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Dynamics in a Delayed Eco-epidemiological Model with Disease in the Pest*

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Abstract In ecology, it is of great significance to research the influence of gestation period on the dynamics of eco-epidemiology. In the paper, we establish and explore a delayed predator-pest model with disease in pest. We first analyze the existence and local stability of each equilibrium of the model. Then, we investigate the existence of Hopf bifurcation at the coexistence equilibrium. Moreover, we calculate the normal form to examine the properties of Hopf bifurcation. Some numerical simulations are conducted to verify the theoretical results obtained and explore how the delay affects the biomass of pest. Our findings may contribute to a better understanding of the mechanisms of interaction between species in eco-epidemiology. At the same time, this study also provides an insightful perspective into the control of pests in ecosystems.

Keywords Predator-pest model, eco-epidemiology, gestation delay, Hopf bifurcation, normal form

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1. Introduction

In agriculture, organisms that cause harm to agriculture by destroying crops or parasitizing livestock are referred to as pests. They can directly or indirectly cause damage to the human population. It has been noticed that there are many animals and birds that feed on pests but do not affect the development of ecosystem, and therefore could be employed as an ecological method for the control of pest. This has been investigated extensively by constructing mathematical models in many recent research works [1–4]. On the other hand, some diseases may have a direct impact on pests, which can serve as a biological control to indirectly reduce pest populations [5–8]. In fact, once the pests are infected, they will become less mobile and their escape responses are weakened. Additionally because the lifestyle of the infected pests has changed and they live in habitats accessible to predators, and therefore can be easily caught by predators [9,10]. Thus, a profound comprehension of predation mechanisms is essential for elucidating the principles governing the eco-epidemiological system. Bhattacharyya et al. [7] proposed a mathematical model for pest management under virus infection which provides insights into the interaction between infected pests and predator in the epidemiological dynamics. Joly et al. [11]

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have demonstrated that severe infection of Echinococcus granulosus makes moose vulnerable to capture by wolves, suggesting that the disease plays a remarkable role in regulating moose populations.

Eco-epidemiology is considered as a very significant field of research in mathematical biology that simultaneously takes into consideration the ecological and epidemiological factors from both mathematical and ecological perspectives. The mathematical modelling researches based on the eco-epidemiological frameworks have attracted great attention of many researchers since the pioneering study of Hadeler and Freedman [12] and Chattopadhyay and Arino [13], and there are a large number of works devoted to exploring the impact of disease on ecosystems [14–16]. In eco-epidemiology, researchers explore an ecosystem with contagion either in predator [17–19] or in prey [20–22], or in both populations [23–25]. Li et al. [19] explored a mathematical model with diseased predators in an open environment, and obtained sufficient conditions for the disease to successfully invade the system. Hsieh and Hsiao [23] put forward an eco-epidemiology model with infected prey and infected predator, and their results indicate that the coexistence of organisms is determined by ecological threshold. It is worthy of noting that the eco-epidemiological models usually exhibit more complex dynamical behaviors than those without diseases.

Notice that time delay is an extremely common phenomenon in ecosystems; neglecting time delays implies neglecting the reality [26–28]. When developing an eco-epidemiological model, it is important to keep in mind that predators need to accumulate enough energy and nutrients over a period of time after eating to reproduce successfully, and therefore the reproduction is not instantaneous. It also important to explore the influence of time delay on the dynamics of the model [21, 29–31]. As an example, Xiao and Chen [21] proposed a delayed mathematical model with infected prey, in which the infected prey was assumed to be unable to reproduce and predators need certain time to reproduce after consuming the prey. Their results indicate that the delay may disrupt the stability of the system and lead to the occurrence of Hopf bifurcation. It is worth noting that the pregnant individuals may die during their pregnancy. It is necessary to incorporate this when constructing ecological dynamics models. To the best of our knowledge, few investigations have considered this factor in the eco-epidemiological models. In this paper, by explicitly incorporating the effct of this factor using a survival probability after pregnancy in the delay terms, we study how the time delay affects the dynamics of an eco-epidemiological model.

The organization of this paper is as follows. In Section 2, we present the model and some preliminary results, including the positivity and boundedness of the solutions. In Section 3, we perform the existence and stability analysis of equilibria of the model and derive conditions for the occurrence of Hopf bifurcation at the coexisting equilibrium. In Section 4, we compute the normal form to determine the nature of Hopf bifurcation. Some numerical simulations are given to confirm the theoretical results obtained in Section 5. Finally, we briefly summarize the paper in Section 6.

2. The model and some preliminaries

Assume that there is a disease spreading in a pest population, and that the infected individuals are too vulnerable to compete for the resources with susceptibles. Meantime, they can be easily preyed once encountering the predators. To explore the combined effects of predators and disease in the control of pests, in Kar et al. [4], the authors proposed an eco-epidemiological model as follows:

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = rS(1-\frac{S}{K}) - \beta IS - \frac{\gamma SP}{a+S}, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta IS - mIP - \mu I, \\ \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\alpha_1 \gamma SP}{a+S} + \alpha_2 mIP - \delta P, \end{cases} \tag{2.1}$$

where $S(t)$, $I(t)$, $P(t)$ are the numbers of susceptible pest, infected pest and predators at time t, respectively. All the parameters are nonnegative. r is the intrinsic birth rate of susceptible pest, K is the carrying capacity of the pest, β is the transmission rate of disease, a is the half-saturation constant, α_1 and α_2 stand for the conversion factors, and μ and δ are respectively the natural death rates of infected pest and predators; mI and $\frac{\gamma S}{a+S}$ are the response functions describing respectively the capture rates of predators to the infected and susceptible pests, where m and γ denote the corresponding maximum capture rates. See the flow diagram of the model in Fig. 1. The authors have performed a detailed stability analysis of the model, and indicate that the spread of disease in the pest population can induce the occurrence of Hopf bifurcation.

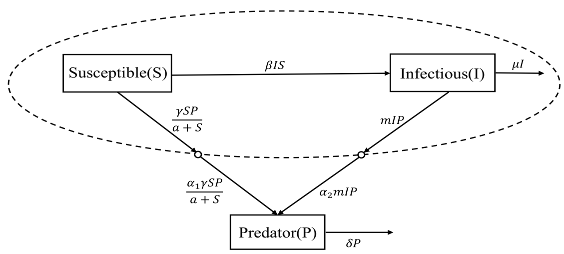


Figure 1. The flow diagram of model (2.1)

Notice that model (2.1) has not considered the time required for the conversion from the pest preyed to predators, and therefore fails to comprehensively describe the interactions between the pest and predators. Hence, it appears more appropriate to incorporate it into model (2.1). By regarding the time required for the predators to reproduce their offsprings after eating pests as a time delay τ, we propose the following model:

$$\begin{cases} \frac{\mathrm{d}S}{\mathrm{d}t} = rS(1-\frac{S}{K}) - \beta IS - \frac{\gamma SP}{a+S}, \\ \frac{\mathrm{d}I}{\mathrm{d}t} = \beta IS - mIP - \mu I, \\ \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{\alpha\gamma S(t-\tau)P(t-\tau)}{a+S(t-\tau)} e^{-\delta\tau} + \alpha m I(t-\tau)P(t-\tau)e^{-\delta\tau} - \delta P, \end{cases} \tag{2.2}$$

where the factor $e^{-\delta\tau}$ in the conversion rate term is used to describe the effect of death of predators in the period of their pregnancy. Notice that we have assumed $\alpha_1 = \alpha_2 = \alpha$ in model (2.2) since the susceptible pests and the infected pests are the same species and therefore the same food for predators. From the perspective of biological feasibility, model (2.2) should satisfy the following initial conditions:

$$S(\nu) = \psi_1(\nu), \; I(\nu) = \psi_2(\nu), \; P(\nu) = \psi_3(\nu), \quad \nu \in [-\tau, 0], \tag{2.3}$$

where $(\psi_1(\nu), \psi_2(\nu), \psi_3(\nu)) \in \mathbb{C}_+ := C([-\tau, 0], \mathbb{R}^3_+)$ and $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i \geq 0\}$.

The following two theorems are about the non-negativity and boundedness of the solutions of model (2.2).

Theorem 2.1. *For any initial value $(\psi_1(\nu), \psi_2(\nu), \psi_3(\nu)) \in \mathbb{C}_+$, the solution of model (2.2) will remain nonnegative for all $t \geq 0$.*

Proof. Notice from the first two equations of S and I in model (2.2) that

$$S(t) = \psi_1(0) \exp \int_0^t \left[r\left(1 - \frac{S(\nu)}{K}\right) - \beta I(\nu) - \frac{\gamma P(\nu)}{a + S(\nu)} \right] \mathrm{d}\nu,$$
$$I(t) = \psi_2(0) \exp \int_0^t (\beta S(v) - mP(v) - \mu) \mathrm{d}\nu.$$

Thus, we have that $S(t) \geq 0$ and $I(t) \geq 0$ for all $t > 0$ since $\psi_i(0) \geq 0$, $i = 1, 2$. Notice also that the third equation of P in model (2.2) can be written as

$$P(t)e^{\delta t} = \psi_3(0) + \exp \int_0^t \alpha \left(\frac{\gamma S(\nu - \tau) P(\nu - \tau)}{a + S(\nu - \tau)} + mI(\nu - \tau)P(\nu - \tau) \right) e^{-\delta(\tau + \nu)} \mathrm{d}\nu.$$

Thus we have that $P(t) \geq 0$ for $t \in [0, \tau]$ since $\psi_3(\nu) \geq 0$ for $\nu \in [0, \tau]$. Performing the similar arguments, we can further prove in turn that $P(t) \geq 0$ for $t \in [(n-1)\tau, n\tau]$, $n = 1, 2, \cdots$. The proof is thus completed. □

Theorem 2.2. *The solutions of model (2.2) with initial values in $\mathbb{C}_+$ are uniformly ultimately bounded.*

Proof. Consider the function defined by

$$V(t) = e^{-\delta\tau} S(t - \tau) + e^{-\delta\tau} I(t - \tau) + \frac{1}{\alpha} P(t).$$

Differentiating V with respect to t along the solution of system (2.2) for $t \geq \tau$, we obtain

$$\frac{\mathrm{d}V}{\mathrm{d}t} = e^{-\delta\tau} \left[rS(t - \tau) \left(1 - \frac{S(t - \tau)}{K}\right) - \mu I(t - \tau) \right] - \frac{\delta}{\alpha} P(t).$$

Now taking κ such that $0 < \kappa < \min\{\mu, \delta\}$, we can compute that

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t} + \kappa V &\leq e^{-\delta\tau} \left\{ S(t - \tau) \left[r\left(1 - \frac{S(t - \tau)}{K}\right) + \kappa \right] - (\mu - \kappa) I(t - \tau) \right\} - \frac{\delta - \kappa}{\alpha} P(t) \\
&\leq e^{-\delta\tau} S(t - \tau) \left[r\left(1 - \frac{S(t - \tau)}{K}\right) + \kappa \right] \\
&\leq e^{-\delta\tau} \frac{K(r + \kappa)^2}{4r} := M.
\end{aligned} \tag{2.4}$$

Applying the comparison theorem, it then follows from (2.4) that

$$\limsup_{t \to \infty} V(S, I, P) \leq \frac{M}{\kappa}.$$

The proof is thus completed. □

3. Existence and stability of equilibria

In this section, we first perform the analysis of the existence of equilibria of model (2.2), then consider the stability of each equilibrium when it exists.

3.1. Existence of equilibriua

In the absence of time delay, i.e., $\tau = 0$, in Kar et al. [4], the authors have performed a detailed analysis on the existence of equilibria of model (2.1). In the presence of delay, i.e., $\tau \neq 0$, we can easily obtain that there are five possible feasible equilibria for model (2.2) by just assuming that $\alpha_i = \alpha$, i=1,2 and using $\alpha e^{-\delta\tau}$ instead of α in their analysis, namely,

(i) the trivial equilibrium $E_1\,(0,0,0)$;

(ii) the boundary equilibrium $E_2\,(K,0,0)$;

(iii) the predator free equilibrium $E_3\,(S_3, I_3, 0)$, where $S_3 = \frac{\mu}{\beta}$ and $I_3 = \frac{r(\beta K-\mu)}{\beta^2 K}$, which is feasible for $\beta > \frac{\mu}{K}$;

(iv) the infected pest free equilibrium $E_4\,(S_4, 0, P_4)$, where $S_4 = \frac{a\delta}{\alpha\gamma e^{-\delta\tau}-\delta}$ and $P_4 = \frac{ar\alpha e^{-\delta\tau}\left(K\alpha\gamma e^{-\delta\tau}-(a+K)\delta\right)}{K(\alpha\gamma e^{-\delta\tau}-\delta)^2}$, which is feasible for $\frac{\alpha\gamma K}{a+K}e^{-\delta\tau} > \delta$;

(v) the coexistence equilibrium $E^*\,(S^*, I^*, P^*)$, where

$$P^* = \frac{\beta S^* - \mu}{m},\ I^* = \frac{1}{\alpha_2 m e^{-\delta\tau}}\left(\delta - \frac{\alpha\gamma S^*}{a+S^*}e^{-\delta\tau}\right),$$

which is feasible provided

$$\frac{\mu}{\beta} < S^* < \frac{a\delta}{\alpha\gamma e^{-\delta\tau}-\delta} \quad \text{and} \quad \alpha > \frac{\delta}{\gamma e^{-\delta\tau}}. \tag{3.1}$$

Here S^* is a positive root of the equation

$$r_1 S^2 + r_2 S + r_3 = 0, \tag{3.2}$$

with

$$\begin{aligned} r_1 &= rm\alpha e^{-\delta\tau},\\ r_2 &= arm\alpha e^{-\delta\tau} - Krm\alpha e^{-\delta\tau} + K\beta\delta,\\ r_3 &= Ka\beta\delta - K\alpha\gamma\mu e^{-\delta\tau} - Karm\alpha e^{-\delta\tau}. \end{aligned}$$

Now we analyze the existence of S^*. There are two cases:

(a) If $r_3 < 0$, then Eq. (3.2) has a unique positive root:

$$S^* = \frac{-r_2 + \sqrt{r_2^2 - 4r_1r_3}}{2r_1}.$$

(b) If $r_2 < 0$, $r_3 > 0$ and $r_2^2 > 4r_1r_3$, then Eq. (3.2) has two positive roots:

$$S_1^* = \frac{-r_2 - \sqrt{r_2^2 - 4r_1r_3}}{2r_1} \text{ and } S_2^* = \frac{-r_2 + \sqrt{r_2^2 - 4r_1r_3}}{2r_1}.$$

To summarize, under the conditions given in (3.1), model (2.2) exists a unique coexistence equilibrium if $r_3 < 0$, and two coexistence equilibria provided $r_2 < 0$, $r_3 > 0$ and $r_2^2 > 4r_1r_3$ hold.

3.2. Stability of equilibria

In the subsection, we will calculate the characteristic equation of model (2.2) at each of its equilibria and analyze their local stabilities based on it.

Assume that $\bar{E}(\bar{S},\bar{I},\bar{P})$ is an equilibrium of model (2.2). Its associated characteristic equation can be described as follows:

$$\mathrm{Det}\left(\lambda E - M - e^{-\lambda\tau}N\right) = 0, \tag{3.3}$$

where E is the identity matrix of order 3, and

$$M = \begin{pmatrix} r - \frac{2r\bar{S}}{K} - \beta\bar{I} - \frac{\gamma\bar{P}}{a+\bar{S}} + \frac{\gamma\bar{S}\bar{P}}{\left(a+\bar{S}\right)^2} & -\beta\bar{S} & -\frac{\gamma\bar{S}}{a+\bar{S}} \\ \beta\bar{I} & \beta\bar{S} - m\bar{P} - \mu & -m\bar{I} \\ 0 & 0 & -\delta \end{pmatrix},$$

$$N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a\alpha\gamma\bar{P}}{\left(a+\bar{S}\right)^2}e^{-\delta\tau} & \alpha m\bar{P}e^{-\delta\tau} & \left(\frac{\alpha\gamma\bar{S}}{a+\bar{S}} + \alpha m\bar{I}\right)e^{-\delta\tau} \end{pmatrix}.$$

3.2.1. Stability of equilibrium E_1

At $E_1\,(0,0,0)$, Eq. (3.3) becomes

$$(\lambda - r)(\lambda + \mu)(\lambda + \delta) = 0.$$

Its associated three characteristic roots are

$$\lambda_1 = r > 0,\ \lambda_2 = -\mu < 0,\ \lambda_3 = -\delta < 0.$$

Thus, E_1 is always an unstable saddle.

3.2.2. Stability of equilibrium E_2

At $E_2\,(K,0,0)$, Eq. (3.3) becomes

$$(\lambda + r)(\lambda + \mu - \beta K)\left(\lambda + \delta - \frac{\alpha\gamma K}{a+K}e^{-(\delta+\lambda)\tau}\right) = 0. \tag{3.4}$$

Obviously, $\lambda_1 = -r < 0$ and $\lambda_2 = \beta K - \mu$ are two roots of Eq. (3.4), and the other roots satisfy

$$g_1(\lambda) := \lambda + \delta - \frac{\alpha\gamma K}{a+K}e^{-(\delta+\lambda)\tau} = 0.$$

If $\beta K - \mu > 0$, then $\lambda_2 = \beta K - \mu > 0$, and hence E_2 is unstable.

In what follows, we assume that $\beta K - \mu < 0$. Then $g_1(\lambda) = 0$ means that

$$\lambda + \delta = \frac{\alpha\gamma K}{a+K}e^{-(\delta+\lambda)\tau}.$$

For any solution of $g_1(\lambda) = 0$, if we have $\Re(\lambda) \geq 0$, then

$$\delta \leq |\lambda + \delta| = \left|\frac{\alpha\gamma K}{a+K}e^{(-\delta+\lambda)\tau}\right| = \frac{\alpha\gamma K}{a+K}e^{-\delta\tau}\left|e^{-\lambda\tau}\right| \leq \frac{\alpha\gamma K}{a+K}e^{-\delta\tau}.$$

Therefore, E_2 is locally asymptotically stable if and only if

$$0 < K < \min\left\{\frac{\mu}{\beta}, \frac{a\delta}{\alpha\gamma e^{-\delta\tau} - \delta}\right\}. \tag{3.5}$$

3.2.3. Stability of equilibrium E_3

At $E_3\,(S_3, I_3, 0)$, Eq. (3.3) becomes

$$\left[\lambda + \delta - \left(\frac{\alpha\gamma S_3}{a + S_3} + \alpha m I_3\right) e^{-(\delta+\lambda)\tau}\right] \left(\lambda^2 + \frac{rS_3}{K}\lambda + \beta^2 S_3 I_3\right) = 0.$$

Obviously, the equation $\lambda^2 + \frac{rS_3}{K}\lambda + \beta^2 S_3 I_3 = 0$ has two roots with negative real parts. We need only consider the equation

$$g_2(\lambda) := \lambda + \delta - \left(\frac{\alpha\gamma S_3}{a + S_3} + \alpha m I_3\right) e^{-(\delta+\lambda)\tau} = 0,$$

which can be rewritten as

$$\lambda + \delta = \left(\frac{\alpha\gamma S_3}{a + S_3} + \alpha m I_3\right) e^{-(\delta+\lambda)\tau}.$$

For any solution of $g_2(\lambda) = 0$, if we have $\Re(\lambda) \geq 0$, then

$$\delta \leq |\lambda + \delta| = \left|\left(\frac{\alpha\gamma S_3}{a + S_3} + \alpha m I_3\right) e^{-(\delta+\lambda)\tau}\right| = \left(\frac{\alpha\gamma S_3}{a + S_3} + \alpha m I_3\right) e^{-\delta\tau}.$$

Hence, E_3 is locally asymptotically stable if and only if

$$\delta > \alpha\left(\frac{\gamma S_3}{a + S_3} + m I_3\right) e^{-\delta\tau}. \tag{3.6}$$

3.2.4. Stability of equilibrium E_4

At $E_4\,(S_4, 0, P_4)$, Eq. (3.3) becomes

$$(\beta S_4 - m P_4 - \mu - \lambda)\, g_3(\lambda, \tau) = 0, \tag{3.7}$$

where

$$\begin{aligned} g_3(\lambda, \tau) :=& \lambda^2 + \left(\delta + \frac{rS_4}{K} - \frac{\gamma S_4 P_4}{(a + S_4)^2}\right)\lambda + \delta\left(\frac{rS_4}{K} - \frac{\gamma S_4 P_4}{(a + S_4)^2}\right) \\ &- \left(\lambda\delta + \frac{rS_4}{K}\delta - \frac{\gamma S_4 P_4}{(a + S_4)^2}\delta - \frac{a\alpha\gamma^2 S_4 P_4}{(a + S_4)^3}\right) e^{-\delta\tau} e^{-\lambda\tau}. \end{aligned} \tag{3.8}$$

Obviously, Eq. (3.7) has a root $\lambda_1 = \beta S_4 - m P_4 - \mu$, which is positive if $\beta S_4 > m P_4 + \mu$. In this case, E_4 is unstable.

In what follows, we assume that $\beta S_4 < m P_4 + \mu$. We need only to consider the roots of Eq. (3.8). Notice that if $\frac{r}{K} - \frac{\gamma P_4}{(a+S_4)^2} > 0$. Then $\delta\left(\frac{rS_4}{K} - \frac{\gamma S_4 P_4}{(a+S_4)^2}\right)\left(1 - e^{-\delta\tau}\right)$

$+\frac{a\alpha\gamma^2 S_4 P_4}{(a+S_4)^3}e^{-\delta\tau} > 0$, and hence $\lambda = 0$ is not a root of Eq. (3.8). Notice also that when $\tau = 0$, (3.8) becomes

$$g_3(\lambda, 0) = \lambda^2 + \left(\frac{rS_4}{K} - \frac{\gamma S_4 P_4}{(a+S_4)^2}\right)\lambda + \frac{a\alpha\gamma^2 S_4 P_4}{(a+S_4)^3}. \tag{3.9}$$

It then follows (3.9) that if $\frac{r}{K} - \frac{\gamma P_4}{(a+S_4)^2} > 0$, the equation $g_3(\lambda, 0) = 0$ has two negative real roots.

Assume that $\lambda = i\omega$ with $\omega > 0$ is a purely imaginary root of (3.8). Then we have

$$\begin{aligned} -\omega^2 + i\omega\left(\delta + \frac{rS_4}{K} - \frac{\gamma S_4 P_4}{(a+S_4)^2}\right) + \delta\left(\frac{rS_4}{K} - \frac{\gamma S_4 P_4}{(a+S_4)^2}\right) \\ - \left(i\omega\delta + \frac{rS_4}{K}\delta - \frac{\gamma S_4 P_4}{(a+S_4)^2}\delta - \frac{a\alpha\gamma^2 S_4 P_4}{(a+S_4)^3}\right)e^{-\delta\tau}e^{-i\omega\tau} = 0. \end{aligned}$$

Applying the modulus operation to both sides of the above, we obtain

$$\begin{aligned} &\omega^4 + \left[\left(\delta + \frac{rS_4}{K}\right)^2 - \left(\frac{\gamma S_4 P_4}{(a+S_4)^2}\right)^2 - \left(\delta e^{-\delta\tau}\right)^2\right]\omega^2 \\ &+ \delta^2\left(\frac{rS_4}{K} - \frac{\gamma S_4 P_4}{(a+S_4)^2}\right)^2\left(1 - \left(e^{-\delta\tau}\right)^2\right) + \left(\frac{a\alpha\gamma^2 S_4 P_4}{(a+S_4)^3}e^{-\delta\tau}\right)^2 = 0. \end{aligned}$$

Letting $\nu = \omega^2$ yields

$$\begin{aligned} &\nu^2 + \left[\left(\delta + \frac{rS_4}{K}\right)^2 - \left(\frac{\gamma S_4 P_4}{(a+S_4)^2}\right)^2 - \left(\delta e^{-\delta\tau}\right)^2\right]\nu \\ &+ \delta^2\left(\frac{rS_4}{K} - \frac{\gamma S_4 P_4}{(a+S_4)^2}\right)^2\left(1 - \left(e^{-\delta\tau}\right)^2\right) + \left(\frac{a\alpha\gamma^2 S_4 P_4}{(a+S_4)^3}e^{-\delta\tau}\right)^2 = 0. \end{aligned} \tag{3.10}$$

It is easy to check that if $\frac{r}{K} - \frac{\gamma P_4}{(a+S_4)^2} > 0$, then (3.10) exists no positive real roots. That is, there is no root $\lambda = i\omega$ with $\omega > 0$ for (3.8), which implies that the root of (3.8) cannot cross the purely imaginary axis. Therefore, all roots of (3.8) have negative real parts provided $\frac{r}{K} - \frac{\gamma P_4}{(a+S_4)^2} > 0$. As a result, E_4 is locally asymptotically stable if and only if

$$\beta S_4 < mP_4 + \mu \quad \text{and} \quad \frac{r}{K} - \frac{\gamma P_4}{(a+S_4)^2} > 0. \tag{3.11}$$

Summarizing the above, we have the following theorem.

Theorem 3.1. *For model* (2.2), *we have the following results:*

(i) The trivial equilibrium E_1 is always unstable, and it is a saddle point.

(ii) The boundary equilibrium E_2 is asymptotically stable if $0 < K < \min\left\{\frac{\mu}{\beta}, \frac{a\delta}{\alpha\gamma e^{-\delta\tau} - \delta}\right\}$.

(iii) The predator free equilibrium E_3 is asymptotically stable if and only if $\delta > \alpha\left(\frac{\gamma S_3}{a+S_3} + mI_3\right)e^{-\delta\tau}$.

(iv) The infected pest free equilibrium E_4 is asymptotically stable if and only if $\beta S_4 < mP_4 + \mu$ *and* $\frac{r}{K} - \frac{\gamma P_4}{(a+S_4)^2} > 0$.

Remark 3.1. Notice that in the absence of delay, Kar et al. [4] have obtained the corresponding stability result for each bounded equilibrium of model (2.1), which are consistent with our obtained results in Theorem 3.1 when $\tau = 0$ and $\alpha_i = \alpha, i = 1, 2$.

3.2.5. Stability of equilibrium E^*

At $E^*(S^*, I^*, P^*)$, we can easily compute that

$$M = \begin{pmatrix} M_1 & M_2 & M_3 \\ M_4 & 0 & M_5 \\ 0 & 0 & M_6 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ N_1 & N_2 & N_3 \end{pmatrix},$$

where

$$\begin{aligned} &M_1 = -\frac{rS^*}{K} + \frac{\gamma S^* P^*}{(a+S^*)^2},\ M_2 = -\beta S^*,\ M_3 = -\frac{\gamma S^*}{a+S^*}, \\ &M_4 = \beta I^*,\ M_5 = -mI^*,\ M_6 = -\delta, \\ &N_1 = \frac{a\alpha_1\gamma P^*}{(a+S^*)^2}e^{-\delta\tau},\ N_2 = \alpha_2 mP^* e^{-\delta\tau},\ N_3 = \left(\frac{\alpha_1\gamma S^*}{a+S^*} + \alpha_2 mI^*\right)e^{-\delta\tau}. \end{aligned}$$

The associated characteristic equation of model (2.2) at $E^*(S^*, I^*, P^*)$ is

$$\Gamma(\lambda,\tau) := E(\lambda,\tau) + F(\lambda,\tau)e^{-\lambda\tau} = 0, \tag{3.12}$$

where

$$\begin{aligned} E(\lambda,\tau) &= \lambda^3 + A_2(\tau)\lambda^2 + A_1(\tau)\lambda + A_0(\tau), \\ F(\lambda,\tau) &= B_2(\tau)\lambda^2 + B_1(\tau)\lambda + B_0(\tau), \end{aligned}$$

and

$$\begin{aligned} &A_2(\tau) = \frac{rS^*}{K} - \frac{\gamma S^* P^*}{(a+S^*)^2}, \quad A_1(\tau) = \left(\frac{rS^*}{K} - \frac{\gamma S^* P^*}{(a+S^*)^2}\right)\delta + \beta^2 S^* I^*, \\ &A_0(\tau) = \delta\beta^2 S^* I^*, \quad B_2(\tau) = \delta - \alpha\left(\frac{\gamma S^*}{a+S^*} + mI^*\right)e^{-\delta\tau}, \\ &B_1(\tau) = \alpha e^{-\delta\tau}\left(m^2 I^* P^* + \frac{a\gamma^2 S^* P^*}{(a+S^*)^3}\right) - \left(\frac{rS^*}{K} - \frac{\gamma S^* P^*}{(a+S^*)^2}\right)\delta, \\ &B_0(\tau) = \alpha e^{-\delta\tau} S^* I^* P^* \left[m^2\left(\frac{r}{K} - \frac{\gamma P^*}{(a+S^*)^2}\right) + \frac{m\beta\gamma}{(a+S^*)^2}S^*\right] - \delta\beta^2 S^* I^*. \end{aligned}$$

When $\tau = 0$, Eq. (3.12) becomes

$$\lambda^3 + C_1\lambda^2 + C_2\lambda + C_3 = 0, \tag{3.13}$$

where

$$C_1 = A_2(0) + B_2(0) = \left(\frac{r}{K} - \frac{\gamma P^*}{(a+S^*)^2}\right) S^*,$$
$$C_2 = A_1(0) + B_1(0) = \alpha m^2 I^* P^* + \beta^2 S^* I^* + \frac{a\alpha\gamma^2 S^* P^*}{(a+S^*)^3},$$
$$C_3 = A_0(0) + B_0(0) = \alpha S^* I^* P^* \left[m^2\left(\frac{r}{K} - \frac{\gamma P^*}{(a+S^*)^2}\right) + \frac{m\beta\gamma}{(a+S^*)^2} S^*\right].$$

Obviously, if $\frac{r}{K} > \frac{\gamma P^*}{(a+S^*)^2}$, then $C_i > 0$, $i = 1, 2, 3$. Moreover, we can compute that

$$C_1C_2 - C_3 = S^{*2}\left[\left(\frac{r}{K} - \frac{\gamma P^*}{(a+S^*)^2}\right)\left(\beta^2 I^* + \frac{a\alpha\gamma^2 P^*}{(a+S^*)^3}\right) - \frac{m\alpha\beta\gamma}{(a+S^*)^2} I^* P^*\right],$$

which is positive if and only if

$$\frac{r}{K} > \frac{m\alpha\beta\gamma I^* P^* (a+S^*)^2}{\beta^2 I^* (a+S^*)^3 + a\alpha\gamma^2 P^*} + \frac{\gamma P^*}{(a+S^*)^2}. \tag{3.14}$$

According to the Routh-Hurwitz criterion, we know that all the roots of Eq. (3.13) have negative real parts provided (3.14) holds. Thus, we can have the following result.

Theorem 3.2. *When $\tau = 0$, the coexistence equilibrium E^* is asymptotically stable if and only if* (3.14) *holds.*

Notice that if the stability of E^* changes, then Eq. (3.12) must have purely imaginary roots at some value of τ, which is one of its necessary conditions. Since E^* is the function of delay τ, both $E(\lambda, \tau)$ and $F(\lambda, \tau)$ are dependent on τ. That is, the coefficients of equation (3.12) depend on τ. In the following, we will adopt the approach proposed by Berreta and Kuang [33] to prove the existence of purely imaginary roots of characteristic equation (3.12).

Assume that $[0, \tau_{\max})$ is the maximum interval of delay τ where E^* exists. For $\tau \in [0, \tau_{\max})$, we make the following assumptions:

(a) $E(0,\tau) + F(0,\tau) = A_0(\tau) + B_0(\tau) \neq 0$.

(b) $E(i\omega,\tau) + F(i\omega,\tau) = A_0(\tau) + B_0(\tau) - \omega^2(A_2(\tau) + B_2(\tau))$
$+ i\omega\left(A_1(\tau) + B_1(\tau) - \omega^2\right) \neq 0$.

(c) $\lim_{|\lambda|\to\infty}\left|\frac{F(\lambda,\tau)}{E(\lambda,\tau)}\right| = \lim_{|\lambda|\to\infty}\frac{B_2(\tau)\lambda^2 + B_1(\tau)\lambda + B_0(\tau)}{\lambda^3 + A_2(\tau)\lambda^2 + A_1(\tau)\lambda + A_0(\tau)} = 0 < 1$.

(d) $G(\omega,\tau) = |E(i\omega,\tau)|^2 - |F(i\omega,\tau)|^2$ has at most finite positive solutions.

(e) Any positive solution of $G(\omega,\tau) = 0$ is continuously differentiable in τ.

Assume that $\lambda(\tau) = i\beta(\tau)$, $\beta(\tau) > 0$, is a characteristic root of Eq. (3.12). By substituting it in (3.12) and separating the real and imaginary parts, we obtain

$$\begin{cases} A_2\beta^2 - A_0 = \left(-B_2\beta^2 + B_0\right)\cos(\beta\tau) + B_1\beta\sin(\beta\tau), \\ \beta^3 - A_1\beta = B_1\beta\cos(\beta\tau) - \left(-B_2\beta^2 + B_0\right)\sin(\beta\tau), \end{cases} \tag{3.15}$$

which gives

$$\begin{cases} \sin(\beta\tau) = \dfrac{B_2\beta^5 + (A_2B_1 - A_1B_2)\beta^3 + (A_1B_0 - A_0B_1)\beta}{B_2^2\beta^4 + (B_1^2 - 2B_0B_2)\beta^2 + B_0^2}, \\ \cos(\beta\tau) = \dfrac{(B_1 - A_2B_2)\beta^4 + (A_2B_0 - A_1B_1 + A_0B_2)\beta^2 - A_0B_0}{B_2^2\beta^4 + (B_1^2 - 2B_0B_2)\beta^2 + B_0^2}. \end{cases} \tag{3.16}$$

Squaring and adding both sides of the two equations in (3.15), we obtain

$$\beta^6 + q_2\beta^4 + q_1\beta^2 + q_0 = 0, \tag{3.17}$$

where

$$q_2 = A_2^2 - 2A_1 - B_2^2,\ q_1 = A_1^2 - 2A_0A_2 + 2B_0B_2 - B_1^2,\ q_0 = A_0^2 - B_0^2.$$

If we put $v = \beta^2$, then Eq. (3.17) becomes

$$R(v) := v^3 + q_2v^2 + q_1v + q_0 = 0. \tag{3.18}$$

Notice that

$$R(0) = q_0,\ \lim_{v\to\infty} R(v) = \infty \ \text{ and } \ R'(v) = 3v^2 + 2q_2v + q_1.$$

If $q_0 \geq 0$ and $\Delta = 4(q_2^2 - 3q_1) > 0$, $R'(v) = 0$ has two real solutions:

$$v_+ = \frac{-q_2 + \sqrt{q_2^2 - 3q_1}}{3},\quad v_- = \frac{-q_2 - \sqrt{q_2^2 - 3q_1}}{3}. \tag{3.19}$$

Moreover,

$$R''(v_+) = 2\sqrt{\Delta} > 0,\ R''(v_-) = -2\sqrt{\Delta} < 0.$$

Obviously, v_+ and v_- are respectively the local minimum point and maximum point of $R(v)$. Song et al. [34] have performed a detailed analysis on the solutions of (3.18) and obtained the following lemma.

Lemma 3.1. *For the cubic polynomial equation* (3.18), *it has*

(i) at least one positive solution if $q_0 < 0$*;*

(ii) no positive solutions if $q_0 \geq 0$ *and* $\Delta = 4\left(q_2^2 - 3q_1\right) \leq 0$*;*

(iii) positive solutions if $q_0 \geq 0$, $\Delta = 4\left(q_2^2 - 3q_1\right) > 0$, $v_+ > 0$, *and* $R(v_+) \leq 0$.

In what follows, without loss of generality, the coefficients in $R(v)$ are assumed to satisfy the following condition.

(H1) $q_0 < 0$ or $q_0 \geq 0,\ \Delta > 0,\ v_+ > 0$ and $R(v_+) < 0$.

Notice that β can be expressed by Eq. (3.17) as an implicit function in terms of τ. The solution could cross the imaginary axis a finite number of times since Eq. (3.17) exists at most a finite number of real positive solutions for each τ.

Let $\Lambda = \{\tau : \tau > 0$ and $\beta(\tau)$ is a positive solution of (3.17)$\}$. Then, when $\tau \in \Lambda^C := [0, \tau_{\max})\backslash\Lambda$, the stability will not change because there is no positive solution for Eq. (3.17) (Chakraborty et al. [35]).

On the other hand, when $\tau \in \Lambda$, we define the angle $\varphi(\tau) \in (0, 2\pi)$ for the solution of (3.17). We write the following

$$\begin{cases} \sin(\varphi(\tau)) = \dfrac{B_2\beta^5 + (A_2B_1 - A_1B_2)\beta^3 + (A_1B_0 - A_0B_1)\beta}{B_2^2\beta^4 + (B_1^2 - 2B_0B_2)\beta^2 + B_0^2} = \dfrac{\xi_1}{|F(i\beta,\tau)|^2}, \\ \cos(\varphi(\tau)) = \dfrac{(B_1 - A_2B_2)\beta^4 + (A_2B_0 - A_1B_1 + A_0B_2)\beta^2 - A_0B_0}{B_2^2\beta^4 + (B_1^2 - 2B_0B_2)\beta^2 + B_0^2} = \dfrac{\xi_2}{|F(i\beta,\tau)|^2}, \end{cases} \tag{3.20}$$

where ξ_1 and ξ_2 are continuously differentiable functions of τ such that $\xi_1^2 + \xi_2^2 = |F(i\beta,\tau)|^4$ and $|F(i\beta,\tau)|^2 = |E(i\beta,\tau)|^2$. Substituting $\beta = \beta(\tau)$ in (3.20), $\varphi(\tau) \in (0, 2\pi)$ can be expressed as

$$\varphi(\tau) = \begin{cases} \arctan\left(-\dfrac{\xi_1}{\xi_2}\right), & \text{if } \sin(\varphi(\tau)) > 0,\ \cos(\varphi(\tau)) > 0; \\ \dfrac{\pi}{2}, & \text{if } \sin(\varphi(\tau)) = 1,\ \cos(\varphi(\tau)) = 0; \\ \pi + \arctan\left(-\dfrac{\xi_1}{\xi_2}\right), & \text{if } \cos(\varphi(\tau)) < 0; \\ \dfrac{3\pi}{2}, & \text{if } \sin(\varphi(\tau)) = -1,\ \cos(\varphi(\tau)) = 0; \\ 2\pi + \arctan\left(-\dfrac{\xi_1}{\xi_2}\right), & \text{if } \sin(\varphi(\tau)) < 0. \end{cases}$$

When $\tau \in \Lambda$, $\varphi(\tau)$ is continuous at τ. Besides, when $\varphi(\tau) \in (0, 2\pi), \tau \in \Lambda$, $\varphi(\tau)$ is also differentiable at τ. Noticing equation (3.20) and "$\varphi(\tau)$" defined above for $\tau \in \Lambda$, we obtain

$$\beta(\tau)\tau = \varphi(\tau) + 2m\pi,\ m \in \mathbb{N}_0.$$

Let $\tau_m : \Lambda \to \mathbb{R}_0^+$ be the maps

$$\tau_m(\tau) = \frac{1}{\beta(\tau)}(\varphi(\tau) + 2m\pi), m \in \mathbb{N}_0, \tau \in \Lambda,$$

where $\beta(\tau)$ is a positive solution of (3.17). We further introduce the function $L_m : \Lambda \to \mathbb{R}$ by

$$L_m(\tau) = \tau - \tau_m(\tau),\ \tau \in \Lambda,\ m \in \mathbb{N}_0,$$

which is continuously differentiable at τ. For the numbers of $\tau \in \Lambda$, $L_m(\tau) = 0$, $m \in \mathbb{N}_0$, stability may change if the transversality condition holds [35]. We introduce the following theorem from Berreta and Kuang [33].

Theorem 3.3. *Let $\beta(\tau)$ be the positive real solution of equation* (3.12) *for $\tau \in \Lambda \subseteq \mathbb{R}_0^+$, and for some $\tau^* \in \Lambda$,*

$$L_m(\tau^*) = 0,\ \textit{for some } m \in \mathbb{N}_0.$$

At $\tau = \tau^*$, the pair of conjugate pure imaginary solutions (i.e. $\pm i\beta(\tau^*)$) of equation (3.12) *crosses the imaginary axis from left to right (right to left) if $\chi(\tau^*) > 0$ ($\chi(\tau^*) < 0$), where*

$$\chi(\tau^*) = \text{sign}\left\{R'_\beta(\beta(\tau^*), \tau^*)\right\} \text{sign}\left\{\left.\frac{\mathrm{d}L_m(\tau)}{\mathrm{d}\tau}\right|_{\tau=\tau^*}\right\},$$

where $R'_\beta(\beta(\tau^*), \tau^*) := \partial_\beta R(\beta(\tau^*), \tau^*)$ *is the partial derivatives with respect to* β. *Furthermore, assume that* (3.14) *holds and the positive equilibrium exists. The coexistence equilibrium* E^* *is asymptotically stable for* $0 \le \tau \le \tau^*$ *and unstable for* $\tau \ge \tau^*$. *If* $\chi(\tau^*) \ne 0$, *the model* (2.2) *will occur a Hopf bifurcation around the coexistence equilibrium* E^*.

4. Direction and stability of Hopf-bifurcation

In the preceding section, we demonstrated that model (2.2) undergoes a Hopf bifurcation at E^* at $\tau = \tau^*$. Subsequently, we will employ the normal method of Hassard [36] and the central manifold theory to formulate the expressions for ascertaining the characteristic of the Hopf bifurcation at $\tau = \tau^*$.

The details of the derivations of quantities are given in Appendix A. Consequently, g_{11}, g_{20}, g_{02} and g_{21} can be computed. Consequently, the following values can be straightforwardly computed as:

$$\begin{aligned} c_1(0) &= \frac{i}{2\omega^*\tau^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\Re\{c_1(0)\}}{\Re\{\lambda'(\tau^*)\}}, \\ \beta_2 &= 2\Re\{c_1(0)\}, \\ T_2 &= -\frac{\Im\{c_1(0)\} + \mu_2\Im\{\lambda'(\tau^*)\}}{\omega^*\tau^*}. \end{aligned}$$

Theorem 4.1. *For model* (2.2), *the Hopf bifurcation is supercritical (resp. subcritical) if* $\mu_2 > 0$ *(resp.* $\mu_2 < 0$*). The periodic solutions are stable (resp. unstable) if* $\beta_2 > 0$ *(resp.* $\beta_2 < 0$*). The period increases (resp. decreases) if* $T_2 > 0$ *(resp.* $T_2 < 0$*).*

5. Numerical simulation

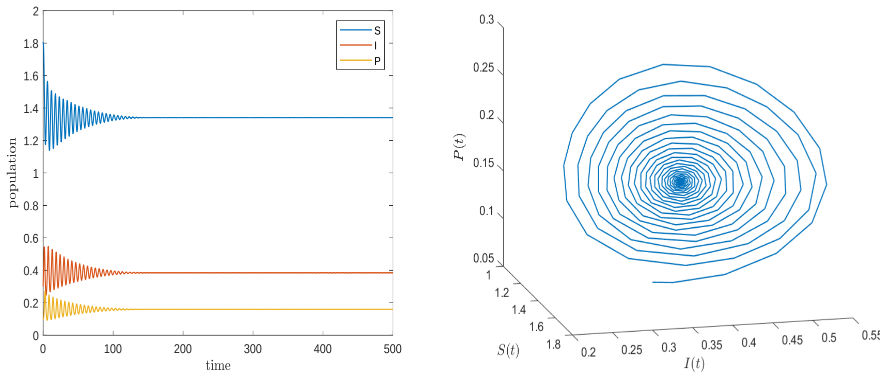
In the following, numerical simulations are performed to certify the accuracy of the previous conclusion and to further understand the behavior of the system.

5.1. Non-delayed system

To verify the properties of model (2.1), let the parameter values be:

$$\begin{aligned} &r = 1, \quad K = 3, \quad \beta = 0.55, \quad \gamma = 3, \quad a = 1, \\ &m = 4, \quad \mu = 0.1, \quad \alpha = 0.9, \quad \delta = 3. \end{aligned} \tag{5.1}$$

For the above set of parameters (5.1), by verifying that the conditions for the existence of interior equilibrium point hold and the interior equilibrium point can be calculated as $E^*(1.70808, 0.36028, 0.20986)$. As shown in Fig. 2, the interior equilibrium E^* of the model (2.1) is asymptotically stable. This figure further shows that the population densities of predator and pest stabilize at their respective equilibrium levels after fluctuating for a small period of time.



(a) (b)

Figure 2. Local asymptotic stability of model (2.1)

5.2. Delayed system

In the following, we select the following parameter values to verify the impact of gestation delay on model (2.2):

$$\begin{aligned} &r = 1, \quad K = 3, \quad \beta = 0.5, \quad \gamma = 1, \quad a = 1, \\ &m = 2, \quad \mu = 0.1, \quad \alpha = 0.9, \quad \delta = 1.5. \end{aligned} \tag{5.2}$$

For the above set of parameters (5.2), the delay-dependent interior equilibrium point exists. Further, by verifying that the transversality condition (3.14) also holds, so that the system will exhibit the Hopf bifurcation with respect to τ. By simple calculation, we obtain $\tau^* = 0.097$ and $\tau^{**} = 0.359$.

When $\tau = 0.08$, we can get the interior equilibrium $E^*(1.64729, 0.62845, 0.36182)$. When the gestation delay $\tau < \tau^*$, the interior equilibrium E^* is asymptotically stable(see Fig. 3). When $\tau = 0.25$, we can also obtain the interior equilibrium $E^*(1.24799, 0.93491, 0.26199)$. However, the interior equilibrium E^* becomes unstable once τ passes through τ^*, and the system (2.2) undergoes a Hopf-bifurcation by Theorem 3.3 (see Fig. 4).

It is obvious that system (2.2) undergoes a transition from stability to instability near the interior equilibrium point E^* at $\tau = \tau^* = 0.097$. When $\tau < \tau^*$, the predator biomass and the susceptible pest biomass decreases as the gestation delay increases, while the infected pest biomass increases as the gestation delay increases (see Fig. 5). This phenomenon might potentially be ascribed to a decrease in the predator population, which subsequently causes an increase in the biomass of infected pests, thereby enabling the infection of more susceptible pests by the infected ones.

By the algorithm given in Sect. 4, we get

$$c_1(0) = -0.0596 - 0.1602i, \ \mu_2 = 1.1509 > 0, \ \beta_2 = -0.1193 < 0, \ T_2 = 1.9238 > 0.$$

The Hopf bifurcation is supercritical and the bifurcated periodic solution is stable.

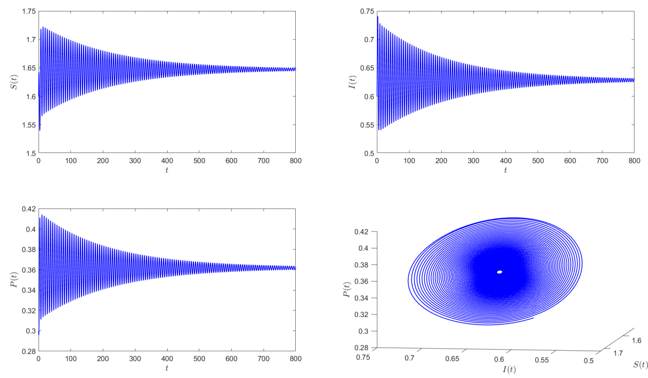


Figure 3. E^* is asymptotically stable when $\tau = 0.08 < \tau^*$

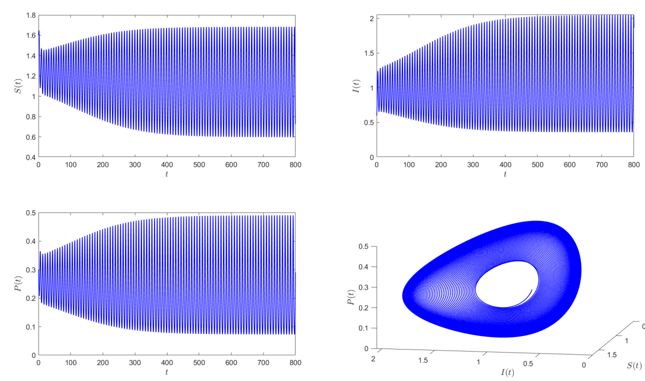


Figure 4. E^* is unstable when $\tau^* < \tau = 0.25 < \tau^{**}$

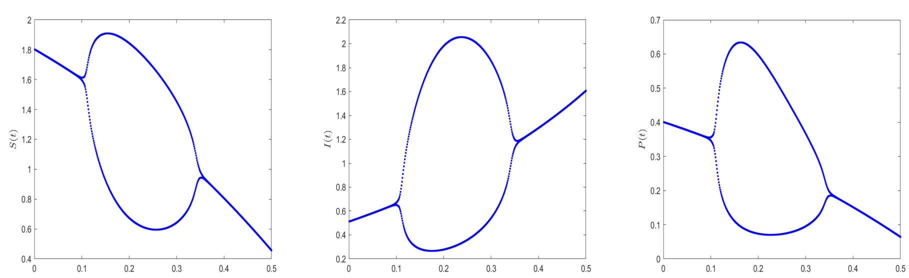


Figure 5. Bifurcation diagram of S, I and P

6. Discussion

Studies have shown that infectious diseases are one of the main reasons for the death of pest population and play a leading role in the extinction of pests in more severe cases. However, research in this aspect is not yet complete. Eco-epidemiology dynamic model is a comparatively significant tool in population dynamics, which reveals complex mechanisms among species. In this paper, we take into account a predator-pest eco-epidemiological system with gestation delay. Firstly, we have attained the nature of the solutions, which are prerequisites to ensure that the system is biologically meaningful. Next, we have analyzed the existence and stability of all the equilibriums. Further, we explore the existence of the Hopf bifurcation of E^*. Moreover, the stability, the direction and the periodic solution of Hopf bifurcation are determined.

Numerically, we verify the above analytical results. It is found that gestation delay changes the stability behavior of system (2.2) into instability, and thereafter a Hopf bifurcation occurs. This suggests that gestation delay plays an important role in maintaining the stability of pest and predator population. Specifically, it can be seen from Fig. 3 that when gestation period is below the critical threshold τ^*, the coexistence equilibrium E^* is stable. However, when gestation period exceeds the critical threshold τ^*, the phenomenon of population fluctuation occurs (see Fig. 4). This phenomenon means that the spread of epidemics among pests is periodic. It is also revealed that repeated outbreak of epidemics might be caused by interactions between diseased pest and predator.

In addition, we investigate the effects of different predator gestation period τ on population biomass. Numerical simulations show that when $\tau < \tau^*$, the predator biomass decreases with the increase of gestation period since a longer gestation period decelerates the growth of predator population (see Fig. 5). When gestation period τ increases, the infected pest biomass increases, while the susceptible pest biomass decreases(see Fig. 5). This phenomenon could potentially be attributed to a decline in the predator population, which subsequently leads to an escalation in the biomass of infected pests, thereby facilitating the infection of more susceptible pests by the infected ones. Hence it is an extremely effective approach for pest control to combine using infection to target pest and using biological predator population with small gestation period. Our studies findings provide significant ecological insights into the pest control in agriculture and ecosystem management.

Future, making some research on the existence of Turing instability in a diffusive eco-epidemiological system will be meaningful. Exploring the above questions is both interesting and challenging, and we will investigate them further as open problems.

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A. Computation of the Coefficients μ_2, β_2, T_2

For convenience, let

$$S_1(t) = S(t) - S^*,\ I_1(t) = I(t) - I^*,\ P_1(t) = P(t) - P^*,$$

and still denote $S_1(t),\ I_1(t),\ P_1(t)$ by $S(t),\ I(t),\ P(t)$. Let $\tau = \tau^* + \mu,\ \mu \in \mathbb{R}$. Then $\mu = 0$ is the Hopf bifurcation value of model (2.2) at the equilibrium E^*, further normalize the delay with scaling $t \mapsto \left(\frac{t}{\tau}\right)$. Drop the bar for simplicity of notation, then model (2.2) is transformed into a functional differential equation in $\mathbb{C} = \mathbb{C}\left([-1,0], \mathbb{R}^3\right)$ as

$$\dot{U}(t) = L_\mu U_t + f(\mu, U_t), \tag{A.1}$$

where $U(t) = (S_1(t), I_1(t), P_1(t))^T \in \mathbb{R}^3$, and $U_t = U_t(\theta) = U(t+\theta) = (S_1(t+\theta), I_1(t+\theta), P_1(t+\theta))^T \in \mathbb{C}$, and $L_\mu : \mathbb{C} \to \mathbb{R}^3$, $f : \mathbb{R} \times \mathbb{C} \to \mathbb{R}^3$ are given respectively by

$$L_\mu(\phi) = (\tau^* + \mu)\left[M\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\\ \phi_3(0)\end{pmatrix} + N\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\\ \phi_3(-1)\end{pmatrix}\right],$$

and

$$f(\mu, \phi) = (\tau^* + \mu)\begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix},$$

where

$$\begin{aligned} f_1 =& l_1\phi_1^2(0) + l_2\phi_1(0)\phi_2(0) + l_3\phi_1(0)\phi_3(0) + l_4\phi_1^3(0) + l_5\phi_1^2(0)\phi_3(0) + \cdots, \\ f_2 =& m_1\phi_1(0)\phi_2(0) + m_2\phi_2(0)\phi_3(0) + \cdots, \\ f_3 =& n_1\phi_1^2(-1) + n_2\phi_1(-1)\phi_3(-1) + n_3\phi_2(-1)\phi_3(-1) \\ & + n_4\phi_1^3(-1) + n_5\phi_1^2(-1)\phi_3(-1) + \cdots, \end{aligned}$$

and

$$\begin{aligned} l_1 &= -\frac{r}{K} + \frac{\gamma P^*(a - S^*)}{(a+S^*)^3}, & l_2 &= -\beta, & l_3 &= -\frac{a\gamma}{(a+S^*)^2}, \\ l_4 &= \frac{2\gamma P^*(S^* - 2a)}{(a+S^*)^4}, & l_5 &= \frac{\gamma(a - S^*)}{(a+S^*)^3} & m_1 &= \beta, \\ m_2 &= -m, & n_1 &= -\frac{2\alpha\gamma P^*}{(a+S^*)^3}e^{-\delta\tau}, & n_2 &= \frac{\alpha\gamma}{(a+S^*)^2}e^{-\delta\tau}, \end{aligned}$$

$$n_3 = m\alpha e^{-\delta\tau}, \qquad n_4 = \frac{6\alpha\gamma P^*}{(a+S^*)^4} e^{-\delta\tau}, \qquad n_5 = -\frac{2\alpha\gamma}{(a+S^*)^3} e^{-\delta\tau}.$$

By the Riesz representation theorem, there exists a 3×3 matrix function $\rho(\theta, \mu)$, $(-1 \leq \theta \leq 0)$ whose components are of bounded variation function such that

$$L_\mu(\phi) = \int_{-1}^{0} \mathrm{d}\rho(\theta, \mu)\phi(\theta) \text{ for } \phi \in \mathbb{C}\left([-1, 0], \mathbb{R}^3\right). \tag{A.2}$$

In fact, we can choose

$$\begin{aligned} \rho(\theta, \mu) = {} & (\tau^* + \mu) \begin{pmatrix} -\frac{rS^*}{K} + \frac{\gamma S^* P^*}{(a+S^*)^2} & -\beta S^* & -\frac{\gamma S^*}{a+S^*} \\ \beta I^* & 0 & -mI^* \\ 0 & 0 & -\delta \end{pmatrix} \delta(\theta) \\ & - (\tau^* + \mu) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{a\alpha\gamma P^*}{(a+S^*)^2} e^{-\delta\tau} & \alpha m P^* e^{-\delta\tau} & \left(\frac{\alpha\gamma S^*}{a+S^*} + \alpha m I^*\right) e^{-\delta\tau} \end{pmatrix} \delta(\theta + 1), \end{aligned} \tag{A.3}$$

where δ is the Dirac delta function.

Next, for $\phi \in \mathbb{C}^1\left([-1, 0], \mathbb{R}^3\right)$, we define the operator $A(\mu)$ as

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{\mathrm{d}\phi(\theta)}{\mathrm{d}\theta}, & \text{if } \theta \in [-1, 0), \\ \displaystyle\int_{-1}^{0} \mathrm{d}\rho(s, \mu)\phi(s) = L_\mu(\phi), & \text{if } \theta = 0, \end{cases} \tag{A.4}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \text{if } \theta \in [-1, 0), \\ f(\mu, \phi), & \text{if } \theta = 0. \end{cases} \tag{A.5}$$

Since $\frac{\mathrm{d}U_t}{\mathrm{d}\theta} = \frac{\mathrm{d}U_t}{\mathrm{d}t}$, then (A.1) is equivalent to the following operator equation

$$\dot{U}_t = A(\mu)U_t + R(\mu)U_t. \tag{A.6}$$

For $\psi \in \mathbb{C}^1\left([-1, 0], \left(\mathbb{R}^3\right)^*\right)$, define

$$A^*(\mu)\psi(s) = \begin{cases} -\dfrac{\mathrm{d}\psi(s)}{\mathrm{d}s}, & \text{if } s \in [-1, 0), \\ \displaystyle\int_{-1}^{0} \mathrm{d}\rho^T(t, 0)\psi(-t), & \text{if } s = 0, \end{cases} \tag{A.7}$$

and we also define a bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0) \cdot \phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta) \mathrm{d}\rho(\theta)\phi(\xi) \mathrm{d}\xi, \tag{A.8}$$

where $\rho(\theta) = \rho(\theta, 0)$. Here, for a and b in $\mathbb{C}^n$, $a \cdot b$ means $\sum_{i=1}^{n} a_i b_i$, where a_i and b_i are the components of the vectors a and b, respectively. Then $A(0)$ and $A^*(0)$ are adjoint operators. Furthermore, $\langle \psi, A\phi \rangle = \langle A^*\psi, \phi \rangle$.

Note that the above scaling transformation, the corresponding characteristic exponents and the associated frequencies are transformed into $\tau\lambda$ and $\tau\omega$, respectively. Hence, when $\mu = 0$, $\pm i\omega^*\tau^*$ are the eigenvalues of $A(0)$ and therefore they are also eigenvalues of $A^*(0)$. Let $q(\theta)$ be the eigenvector for $A(0)$ corresponding to $i\omega^*\tau^*$ and $q^*(\theta)$ be the eigenvector for $A^*(0)$ corresponding to $-i\omega^*\tau^*$. Then we have

$$A(0)q(\theta) = i\omega^*\tau^* q(\theta), \tag{A.9}$$

$$A^*(0)q^*(s) = -i\omega^*\tau^* q^*(s). \tag{A.10}$$

From (A.4), we can rewrite (A.9) as follows

$$\begin{cases} \dfrac{\mathrm{d}q(\theta)}{\mathrm{d}\theta} = i\omega^*\tau^* q(\theta), & \text{if } \theta \in [-1, 0), \\ L_0 q(0) = i\omega^*\tau^* q(\theta), & \text{if } \theta = 0. \end{cases} \tag{A.11}$$

Using (A.11), we have

$$q(\theta) = V e^{i\omega^*\tau^*\theta}, \ \theta \in [-1, 0], \tag{A.12}$$

where $V = (v_1, v_2, v_3)^T$ is an undetermined constant vector, and from (A.11), the constant vector V must satisfy

$$\left(M + N e^{-i\omega^*\tau^*} - i\omega^* I\right) V = 0,$$

i.e.,

$$\begin{pmatrix} M_1 - i\omega^* & M_2 & M_3 \\ M_4 & -i\omega^* & M_5 \\ N_1 e^{-i\omega^*\tau^*} & N_2 e^{-i\omega^*\tau^*} & M_6 + N_3 e^{-i\omega^*\tau^*} - i\omega^* \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = 0,$$

where I denotes the 3×3 identity matrix, and the above algebraic equation has an infinite number of solutions. Without loss of generality, setting $v_1 = 1$, we have

$$v_2 = \frac{M_3 M_4 - M_1 M_5 + i\omega^* M_5}{M_2 M_5 + i\omega^* M_3},$$
$$v_3 = \frac{N_1 (M_2 M_5 + i\omega^* M_3) + N_2 (M_3 M_4 - M_1 M_5 + i\omega^* M_5)}{(M_2 M_5 + i\omega^* M_3)(i\omega^* - N_3 e^{-i\omega^*\tau^*} - M_6)}.$$

Similarly, from (A.7), we rewrite (A.10) as follows

$$\begin{cases} \dfrac{\mathrm{d}q^*(s)}{\mathrm{d}s} = i\omega^*\tau^* q^*(s), & \text{if } s \in (0, 1], \\ \displaystyle\int_{-1}^{0} \mathrm{d}\rho^T(t, 0)\varphi(-t) = \tau^* M^T \varphi(0) + \tau^* N^T \varphi(1) = -i\omega^*\tau^* q(0), & \text{if } s = 0. \end{cases} \tag{A.13}$$

Using (A.13), we have

$$q^*(s) = P V^* e^{i\omega^*\tau^* s}, \ s \in [0, 1], \tag{A.14}$$

where P and $V^* = (v_1^*, v_2^*, v_3^*)$ are an undetermined constant and constant vector respectively, and from (A.13), the constant vector V^* must satisfy

$$\left(M^T + N^T e^{i\omega^*\tau^*} + i\omega^* I\right) V^* = 0,$$

i.e.,

$$\begin{pmatrix} M_1 + i\omega^* & M_4 & N_1 e^{-i\omega^*\tau^*} \\ M_2 & i\omega^* & N_2 e^{-i\omega^*\tau^*} \\ M_3 & M_5 & M_6 + N_3 e^{-i\omega^*\tau^*} + i\omega^* \end{pmatrix} \begin{pmatrix} v_1^* \\ v_2^* \\ v_3^* \end{pmatrix} = 0,$$

where I denotes the 3×3 identity matrix, and the above algebraic equation has an infinite number of solutions. Without loss of generality, setting $v_1^* = 1$, we have

$$\begin{aligned} v_2^* &= \frac{M_2 N_1 - M_1 N_2 - i\omega^* N_2}{M_4 N_2 + i\omega^* N_1}, \\ v_3^* &= \frac{M_3 \left(M_4 N_2 - i\omega^* N_1\right) + M_5 \left(M_2 N_1 - M_1 N_2 - i\omega^* N_2\right)}{\left(i\omega^* N_1 - M_4 N_2\right)\left(i\omega^* + N_3 e^{-i\omega^*\tau^*} + M_6\right)}. \end{aligned}$$

From (A.8), we have

$$\begin{aligned} \langle q^*(s), q(\theta)\rangle =& \bar{q}^*(0)q(0) - \int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} \bar{q}^*(\xi-\theta)\,\mathrm{d}\rho(\theta)\,q(\xi)\,\mathrm{d}\xi \\ =& \bar{P}\left(1, \bar{v}_2^*, \bar{v}_3^*\right)\left(1, v_2, v_3\right)^T \\ &- \bar{P}\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\left(1, \bar{v}_2^*, \bar{v}_3^*\right) e^{-i\omega^*\tau^*(\xi-\theta)}\mathrm{d}\rho(\theta)\left(1, v_2, v_3\right)^T e^{i\omega^*\tau^*\xi}\mathrm{d}\xi \\ =& \bar{P}\left(1, \bar{v}_2^*, \bar{v}_3^*\right)\left(1, v_2, v_3\right)^T - \bar{q}^{*T}(0)\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta} e^{i\omega^*\tau^*\theta}\mathrm{d}\xi\mathrm{d}\rho(\theta)\,q(0) \\ =& \bar{P}\left(1, \bar{v}_2^*, \bar{v}_3^*\right)\left(1, v_2, v_3\right)^T - \bar{q}^{*T}(0)\int_{\theta=-1}^{0} \xi e^{i\omega^*\tau^*\theta}\Big|_{\xi=0}^{\theta}\mathrm{d}\rho(\theta)\,q(0) \\ =& \bar{P}\left(1, \bar{v}_2^*, \bar{v}_3^*\right)\left(1, v_2, v_3\right)^T - \bar{q}^{*T}(0)\int_{\theta=-1}^{0} \theta e^{i\omega^*\tau^*\theta}\mathrm{d}\rho(\theta)q(0) \\ =& \bar{P}(1 + v_2\bar{v}_2^* + v_3\bar{v}_3^*) + \bar{q}^{*T}(0)\tau^*\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ N_1 & N_2 & N_3 \end{pmatrix} e^{-i\omega^*\tau^*}q(0) \\ =& \bar{P}\left[1 + v_2\bar{v}_2^* + v_3\bar{v}_3^* + \tau^*\bar{v}_3^* e^{-i\omega^*\tau^*}\left(N_1 + v_2 N_2 + v_3 N_3\right)\right] \\ =& \bar{P}\left[1 + v_2\bar{v}_2^* + v_3\bar{v}_3^*\right. \\ &\left. + \tau^*\bar{v}_3^* e^{-i\omega^*\tau^*}\left(\frac{a\alpha\gamma P^*}{(a+S^*)^2}e^{-\delta\tau} + v_2\alpha m P^* e^{-\delta\tau} + v_3\delta\right)\right]. \end{aligned}$$

Next, one can choose P as

$$\bar{P} = \frac{1}{1 + v_2\bar{v}_2^* + v_3\bar{v}_3^* + \tau^*\bar{v}_3^* e^{-i\omega^*\tau^*}\left(\frac{a\alpha\gamma P^*}{(a+S^*)^2}e^{-\delta\tau} + v_2\alpha m P^* e^{-\delta\tau} + v_3\delta\right)}, \quad \text{(A.15)}$$

to normalize $q(\theta)$ and $q^*(s)$ by the condition $\langle q^*(s), q(\theta)\rangle = 1$. Furthermore, $\langle q^*(s), \bar{q}(\theta)\rangle = 0$.

Using the same notations as in Hassard et al. [36], we first compute the coordinates to describe the center manifold C_0 at $\mu = 0$. Let X_t be the solution of (A.1) when $\mu = 0$. Define

$$\begin{aligned} z(t) &= \langle q^*(s), X_t\rangle, \\ W(t,\theta) &= X_t(\theta) - z(t)\,q(\theta) - \bar{z}(t)\,\bar{q}(\theta) \\ &= X_t(\theta) - 2Re\{z(t)\,q(\theta)\}. \end{aligned} \tag{A.16}$$

On the center manifold C_0 we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta), \tag{A.17}$$

where

$$\begin{aligned} W(z(t), \bar{z}(t), \theta) &= W(z(t), \bar{z}(t)) \\ &= W_{20}(\theta)\frac{z^2(t)}{2} + W_{11}(\theta)\,z(t)\bar{z}(t) \\ &\quad + W_{02}(\theta)\frac{\bar{z}^2(t)}{2} + W_{30}(\theta)\frac{z^3(t)}{6} + \cdots. \end{aligned} \tag{A.18}$$

$z(t)$ and $\bar{z}(t)$ are local coordinates for center manifold C_0 in the directions of q^* and $\bar{q}^*$. Note that W is real if X_t is. We shall deal with real solutions only. For the solution $X_t \in C_0$ of (A.1), since $\mu = 0$, we have

$$\begin{aligned} \dot{z}(t) &= \left\langle q^*, \dot{X}_t \right\rangle = \langle q^*, A(0)\,X_t + R(0)\,X_t\rangle \\ &= \langle q^*, A(0)\,X_t\rangle + \langle q^*, R(0)\,X_t\rangle = \langle A^*(0)\,q^*, X_t\rangle + \bar{q}^*(0)\cdot f(0, X_t) \\ &= i\omega^*\tau^* z(t) + \bar{q}^*(0)\cdot f(0, W(z(t), \bar{z}(t), 0)) + 2Re\{z(t)\,q(\theta)\} \\ &\triangleq i\omega^*\tau^* z(t) + \bar{q}^*(0)\cdot f_0(z(t), \bar{z}(t)). \end{aligned} \tag{A.19}$$

We rewrite this equation as

$$\dot{z}(t) = i\omega^*\tau^* z(t) + g(z(t), \bar{z}(t)), \tag{A.20}$$

where

$$g(z(t), \bar{z}(t)) = \bar{q}^*(0)\cdot f_0(z(t), \bar{z}(t)),$$

and expand $g(z(t), \bar{z}(t))$ in powers of $z(t)$ and $\bar{z}(t)$, that is

$$g(z(t), \bar{z}(t)) = g_{20}\frac{z^2(t)}{2} + g_{11}z(t)\bar{z}(t) + g_{02}\frac{\bar{z}^2(t)}{2} + g_{21}\frac{z^2(t)\,\bar{z}(t)}{2} + \cdots. \tag{A.21}$$

Noticing

$$X_t(\theta) = (X_{1t}(\theta), X_{2t}(\theta), X_{3t}(\theta)) = W(t,\theta) + zq(\theta) + \bar{z}\bar{q}(\theta),$$

and

$$q(\theta) = Ve^{i\omega^*\tau^*\theta} = (1, v_2, v_3)^T e^{i\omega^*\tau^*\theta},$$

we have

$$X_{1t}(0) = z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)\,z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right),$$

$$
\begin{aligned}
X_{2t}(0) =& zv_2 + \bar{z}\bar{v}_2 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right),\\
X_{3t}(0) =& zv_3 + \bar{z}\bar{v}_3 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right),\\
X_{1t}(-1) =& ze^{-i\omega^*\tau^*} + \bar{z}e^{i\omega^*\tau^*} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z}\\
&+ W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right),\\
X_{2t}(-1) =& zv_2e^{-i\omega^*\tau^*} + \bar{z}\bar{v}_2e^{i\omega^*\tau^*} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z}\\
&+ W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right),\\
X_{3t}(-1) =& zv_3e^{-i\omega^*\tau^*} + \bar{z}\bar{v}_3e^{i\omega^*\tau^*} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z}\\
&+ W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right).
\end{aligned}
$$

Thus, from (A.21), we have

$$
\begin{aligned}
g(z,\bar{z}) =& \bar{q}^*(0)\cdot f_0(z(t),\bar{z}(t)) = \bar{q}^*(0)\cdot f_0(0,X_t(t))\\
=& \tau^*\bar{P}\left(1,\bar{v}_2^*,\bar{v}_3^*\right)\begin{pmatrix} l_1X_{1t}^2(0) + l_2X_{1t}(0)X_{2t}(0)\\ +l_3X_{1t}(0)X_{3t}(0) + l_4X_{1t}^3(0) + l_5X_{1t}^2(0)X_{3t}(0)\\ m_1X_{1t}(0)X_{2t}(0) + m_2X_{2t}(0)X_{3t}(0)\\ n_1X_{1t}^2(-1) + n_2X_{1t}(-1)X_{3t}(-1) + n_3X_{2t}(-1)X_{3t}(-1)\\ +n_4X_{1t}^3(-1) + n_5X_{1t}^2(-1)X_{3t}(-1)\end{pmatrix}\\
=& \tau^*\bar{P}\left\{l_1X_{1t}^2(0) + l_2X_{1t}(0)X_{2t}(0) + l_3X_{1t}(0)X_{3t}(0)\right.\\
&\left.+l_4X_{1t}^3(0) + l_5X_{1t}^2(0)X_{3t}(0)\right\} + \tau^*\bar{P}\bar{v}_2^*\left\{m_1X_{1t}(0)X_{2t}(0)\right.\\
&\left.+m_2X_{2t}(0)X_{3t}(0)\right\} + \tau^*\bar{P}\bar{v}_3^*\left\{n_1X_{1t}^2(-1) + n_2X_{1t}(-1)X_{3t}(-1)\right.\\
&\left.+n_3X_{2t}(-1)X_{3t}(-1) + n_4X_{1t}^3(-1) + n_5X_{1t}^2(-1)X_{3t}(-1)\right\}\\
=& \tau^*\bar{P}\left\{l_1\left[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]^2\right.\\
&+ l_2\left[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]\\
&\times\left[zv_2 + \bar{z}\bar{v}_2 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]\\
&+ l_3\left[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]\\
&\times\left[zv_3 + \bar{z}\bar{v}_3 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]\\
&+ l_4\left[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]^3
\end{aligned}
$$

$$+ l_5 \left[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]^2$$
$$\times \left[zv_3 + \bar{z}\bar{v}_3 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z} + W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]\Bigg\}$$
$$+ \tau^*\bar{P}\bar{v}_2^*\Bigg\{m_1\left[z + \bar{z} + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2}\right.$$
$$\left.+O\left(|(z,\bar{z})|^3\right)\right] \times \left[zv_2 + \bar{z}\bar{v}_2 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2}\right.$$
$$\left.+O\left(|(z,\bar{z})|^3\right)\right] + m_2\left[zv_2 + \bar{z}\bar{v}_2 + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z}\right.$$
$$\left.+W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right] \times \left[zv_3 + \bar{z}\bar{v}_3 + W_{20}^{(3)}(0)\frac{z^2}{2} + W_{11}^{(3)}(0)z\bar{z}\right.\Bigg\}$$
$$\left.+W_{02}^{(3)}(0)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right] + \tau^*\bar{P}\bar{v}_3^*\Bigg\{n_1\left[ze^{-i\omega^*\tau^*} + \bar{z}e^{i\omega^*\tau^*}\right.$$
$$\left.+W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]^2$$
$$+ n_2\left[ze^{-i\omega^*\tau^*} + \bar{z}e^{i\omega^*\tau^*} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z}\right.$$
$$\left.+W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right] \times \left[zv_3e^{-i\omega^*\tau^*} + \bar{z}\bar{v}_3e^{i\omega^*\tau^*}\right.$$
$$\left.+W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]$$
$$+ n_3\left[zv_2e^{-i\omega^*\tau^*} + \bar{z}\bar{v}_2e^{i\omega^*\tau^*} + W_{20}^{(2)}(-1)\frac{z^2}{2} + W_{11}^{(2)}(-1)z\bar{z}\right.$$
$$\left.+W_{02}^{(2)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right] \times \left[zv_3e^{-i\omega^*\tau^*} + \bar{z}\bar{v}_3e^{i\omega^*\tau^*}\right.$$
$$\left.+W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z} + W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]$$
$$+ n_4\left[ze^{-i\omega^*\tau^*} + \bar{z}e^{i\omega^*\tau^*} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z}\right.$$
$$\left.+W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]^3 + n_5\left[ze^{-i\omega^*\tau^*} + \bar{z}e^{i\omega^*\tau^*}\right.$$
$$\left.+W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]^2$$
$$\times\left[zv_3e^{-i\omega^*\tau^*} + \bar{z}\bar{v}_3e^{i\omega^*\tau^*} + W_{20}^{(3)}(-1)\frac{z^2}{2} + W_{11}^{(3)}(-1)z\bar{z}\right.$$
$$\left.+W_{02}^{(3)}(-1)\frac{\bar{z}^2}{2} + O\left(|(z,\bar{z})|^3\right)\right]\Bigg\}.$$

Comparing the coefficients with (A.21) we get

$$g_{20} = 2\tau^*\bar{P}\Big\{l_1 + l_2v_2 + l_3v_3 + m_1v_2\bar{v}_2^* + m_2v_2v_3\bar{v}_2^* + n_1\bar{v}_3^*e^{-2i\omega^*\tau^*}$$

$$
\begin{aligned}
&+n_2 v_3 \bar{v}_3^* e^{-2i\omega^*\tau^*} + n_3 v_2 v_3 \bar{v}_3^* e^{-2i\omega^*\tau^*} \Big\} \\
&= 2\tau^* \bar{P} \left\{ l_1 + l_2 v_2 + l_3 v_3 + \bar{v}_2^* \left[v_2 \left(m_1 + m_2 v_3 \right) \right] + \bar{v}_3^* \left[v_3 e^{-2i\omega^*\tau^*} \left(n_2 + n_3 v_2 \right) \right] \right\}, \\
g_{11} &= \tau^* \bar{P} \left\{ 2l_1 + l_2 v_2 + l_2 \bar{v}_2 + l_3 v_3 + l_3 \bar{v}_3 + v_2 \bar{v}_2^* m_1 + \bar{v}_2 \bar{v}_2^* m_1 + v_2 \bar{v}_2^* \bar{v}_3 m_2 \right. \\
&\left. + \bar{v}_2 \bar{v}_2^* v_3 m_2 + 2\bar{v}_3^* n_1 + v_3 \bar{v}_3^* n_2 + \bar{v}_3 \bar{v}_3^* n_2 + v_2 \bar{v}_3 \bar{v}_3^* n_3 + \bar{v}_2 v_3 \bar{v}_3^* n_3 \right\} \\
&= 2\tau^* \bar{P} \left\{ \frac{1}{2} \left[2l_1 + l_2 \left(v_2 + \bar{v}_2 \right) + l_3 \left(v_3 + \bar{v}_3 \right) \right] + \frac{1}{2} \bar{v}_2^* \left[m_1 \left(v_2 + \bar{v}_2 \right) \right. \right. \\
&\left. \left. + m_2 \left(v_2 \bar{v}_3 + \bar{v}_2 v_3 \right) \right] + \frac{1}{2} \bar{v}_3^* \left[2n_1 + n_2 \left(v_3 + \bar{v}_3 \right) + n_3 \left(v_2 \bar{v}_3 + \bar{v}_2 v_3 \right) \right] \right\}, \\
g_{02} &= 2\tau^* \bar{P} \left\{ l_1 + l_2 \bar{v}_2 + l_3 \bar{v}_3 + m_1 \bar{v}_2 \bar{v}_2^* + m_2 \bar{v}_2 \bar{v}_3 \bar{v}_2^* + n_1 \bar{v}_3^* e^{2i\omega^*\tau^*} \right. \\
&\left. + n_2 \bar{v}_3 \bar{v}_3^* e^{2i\omega^*\tau^*} + n_3 \bar{v}_2 \bar{v}_3 \bar{v}_3^* e^{2i\omega^*\tau^*} \right\} \\
&= 2\tau^* \bar{P} \left\{ l_1 + l_2 \bar{v}_2 + l_3 \bar{v}_3 + \bar{v}_2^* \left[\bar{v}_2 \left(m_1 + m_2 \bar{v}_3 \right) \right] \right. \\
&\left. + \bar{v}_3^* \left[e^{2i\omega^*\tau^*} \left(n_1 + n_2 \bar{v}_3 + n_3 \bar{v}_2 \bar{v}_3 \right) \right] \right\}, \\
g_{21} &= 2\tau^* \bar{P} \left\{ l_1 \left(W_{20}^{(1)}(0) + 2W_{11}^{(1)}(0) \right) + l_2 \left(\frac{1}{2} \bar{v}_2 W_{20}^{(1)}(0) + \frac{1}{2} W_{20}^{(2)}(0) \right. \right. \\
&\left. + v_2 W_{11}^{(1)}(0) + W_{11}^{(2)}(0) \right) + l_3 \left(\frac{1}{2} \bar{v}_3 W_{20}^{(1)}(0) + \frac{1}{2} W_{20}^{(3)}(0) + v_3 W_{11}^{(1)}(0) \right. \\
&\left. + W_{11}^{(3)}(0) \right) + 3l_4 + l_5 \left(\bar{v}_3 + 2v_3 \right) + m_1 \bar{v}_2^* \left(\frac{1}{2} \bar{v}_2 W_{20}^{(1)}(0) + \frac{1}{2} W_{20}^{(2)}(0) \right. \\
&\left. + v_2 W_{11}^{(1)}(0) + W_{11}^{(2)}(0) \right) + m_2 \bar{v}_2^* \left(\frac{1}{2} \bar{v}_3 W_{20}^{(2)}(0) + \frac{1}{2} \bar{v}_2 W_{20}^{(3)}(0) \right. \\
&\left. + v_3 W_{11}^{(2)}(0) + v_2 W_{11}^{(3)}(0) \right) + n_1 \bar{v}_3^* \left(2e^{-i\omega^*\tau^*} W_{11}^{(1)}(-1) + e^{i\omega^*\tau^*} W_{20}^{(1)}(-1) \right) \\
&+ n_2 \bar{v}_3^* \left(\frac{1}{2} \bar{v}_3 e^{i\omega^*\tau^*} W_{20}^{(1)}(-1) + \frac{1}{2} e^{i\omega^*\tau^*} W_{20}^{(3)}(-1) + v_3 e^{-i\omega^*\tau^*} W_{11}^{(1)}(-1) \right. \\
&\left. + e^{-i\omega^*\tau^*} W_{11}^{(3)}(-1) \right) + n_3 \bar{v}_3^* \left(\frac{1}{2} \bar{v}_3 e^{i\omega^*\tau^*} W_{20}^{(2)}(-1) + \frac{1}{2} \bar{v}_2 e^{i\omega^*\tau^*} W_{20}^{(3)}(-1) \right. \\
&\left. + v_3 e^{-i\omega^*\tau^*} W_{11}^{(2)}(-1) + v_2 e^{-i\omega^*\tau^*} W_{11}^{(3)}(-1) \right) + 3n_4 \bar{v}_3^* e^{-i\omega^*\tau^*} \\
&\left. + n_5 \bar{v}_3^* e^{-i\omega^*\tau^*} \left(2v_3 + \bar{v}_3 \right) \right\}.
\end{aligned}
$$

Since there are $W_{20}(\theta)$ and $W_{11}(\theta)$ in g_{21}, we still need to compute them. From (A.1), (A.16) and (A.20), we have

$$
\begin{aligned}
\dot{W} &= \dot{X}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\
&= \begin{cases} AW - 2Re\left\{ \bar{q}^*(0) \cdot f_0(z, \bar{z}) q(\theta) \right\}, & \text{if } \theta \in [-1, 0), \\ AW - 2Re\left\{ \bar{q}^*(0) \cdot f_0(z, \bar{z}) q(\theta) \right\} + f_0(z, \bar{z}), & \text{if } \theta = 0. \end{cases} \\
&\triangleq AW + H(z, \bar{z}, \theta),
\end{aligned}
$$

where

$$
H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots. \tag{A.22}
$$

On the other hand, on C_0 near to the origin, we have

$$\dot{W} = W_z \dot{z} + W_{\bar{z}} \dot{\bar{z}}.$$

Using (A.18) and (A.20) to replace W_z and $\dot{z}$ and their conjugates by their power series expansions, comparing the coefficients with the right hand side of (A.22), we obtain

$$(2i\omega^* \tau^* I - A) W_{20}(\theta) = H_{20}(\theta), \quad -AW_{11}(\theta) = H_{11}(\theta), \tag{A.23}$$

where I denotes the 3×3 identity matrix. From (A.22), we know that for $\theta \in [1, 0)$,

$$\begin{aligned} H(z, \bar{z}, \theta) &= -\bar{q}^*(0) \cdot f_0(z, \bar{z}) q(\theta) - q^*(0) \cdot \bar{f}_0(z, \bar{z}) \bar{q}(\theta) \\ &= -g(z, \bar{z}) q(\theta) - \bar{g}(z, \bar{z}) \bar{q}(\theta). \end{aligned} \tag{A.24}$$

Substituting (A.21) into (A.22) gives

$$\begin{aligned} H(z, \bar{z}, \theta) =& [-g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta)] \frac{z^2}{2} + [-g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta)] z\bar{z} \\ &+ [-g_{02} q(\theta) - \bar{g}_{20} \bar{q}(\theta)] \frac{\bar{z}^2}{2} + \cdots . \end{aligned} \tag{A.25}$$

Comparing the coefficients in (A.25) with those in (A.22) gives that

$$H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta), \quad H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta). \tag{A.26}$$

From (A.23), (A.26) and the definition of A, we have

$$\dot{W}_{20}(\theta) = 2i\omega^* \tau^* W_{20}(\theta) + g_{20} q(\theta) + \bar{g}_{02} \bar{q}(\theta). \tag{A.27}$$

Notice that

$$q(\theta) = (v_1, v_2, v_3)^T e^{i\omega^* \tau^* \theta} = q(0) e^{i\omega^* \tau^* \theta}.$$

Hence, using the method of variation of constants, the solution of (A.27) is given by

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^* \tau^*} q(0) e^{i\omega^* \tau^* \theta} + \frac{i\bar{g}_{02}}{3\omega^* \tau^*} \bar{q}(0) e^{-i\omega^* \tau^* \theta} + E_1 e^{2i\omega^* \tau^* \theta}, \tag{A.28}$$

where $E_1 = \left(E_1^{(1)}, E_1^{(2)}, E_1^{(3)}\right) \in \mathbb{R}^3$ is a constant vector.

Similarly, from (A.23), (A.26) and the definition of A, we have

$$\dot{W}_{11}(\theta) = g_{11} q(\theta) + \bar{g}_{11} \bar{q}(\theta), \tag{A.29}$$

and

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega^* \tau^*} q(0) e^{i\omega^* \tau^* \theta} + \frac{i\bar{g}_{11}}{\omega^* \tau^*} \bar{q}(0) e^{-i\omega^* \tau^* \theta} + E_2, \tag{A.30}$$

where $E_2 = \left(E_2^{(1)}, E_2^{(2)}, E_2^{(3)}\right) \in \mathbb{R}^3$ is also a constant vector.

In what follows, we seek appropriate constant vectors E_1 and E_2 in (A.28) and (A.30), respectively. From (A.23) and the definition of A, we know that when $\theta = 0$,

$$A(0) W_{20}(\theta) = \int_{-1}^{0} \mathrm{d}\rho(\theta) W_{20}(\theta) = 2i\omega^* \tau^* W_{20}(0) - H_{20}(0), \tag{A.31}$$

and

$$A(0)W_{11}(\theta)=\int_{-1}^{0}\mathrm{d}\rho(\theta)W_{11}(\theta)=-H_{11}(0), \tag{A.32}$$

where $\rho(\theta)=\rho(\theta,0)$. And from (A.22), we can obtain when $\theta=0$,

$$H_{20}(0)=-g_{20}q(0)-\bar{g}_{02}\bar{q}(0)+2\tau^*\left(H_1,H_2,H_3\right)^T, \tag{A.33}$$

$$H_{11}(0)=-g_{11}q(0)-\bar{g}_{11}\bar{q}(0)+2\tau^*\left(P_1,P_2,P_3\right)^T, \tag{A.34}$$

where

$$\begin{aligned}H_1=&l_1+l_2v_2+l_3v_3,\quad H_2=v_2(m_1+m_2v_3),\quad H_3=v_3e^{-2i\omega^*\tau^*}(n_2+n_3v_2),\\ P_1=&\frac{1}{2}[2l_1+l_2(v_2+\bar{v}_2)+l_3(v_3+\bar{v}_3)],\quad P_2=\frac{1}{2}[m_1(v_2+\bar{v}_2)+m_2(v_2\bar{v}_3+\bar{v}_2v_3)],\\ P_3=&\frac{1}{2}[2n_1+n_2(v_3+\bar{v}_3)+n_3(v_2\bar{v}_3+\bar{v}_2v_3)].\end{aligned}$$

Noticing that

$$\left(i\omega^*\tau^*I-\int_{-1}^{0}e^{i\omega^*\tau^*\theta}\mathrm{d}\rho(\theta)\right)q(0)=0,\quad\left(-i\omega^*\tau^*I-\int_{-1}^{0}e^{-i\omega^*\tau^*\theta}\mathrm{d}\rho(\theta)\right)\bar{q}(0)=0,$$

and substituting (A.28) and (A.33) into (A.30), we have

$$\left(2i\omega^*\tau^*I-\int_{-1}^{0}e^{2i\omega^*\tau^*\theta}\mathrm{d}\rho(\theta)\right)E_1=2\tau^*\left(H_1,H_2,H_3\right)^T.$$

From the definition of A, we have

$$\int_{-1}^{0}e^{2i\omega^*\tau^*\theta}\mathrm{d}\rho(\theta)=A(\mu)e^{2i\omega^*\tau^*\theta}=L_\mu\left(e^{2i\omega^*\tau^*\theta}\right).$$

Therefore, when $\mu=0$, we have

$$\int_{-1}^{0}e^{2i\omega^*\tau^*\theta}\mathrm{d}\rho(\theta)=\tau^*\begin{pmatrix}M_1&M_2&M_3\\M_4&0&M_5\\0&0&M_6\end{pmatrix}+\tau^*\begin{pmatrix}0&0&0\\0&0&0\\N_1&N_2&N_3\end{pmatrix}e^{-2i\omega^*\tau^*}.$$

That is,

$$\begin{pmatrix}2i\omega^*-M_1&-M_2&-M_3\\-M_4&2i\omega^*&-M_5\\-N_1e^{-2i\omega^*\tau^*}&-N_2e^{-2i\omega^*\tau^*}&2i\omega^*-M_6-N_3e^{-2i\omega^*\tau^*}\end{pmatrix}E_1=2\begin{pmatrix}H_1\\H_2\\H_3\end{pmatrix}.$$

It follows that

$$E_1^{(1)}=\frac{\triangle_{11}}{\triangle_1},\ E_1^{(2)}=\frac{\triangle_{12}}{\triangle_1},\ E_1^{(3)}=\frac{\triangle_{13}}{\triangle_1},$$

where

$$\triangle_1=\mathrm{Det}\begin{pmatrix}2i\omega^*-M_1&-M_2&-M_3\\-M_4&2i\omega^*&-M_5\\-N_1e^{-2i\omega^*\tau^*}&-N_2e^{-2i\omega^*\tau^*}&2i\omega^*-V_6-N_3e^{-2i\omega^*\tau^*}\end{pmatrix},$$

$$\triangle_{11} = 2\mathrm{Det}\begin{pmatrix} H_1 & -M_2 & -M_3 \\ H_2 & 2i\omega^* & -M_5 \\ H_3 & -N_2 e^{-2i\omega^*\tau^*} & 2i\omega^* - V_6 - N_3 e^{-2i\omega^*\tau^*} \end{pmatrix},$$

$$\triangle_{12} = 2\mathrm{Det}\begin{pmatrix} 2i\omega^* - M_1 & H_1 & -M_3 \\ -M_4 & H_2 & -M_5 \\ -N_1 e^{-2i\omega^*\tau^*} & H_3 & 2i\omega^* - V_6 - N_3 e^{-2i\omega^*\tau^*} \end{pmatrix},$$

$$\triangle_{13} = 2\mathrm{Det}\begin{pmatrix} 2i\omega^* - M_1 & -M_2 & H_1 \\ -M_4 & 2i\omega^* & H_2 \\ -N_1 e^{-2i\omega^*\tau^*} & -N_2 e^{-2i\omega^*\tau^*} & H_3 \end{pmatrix}.$$

Similarly, substituting (A.30) and (A.34) into (A.32), we can obtain

$$\left(\int_{-1}^{0} \mathrm{d}\rho(\theta)\right) E_2 = -2\tau^* \left(P_1, P_2, P_3\right)^T.$$

From the definition of A, we have

$$\int_{-1}^{0} \mathrm{d}\rho(\theta) = A(\mu) = L_\mu,$$

therefore, when $\mu = 0$, we have

$$\int_{-1}^{0} \mathrm{d}\rho(\theta) = \tau^* \begin{pmatrix} M_1 & M_2 & M_3 \\ M_4 & 0 & M_5 \\ 0 & 0 & M_6 \end{pmatrix} + \tau^* \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ N_1 & N_2 & N_3 \end{pmatrix},$$

thus, we have

$$\begin{pmatrix} M_1 & M_2 & M_3 \\ M_4 & 0 & M_5 \\ N_1 & N_2 & M_6 + N_3 \end{pmatrix} E_2 = -2 \begin{pmatrix} P_1 \\ P_2 \\ P_3 \end{pmatrix}.$$

It follows that

$$E_2^{(1)} = \frac{\triangle_{21}}{\triangle_2},\ E_2^{(2)} = \frac{\triangle_{22}}{\triangle_2},\ E_2^{(3)} = \frac{\triangle_{23}}{\triangle_2},$$

where

$$\triangle_2 = \mathrm{Det}\begin{pmatrix} M_1 & M_2 & M_3 \\ M_4 & 0 & M_5 \\ N_1 & N_2 & M_6 + N_3 \end{pmatrix}, \qquad \triangle_{21} = 2\mathrm{Det}\begin{pmatrix} P_1 & M_2 & M_3 \\ P_2 & 0 & M_5 \\ P_3 & N_2 & M_6 + N_3 \end{pmatrix},$$

$$\triangle_{22} = 2\mathrm{Det}\begin{pmatrix} M_1 & P_1 & M_3 \\ M_4 & P_2 & M_5 \\ N_1 & P_3 & M_6 + N_3 \end{pmatrix}, \qquad \triangle_{23} = 2\mathrm{Det}\begin{pmatrix} M_1 & M_2 & P_1 \\ M_4 & 0 & P_2 \\ N_1 & N_2 & P_3 \end{pmatrix}.$$