22} & & & & & & \\ \hline & -C_3 & & H & & & & & \\ \hline & & & & E_{28} & & & & \\ \hline & & & & & E_6 & & & \\ \hline & & & & & & E_{22} & & \\ \hline & & & & & -C_3 & & H & \\ \hline & & & & & & & & E_6 \end{array} \right), (H = E_6)$$

is invertible, $g_{39} = f_{39}, g_{40} = f_{40}, g_{41} = f_{41}, g_{42} = f_{42}, g_{43} = f_{43}, g_{44} = f_{44}$, $g_{101} = f_{101}, g_{102} = f_{102}, g_{103} = f_{103}, g_{104} = f_{104}, g_{105} = f_{105}, g_{106} = f_{106}$ is a basis of a space complementary to $M_2^1(H_{4+3}^2)$. That is

$$H_{4+3}^2 = M_2^1(H_{4+3}^2) \oplus W,$$

where W is the subspace spanned by

$$\begin{pmatrix} 0 \\ x_1\mu_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu_1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1\mu_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu_2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1\mu_3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu_3 \\ 0 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ x_3\mu_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_4\mu_1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_3\mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_4\mu_2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_3\mu_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ x_4\mu_3 \end{pmatrix}.$$

(2) Write $f_2^1(x,0,\mu)=\sum_{k=1}^{112}a_kf_k=\sum_{k=1}^{112}y_kg_k$. Then

$$
\begin{pmatrix} y_1\\ y_2\\ \vdots\\ \vdots\\ y_{112}\end{pmatrix}
=\left(\begin{array}{c|c|c|c|c|c|c|c|c}
E_{10} & & & & & & & & \\ \hline
 & E_6 & & & & & & & \\ \hline
 & & E_{22} & & & & & & \\ \hline
 & -C_3 & & E_6 & & & & & \\ \hline
 & & & & E_{28} & & & & \\ \hline
 & & & & & E_6 & & & \\ \hline
 & & & & & & E_{22} & & \\ \hline
 & & & & & -C_3 & & E_6 & \\ \hline
 & & & & & & & & E_6
\end{array}\right)^{-1}
\begin{pmatrix} a_1\\ a_2\\ \vdots\\ \vdots\\ a_{112}\end{pmatrix}
$$

$$
=\left(\begin{array}{c|c|c|c|c|c|c|c|c}
E_{10} & & & & & & & & \\ \hline
 & E_6 & & & & & & & \\ \hline
 & & E_{22} & & & & & & \\ \hline
 & C_3 & & E_6 & & & & & \\ \hline
 & & & & E_{28} & & & & \\ \hline
 & & & & & E_6 & & & \\ \hline
 & & & & & & E_{22} & & \\ \hline
 & & & & & C_3 & & E_6 & \\ \hline
 & & & & & & & & E_6
\end{array}\right)
\begin{pmatrix} a_1\\ a_2\\ \vdots\\ \vdots\\ a_{112}\end{pmatrix}.
$$

Thus

$$
\begin{cases}
y_{39}=a_{39},\\
y_{40}=a_{40}+a_{11},\\
y_{41}=a_{41},\\
y_{42}=a_{42}+a_{13},\\
y_{43}=a_{43},\\
y_{44}=a_{44}+a_{15},\\
y_{101}=a_{101},\\
y_{102}=a_{102}+a_{73},\\
y_{103}=a_{103},\\
y_{104}=a_{104}+a_{75},\\
y_{105}=a_{105},\\
y_{106}=a_{106}+a_{77},\\
y_i=a_i, \qquad i=1,2,\cdots,38,45,46,\cdots,100,107,108,\cdots,112.
\end{cases}
$$

Since $\sum_{i=1}^{38}y_ig_i+\sum_{i=45}^{100}y_ig_i+\sum_{i=107}^{112}y_ig_i\in M_2^1(H_{4+3}^2)$, there exists $P(x,\mu)\in H_{4+3}^2$

such that

$$f_2^1(x,0,\mu) = M_2^1(P(x,\mu)) + \begin{pmatrix} 0 \\ b_{11}^2 x_1\mu_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (b_{21}^2 + b_{11}^1)x_2\mu_1 \\ 0 \\ 0 \end{pmatrix} +$$

$$\begin{pmatrix} 0 \\ b_{12}^2 x_1\mu_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (b_{22}^2 + b_{12}^1)x_2\mu_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b_{13}^2 x_1\mu_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ (b_{23}^2 + b_{13}^1)x_2\mu_3 \\ 0 \\ 0 \end{pmatrix} +$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ b_{31}^4 x_3\mu_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (b_{41}^4 + b_{31}^3)x_4\mu_1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ b_{32}^4 x_3\mu_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (b_{42}^4 + b_{32}^3)x_4\mu_2 \end{pmatrix} +$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ b_{33}^4 x_3\mu_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ (b_{43}^4 + b_{33}^3)x_4\mu_3 \end{pmatrix}.$$

Remark 6.1. In general, there exist infinitely many spaces witch are complementary to the space $M_2^1(H_{4+3}^2)$.

□

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