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Local Existence for the Generalized Navier-Stokes-Maxwell Equations*

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Abstract In this paper, we establish the local existence for the generalized Navier-Stokes-Maxwell system with the fractional velocity dissipative term $\Lambda^{2\alpha}u$ and fractional magnetic dissipative term $\Lambda^{2\beta}B$. Moreover, we establish the global existence of strong solutions to this generalized model.

Keywords Navier-Stokes-Maxwell system, fractional dissipative, local existence, global existence

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1. Introduction

In this paper, we consider the following Cauchy problem for the Navier-Stokes-Maxwell system:

$$\begin{cases} u_t + u\cdot\nabla u + \nabla P + \Lambda^{2\alpha}u = j\times B, \\ E_t - \mathrm{curl}B = -j, \\ B_t + \mathrm{curl}E + \Lambda^{2\beta}B = 0, \\ \mathrm{div}u = 0, \mathrm{div}B = 0, \\ u(x,0) = u_0(x), B(x,0) = B_0(x), E(x,0) = E_0(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}^3$ and $t > 0$. $u = u(x,t), B = B(x,t)$ and $E = E(x,t)$ denote the velocity field, the electric field and the magnetic field of the fluid, respectively. P denotes the scalar pressure and j denotes the electric current density which is given by Ohm's law. Moreover,

$$j = \sigma(E + u\times B), \tag{1.2}$$

where $\sigma > 0$ denotes the electric resistivity. For simplicity, we set $\sigma = 1$. The fractional Laplacian operator $\Lambda^\alpha = (-\Delta)^{\alpha/2}$ is defined through the Fourier transform

$$\widehat{\Lambda^\alpha f}(\xi) = |\xi|^\alpha \widehat{f}(\xi),$$

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where the Fourier transform is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\, dx.$$

When $\alpha = 1,\ \beta = 0$, the equations (1.1) are reduced to the classical Navier-Stokes-Maxwell equations. Masmoudi [1] proved the global existence and uniqueness of strong solutions if the initial data $u_0,\ E_0,\ B_0 \in L^2(\mathbb{R}^2) \times H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2)$. Using energy estimates and Brezis-Gallouet inequality, Kang and Lee [2] reproved the global existence of regular solutions to the 2D system and obtained a blow-up criterion. In [3], by using Fujita-Kato's method, Masmoudi and Yoneda proved the local existence of the solutions and the loss of smoothness for three dimensional large periodic initial data. Ibrahim and Keraani [4] showed the global existence of the strong solution provided that the initial data $\|u_0\|_{\dot{B}^{1/2}_{2,1}} + \|E_0\|_{\dot{H}^{1/2}} + \|B_0\|_{\dot{H}^{1/2}}$ is small enough. Germain et al. [5] simplified the proof in [4] and lowed the regularity of the initial velocity field in the $\dot{H}^{1/2}(\mathbb{R}^3)$ by using $L^2_t(L^\infty_x)$ estimate on the velocity field. Arsenio and Isabelle [6] proved that global solutions exist under the assumption that the initial velocity and electromagnetic fields have finite energy, and the initial electromagnetic field is small in $\dot{H}^s(\mathbb{R}^n)$ with $s \in [\frac{1}{2}, \frac{3}{2})$. As for the generalized Navier-Stokes-Maxwell system, Jiang [7] proved the global existence and uniqueness of strong solution when $\alpha \geq \frac{3}{2},\ \beta = 0$. In addition, there are many regularity criteria results for the equations (1.1) in [8–11].

Now we state our main theorems as follows:

Theorem 1.1. *Assume the initial data* $u_0, E_0, B_0 \in H^s(\mathbb{R}^3)$ *satisfying* $\mathrm{div} u_0 = \mathrm{div} B_0 = 0$ *for* $s > \max\left\{\frac{5}{2} - 2\alpha, \frac{3}{2} - \alpha, \frac{3}{2} - \beta, 0\right\}$*, then there exists a time* $T_* > 0$ *such that the equations* (1.1) *have a unique solution* (u, E, B) *with*

$$u \in L^\infty(0, T_*; H^s(\mathbb{R}^3)) \cap L^2(0, T_*;\ H^{s+\alpha}(\mathbb{R}^3));$$

$$B \in L^\infty(0, T_*; H^s(\mathbb{R}^3)) \cap L^2(0, T_*;\ H^{s+\beta}(\mathbb{R}^3));$$

$$E \in L^\infty(0, T_*; H^s(\mathbb{R}^3)).$$

Moreover, we could obtain $u,\ E,\ B \in C_w([0, T_*];\ H^s(\mathbb{R}^3))$.

Theorem 1.2. *Assume* $\alpha \geq \frac{5}{4}$, $\beta \geq \frac{7}{4}$, $u_0, E_0, B_0 \in H^s(\mathbb{R}^3)$, $s > \max\left\{\frac{5}{2} - 2\alpha, \frac{3}{2} - \alpha, \frac{3}{2} - \beta, 0\right\}$ *with* $\mathrm{div} u_0 = \mathrm{div} B_0 = 0$*, then the Navier-Stokes-Maxwell system* (1.1) *has a global classical solution.*

The organization of this paper is presented as follows. Firstly, we introduce some lemmas in Section 2. Secondly, we construct the approximate solutions and prove the local wellposedness in Section 3 by using Fourier truncation method. Finally, in Section 4, we justify the global existence to the system (1.1).

2. Preliminaries

In this section, we recall some elementary lemmas which will be used in our proof.

Lemma 2.1. *[12] Define the Fourier truncation* S_R *as follows:*

$$\widehat{S_R f}(\xi) = 1_{B_R(\xi)} \hat{f}(\xi) = \begin{cases} \hat{f}(\xi), & |\xi| \leq R, \\ 0, & |\xi| > R, \end{cases}$$

which satisfies

$$\|S_R f - f\|_{H^s} \le C\frac{1}{R^k}\|f\|_{H^{s+k}}, \tag{2.1}$$

$$\|S_R f - S_{R'} f\|_{H^s} \le C\max\{\frac{1}{R^k}, \frac{1}{R'^k}\}\|f\|_{H^{s+k}}. \tag{2.2}$$

Lemma 2.2. *[13, 14] (Gagliardo-Nirenberg inequality). Let $u \in L^q(\mathbb{R}^n)$ and its derivatives of order m, $D^m u \in L^r$, $1 \le q,\ r \le \infty$. For the derivatives $D^j u$, $0 \le j < m$, the following inequalities hold,*

$$\|D^j u\|_p \le C\|D^m u\|_r^{\alpha}\|u\|_q^{1-\alpha}, \tag{2.3}$$

where

$$\frac{1}{p} = \frac{j}{n} + \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q},$$

for all α in the interval $\frac{j}{m} \le \alpha \le 1$.

Lemma 2.3. *[15, 16] (Kato-Ponce inequality) Let $s > 0$, $p \in (1, \infty)$. Assume that $f \in W^{1,p_1} \cap W^{s,q_2}$, $g \in L^{p_2} \cap W^{s,q_1}$. Then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_p \le C\left(\|\nabla f\|_{p_1}\left\|\Lambda^{s-1}g\right\|_{q_1} + \|g\|_{p_2}\|\Lambda^s f\|_{q_2}\right), \tag{2.4}$$

and

$$\|\Lambda^s(fg)\|_p \le C\left(\|f\|_{p_1}\|\Lambda^s g\|_{q_1} + \|g\|_{p_2}\|\Lambda^s f\|_{q_2}\right), \tag{2.5}$$

with $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$.

3. Local existence

First of all, we give some uniform bounds to the equations (1.1). Then using the Fourier truncation method and energy method, we construct the approximate solutions and obtain the local existence.

3.1. Uniform bounds

Proposition 3.1. *For any $\alpha,\ \beta \ge 0$, suppose that $(u_0,\ E_0,\ B_0) \in H^s(\mathbb{R}^3)$ with $s > \max\left\{\frac{5}{2} - 2\alpha,\ \frac{3}{2} - \alpha,\ \frac{3}{2} - \beta,\ 0\right\}$ satisfies the divergence condition $\mathrm{div}u_0 = \mathrm{div}B_0 = 0$. Then there exists a time T_* such that*

$$\sup_{t\in[0,T_*]}\left[\|u\|^2_{H^s(\mathbb{R}^3)} + \|E\|^2_{H^s(\mathbb{R}^3)} + \|B\|^2_{H^s(\mathbb{R}^3)}\right](t)$$

and

$$\int_0^{T_*}\left[\|u\|^2_{H^{s+\alpha}(\mathbb{R}^3)}(t) + \|B\|^2_{H^{s+\beta}(\mathbb{R}^n)}(t)\right]dt$$

are bounded uniformly in R.

Proof. Taking the inner product of $(1.1)_{1,2,3}$ with u, E, B in $L^2\left(\mathbb{R}^3\right)$ respectively and gathering together, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|u\|_{L^2}^2+\|E\|_{L^2}^2+\|B\|_{L^2}^2)+\|\Lambda^\alpha u\|_{L^2}^2+\|\Lambda^\beta b\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}(j\times B)\cdot udx+\int_{\mathbb{R}^3}\mathrm{curl}B\cdot Edx-\int_{\mathbb{R}^3}j\cdot Edx-\int_{\mathbb{R}^3}\mathrm{curl}E\cdot Bdx\\
&=-\int_{\mathbb{R}^3}(u\times B)\cdot jdx-\int_{\mathbb{R}^3}j\cdot Edx\\
&=-\int_{\mathbb{R}^3}(u\times B)\cdot jdx-\int_{\mathbb{R}^3}j\cdot(j-u\times B)dx\\
&=-\int_{\mathbb{R}^3}|j|^2dx=-\|j\|_{L^2}^2,
\end{aligned}$$

where we have used the Ohm's law (1.2). After integrating in time, we yield the basic energy estimate

$$\begin{aligned}
&\|u\|_{L^2}^2+\|E\|_{L^2}^2+\|B\|_{L^2}^2+2\int_0^{T_*}\|\Lambda^\alpha u\|_{L^2}^2+\|\Lambda^\beta B\|_{L^2}^2+\|j\|_{L^2}^2dt\\
&=\|u_0\|_{L^2}^2+\|E_0\|_{L^2}^2+\|B_0\|_{L^2}^2.
\end{aligned}\tag{3.1}$$

Applying Λ^s to $(1.1)_{1,2,3}$ and multiplying by $\Lambda^s u, \Lambda^s E$ and $\Lambda^s B$, respectively, and taking $\mathrm{div}u=\mathrm{div}B=0$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\Lambda^s u\|_{L^2}^2+\|\Lambda^s B\|_{L^2}^2+\|\Lambda^s E\|_{L^2}^2\right)+\|\Lambda^{s+\alpha}u\|_{L^2}^2+\|\Lambda^{s+\beta}B\|_{L^2}^2+\|\Lambda^s j\|_{L^2}^2\\
&=\int_{\mathbb{R}^3}-\Lambda^s[(u\cdot\nabla)u]\cdot\Lambda^s u+\Lambda^s(j\times B)\cdot\Lambda^s u+\Lambda^s j\cdot\Lambda^s(u\times B)dx\\
&:=I_1+I_2+I_3.
\end{aligned}$$

Using the same method in [17], we have

$$I_1\le\frac{1}{4}\|u\|_{H^{s+\alpha}}^2+C\|u\|_{H^s}^{\frac{2}{1-\theta_0}},$$

where $\theta_0=\min\{\frac{5-2\alpha}{2(s+\alpha)},\frac{4s}{4s+5}\}$, which implies $s>\max\{\frac{5}{2}-2\alpha,0\}$.

The estimate of I_2 is split up into the following cases:

When $\beta<\frac{3}{2}$,

$$\begin{aligned}
I_2&\le\|\Lambda^s(j\times B)\|_{L^2}\|\Lambda^s u\|_{L^2}\\
&\le(\|\Lambda^s j\|_{L^2}\|B\|_{L^\infty}+\|j\|_{L^{\frac{3}{\beta}}}\|\Lambda^s B\|_{L^{\frac{6}{3-2\beta}}})\|\Lambda^s u\|_{L^2}\\
&\le(\|\Lambda^s j\|_{L^2}\|B\|_{L^2}^{1-\theta_1}\|\Lambda^{s+\beta}B\|_{L^2}^{\theta_1}+\|j\|_{L^2}^{1-\theta_2}\|\Lambda^s j\|_{L^2}^{\theta_2}\|\Lambda^{s+\beta}B\|_{L^2})\|\Lambda^s u\|_{L^2}\\
&\le\frac{1}{4}\|\Lambda^s j\|_{L^2}^2+\frac{1}{4}\|\Lambda^{s+\beta}B\|_{L^2}^2+C(\|j\|_{L^2}^2+1)\|\Lambda^s u\|_{L^2}^{\frac{2}{1-\theta_3}},
\end{aligned}$$

where $\theta_1=\frac{3}{2(s+\beta)}$, $\theta_2=\frac{3-2\beta}{2s}$ and $\theta_3=\min\{\theta_1,\theta_2\}$, which implies $s>\frac{3}{2}-\beta$.

When $\beta\ge\frac{3}{2}$, using the embedding $H^s(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, $s>\frac{3}{2}$ and Gagliardo-Nirenberg inequality $\|B\|_{L^\infty}\le\|\Lambda^\beta B\|_{L^2}$, $\beta=\frac{3}{2}$, we have

$$I_2\le(\|\Lambda^s j\|_{L^2}\|B\|_{L^\infty}+\|j\|_{L^2}\|\Lambda^s B\|_{L^\infty})\|\Lambda^s u\|_{L^2}$$

$$
\begin{aligned}
&\leq (\|\Lambda^s j\|_{L^2}\|B\|_{H^\beta} + \|j\|_{L^2}\|B\|_{H^{s+\beta}})\|\Lambda^s u\|_{L^2} \\
&\leq \frac{1}{4}\|\Lambda^s j\|_{L^2}^2 + \frac{1}{4}\|B\|_{H^{s+\beta}}^2 + C(\|j\|_{L^2}^2 + \|B\|_{H^\beta}^2)\|\Lambda^s u\|_{L^2}^2.
\end{aligned}
$$

Similarly,

$$
I_3 \leq \frac{1}{4}\|\Lambda^s j\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{s+\beta} B\|_{L^2}^2 + C\|\Lambda^s u\|_{L^2}^{\frac{2}{1-\theta_1}} + C\|\Lambda^s B\|_{L^2}^{\frac{2}{1-\theta_4}},
$$

where we have used the Gagliardo-Nirenberg inequalities:

$$
\begin{aligned}
\|B\|_{L^\infty} &\leq \|B\|_{L^2}^{1-\theta_1}\|\Lambda^{s+\beta}B\|_{L^2}^{\theta_1},\ \theta_1 = \frac{3}{2(s+\beta)}, \\
\|u\|_{L^\infty} &\leq \|u\|_{L^2}^{1-\theta_4}\|\Lambda^{s+\alpha}u\|_{L^2}^{\theta_4},\ \ \theta_4 = \frac{3}{2(s+\alpha)}.
\end{aligned}
$$

Combining the above estimates together, we have

$$
\begin{aligned}
&\frac{d}{dt}\left(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s B\|_{L^2}^2 + \|\Lambda^s E\|_{L^2}^2\right) + \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \|\Lambda^{s+\beta} B\|_{L^2}^2 + \|\Lambda^s j\|_{L^2}^2 \\
&\leq C(\|j\|_{L^2}^2 + \|B\|_{H^\beta}^2 + 1)(\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s B\|_{L^2}^2 + \|\Lambda^s E\|_{L^2}^2 + 1)^N,
\end{aligned}
$$

where $N = \frac{1}{1-\theta}$, and $\theta = \min\{\theta_0, \theta_3, \theta_4\}$.

A standard Gronwall's inequality shows that

$$
\|\Lambda^s u\|_{L^2}^2 + \|\Lambda^s B\|_{L^2}^2 + \|\Lambda^s E\|_{L^2}^2 + \int_0^{T_*}(\|\Lambda^{s+\alpha} u\|_{L^2}^2 + \|\Lambda^{s+\beta} B\|_{L^2}^2 + \|\Lambda^s j\|_{L^2}^2) \leq C(T).
$$

□

Based on the above uniform bounds, we employ Fourier cutoffs to construct the approximate equations of (1.1).

3.2. Approximate solutions

Now we consider the following truncated Navier-Stokes-Maxwell equations :

$$
\begin{cases}
u_t^R = -\Lambda^{2\alpha} u^R - \nabla P^R - S_R[(u^R \cdot \nabla)u^R] + S_R[j^R \times B^R], \\
E_t^R = \mathrm{curl} B^R - S_R j^R, \\
B_t^R = -\mathrm{curl} E^R - \Lambda^{2\beta} B^R, \\
\mathrm{div} u^R = 0,\ \mathrm{div} B^R = 0, \\
(u^R, B^R, E^R)(x,0) = (S_R u_0, S_R B_0, S_R E_0).
\end{cases} \tag{3.2}
$$

Set $X^R = (u^R,\ E^R,\ B^R)^T, X_1^R = (u_1^R,\ E_1^R,\ B_1^R)^T, X_2^R = (u_2^R,\ B_2^R,\ B_2^R)^T$, then the truncated Navier-Stokes-Maxwell equations (3.2) could be reformulated as

$$
\begin{aligned}
&\frac{dX^R}{dt} = F(X^R), \\
&\left(u^R, E^R, B^R\right)(x,0) = (S_R u_0, S_R E_0, S_R B_0) = X_0^T,
\end{aligned}
$$

where

$$F(X^R)=\begin{pmatrix} -\Lambda^{2\alpha}u^R-PS_R\left[\left(u^R\cdot\nabla\right)u^R\right]+PS_R\left[j^R\times B^R\right]\\ \mathrm{curl}B^R-PS_Rj^R\\ \mathrm{curl}E^R-\Lambda^{2\beta}B^R\end{pmatrix}, \tag{3.3}$$

and P denotes the Leray projection operator which projects functions onto the space of divergence-free functions. Taking $V^s=\left\{f\in H^s\left(\mathbb{R}^n\right):\mathrm{div}f=0,\ \mathrm{supp}\hat{f}\subset B_R\right\}$. By using (3.1), (3.2) and the fact of $\|S_Rf\|_{H^s}\le CR^s\|f\|_2$, we have

$$\left\|F\left(X_1^R\right)-F\left(X_2^R\right)\right\|_{H^s}\le C\left(\|X_0\|_2,\ R,\ n\right)\left\|X_1^R-X_2^R\right\|_{H^s}. \tag{3.4}$$

Thus, F is locally Lipschitz continuous on any open set $V^s\times V^s\times V^s$. According to the Picard's theorem, given any initial condition $X_0^R\in V^s\times V^s$, there exists a unique solution $X^R\in C^1([0,T_R);\ V^s)\times C^1([0,T_R);\ V^s)\times C^1([0,T_R);\ V^s)$ for some $T_R>0$.

In order to establish the continuity of $u,\ E,\ B$ on the interval $[0,T_*]$ with values in the weak topology of $H^s(\mathbb{R}^3)$, we need the strong convergence of the sequence $\left(u^R,b^R\right)$ in $L^\infty\left(0,T_*;L^2\left(\mathbb{R}^n\right)\right)\times L^\infty\left(0,T_*;L^2\left(\mathbb{R}^n\right)\right)$.

Proposition 3.2. *The family* $\left(u^R,E^R,B^R\right)$ *of solutions of* (3.2) *are Cauchy sequences (as* $R\to\infty$*) in* $L^\infty\left(0,T_*;L^2\left(\mathbb{R}^n\right)\right)$.

Proof. Without loss of generality, we assume $R'>R\ge1$. Taking the difference between the equations, we get

$$\left(u^R-u^{R'}\right)_t=-\Lambda^{2\alpha}\left(u^R-u^{R'}\right)-\nabla\left(P^R-P^{R'}\right)-S_R[(u^R\cdot\nabla)u^R-j^R\times B^R]$$
$$+S_{R'}[(u^{R'}\cdot\nabla)u^{R'}-j^{R'}\times B^{R'}], \tag{3.5}$$
$$\left(E^R-E^{R'}\right)_t=\mathrm{curl}\left(B^R-B^{R'}\right)-S_Rj^R+S_{R'}j^{R'}, \tag{3.6}$$
$$\left(B^R-B^{R'}\right)_t=-\mathrm{curl}\left(E^R-E^{R'}\right)-\Lambda^{2\beta}\left(B^R-B^{R'}\right). \tag{3.7}$$

Taking the inner product of (3.5)-(3.7) with $u^R-u^{R'},E^R-E^{R'},B^R-B^{R'}$ respectively, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\left\|u^R-u^{R'}\right\|_{L^2}^2+\left\|E^R-E^{R'}\right\|_{L^2}^2+\left\|B^R-B^{R'}\right\|_{L^2}^2\right)\\&+\left\|\Lambda^\alpha\left(u^R-u^{R'}\right)\right\|_{L^2}^2+\left\|\Lambda^\beta\left(B^R-B^{R'}\right)\right\|_{L^2}^2\\=&<S_R\left(j^R\times B^R\right)-S_{R'}\left(j^{R'}\times B^{R'}\right),u^R-u^{R'}>\\&-<S_R\left[\left(u^R\cdot\nabla\right)u^R\right]-S_{R'}\left[\left(u^{R'}\cdot\nabla\right)u^{R'}\right],u^R-u^{R'}>\\&-<S_Rj^R-S_{R'}j^{R'},E^R-E^{R'}>\\:=&I_1+I_2+I_3.\end{aligned}$$

Now we estimate the terms I_i $(i=1,2,3)$ one by one.

Using the same method in [17], we have

$$I_2 \le \frac{C}{R^s}\|u^R\|_{H^\alpha}^2 + \frac{1}{4}\left\|u^R - u^{R'}\right\|_{H^\alpha}^2 + C\|u^R\|_{H^{s+\alpha}}^2\left\|u^R - u^{R'}\right\|_{L^2}^2.$$

We split I_1 into three parts:

$$\begin{aligned} I_1 &= \left\langle S_R\left(j^R \times B^R\right) - S_{R'}\left(j^{R'} \times B^{R'}\right), u^R - u^{R'}\right\rangle \\ &= \left\langle \left(S_R - S_{R'}\right)\left(j^R \times B^R\right), u^R - u^{R'}\right\rangle \\ &\quad + \left\langle S_{R'}\left[\left(j^R - j^{R'}\right) \times B^R\right], u^R - u^{R'}\right\rangle \\ &\quad + \left\langle S_{R'}\left[j^{R'} \times \left(B^R - B^{R'}\right)\right], u^R - u^{R'}\right\rangle \\ &:= I_{11} + I_{12} + I_{13}. \end{aligned}$$

For the first term, using the embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3),\ s > \frac{3}{2}$, we have

$$\begin{aligned} I_{11} &\le \frac{1}{R^s}\left\|u^R - u^{R'}\right\|_{H^s}\left\|j^R \times B^R\right\|_{L^2} \\ &\le \frac{1}{R^s}\left(\left\|u^R\right\|_{H^s} + \left\|u^{R'}\right\|_{H^s}\right)\left\|j^R\right\|_{L^2}\left\|B^R\right\|_{L^\infty} \\ &\le \frac{C}{R^s}\left\|j^R\right\|_{L^2}\left\|B^R\right\|_{H^{s+\beta}} \\ &\le \frac{C}{R^s}\left(\left\|j^R\right\|_{L^2}^2 + \left\|B^R\right\|_{H^{s+\beta}}^2\right). \end{aligned}$$

Next we estimate the other terms:

$$\begin{aligned} I_{12} &\le \left\|\left(j^R - j^{R'}\right) \times B^R\right\|_{L^2}\left\|u^R - u^{R'}\right\|_{L^2} \\ &\le \left\|j^R - j^{R'}\right\|_{L^2}\left\|B^R\right\|_{L^\infty}\left\|u^R - u^{R'}\right\|_{L^2} \\ &\le \left\|j^R - j^{R'}\right\|_{L^2}\left\|B^R\right\|_{H^{s+\beta}}\left\|u^R - u^{R'}\right\|_{L^2} \\ &\le \frac{1}{4}\left\|j^R - j^{R'}\right\|_{L^2}^2 + C\left\|B^R\right\|_{H^{s+\beta}}^2\left\|u^R - u^{R'}\right\|_{L^2}^2. \end{aligned}$$

The estimates of I_{13} are split up into the following two cases:
When $\beta < \frac{3}{2}$,

$$\begin{aligned} I_{13} &\le \left\|j^{R'} \times \left(B^R - B^{R'}\right)\right\|_{\frac{3}{3-\beta}}\left\|u^R - u^{R'}\right\|_{\frac{3}{\beta}} \\ &\le \left\|j^{R'}\right\|_{L^2}\left\|B^R - B^{R'}\right\|_{\frac{6}{3-2\beta}}\left\|u^R - u^{R'}\right\|_{L^2}^{1-\theta}\left\|\Lambda^s\left(u^R - u^{R'}\right)\right\|_{L^2}^{\theta} \\ &\le \left\|j^{R'}\right\|_{L^2}\left\|\Lambda^\beta\left(B^R - B^{R'}\right)\right\|_{L^2}\left\|u^R - u^{R'}\right\|_{L^2}^{1-\theta}\left(\left\|\Lambda^s u^R\right\|_{L^2} + \left\|\Lambda^s u^{R'}\right\|_{L^2}\right)^{\theta} \\ &\le \frac{1}{2}\left\|\Lambda^\beta\left(B^R - B^{R'}\right)\right\|_{L^2}^2 + C\left\|j^{R'}\right\|_{L^2}^2\left(\left\|u^R - u^{R'}\right\|_{L^2}^2 + 1\right), \end{aligned}$$

where $\theta = \frac{3-2\beta}{2s}$, which implies $s > \frac{3}{2} - \beta$.

When $\beta \geq \frac{3}{2}$,

$$\begin{aligned} I_{13} &\leq \left\| j^{R'} \times \left(B^R - B^{R'} \right) \right\|_{L^2} \left\| u^R - u^{R'} \right\|_{L^2} \\ &\leq \left\| j^{R'} \right\|_{L^2} \left\| B^R - B^{R'} \right\|_{L^\infty} \left\| u^R - u^{R'} \right\|_{L^2} \\ &\leq \left\| j^{R'} \right\|_{L^2} \left\| B^R - B^{R'} \right\|_{H^\beta} \left\| u^R - u^{R'} \right\|_{L^2} \\ &\leq \frac{1}{2} \left\| B^R - B^{R'} \right\|_{H^\beta}^2 + C \left\| j^{R'} \right\|_{L^2}^2 \left\| u^R - u^{R'} \right\|_{L^2}^2, \end{aligned}$$

where we have made use of Sobolev embedding $H^s(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, $s > \frac{3}{2}$ and Gagliardo-Nirenberg inequality $\|B^R - B^{R'}\|_{L^\infty} \leq \|\Lambda^\beta (B^R - B^{R'})\|_{L^2}$ when $\beta = \frac{3}{2}$.

We split I_3 into two parts:

$$\begin{aligned} I_3 &= -\left\langle S_R j^R - S_{R'} j^{R'}, E^R - E^{R'} \right\rangle \\ &= -\left\langle (S_R - S_{R'}) j^R, E^R - E^{R'} \right\rangle - \left\langle S_{R'} (j_R - j_{R'}), E^R - E^{R'} \right\rangle \\ &:= I_{31} + I_{32}. \end{aligned}$$

For the term I_{31}, we have

$$\begin{aligned} \|I_{31}\| &\leq \frac{1}{R^s} \|j^R\|_{H^s} \left\| E^R - E^{R'} \right\|_{L^2} \\ &\leq \frac{1}{R^s} \|j^R\|_{H^s}^2 + C \|E^R - E^{R'}\|_{L^2}^2. \end{aligned}$$

By using the Ohm's law (1.2), we obtain

$$\begin{aligned} I_{32} &= -\left\langle j^R - j^{R'}, E^R - E^{R'} \right\rangle \\ &= -\left\langle j^R - j^{R'}, j^R - j^{R'} - u^R \times B^R + u^{R'} \times B^{R'} \right\rangle. \\ &= -\left\| j^R - j^{R'} \right\|_{L^2}^2 + \left\langle j^R - j^{R'}, u^R \times B^R - u^{R'} \times B^{R'} \right\rangle \\ &= -\left\| j^R - j^{R'} \right\|_{L^2}^2 + III. \end{aligned}$$

For III, we have

$$\begin{aligned} III =& \left\langle j^R - j^{R'}, \left(u^R - u^{R'} \right) \times B^R \right\rangle + \left\langle j^R - j^{R'}, u^{R'} \times \left(B^R - B^{R'} \right) \right\rangle \\ \leq& \left\| j^R - j^{R'} \right\|_{L^2} \left\| u^R - u^{R'} \right\|_{L^2} \|B^R\|_{L^\infty} + \left\| j^R - j^{R'} \right\|_{L^2} \left\| u^{R'} \right\|_{L^\infty} \left\| B^R - B^{R'} \right\|_{L^2} \\ \leq& \left\| j^R - j^{R'} \right\|_{L^2} \left\| u^R - u^{R'} \right\|_{L^2} \|B^R\|_{H^{s+\beta}} \\ &+ \left\| j^R - j^{R'} \right\|_{L^2} \left\| u^{R'} \right\|_{H^{s+\alpha}} \left\| B^R - B^{R'} \right\|_{L^2} \\ \leq& \frac{1}{4} \left\| j^R - j^{R'} \right\|_{L^2}^2 + C \|B^R\|_{H^{s+\beta}}^2 \left\| u^R - u^{R'} \right\|_{L^2}^2 + C \|u^{R'}\|_{H^{s+\alpha}}^2 \left\| B^R - B^{R'} \right\|_{L^2}^2. \end{aligned}$$

Putting the above estimates together and setting

$$Y(t) = \left\| u^R - u^{R'} \right\|_{L^2}^2 + \left\| E^R - E^{R'} \right\|_{L^2}^2 + \left\| B^R - B^{R'} \right\|_{L^2}^2,$$

we have

$$\frac{d}{dt}Y(t) + \left\|\Lambda^{\alpha}\left(u^R - u^{R'}\right)\right\|_{L^2}^2 + \left\|\Lambda^{\beta}\left(B^R - B^{R'}\right)\right\|_{L^2}^2 + \left\|j^R - j^{R'}\right\|_{L^2}^2$$
$$\leq \frac{C}{R^s}\left(\|u^R\|_{H^\alpha}^2 + \|B^R\|_{H^{s+\beta}}^2 + \|j^R\|_{H^s}^2\right)$$
$$+C(\|j^{R'}\|_{L^2}^2 + \|u^R\|_{H^{s+\alpha}}^2 + \|B^R\|_{H^{s+\beta}}^2 + 1)(Y(t)+1).$$

Remind that

$$\sup_{t\in[0,T_*]}\left[\|u\|_{H^s(\mathbb{R}^3)}^2 + \|E\|_{H^s(\mathbb{R}^3)}^2 + \|B\|_{H^s(\mathbb{R}^3)}^2\right](t),$$
$$\int_0^{T_*}\left[\|u\|_{H^\alpha(\mathbb{R}^3)}^2(t) + \|B\|_{H^\beta(\mathbb{R}^n)}^2(t) + \|j\|_{L^2}^2\right](t)dt,$$
$$\int_0^{T_*}\left[\|u\|_{H^{s+\alpha}(\mathbb{R}^3)}^2 + \|B\|_{H^{s+\beta}(\mathbb{R}^n)}^2 + +\|\Lambda^s j\|_{L^2}^2\right](t)dt$$

are bounded for all $t \in [0, T_*]$, using the Gronwall's inequality, we see that

$$\sup_{t\in[0,T_*]} Y(t) \leq \frac{C}{R^s}.$$

The right hand tends to zero as $R,\ R' \to \infty$. The proof of Proposition 3.2 is completed.

□

3.3. Proof of Theorem 1.1

Proof of Theorem 1.1 According to Proposition 3.1, using the Banach-Alaoglu theorem, we can extract a sequence that converges weakly to $u,\ E,\ B \in L^2(0, T_*; H^s(\mathbb{R}^3))$. Moreover, for each $t \in [0, T_*]$, the subsequence is uniformly bounded in $H^s(\mathbb{R}^3)$, so it has a subsequence that converges weakly in $L^\infty(0, T_*; H^s(\mathbb{R}^3))$. Hence, we get $u,\ E,\ B \in L^\infty(0, T_*; H^s(\mathbb{R}^3))$.

From Propositons 3.1 and 3.2, u^R, E^R, B^R converge strongly in $L^\infty(0, T_*; L^2(\mathbb{R}^3))$. By interpolation, we obtain $u^R,\ E^R,\ B^R \to u,\ E,\ B$ strongly in $L^\infty(0, T_*; H^{s'}(\mathbb{R}^3))$ for any $0 < s' < s$. By using the standard argument in Majda and Bertozzi [18](proof of Theorem 3.4), we have $u,\ E,\ B \in C_w([0, T_*]; H^s(\mathbb{R}^3))$. □

4. Global existence

In this section, we will complete the proof of Theorem 1.2.
Proof of Theorem 1.2 Actually, we only need to show the case $\alpha = \frac{5}{4}$ and $\beta = \frac{7}{4}$.

Multiplying the first equation of (1.1) by $-\Delta u,\ -\Delta E,\ -\Delta B$ respectively, integrating the results on $\mathbb{R}^3$ and using integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|(\nabla u,\ \nabla E,\ \nabla B)\|_{L^2}^2 + \|\Lambda^{\alpha+1}u,\ \Lambda^{\beta+1}B\|_{L^2}^2$$
$$= \int_{\mathbb{R}^3}[(u\cdot\nabla)u]\cdot\Delta u dx - \int_{\mathbb{R}^3}(j\times B)\cdot\Delta u dx$$

$$
\begin{aligned}
&-\int_{\mathbb{R}^3}\mathrm{curl}B\cdot\Delta Edx+\int_{\mathbb{R}^3}j\cdot\Delta Edx+\int_{\mathbb{R}^3}\mathrm{curl}E\cdot\Delta Bdx\\
=&-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left[(\partial_i u\cdot\nabla)\,u\right]\cdot\partial_i udx+\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i(j\times B)\cdot\partial_i udx\\
&-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i j\cdot\partial_i Edx.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i j\cdot\partial_i Edx&=-\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i j\cdot\partial_i(j-u\times B)dx\\
&=-\|\nabla j\|_{L^2}^2+\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i(u\times B)\cdot\partial_i jdx,
\end{aligned}
$$

we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|(\nabla u,\ \nabla E,\ \nabla B)\|_{L^2}^2+\|(\Lambda^{\alpha+1}u,\ \Lambda^{\beta+1}B,\ \nabla j)\|_{L^2}^2\\
=&\ -\sum_{i=1}^{3}\int_{\mathbb{R}^3}\left[(\partial_i u\cdot\nabla)\,u\right]\cdot\partial_i udx+\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i(j\times B)\cdot\partial_i udx\\
&+\sum_{i=1}^{3}\int_{\mathbb{R}^3}\partial_i(u\times B)\cdot\partial_i jdx\\
:=&\ I_1+I_2+I_3.
\end{aligned}
\tag{4.1}
$$

The estimates of I_1 are split up into the following two cases:

When $\alpha<\frac{5}{2}$,

$$
\begin{aligned}
|I_1|&\le\|u\|_{\frac{3}{\alpha-1}}\|\nabla u\|_{L^2}\|\Delta u\|_{\frac{6}{5-2\alpha}}\\
&\le\|u\|_{L^2}^{1-\theta}\|\Lambda^\alpha u\|_{L^2}^{\theta}\|\nabla u\|_{L^2}\|\Lambda^{\alpha+1}u\|_{L^2}\\
&\le\frac{1}{2}\|\Lambda^{\alpha+1}u\|_{L^2}^2+C\|\Lambda^\alpha u\|_{L^2}^{2\theta}\|\nabla u\|_{L^2}^2,
\end{aligned}
$$

where we have used the Gagliardo-Nirenberg inequality:

$$
\frac{\alpha-1}{3}=\theta\left(\frac{1}{2}-\frac{\alpha}{3}\right)+\frac{1-\theta}{2},\ 0\le\theta\le 1,
$$

which implies $\alpha\ge\frac{5}{4}$.

When $\alpha\ge\frac{5}{2}$, using the embedding $H^q(\mathbb{R}^3)\hookrightarrow L^\infty(\mathbb{R}^3)$, $q>\frac{3}{2}$ and Gagliardo-Nirenberg inequality $\|u\|_{L^\infty}\le\|\Lambda^{\frac{3}{2}}u\|_{L^2}$, we have

$$
\begin{aligned}
|I_1|&\le\|u\|_{L^2}\|\nabla u\|_{L^2}\|\Delta u\|_{L^\infty}\\
&\le C\|\nabla u\|_{L^2}\|u\|_{H^{\alpha+1}}\\
&\le\frac{1}{2}\|u\|_{H^{\alpha+1}}^2+C\|\nabla u\|_{L^2}^2.
\end{aligned}
$$

Similarly, we can estimate the other terms:

$$\begin{aligned}|I_2| &\le \|j\|_{L^2}\|\nabla u\|_{L^2}\|\nabla B\|_{L^\infty}\\ &\le \|j\|_{L^2}\|\nabla u\|_{L^2}\|\nabla B\|_{L^2}^{1-\delta}\|\Lambda^{\frac{11}{4}}B\|_{L^2}^{\delta}\\ &\le \frac{1}{4}\|\Lambda^{\frac{11}{4}}B\|_{L^2}^2 + C\|\nabla B\|_{L^2}^2 + C\|j\|_{L^2}^2\|\nabla u\|_{L^2}^2\\ |I_3| &\le \|u\|_{L^2}\|\nabla j\|_{L^2}\|\nabla B\|_{L^\infty},\\ &\le \|u\|_{L^2}\|\nabla j\|_{L^2}\|\nabla B\|_{L^2}^{1-\delta}\|\Lambda^{\frac{11}{4}}B\|_{L^2}^{\delta}\\ &\le \frac{1}{2}\|\nabla j\|_{L^2}^2 + \frac{1}{4}\|\Lambda^{\frac{11}{4}}B\|_{L^2}^2 + C\|\nabla B\|_{L^2}^2,\end{aligned}$$

where

$$0 = \delta\left(\frac{1}{2} - \frac{\frac{7}{4}}{3}\right) + \frac{1-\delta}{2},$$

with $\delta = \frac{1}{7}$.

Putting the above estimates together, we complete the proof by using the Gronwall's inequality. □

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