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Traveling Wave Solutions of Some *abcd*-Water Wave Models Describing Small Amplitude, Long Wavelength Gravity Waves on the Surface of Water*

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Abstract For some *abcd*-water wave models describing small amplitude, long wavelength gravity waves on the surface of water, in this paper, by using the method of dynamical systems to analyze corresponding traveling wave systems and find the bifurcations of phase portraits, the dynamical behavior of systems can be derived. Under some given parameter conditions, for a wave component, the existence of periodic wave solutions, solitary wave solutions, kink and anti-kink wave solutions as well as compacton families can be proved. Possible exact explicit parametric representations of the traveling wave solutions are given.

Keywords Pseudo-peakon, solitary wave, kink and anti-kink wave, compacton family, planar dynamical system, *abcd*-water wave models

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1. Introduction

More recently, in [5], the authors studied exact Jacobi elliptic solutions for the following *abcd*-system:

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x + a w_{xxx} - b\eta_{xxt} = 0, \\ &w_t + \eta_x + w w_x + c\eta_{xxx} - d w_{xxt} = 0, \end{aligned} \tag{1.1}$$

where a, b, c, and d are real constants and $\theta \in [0,1]$ that satisfy

$$a + b = \frac{1}{2}\left(\theta^2 - \frac{1}{3}\right), \quad c + d = \frac{1}{2}(1-\theta^2), \quad a+b+c+d = \frac{1}{3}. \tag{1.2}$$

System (1.1) was introduced by Bona et al. in [1] and [2] to describe the wave motion of small amplitude, long wavelength gravity waves on the surface of water. The functions $\eta(x,t)$ and $w(x,t)$ are real valued and $x, t \in R$.

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A traveling-wave solution to the system (1.1) is a vector solution $(\eta(x,t), w(x,t))$ of the form

$$\eta(x,t) = \phi(x - \sigma t) = \phi(\xi),\ w(x,t) = \psi(x - \sigma t) = \psi(\xi),\ \xi = x - \sigma t, \tag{1.3}$$

where σ denotes the speed of the waves.

Substituting (1.3) into (1.1) and integrating the obtained system once, we have

$$\begin{aligned} &-\sigma\phi + \psi + \phi\psi + a\psi'' + b\sigma\phi'' = g_1, \\ &-\sigma\psi + \phi + \frac{1}{2}\psi^2 + c\phi'' + d\sigma\psi'' = g_2, \end{aligned} \tag{1.4}$$

where "′" denotes the derivative in ξ, g_1 and g_2 are two integral constants. System (1.4) can be written as:

$$\begin{aligned} \left(1 - \frac{\psi}{\sigma}\right)\phi - b\phi'' &= g_1 + \frac{1}{\sigma}(a\psi'' + \psi), \\ \phi + c\phi'' &= g_2 - d\sigma\psi'' - \frac{1}{2}\psi^2 + \sigma\psi. \end{aligned} \tag{1.5}$$

(1.5) implies that if $c \neq 0$ and $a^2 + b^2 \neq 0$,

$$\phi = F(\psi, \psi'') = \frac{b(g_2 - d\sigma\psi'' - \frac{1}{2}\psi^2 + \sigma\psi) + c(g_1 + \frac{1}{\sigma}(a\psi'' + \psi))}{(b+c) - \frac{c}{\sigma}\psi}. \tag{1.6}$$

Notice that

$$\frac{d\phi}{d\xi} = \frac{\partial F}{\partial \psi}\psi' + \frac{\partial F}{\partial \psi''}\psi''',\quad \frac{d^2\phi}{d\xi^2} = \frac{\partial^2 F}{\partial \psi^2}(\psi')^2 + \frac{\partial F}{\partial \psi}\psi'' + \frac{\partial^2 F}{(\partial \psi'')^2}(\psi''')^2 + \frac{\partial F}{\partial \psi''}\psi''''. \tag{1.7}$$

Generally, substituting (1.6) and (1.7) into the second equation of (1.5), we obtain a fourth order ordinary differential equation about the variable ψ, because it contains a fourth order derivative ψ'''' with respect to ξ.

If $c \neq 0, a = b = 0$ or $a = d = 0$, then, we can obtain a second order traveling wave differential equation.

Obviously, to investigate the traveling wave solutions of the PDE system (1.1), we must study the all solutions of the corresponding ordinary differential equation (traveling system). [5] did not discuss the dynamics of solutions for the traveling system of system (1.1). Therefore, the conclusions in their paper are not complete and some results are incorrect.

By choosing specific values for the parameters a, b, c, and d, the system (1.1) includes a wide range of other systems that have been derived over the last few decades such as the classical Boussinesq system, the Kaup system, the coupled Benjamin-Bona-Mahony system (BBM-system), the coupled Korteweg-de Vries system (KdV-system), the Bona-Smith system, and the integrable version of Boussinesq system. In particular, these specializations are (see Bona et al. [1]):

(i) Classical Boussinesq system ($\theta^2 = \frac{1}{3}, a = b = c = 0, d = \frac{1}{3}$) :

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x = 0, \\ &w_t + \eta_x + ww_x - \frac{1}{3}w_{xxt} = 0. \end{aligned} \tag{1.8}$$

(ii) Coupled BBM-KdV system ($\theta^2 = \frac{2}{3}, a = d = 0, b = c = \frac{1}{6}$):

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x - \frac{1}{6}\eta_{xxt} = 0, \\ &w_t + \eta_x + ww_x + \frac{1}{6}\eta_{xxx} = 0. \end{aligned} \tag{1.9}$$

(iii) Coupled KdV-BBM system ($\theta^2 = \frac{3}{2}, b = c = 0, a = d = \frac{1}{6}$):

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x + \frac{1}{6}w_{xxx} = 0, \\ &w_t + \eta_x + ww_x - \frac{1}{6}w_{xxt} = 0. \end{aligned} \tag{1.10}$$

(iv) Kaup system ($\theta^2 = 1, b = c = d = 0, a = \frac{1}{3}$):

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x + \frac{1}{3}w_{xxx} = 0, \\ &w_t + \eta_x + ww_x = 0. \end{aligned} \tag{1.11}$$

(v) Coupled BBM-system ($\theta^2 = \frac{2}{3}, a = c = 0, b = d = \frac{1}{6}$):

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x - \frac{1}{6}\eta_{xxt} = 0, \\ &w_t + \eta_x + ww_x - \frac{1}{6}w_{xxt} = 0. \end{aligned} \tag{1.12}$$

(vi) Bona-Smith system $\left(\theta = \frac{(\frac{4}{3}-\mu)}{(2-\mu)}, \mu < 0,\ arbitrary, a = 0, b = d\right)$:

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x - b\eta_{xxt} = 0, \\ &w_t + \eta_x + ww_x + c\eta_{xxx} - bw_{xxt} = 0. \end{aligned} \tag{1.13}$$

(vii) Coupled KdV-system ($\theta^2 = \frac{2}{3}, b = d = 0, a = c = \frac{1}{6}$) :

$$\begin{aligned} &\eta_t + w_x + (w\eta)_x + \frac{1}{6}w_{xxx} = 0, \\ &w_t + \eta_x + ww_x + \frac{1}{6}\eta_{xxx} = 0. \end{aligned} \tag{1.14}$$

In this paper, we generally investigate the traveling wave solutions for systems (1.8)-(1.11) by using the method of dynamical systems. We show that depending on the changes of some parameters, system (1.1) with respect to the variable $w = \psi(\xi)$ has families of periodic wave solutions, solitay wave solutions, kink and anti-kink wave solutions as well as compacton solution families. Under some special parameter conditions, we can find the exact parametric representations for the traveling wave solutions.

The article is organized as follows. In sections 2-5, we discuss respectively the dynamics and possible exact solutions of the corresponding traveling wave systems of equations (1.8)-(1.11). In section 6, we give the traveling system for the $w-$component of equations (1.12) and show that it is a non-integrable equation.

2. The dynamics and possible exact solutions of the traveling wave system of the classical Boussinesq system (1.8)

Substituting (1.3) to (1.8) and integrating the obtained system once, we have

$$\begin{aligned} & -\sigma\phi+\psi+\phi\psi=g_1, \\ & -\sigma\psi+\phi+\frac{1}{2}\psi^2+\frac{1}{3}\sigma\psi''=g_2, \end{aligned} \tag{2.1}$$

where "′" denotes the derivative in ξ. We see from the first equation of (2.1) that $\phi=\frac{g_1-\psi}{\psi-\sigma}=\frac{g_1-\sigma}{\psi-\sigma}-1$. Thus, by the second equation of (2.1), we obtain

$$\frac{1}{3}\sigma\psi''-\sigma\psi+\frac{1}{2}\psi^2+\frac{g_1-\psi}{\psi-\sigma}-g_2=0, \tag{2.2}$$

which is equivalent to the planar dynamical system as follows:

$$\frac{d\psi}{d\xi}=y, \quad \frac{dy}{d\xi}=\frac{-3\psi^3+9\sigma\psi^2+\alpha\psi+\beta}{2\sigma(\psi-\sigma)}, \tag{2.3}$$

where $\alpha=6(-\sigma^2+g_2-1), \beta=6(-g_2\sigma+g_1)$.

System (2.3) is called a singular traveling wave system of the first kind defined by [6], [7]. Because it defines a piecewise smooth vector field in two-half phase planes in the two sides of the singular straight line $\psi=\sigma$ and near this singular straight line, the solution orbits have two time scales. Therefore, one must understand the dynamical behavior of the solutions of singular nonlinear traveling wave systems.

System (2.3) has the first integral

$$\begin{aligned} & H_1(\psi,y) \\ & =\frac{1}{2}y^2-\left(3\sigma^2+\frac{1}{2}\alpha+\frac{1}{2\sigma}\beta\right)\ln|\psi-\sigma|+\frac{1}{2\sigma}\psi^3-\frac{3}{2}\psi^2-\left(3\sigma+\frac{1}{2\sigma}\alpha\right)\psi=h. \end{aligned} \tag{2.4}$$

We next consider the associated regular system of system (2.2) as follows:

$$\frac{d\psi}{d\zeta}=2\sigma(\psi-\sigma)y, \qquad \frac{dy}{d\zeta}=-3\psi^3+9\sigma\psi^2+\alpha\psi+\beta, \tag{2.5}$$

where $d\xi=2\sigma(\psi-\sigma)d\zeta$. For $\psi\neq\sigma$, this system has the same first integral as system (2.3). The dynamics of systems (2.3) and (2.5) are different in the neighborhood of the straight line $\psi=\sigma$. Specially, under some parameter conditions, the variable ζ is a fast variable while the variable ξ is a slow variable in the sense of the geometric singular perturbation theory (see [7]).

Write that $f(\psi)=3\left(\psi^3-3\sigma\psi^2-\frac{1}{3}\alpha\psi-\frac{1}{3}\beta\right), G(\alpha,\sigma)=729\sigma^6+243\alpha\sigma^4+27\alpha^2\sigma^2+\alpha^3$. For a given parameter pair (α,σ), when $S(\sigma,\alpha,\beta)=\frac{1}{36}\beta^2+\frac{1}{18}\sigma(6\sigma^2+\alpha)\beta-\frac{1}{108}\alpha^2\sigma^2-\frac{1}{729}\alpha^3<0$, i.e.

$$\beta\in\left(-6\sigma(\sigma^2+\alpha)-\frac{2}{9}\sqrt{G(\alpha,\sigma)},-6\sigma(\sigma^2+\alpha)+\frac{2}{9}\sqrt{G(\alpha,\sigma)}\right), \tag{2.6}$$

the function $f(\psi)$ has three real zeros $\psi_j, j = 1, 2, 3$. It means that system (2.5) has three equilibrium points $E_j(\psi_j, 0)$.

Especially, when $\alpha = -6\sigma^2, \beta = 0$, i.e., $g_1 = 1, g_2 = \sigma$, system (2.3) becomes

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{3}{2\sigma}\psi(2\sigma - \psi), \tag{2.7}$$

with

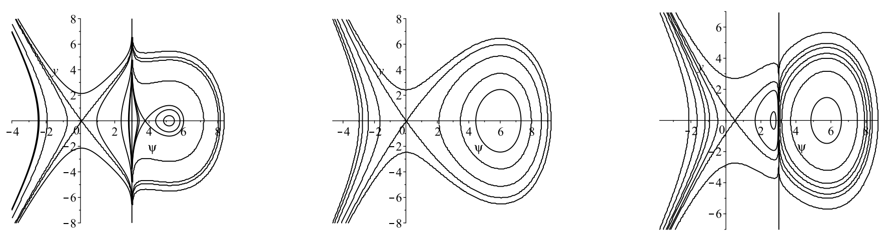
$$H_1(\psi, y) = \frac{1}{2}y^2 + \frac{1}{2\sigma}\psi^3 - \frac{3}{2}\psi^2 = h. \tag{2.8}$$

Let $M(\psi_j, 0)$ be the coefficient matrix of the linearized system of system (2.5) at an equilibrium point $E_j(\psi_j, 0)$. and $J(\psi_j, 0) = \det\mathrm{M}(\psi_j, 0)$. We have $J(\psi_j, 0) = 2\sigma(\psi_j - \sigma)f'(\psi_j)$.

By the theory of planar dynamical systems, for an equilibrium point of a planar integrable system, if $J < 0$, then the equilibrium point is a saddle point; if $J > 0$ and $\mathrm{Trace}(M(\psi_j, 0))^2 - 4J(\psi_j, 0) < 0$, then it is a center point; if $J > 0$ and $(\mathrm{Trace}(M(\psi_j, 0)))^2 - 4J(\psi_j, 0) > 0$, then it is a node; if $J = 0$ and the Poincaré index of the equilibrium point is 0, then it is a cusp.

Write that $h_j = H_1(\psi_j, 0)$, where $H_1(\psi, y)$ is given by (2.4).

By using the above information to do the qualitative analysis, when there exist three equilibrium points of system (2.5), we have the following bifurcations of the phase portraits of system (2.5) which are shown in Figure 1.



(a) $0 < \psi_1 < \sigma < \psi_2 < \psi_3$. (b) $\alpha = -6\sigma^2, \beta = 0$. (c) $0 < \psi_1 < \psi_2 < \sigma < \psi_3$.

Figure 1. The bifurcations of phase portraits of system (2.3).

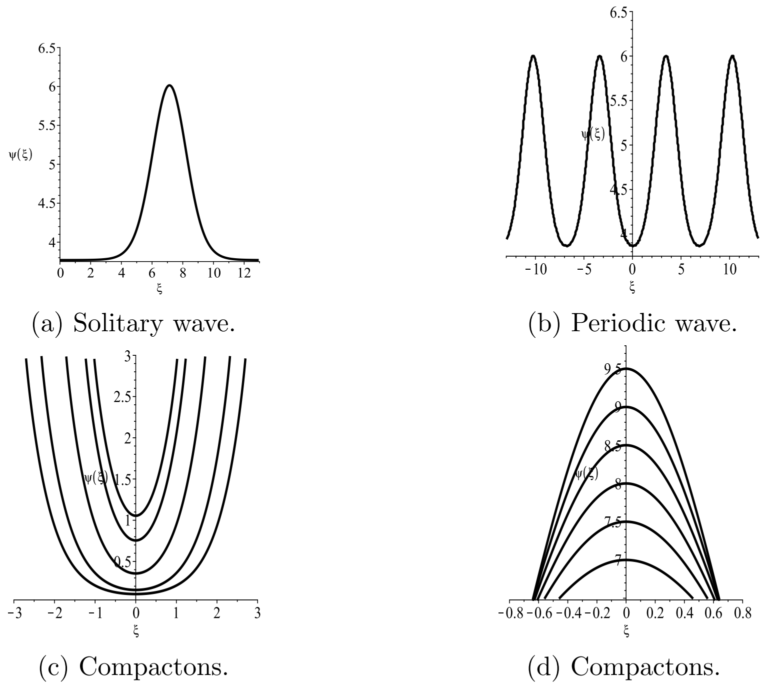
We see from Figure 1(a) and Figure 1(c) that the following two conclusions hold.

Theorem 2.1. *For a given parameter pair* (α, σ) *and* β *satisfying* (2.6) *such that system* (2.3) *has phase portrait Figure 1(a).*

(i) Corresponding to the homoclinic orbit defined by $H_1(\psi, y) = h_2$*, the* w-*component of the classical Boussinesq system* (1.8) *has a smooth solitary wave solution (see Figure 2(a)).*

(ii) Corresponding to the family of periodic orbits defined by $H_1(\psi, y) = h, h \in (h_3, h_2)$*,* $w-$*component of the classical Boussinesq system* (1.8) *has a smooth family of periodic wave solutions (see Figure 2(b)).*

(iii) Corresponding to the two families of open orbits defined by $H_1(\psi, y) = h, h \in (h_2, h_1)$*, which tend to the singular straight line* $\psi = \sigma$ *when* $|y| \to \infty$*,* $w-$*component of the classical Boussinesq system* (1.8) *has two smooth compacton solution familes (see Figure 2(c) and (d)).*



(a) Solitary wave. (b) Periodic wave.

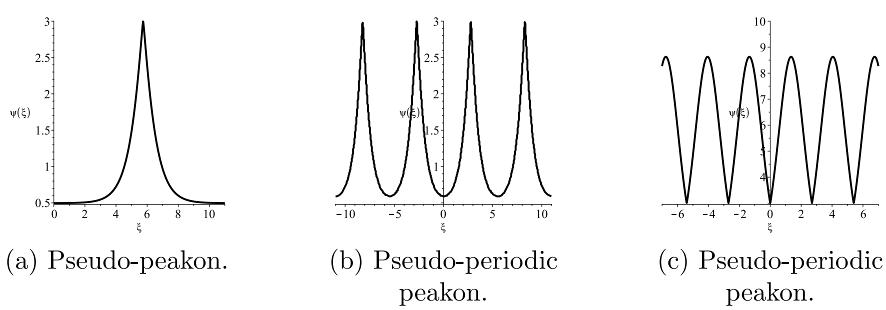
(c) Compactons. (d) Compactons.

Figure 2. The wave profiles of the orbits in Fig. 1 (a).

Theorem 2.2. *For a given parameter pair* (α, σ) *and* β *satisfying* (2.6) *such that system* (2.3) *has the phase portrait given by Figure 1(c).*

(i) Corresponding to the homoclinic orbit defined by $H_1(\psi, y) = h_1$, $w-$*component of the classical Boussinesq system* (1.8) *has a pseudo-peakon solution (see Figure 3(a)).*

(ii) Corresponding to the two families of periodic orbits defined by $H_1(\psi, y) = h, h \in (h_1 - \epsilon, h_1), 0 < \epsilon \ll 1$, *which enclose the singular points* $E_2(\psi_2, 0)$ *and* $E_3(\psi_3, 0)$, *respectively,* $w-$*component of the classical Boussinesq system* (1.8) *has two families of pseudo-periodic peakon solutions (see Figure 3(b) and (c)).*



(a) Pseudo-peakon. (b) Pseudo-periodic peakon. (c) Pseudo-periodic peakon.

Figure 3. The wave profiles of the orbits in Fig. 1 (c).

For the homoclinic orbit in Figure 1(b) defined by $H_1(\psi, y) = 0$ in (2.8), we have $y^2 = 3\psi^2 - \frac{1}{\sigma}\psi^3$. By using the first equation of system (2.7), we have

$\frac{1}{\sqrt{\sigma}}\xi = \int_{\psi}^{3\sigma} \frac{d\psi}{\psi\sqrt{3\sigma-\psi}}$. Thus, we obtain the following exact solitary wave solution of the $w-$component and an unbounded solution of $\eta-$component:

$$\begin{aligned} w = \psi(\xi) &= 3\sigma \mathrm{sech}^2\left(\frac{3}{2}\sqrt{\sigma}\xi\right), \\ \eta = \phi(\xi) &= \frac{1-\sigma}{\psi(\xi)-\sigma} - 1 = \frac{(1-\sigma)\cosh^2\left(\frac{3}{2}\sqrt{\sigma}\xi\right)}{\sigma\left(3-\cosh^2\left(\frac{3}{2}\sqrt{\sigma}\xi\right)\right)} - 1. \end{aligned} \tag{2.9}$$

Corresponding to the periodic orbit family defined by (2.8) with $H_1(\psi, y) = h, h \in (-2\sigma^2, 0)$, we see that $y^2 = 2h + 3\psi^3 - \frac{1}{\sigma}\psi^3 = \frac{1}{\sigma}(r_1-\psi)(\psi-r_2)(\psi-r_3), r_3 < 0 < r_2 < 2\sigma < r_1 < 3\sigma$. By using the first equation of system (2.7), we have $\frac{1}{\sqrt{\sigma}}\xi = \int_{\psi}^{r_1} \frac{d\psi}{\sqrt{(r_1-\psi)(\psi-r_2)(\psi-r_3)}}$. It gives rise to the following exact periodic wave solutions of the $w-$component and $\eta-$component:

$$\begin{aligned} w = \psi(\xi) &= r_1 - (r_1 - r_2)\mathrm{sn}^2(\Omega_1\xi, k), \\ \eta = \phi(\xi) &= \frac{1-\sigma}{\psi(\xi)-\sigma} - 1 = \frac{1-\sigma}{r_1 - \sigma - (r_1-r_2)\mathrm{sn}^2(\Omega_1\xi, k)} - 1, \end{aligned} \tag{2.10}$$

where $k^2 = \frac{r_1-r_2}{r_1-r_3}, \Omega_1 = \frac{\sqrt{r_1-r_3}}{2\sqrt{\sigma}}$, $\mathrm{sn}(\xi, k)$ is the Jacobian elliptic function (see [3]).

3. The dynamics of the traveling wave system of the coupled BBM-KdV system (1.9)

Substituting (1.3) to (1.9) and integrating the obtained system once, we have

$$\begin{aligned} &-\sigma\phi + \psi + \phi\psi + \frac{1}{6}\sigma\phi'' = g_1, \\ &-\sigma\psi + \phi + \frac{1}{2}\psi^2 + \frac{1}{6}\phi'' = g_2, \end{aligned} \tag{3.1}$$

where "$'$" denotes the derivative in ξ. (3.1) implies that

$$\phi = F(\psi) = \frac{\frac{1}{2}\sigma\psi^2 - (\sigma^2+1)\psi + g}{\psi - 2\sigma}, \tag{3.2}$$

where $g = g_1 - \sigma g_2$. We next take $g_1 = g_2 = g = 0$. Then, we have

$$\frac{d\phi}{d\psi} = \frac{\sigma(\psi^2 - 4\sigma\psi + 4\sigma^2 + 4)}{2(\psi-2\sigma)^2}, \quad \frac{d^2\phi}{d\psi^2} = -\frac{4\sigma}{(\psi-2\sigma)^3}$$

and

$$\frac{d^2\phi}{d\xi^2} = \frac{4\sigma}{(\psi-2\sigma)^3}\left[\frac{1}{8}\psi^3 - \frac{3}{4}\sigma\psi^2 + \frac{1}{2}(3\sigma+1)\psi + \sigma(\sigma+1) + (\psi')^2\right].$$

Thus, we know from the second equation of (3.1) that

$$\psi'' = \frac{-8\sigma(\psi')^2 + 6\psi(\psi^2 - 3\sigma\psi + 2\sigma^2 - 2)(\psi-2\sigma)^2}{\sigma(\psi-2\sigma)(\psi^2 - 4\sigma\psi + 4\sigma^2 + 4)}, \tag{3.3}$$

which is equivalent to the planar dynamical system as follows:

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{-8\sigma y^2 + 6\psi(\psi^2 - 3\sigma\psi + 2\sigma^2 - 2)(\psi - 2\sigma)^2}{\sigma(\psi - 2\sigma)(\psi^2 - 4\sigma\psi + 4\sigma^2 + 4)}. \tag{3.4}$$

System (3.4) has the first integral as follows:

$$\begin{aligned} H_2(\psi, y) =& \frac{\sigma(\psi - 2\sigma)^4 y^2 + (420 - 108\sigma^2)\psi^3 + (648\sigma^2 - 1464)\sigma\psi^2}{(\psi^2 - 4\sigma\psi - 4(\sigma^2 + 1))^2} \\ &+ \frac{(-1296\sigma^4 + 480\sigma^2 + 1392)\psi + (864\sigma^4 + 1536\sigma^2 + 672)\sigma}{(\psi^2 - 4\sigma\psi - 4(\sigma^2 + 1))^2} \\ &- 4\psi^3 + 6\sigma\psi^2 + 168\psi \\ &+ 240\sigma \ln(\psi^2 - 4\sigma\psi - 4(\sigma^2 + 1)) + (90\sigma^2 - 510)\arctan\left(\frac{1}{2}\psi - \sigma\right) = h. \end{aligned} \tag{3.5}$$

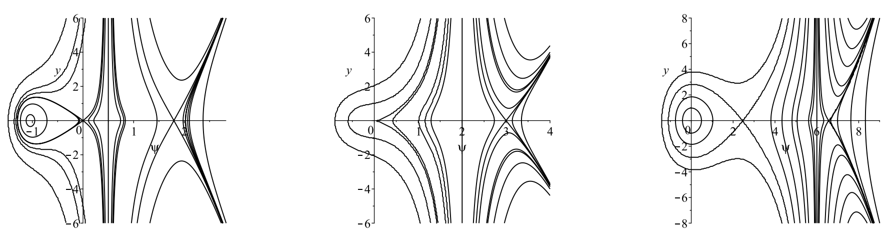
Consider the associated regular system of system (3.4) as follows:

$$\frac{d\psi}{d\zeta} = \sigma y(\psi-2\sigma)(\psi^2-4\sigma\psi+4\sigma^2+4)), \quad \frac{dy}{d\zeta} = -8\sigma y^2+6\psi(\psi^2-3\sigma\psi+2\sigma^2-2)(\psi-2\sigma)^2, \tag{3.6}$$

where $d\xi = (\psi - 2\sigma)(\psi^2 - 4\sigma\psi + 4\sigma^2 + 4))d\zeta$, for $\psi \neq 2\sigma$.

Notice that $\psi^2 - 3\sigma\psi + 2\sigma^2 - 2 = 0$ has two real roots $r_1 = \frac{3}{2}\sigma - \frac{1}{2}\sqrt{\sigma^2 + 8}$ and $r_2 = \frac{3}{2}\sigma + \frac{1}{2}\sqrt{\sigma^2 + 8}$. Clearly, if $\sigma \neq 1$, system (30) has three singular points at $O(0,0), E_1(r_1, 0), E_2(r_2, 0)$. The straight line $\psi = 2\sigma$ is a solution of system (3.6). When $\sigma = 1$, we have $r_1 = 0$.

Write that $h_0 = H_2(0,0), h_j = H_2(r_j, 0), j = 1,2$ and $h_s = H_2(2\sigma, 0)$. Similar to the discussion in section 2, for $\sigma > 0$, we obiain the bifurcations of the phase portraits of system (3.6) which are shown in Figure 4.



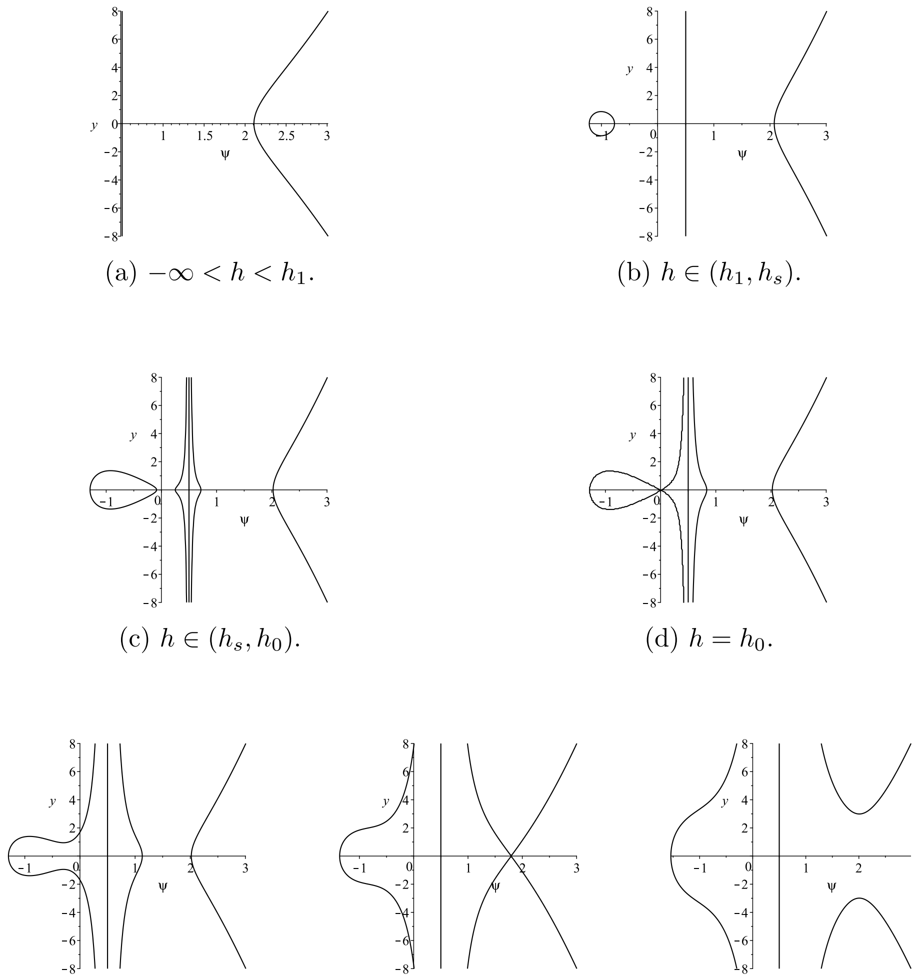
(a) $0 < \sigma < 1, h_1 < h_s < h_0 < h_2$. (b) $\sigma = 1$. (c) $\sigma > 1, h_2 < h_0 < h_s < h_1$.

Figure 4. The bifurcations of the phase portraits of system (3.6).

We see from Figure 4(a) that as h is varied, the level curves defined by $H_2(\psi, y) = h$ are changed which are shown in Figure 5.

Theorem 3.1. *For a given parameter* $0 < \sigma < 1$*, system* (3.4) *has the phase portrait given by Figure 3(a).*

(i) Corresponding to the family of periodic orbits defined by $H_2(\psi, y) = h, h \in (h_1, h_s)$*, which enclose the singular points* $E_1(\psi_1, 0)$*,* $w-$*component of the coupled BBM-KdV system* (1.9) *has a family of periodic wave solutions (see Figure 5(b)).*



(a) $-\infty < h < h_1$.

(b) $h \in (h_1, h_s)$.

(c) $h \in (h_s, h_0)$.

(d) $h = h_0$.

(e) $h \in (h_0, h_2)$.

(f) $h = h_2$.

(g) $h \in (h_2, \infty)$.

Figure 5. The changes of the level curves defined by $H_2(\psi, y) = h$ of system (3.4) in Fig. 4 (a).

(ii) Corresponding to the family of periodic orbits enclosing the singular points $E_1(\psi_1, 0)$ and two open orbit families which tend to the singular straight line $\psi = 2\sigma$ when $|y| \to \infty$, defined by $H_2(\psi, y) = h, h \in (h_s, h_0)$ (see Figure 5(c)), $w-$component of the coupled BBM-KdV system (1.9) *has a family of periodic wave solutions and two families of compacton solutions.*

(iii) Corresponding to the homoclinic orbit enclosing the singular points $E_1(\psi_1, 0)$ and an open orbit tending to the singular straight line $\psi = 2\sigma$ when $|y| \to \infty$, defined by $H_2(\psi, y) = h_0$ (see Figure 5(d)), the $w-$component of the coupled BBM-KdV system (1.9) *has a solitry solution and a compacton solution.*

(iv) Corresponding to two open orbit families tending to the singular straight line $\psi = 2\sigma$ when $|y| \to \infty$, defined by $H_2(\psi, y) = h, h \in (h_0, h_2)$ (see Figure 5(e)), the $w-$component of the coupled BBM-KdV system (1.9) *has two families of compacton solutions.*

(v) Corresponding to an open orbit family tending to the singular straight line $\psi = 2\sigma$ when $|y| \to \infty$, defined by $H_2(\psi, y) = h, h \in [h_2, \infty)$ (see Figure 5(f) and (g)), the $w-$component of the coupled BBM-KdV system (1.9) *has a family of compacton solutions.*

Similarly, for the orbits in Figure 4(b) and (c), we have corresponding conclusions about the traveling wave solutions.

Clearly, we can not use $H_2(\psi, y)$ and the first equation of system (3.4) to calculate any exact explicit solution of system (3.4).

4. The dynamics and possible exact solutions of the traveling wave system of the coupled KdV-BBM system (1.10)

Substituting (1.3) to (1.10) and integrating the obtained system once, we have

$$\begin{aligned} -\sigma\phi + \psi + \phi\psi + \frac{1}{6}\psi'' &= g_1, \\ -\sigma\psi + \phi + \frac{1}{2}\psi^2 + \frac{1}{6}\sigma\psi'' &= g_2. \end{aligned} \tag{4.1}$$

where "$'$" denotes the derivative in ξ. By the second equation of (4.1) we have $\phi = \sigma\psi - \frac{1}{2}\psi^2 - \frac{1}{6}\sigma\psi'' + g_2$. Hence, we see from the first equation of (4.1) that

$$\psi'' = \frac{3\psi^3 - 9\sigma\psi^2 - (6 - 6\sigma^2 + 6g_2)\psi + 6(\sigma g_2 + g_1)}{\sigma^2 + 1 - \sigma\psi}, \tag{4.2}$$

which is equivalent to the planar dynamical system as follows:

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{3\psi^3 - 9\sigma\psi^2 - (6 - 6\sigma^2 + 6g_2)\psi + 6(\sigma g_2 + g_1)}{\sigma^2 + 1 - \sigma\psi}, \tag{4.3}$$

with the singular straight line $\psi = \psi_s = \frac{\sigma^2+1}{\sigma}$.

When $\sigma \neq \frac{\sqrt{3}}{3}$, system (4.3) has the first integral:

$$\begin{aligned} H_3(\psi, y) = &\frac{1}{2}y^2 + \frac{1}{\sigma}\psi^3 + 3\left(\frac{1}{2\sigma^2} - 1\right)\psi^2 + 3\left(\frac{1-\sigma^2(2g_2+3)}{\sigma^3}\right)\psi \\ &+ 3\left(\frac{1}{\sigma^4} - \frac{2}{\sigma^2} + \frac{2g_1}{\sigma} - \frac{2g_2}{\sigma^2}\right)\ln|\sigma\psi - (\sigma^2+1)| = h. \end{aligned} \tag{4.4}$$

Assume that $g_1 = g_2 = 0$. Then, $f(\psi) = \psi^2 - 3\psi\sigma + 2(\sigma^2 - 1) = (\psi - r_1)(\psi - r_2)$, where $r_1 = \frac{3}{2}\sigma - \frac{1}{2}\sqrt{\sigma^2+8}, r_2 = \frac{3}{2}\sigma + \frac{1}{2}\sqrt{\sigma^2+8}$.

When $\sigma = \frac{\sqrt{3}}{3}, r_2 = \frac{\sigma^2+1}{\sigma}$. Hence, system (4.3) becomes that

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = -\sqrt{3}\psi\left(3\psi + \sqrt{3}\right) \tag{4.5}$$

with the first integral

$$H_3(\psi, y) = \frac{1}{2}y^2 + \sqrt{3}\psi^3 + \frac{3}{2}\psi^2 = h. \tag{4.6}$$

Obviously, when $\sigma \neq \frac{\sqrt{3}}{3}$, system (4.3) has three singular points at $E_1(r_1, 0), O(0,0)$ and $E_2(r_2, 0)$. When $\sigma = 1, r_1 = 0, r_2 = 3$. When $\sigma > 1, 0 < r_1 < \frac{\sigma^2+1}{\sigma} < r_2$.

Write that $h_0 = H_3(0,0), h_j = H_2(r_j, 0), j = 1, 2$ and $h_s = H_3(\psi_s, 0)$. Similar to the discussion in section 2, for $\sigma > 0$, with the change of σ, we obtain the bifurcations of the phase portraits of system (4.3) which are shown in Figure 6.

As an example, we consider the changes of the level curves defined by $H_3(\psi, y) = h$ of system (4.3) in Figure 6(e), which are shown in Figure 7.

We see from Figure 7, the following conclusion holds.

Theorem 4.1. *Suppose that $g_1 = g_2 = 0, \sigma > 1$. System* (4.3) *has the phase portrait Figure 6(e).*

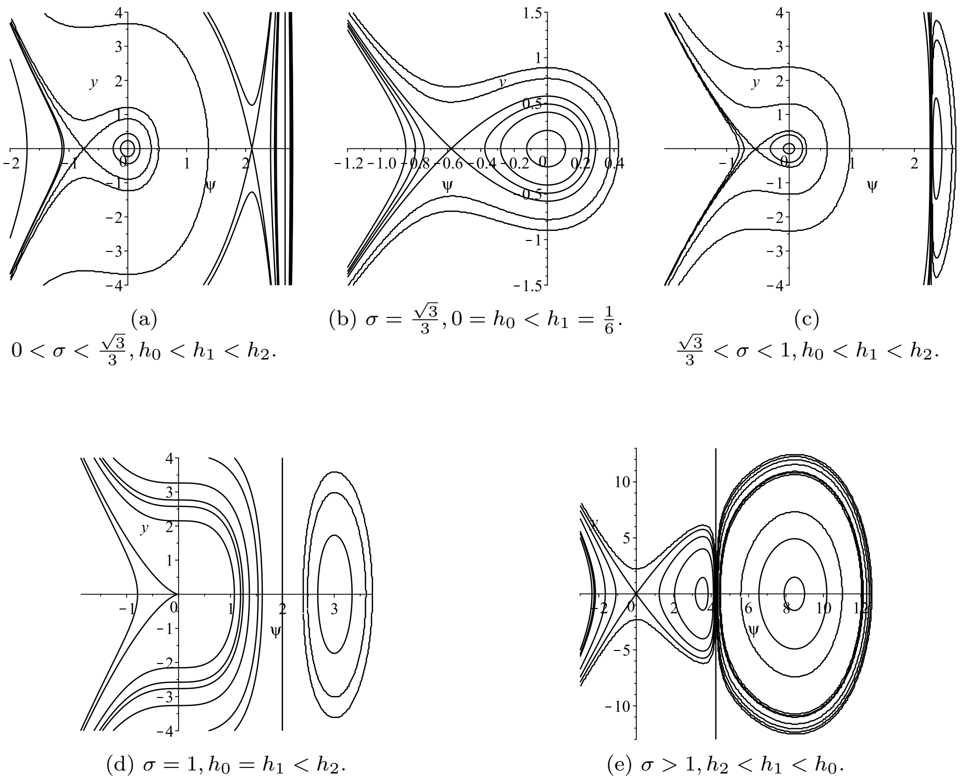
(i) Corresponding to the family of periodic orbits defined by $H_3(\psi, y) = h, h \in (h_2, h_1)$, which enclose the singular point $E_1(r_2, 0)$, $w-$component of the coupled KdV-BBM system (1.10) *has a family of periodic wave solutions.*

(ii) Corresponding to the two families of periodic orbits defined by $H_3(\psi, y) = h, h \in (h_1, h_0)$, which enclose the singular points $E_1(r_1, 0)$ and $E_2(r_2, 0)$, respectively, $w-$component of the coupled KdV-BBM system (1.10) *has two families of periodic wave solutions.*

(iii) Corresponding to the homoclinic orbit and a periodic orbit enclosing the singular point $E_2(r_2, 0)$ defined by $H_1(\psi, y) = h_0$, $w-$component of the coupled KdV-BBM system (1.10) *has a pseudo-peakon solution and a pseudo-periodic peakon solution.*

*(iv) Corresponding to the family of periodic orbits defined by $H_1(\psi, y) = h, h \in (h_0, \infty)$, which enclose the singular points $E_2(r_2, 0)$, $w-$component of the coupled KdV-BBM system ((*1.10) *has a family of pseudo-periodic peakon solutions.*

For the homoclinic orbit in Figure 6(b) defined by $H_3(\psi, y) = \frac{1}{2}y^2 + \sqrt{3}\psi^3 + \frac{3}{2}\psi^2 = \frac{1}{6}$, i.e., $y^2 = 2\sqrt{3}\left(\psi + \frac{\sqrt{3}}{3}\right)^2\left(\frac{\sqrt{3}}{6} - \psi\right)$, by using the first equation of (4.5),



(a) $0 < \sigma < \frac{\sqrt{3}}{3}, h_0 < h_1 < h_2$.

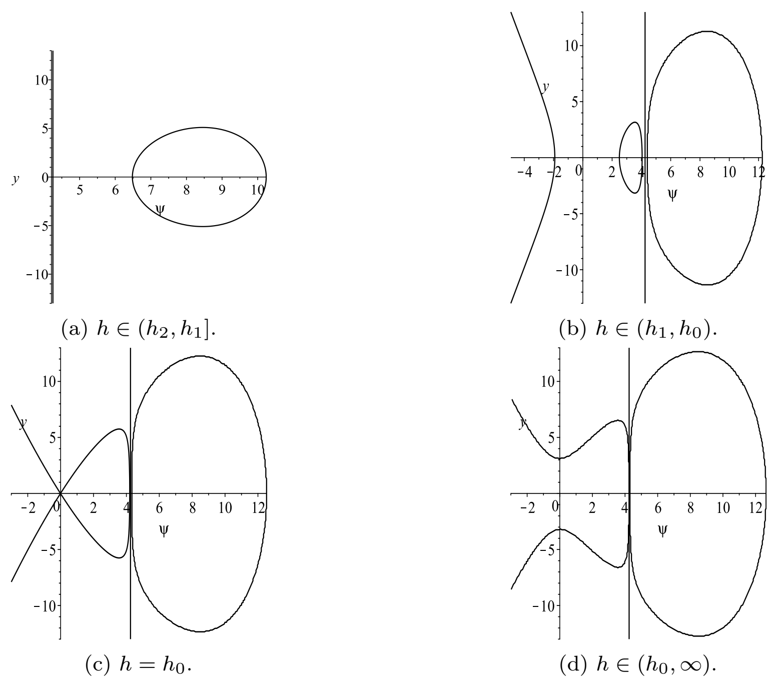
(b) $\sigma = \frac{\sqrt{3}}{3}, 0 = h_0 < h_1 = \frac{1}{6}$.

(c) $\frac{\sqrt{3}}{3} < \sigma < 1, h_0 < h_1 < h_2$.

(d) $\sigma = 1, h_0 = h_1 < h_2$.

(e) $\sigma > 1, h_2 < h_1 < h_0$.

Figure 6. The bifurcations of the phase portraits of system (4.3).



(a) $h \in (h_2, h_1]$.

(b) $h \in (h_1, h_0)$.

(c) $h = h_0$.

(d) $h \in (h_0, \infty)$.

Figure 7. The changes of the level curves defined by $H_3(\psi, y) = h$ of system (4.3) in Fig. 6 (e).

we have $\sqrt{2\sqrt{3}}\xi = \int_{\psi}^{\frac{\sqrt{3}}{6}} \frac{d\psi}{(\psi+\frac{\sqrt{3}}{3})\sqrt{\frac{\sqrt{3}}{6}-\psi}}$. It gives rise to the following exact solution:

$$\begin{aligned} w &= \psi(\xi) = \psi_0(\xi) = -\frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2}\mathrm{sech}^2\left(\frac{1}{2}\sqrt{3}\xi\right), \\ \eta &= \phi(\xi) = \sigma\psi_0(\xi) - \frac{1}{2}(\psi_0(\xi))^2 - \frac{1}{6}\sigma(\psi_0(\xi))''. \end{aligned} \tag{4.7}$$

For the periodic orbit family in Figure 6(b) defined by $H_3(\psi, y) = \frac{1}{2}y^2 + \sqrt{3}\psi^3 + \frac{3}{2}\psi^2 = h, h \in \left(0, \frac{1}{6}\right)$ i.e., $y^2 = 2h - 3\psi^2 - 2\sqrt{3}\psi^3 = 2\sqrt{3}(z_1 - \psi)(\psi - z_2)(\psi - z_3)$, by using the first equation of (4.5), we have the following exact periodic solution:

$$\begin{aligned} w &= \psi(\xi) = \psi_p(\xi) = z_1 - (z_1 - z_2)\mathrm{sn}^2(\Omega_2\xi, k), \\ \eta &= \phi(\xi) = \sigma\psi_p(\xi) - \frac{1}{2}(\psi_p(\xi))^2 - \frac{1}{6}\sigma(\psi_p(\xi))'', \end{aligned} \tag{4.8}$$

where $k^2 = \frac{z_1-z_2}{z_1-z_3}, \Omega_2 = \frac{\sqrt{2\sqrt{3}(z_1-z_3)}}{2}$.

5. The exact solutions and dynamics of the traveling wave solutions of the Kaup system (1.11)

Substituting (1.3) to (1.11) and integrating the obtained system once, we have

$$\begin{aligned} &-\sigma\phi + \psi + \phi\psi + \frac{1}{3}\psi'' = g_1, \\ &-\sigma\psi + \phi + \frac{1}{2}\psi^2 = g_2, \end{aligned} \tag{5.1}$$

where "′" denotes the derivative in ξ. We see from the second equation of (5.1) that $\phi = g_2 + \sigma\psi - \frac{1}{2}\psi^2$. Thus, we obtain from the first equation of (5.1) that

$$\psi'' - \frac{3}{2}\psi^3 + \frac{9}{2}\sigma\psi^2 + 3(g_2 + 1 - \sigma^2)\psi - 3(\sigma g_2 + g_1) = 0. \tag{5.2}$$

Write $\alpha = -2(g_2 + 1 - \sigma), \beta = 2(\sigma g_2 + g_1)$. We next assume that $\beta = 0$, i.e., $g_2 = -\frac{g_1}{\sigma}$. Consider the planar cubic system:

$$\frac{d\psi}{d\xi} = y, \quad \frac{dy}{d\xi} = \frac{3}{2}\psi(\psi^2 - 3\sigma\psi + \alpha) \tag{5.3}$$

with the first integral

$$H_4(\psi, y) = \frac{1}{2}y^2 - \frac{3}{8}\psi^4 + \frac{3}{2}\sigma\psi^3 - \frac{3}{4}\alpha\psi^2 = h. \tag{5.4}$$

When $\Delta = 9\sigma^2 - 4\alpha > 0$, system (5.3) has three singular points $O(0,0)$ and $E_1(r_1, 0), E_2(r_2, 0)$, where $r_1 = \frac{3}{2}\sigma - \frac{1}{2}\sqrt{\Delta}, r_2 = \frac{3}{2}\sigma + \frac{1}{2}\sqrt{\Delta}$.

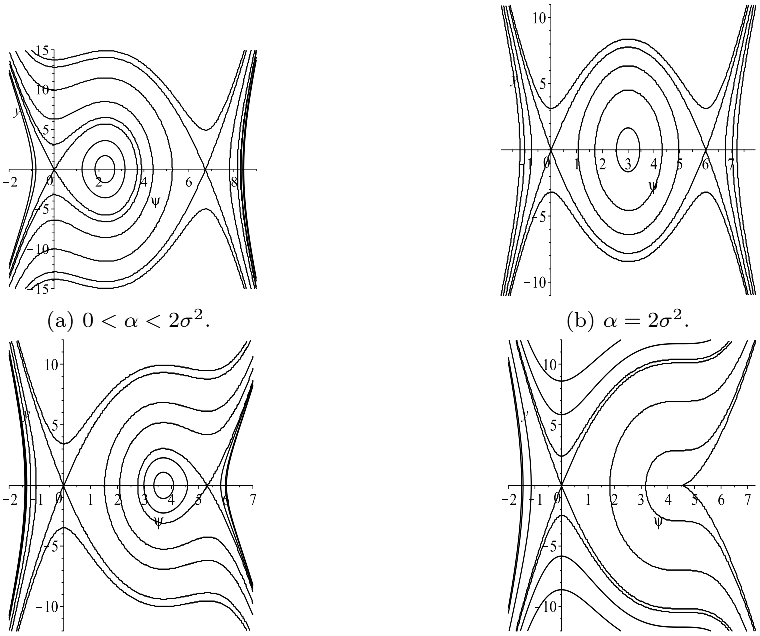
We have $h_0 = H_4(0,0) = 0$ and

$$h_1 = H_4(r_1, 0) = \frac{1}{16}\left[81\sigma^4 - 54\alpha\sigma^2 + 6\alpha^2 + (12\alpha\sigma - 27\sigma^3)\sqrt{\Delta}\right],$$

$$h_2 = H_4(r_2, 0) = \frac{1}{16}\left[81\sigma^4 - 54\alpha\sigma^2 + 6\alpha^2 - (12\alpha\sigma - 27\sigma^3)\sqrt{\Delta}\right].$$

$\alpha = 2\sigma^2$ implies that $h_2 = h_0 = 0$ and $\alpha = \frac{9}{4}\sigma^2$ implies that $\Delta = 0, h_1 = h_2$.

By using the above fact to do qualitative analysis, for a fixed $\sigma > 0$, we have the bifurcations of the phase portraits of system (5.3) for $\alpha > 0$ which are shown in Figure 8.



(a) $0 < \alpha < 2\sigma^2$. (b) $\alpha = 2\sigma^2$.

(c) $2\sigma^2 < \alpha < \frac{9}{4}\sigma^2$. (d) $\alpha = \frac{9}{4}\sigma^2$.

Figure 8. The bifurcations of the phase portraits of system (5.3) for $\alpha > 0$.

We next calculate the exact solutions for system (5.3).

(i) Corresponding to the homoclinic orbit defined by $H_4(\psi, y) = 0$ in Figure 8(a), we have $y^2 = \frac{3}{4}\psi^2(\psi_L - \psi)(\psi_M - \psi)$, where $\psi_M = 2\sigma - \sqrt{4\sigma^2 - 2\alpha}, \psi_L = 2\sigma + \sqrt{4\sigma^2 - 2\alpha}$. By using the first equation of (5.3), we know that $\frac{1}{2}\sqrt{3}\xi = \int_{\psi}^{\psi_M} \frac{d\psi}{\psi\sqrt{(\psi_L - \psi)(\psi_M - \psi)}}$. It gives rise to the following solitaty wave solution of the Kaup system (1.11):

$$\begin{aligned} \psi(\xi) &= \psi_{ho}(\xi) = \frac{2\psi_L\psi_M}{(\psi_L - \psi_M)\cosh\left(\frac{1}{2}\sqrt{3\psi_L\psi_M}\xi\right) + (\psi_M + \psi_L)}, \\ \phi(\xi) &= -\frac{g_1}{\sigma} + \sigma\psi_{ho}(\xi) - \frac{1}{2}\psi_{ho}^2(\xi). \end{aligned} \tag{5.5}$$

(ii) Corresponding to the periodic orbit family defined by $H_4(\psi, y) = h, h \in (h_1, 0)$ in Figure 8(a), we have $y^2 = \frac{3}{4}(z_1 - \psi)(z_2 - \psi)(\psi - z_3)(\psi - z_4), z_4 < 0 < z_3 < \psi_1 < z_2 < \psi_2 < z_1$ and $\frac{1}{2}\sqrt{3}\xi = \int_{z_3}^{\psi} \frac{d\psi}{\sqrt{(z_1-\psi)(z_2-\psi)(\psi-z_3)(\psi-z_4)}}$. Thus, we obtain the parametric representations of the family of periodic wave solutions of the Kaup system (1.11) as follows:

$$\begin{aligned}\psi(\xi) &= \psi_{per}(\xi) = z_4 + \frac{z_3 - z_4}{1 - \hat{\alpha}^2 \mathrm{sn}^2(\Omega_1 \xi, k)},\\ \phi(\xi) &= -\frac{g_1}{\sigma} + \sigma\psi_{per}(\xi) - \frac{1}{2}\psi_{per}^2(\xi),\end{aligned} \tag{5.6}$$

where $\hat{\alpha}^2 = \frac{z_2 - z_3}{z_2 - z_4}, k^2 = \frac{\hat{\alpha}^2(z_1 - z_4)}{z_1 - z_3}, \Omega_1 = \frac{1}{4}\sqrt{3(z_1 - z_3)(z_2 - z_4)}$.

(iii) Corresponding to the heteroclinic orbits defined by $H_4(\psi, y) = 0$ in Figure 8(b), we have $y^2 = \frac{3}{4}\psi^2(2\sigma - \psi)^2$ and $\frac{1}{2}\sqrt{3}\xi = \int_\sigma^\psi \frac{d\psi}{\psi(2\sigma - \psi)}$. Hence, we obtain the parametric representations of a kink wave solution and an anti-kink wave solution of the Kaup system (1.11) as follows:

$$\begin{aligned}\psi(\xi) &= \psi_{kink}(\xi) = \frac{2\sigma}{1 + e^{\mp\sqrt{3}\sigma\xi}},\\ \phi(\xi) &= -\frac{g_1}{\sigma} + \sigma\psi_{kink}(\xi) - \frac{1}{2}\psi_{kink}^2(\xi).\end{aligned} \tag{5.7}$$

(iv) Corresponding to the homoclinic orbit defined by $H_4(\psi, y) = h_2$ in Figure 8(c), we have $y^2 = \frac{3}{4}(r_2 - \psi)^2(\psi - \psi_m)(\psi - \psi_l)$. By using the first equation of (5.3), we know that $\frac{1}{2}\sqrt{3}\xi = \int_{\psi_m}^\psi \frac{d\psi}{(r_2 - \psi)\sqrt{(\psi - \psi_m)(\psi - \psi_l)}}$. It gives rise to the following solitaty wave solution of the Kaup system (1.11):

$$\begin{aligned}\psi(\xi) &= \psi_{hom}(\xi) = r_2 - \frac{2(r_2 - \psi_m)(r_2 - \psi_l)}{(\psi_m - \psi_l)\cosh\left(\frac{1}{2}\sqrt{3(r_2 - \psi_m)(r_2 - \psi_l)}\xi\right) + (2r_2 - \psi_m - \psi_l)},\\ \phi(\xi) &= -\frac{g_1}{\sigma} + \sigma\psi_{hom}(\xi) - \frac{1}{2}\psi_{hom}^2(\xi).\end{aligned} \tag{5.8}$$

(v) Corresponding to the periodic orbit family defined by $H_4(\psi, y) = h, h \in (h_1, 0)$ in Figure 8(b) and $H_4(\psi, y) = h, h \in (h_1, h_2)$ in Figure 8 (c), we have the same parametric representations of the family of periodic wave solutions of the Kaup system (1.11) as (5.6).

To sum up, we obtain the following conclusion.

Theorem 5.1. *(i) The Kaup system* (1.11) *has an exact explicit solitary wave solutions given by* (5.5) *and* (5.8)*;*

(ii) The Kaup system (1.11) *has an exact explicit kink wave and an anti-kink wave solution given by* (5.7)*;*

(iii) The Kaup system (1.11) *has an exact explicit periodic wave solution family given by* (5.6).

6. The traveling wave equation of the coupled BBM system (1.12)

Substituting (1.3) to (1.12) and integrating the obtained system once, we have

$$\begin{aligned} &-\sigma\phi+\psi+\phi\psi+\frac{1}{6}\sigma\phi''=g_1,\\ &-\sigma\psi+\phi+\frac{1}{2}\psi^2+\frac{1}{6}\sigma\psi''=g_2, \end{aligned} \tag{6.1}$$

where "$'$" denotes the derivative in ξ. We see from the second equation of (6.1) that $\phi=g_2+\sigma\psi-\frac{1}{2}\psi^2-\frac{1}{6}\sigma\psi''$. Thus, by taking $g_1=g_2=0$, from the first equation of (6.1), we obtain

$$\psi''''=12\psi''-\frac{6}{\sigma}(\psi')^2-\frac{12}{\sigma}\psi\psi''-\frac{18}{\sigma^2}\psi^3+\frac{54}{\sigma}\psi^2+36\left(\frac{1}{\sigma^2}-1\right). \tag{6.2}$$

Making the transformation $\psi=\sigma(1-Y)$, (6.2) becomes that

$$Y''''=12YY''+6(Y')^2-18Y^3+18\left(1+\frac{2}{\sigma^2}\right)Y-\frac{16}{\sigma^2}. \tag{6.3}$$

Equation (6.3) is a fourth order ordinary differential equation, in [8], [9], [10], by using Cosgrove's higher-order Painleve equations (see [4]), we have obtained a lot of exact explicit solutions for some higher-order traveling wave systems. We notice that equation (6.3) is not the Cosgrove's higher-order Painleve equations (F-III), (F-V) and (F-VI), i.e.,

$$y^{(iv)}=15yy''+\frac{45}{4}(y')^2-15y^3+\alpha y+\beta, \tag{6.4}$$

$$y^{(iv)}=20yy''+10(y')^2-40y^3+\alpha y+\beta, \tag{6.5}$$

$$y^{(iv)}=18yy''+9(y')^2-24y^3+\alpha y^2+\frac{1}{9}\alpha^2 y+\beta. \tag{6.6}$$

Therefore, equation (6.3) is not an integrable system. We can not obtain the exact explicit solution for equation (6.3).

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