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Existence of Weak Solutions for a Kind of Parabolic Steklov Problems

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Abstract Our focus in this study revolves around investigating the following parabolic problem

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega, \ t > 0, \\ \frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega, \ t > 0, \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases}$$

By using the Galerkin approximation and a family of potential wells, we obtain the existence of global solution and finite time blow-up under some suitable conditions. On the other hand, the results for asymptotic behavior of certain solutions with positive initial energy are also given.

Keywords Parabolic problem, global existence, blow-up, asymptotic behavior

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1. Introduction and preliminaries

The motivation for parabolic problems in the context of partial differential equations (PDEs) comes from various fields of physics, engineering and applied sciences. Parabolic equations model phenomena that evolve in time and space, where diffusion plays an important role, such as heat diffusion (heat equation), matter diffusion (diffusion equation), viscous fluid motion (Navier-Stokes equation), wave propagation in a dissipative medium, etc. (see [3, 7–10, 13, 17, 19, 21, 22]). The physical modeling of such equations often involves the numerical resolution of these equations using methods such as the finite difference method, the finite element method or the finite volume method. These methods discretize the continuous equations onto a spatial grid and solve the problem numerically to obtain an approximate solution that represents the physical behavior of the system under study (see [1, 2, 5, 15, 16]).

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In the present article, we mainly study the following Steklov parabolic problem

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = g(u) & \text{on } \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a smooth boundary $\partial\Omega$, and $g(u)$ satisfies the conditions as follows:

$$\text{(C)} \quad \begin{cases} g \in C^1 \text{ and } g(0) = g'(0) = 0, \\ g(u) \text{ is monotone, and is convex for } u > 0, \text{concave for } u < 0, \\ (p+1)G(u) \leqslant ug(u), |ug(u)| \leqslant \mu|G(u)|, \end{cases} \tag{1.2}$$

where

$$G(u) = \int_0^u g(s)\mathrm{d}s,$$

and

$$\begin{cases} 2 < p+1 \leqslant \mu < \infty & \text{if } N = 2, \\ 2 < p+1 \leqslant \mu \leqslant \frac{2(N-1)}{N-2} & \text{if } N \geqslant 3. \end{cases}$$

In the literature, there are several works dealing with Steklov-type parabolic problems (see [11,14,18]). For example, in [11] C. Enache has treated the following quasilinear initial-boundary value problem:

$$\begin{cases} u_t = \operatorname{div}(b(u)\nabla u) + f(u) & \text{in } \Omega,\ t > 0 \\ \frac{\partial u}{\partial n} + \kappa u = 0 & \text{on } \partial\Omega,\ t > 0, \\ u(x,0) = h(x) \geq 0 & \text{in } \Omega. \end{cases}$$

Under the suitable assumptions on the functions b, f and h, the author established a sufficient condition to guarantee the occurrence of the blow-up. Moreover, a lower bound for the blow-up time was obtained.

Also, L. E. Payne and P. W. Schaefer in [18] considered the heat equation subject to a nonlinear boundary condition, i.e.

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = f(u) & \text{on } \partial\Omega,\ t > 0, \\ u(x,0) = g(x) \geq 0 & \text{in } \Omega, \end{cases}$$

where Ω is a bounded smooth convex domain in $\mathbb{R}^3$ and f satisfies the condition

$$0 \leqslant f(s) \leqslant ks^{(n+2)/2}, \quad s > 0,$$

for some positive constants k and $n \geqslant 1$. By using a differential inequality technique, the authors determined a lower bound on the blow-up time for solutions of the heat equation when the solution explosion occurs. In addition, a sufficient condition which implies that blow-up does occur was determined.

In [14], A. Lamaizi et al. have considered the following problem:

$$\begin{cases} u_t - \Delta u + u = 0 & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = \lambda |u|^{p-1}u & \text{on } \partial\Omega \times (0,T), \\ u(x;0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is an open bounded domain for $n \geq 2$ with a smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, u_t denotes the partial derivative with respect to the time variable t and ∇u denotes the one with respect to the space variable x, $\lambda > 0$, and p satisfies

$$(H) \quad \begin{cases} 1 \leq p \leq \frac{n}{n-2} & \text{if } n > 2, \\ 1 \leq p < \infty & \text{if } n = 2. \end{cases}$$

By using the Galerkin approximation, they established the existence of global weak solution and finite time blow-up under some suitable conditions. So, a natural question arises, can we obtain some qualitative results such as the existence and blow up of solutions if we replace the term $\lambda|u|^{p-1}u$ by the function $g(u)$ which satisfies condition (C)? Then, the goal of this article is to give a positive answer to this question. More precisely, we will establish existence and blow up results by applying Galerkin approximation and similar techniques to those used in [14].

Throughout this work, we designate the Lebesgue space $L^p(\Omega)$ by :

$$L^p(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } \int_\Omega |u(x)|^p dx < +\infty \right\}$$

equipped with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

For $p = \infty$, we denote

$$L^\infty(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that ess-}\sup_\Omega |u| < +\infty \right\}$$

with

$$\text{ess-}\sup_\Omega |u| = \inf\{C > 0 \text{ such that } |u(x)| \leq C \text{ a.e. } \Omega\}.$$

Especially, for $p = 2$, the scalar product of $L^2(\Omega)$ will be denoted by $\langle \cdot, \cdot \rangle$ and the scalar product of $L^2(\partial\Omega, \rho)$ will be denoted by $\langle \cdot, \cdot \rangle_0$:

$$\langle u, v \rangle = \int_\Omega uv \, dx, \quad \langle u, v \rangle_0 = \oint_{\partial\Omega} uv \, d\rho.$$

Moreover, usual Sobolev space on Ω is defined by

$$W^{1,2}(\Omega) = \left\{ u \in L^2(\Omega) : |\nabla u| \in L^2(\Omega) \right\},$$

and it is equipped with the norm

$$\|u\|_{1,2}^2 = \|u\|_2^2 + \|\nabla u\|_2^2.$$

Recall the following embedding result.

Lemma 1.1. *(See [4]) The trace operator* $u : W^{1,q}(\Omega) \to L^r(\partial\Omega, \rho)$ *is continuous if and only if*

$$\begin{cases} 1 \leq r \leq q^\partial & \text{if } q \neq N, \\ 1 \leq r < \infty & \text{if } q = N, \end{cases}$$

where

$$q^\partial := \begin{cases} \frac{q(N-1)}{N-q} & \text{if } 1 < q < N, \\ \infty & \text{if } q \geq N. \end{cases}$$

Let X be a Banach space and $T>0$. Denote the following spaces:

$$C([0,T];X)=\{u:[0,T]\longrightarrow X \text{ continue }\},$$

$$L^p(0,T;X)=\left\{u:[0,T]\longrightarrow X \text{ is measurable such that } \int_0^T \|u(t)\|_X^p dt<\infty\right\},$$

equipped with the norm

$$\|u\|_{L^p(0,T;X)}=\left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}},$$

and

$$\begin{aligned}&L^\infty(0,T;X)\\ &=\{u:[0,T]\longrightarrow X \text{ is measurable such that } :\exists C>0;\|u(t)\|_X<C \text{ a.e.t}\},\end{aligned}$$

endowed with the norm

$$\|u\|_{L^\infty(0,T;X)}=\inf\{C>0;\|u(t)\|_X<C \text{ a.e.t}\}.$$

Next, let us introduce some sets and functions as follows

$$B(u)=\frac{1}{2}\|u\|_{1,2}^2-\int_{\partial\Omega}G(u)\mathrm{d}\rho,$$

$$A(u)=\|u\|_{1,2}^2-\int_{\partial\Omega}ug(u)\mathrm{d}\rho,$$

and

$$S=\left\{u\in W^{1,2}(\Omega)\mid A(u)>0,B(u)<h\right\}\cup\{0\},$$

where

$$h=\inf_{u\in Y}B(u),$$

$$Y=\left\{u\in W^{1,2}(\Omega)\mid A(u)=0,\|u\|_{1,2}\neq 0\right\},$$

and

$$U=\left\{u\in W^{1,2}(\Omega)\mid A(u)<0,B(u)<h\right\}.$$

For $\delta>0$ we further define

$$A_\delta(u)=\delta\|u\|_{1,2}^2-\int_{\partial\Omega}ug(u)\mathrm{d}\rho,$$

$$h(\delta)=\inf_{u\in Y_\delta}B(u),$$

$$Y_\delta=\left\{u\in W^{1,2}(\Omega)\mid A_\delta(u)=0,\|u\|_{1,2}\neq 0\right\},$$

$$S_\delta=\left\{u\in W^{1,2}(\Omega)\mid A_\delta(u)>0,B(u)<h(\delta)\right\}\cup\{0\},\quad 0<\delta<b,$$

and

$$U_\delta=\left\{u\in W^{1,2}(\Omega)\mid A_\delta(u)<0,B(u)<h(\delta)\right\}.$$

2. Global existence

In this section, we prove the existence of a global weak solution to our problem. We prepare the proof by a series of lemmas.

Lemma 2.1. *([17]). Let $g(u)$ satisfy* (C). *Then*

1. $|G(u)| \leqslant M|u|^{\mu}$ *for some $M > 0$ and all $u \in \mathbb{R}$.*
2. $G(u) \geqslant N|u|^{p+1}$ *for some $N > 0$ and $|u| \geqslant 1$.*
3. *The equality $u\left(ug'(u) - g(u)\right) \geqslant 0$ holds only for $u = 0$.*

As a result, the following corollary is obtained.

Corollary 2.1. *Let $g(u)$ satisfy* (C). *Then*

1. $|ug(u)| \leqslant \mu M|u|^{\mu}, |g(u)| \leqslant \mu M|u|^{\mu-1}$ *for all $u \in \mathbb{R}$.*
2. $ug(u) \geqslant (p+1)N|u|^{p+1}$ *for $|u| \geqslant 1$.*

Lemma 2.2. *Suppose that $0 < B(u) < h$ for some $u \in W^{1,2}(\Omega), \delta_1 < \delta_2$ are the two roots of equation $h(\delta) = B(u)$. Then the sign of $A_\delta(u)$ does not change for $\delta_1 < \delta < \delta_2$.*

Proof. Arguing by contradiction, we assume that the sign of $A_\delta(u)$ is changeable for $\delta_1 < \delta < \delta_2$, then there exists a $\delta_0 \in (\delta_1, \delta_2)$ such that $A_{\delta_0}(u) = 0$. From $B(u) > 0$ we get $\|u\|_{1,2} \neq 0$, hence $u \in Y_{\delta_0}$, consequently $B(u) \geq h(\delta_0)$, which contradicts

$$B(u) = h\left(\delta_1\right) = h\left(\delta_2\right) < h(\delta_0).$$

□

Lemma 2.3. *Let $g(u)$ satisfy* (C), *$u_0(x) \in W^{1,2}(\Omega), 0 < e < h$ and $\delta_1 < \delta_2$ be the two roots of equation $h(\delta) = e$. Then, all weak solutions $u(t)$ of problem* (1.1) *with $B\left(u_0\right) = e$ belong to S_δ for $\delta_1 < \delta < \delta_2, 0 \leq t < T$, provided $A\left(u_0\right) > 0$.*

Proof. By $B\left(u_0\right) = e, A\left(u_0\right) > 0$ and Lemma 2.2, we can deduce $A_\delta\left(u_0\right) > 0$ and $B\left(u_0\right) < h(\delta)$, i.e., $u_0(x) \in S_\delta$ for $\delta_1 < \delta < \delta_2$. Let $u(t)$ be any weak solution of problem (1.1) with $B\left(u_0\right) = e$ and $A\left(u_0\right) > 0$, and let T be the maximal existence time of $u(t)$. Arguing by contradiction, we suppose that there exists a $\delta_0 \in (\delta_1, \delta_2)$ and $t_0 \in (0, T)$ such that $A_{\delta_0}\left(u\left(t_0\right)\right) = 0, \left\|u\left(t_0\right)\right\|_{1,2} \neq 0$ or $B\left(u\left(t_0\right)\right) = h\left(\delta_0\right)$. From (2.2), we get

$$\int_0^t \|u_\tau\|_2^2 \,\mathrm{d}\tau + B(u) \leq B\left(u_0\right) < h(\delta), \quad \delta_1 < \delta < \delta_2,\ 0 \leq t < T. \tag{2.1}$$

Therefore $B\left(u\left(t_0\right)\right) \neq h\left(\delta_0\right)$. If $A_{\delta_0}\left(u\left(t_0\right)\right) = 0, \left\|u\left(t_0\right)\right\|_{1,2} \neq 0$, then the definition of $h(\delta)$ implies that $B\left(u\left(t_0\right)\right) \geq h\left(\delta_0\right)$, which contradicts (2.1). □

Definition 2.1. Let $T > 0$. A function $u = u(x,t) \in L^\infty\left(0, \infty; W^{1,2}(\Omega)\right) \cap C\left([0,T]; L^2\left(\Omega\right) \times L^2\left(\partial\Omega, \rho\right)\right)$ with $u_t(t) \in L^2\left(0, \infty; L^2(\Omega)\right)$ is said to be a **weak solution** to the problem (1.1) in $\Omega \times [0,T)$, if $u(x,0) = u_0 \in W^{1,2}(\Omega)$, and satisfies

$$\langle u_t, v\rangle + \langle u, v\rangle + \langle \nabla u, \nabla v\rangle = \langle g(u), v\rangle_0, \quad \forall v \in W^{1,2}(\Omega), t \in (0,T).$$

Moreover,

$$\int_0^t \|u_t\|_2^2 \ \mathrm{d}\tau + B(u) \leqslant B\left(u_0\right), \quad \forall t \in [0,T). \tag{2.2}$$

Now, we present the first main result of this paper.

Theorem 2.1. *Let $u_0(x) \in W^{1,2}(\Omega)$ and $g(u)$ satisfy* (C). *Suppose that $0 < B(u_0) < h$ and $A(u_0) > 0$. Then problem* (1.1) *admits a global weak solution $u(t) \in L^\infty\left(0,\infty; W^{1,2}(\Omega)\right) \cap C\left([0,T]; L^2(\Omega) \times L^2(\partial\Omega,\rho)\right)$ with $u_t(t) \in L^2\left(0,\infty; L^2(\Omega)\right)$ and $u(t) \in S$ for $0 \leqslant t < \infty$.*

Proof of Theorem 2.1 The idea of proof is classical. For more information see [12,20]. Let $w_j(x)$ be a system of base functions in $W^{1,2}(\Omega)$. Define the approximate solutions $u_m(x,t)$ of problem (1.1) by

$$u_m(x,t) = \sum_{j=1}^{m} \Phi_{jm}(t) w_j(x), \quad m = 1,2,\dots,$$

verifying

$$\langle u_{mt}, w_s\rangle + \langle u_m, w_s\rangle + \langle \nabla u_m, \nabla w_s\rangle = \langle g(u_m), w_s\rangle_0, \quad s = 1,2,\dots,m, \tag{2.3}$$

$$u_m(x,0) = \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0(x) \quad \text{in } W^{1,2}(\Omega). \tag{2.4}$$

Multiplying (2.3) by $\Phi'_{sm}(t)$ and summing for s yields

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau + B(u_m) \leqslant B(u_0) < h, \quad \forall t \in [0,T), \tag{2.5}$$

and $u_m \in S$ for sufficiently large m and $0 \leqslant t < \infty$ (see the proof of Lemma 2.3).

Combining (2.5) and

$$\begin{aligned} B(u_m) = \frac{1}{2}\|u\|_{1,2}^2 - \int_{\partial\Omega} G(u_m)\,\mathrm{d}\rho &\geqslant \frac{1}{2}\|u\|_{1,2}^2 - \frac{1}{p+1}\int_{\partial\Omega} u_m g(u_m)\,\mathrm{d}\rho \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\|u\|_{1,2}^2 + \frac{1}{p+1}A(u_m) \\ &\geqslant \frac{p-1}{2(p+1)}\|u\|_{1,2}^2, \end{aligned}$$

we obtain

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau + \frac{p-1}{2(p+1)}\|u_m\|_{1,2}^2 < h, \quad 0 \leqslant t < \infty \tag{2.6}$$

for sufficiently large m.

From (2.6), we get

$$\|u_m\|_{1,2}^2 < \frac{2(p+1)}{p-1}h, \quad 0 \leqslant t < \infty, \tag{2.7}$$

$$\|u_m\|_{\mu,\partial\Omega}^2 \leqslant C_*^2\|u_m\|_{1,2}^2 < C_*^2\frac{2(p+1)}{p-1}h, \quad 0 \leqslant t < \infty, \tag{2.8}$$

where C_* is the embedding constant from $W^{1,2}(\Omega)$ into $L^\mu(\partial\Omega,\rho)$.

$$\begin{aligned} \|g(u_m)\|_{q,\partial\Omega}^q &\leqslant \int_{\partial\Omega}\left(\mu M |u_m|^{\mu-1}\right)^q \, \mathrm{d}\rho \\ &= (\mu M)^q \|u_m\|_{\mu,\partial\Omega}^\mu \\ &\leqslant (\mu M)^q C_*^\mu \left(\frac{2(p+1)}{p-1}h\right)^{\mu/2}, \quad q = \frac{\mu}{\mu-1}, \quad 0 \leqslant t < \infty, \end{aligned} \tag{2.9}$$

and

$$\int_0^t \|u_{mt}\|_2^2 \, \mathrm{d}\tau < h, \quad 0 \leqslant t < \infty. \tag{2.10}$$

Therefore, there exist u, ϕ and a subsequence $\{u_v\}$ of $\{u_m\}$ such that,

$u_v \to u$ in $L^\infty\left(0,\infty; W^{1,2}(\Omega)\right)$ weakly star and a.e. in $\Omega \times [0,\infty)$,

$u_{vt} \to u_t$ in $L^2\left(0,\infty; L^2(\Omega)\right)$ weakly star,

$g(u_v) \to \phi$ in $L^\infty(0,\infty; L^q(\partial\Omega))$ weakly star and a.e. in $\partial\Omega \times [0,\infty)$.

Consequently, from Lemma 1.3 in [15] we obtain $\phi = g(u)$. In (2.3) for fixed s letting $m = v \to \infty$ we have

$$\langle u_t, w_s\rangle + \langle u, w_s\rangle + \langle \nabla u, \nabla w_s\rangle = \langle g(u), w_s\rangle_0 \quad \forall s,$$

and

$$\langle u_t, v\rangle + \langle u, v\rangle + \langle \nabla u, \nabla v\rangle = \langle g(u), v\rangle_0 \quad \forall v \in W^{1,2}(\Omega), \quad t > 0.$$

By (2.4), we obtain $u(x,0) = u_0(x)$ in $W^{1,2}(\Omega)$. Then $u(x,t)$ is a global weak solution of problem (1.1).

3. Finite time blow-up

In this section, we prove the blow-up of solutions to problem (1.1) when the initial energy satisfies a certain condition. In order to prove our main result, we will use the following auxiliary results.

Lemma 3.1. *Let $g(u)$ satisfy* (C). *Assume that $A_\delta(u) < 0$, then $\|u\|_{1,2} > z(\delta)$. In particular, if $A(u) < 0$, then $\|u\|_{1,2} > z(1)$,*

where

$$z(\delta) = \left(\frac{\delta}{aC_*^\mu}\right)^{1/(\mu-2)},$$

and

$$a = \sup \frac{ug(u)}{|u|^\mu}.$$

Proof. $A_\delta(u) < 0$ gives

$$\delta\|u\|_{1,2}^2 < \int_{\partial\Omega} ug(u)\mathrm{d}\rho \leqslant a\|u\|_{\mu,\partial\Omega}^\mu \leqslant aC_*^\mu \|u\|_{1,2}^{\mu-2}\|u\|_{1,2}^2. \tag{3.1}$$

Consequently, (3.1) implies $\|u\|_{1,2} > z(\delta)$.

□

Lemma 3.2. *Let $g(u)$ satisfy $(C), u_0(x) \in W^{1,2}(\Omega)$ and $0 < e < h$, where $\delta_1 < \delta_2$ are the two roots of equation $h(\delta) = e$. Suppose that $A(u_0) < 0$, then all weak solutions of problem* (1.1) *with $B(u_0) = e$ belong to U_δ for $\delta \in (\delta_1, \delta_2)$.*

Proof. Let $u(t)$ be any solution of problem (1.1) with $B(u_0) = e$ and $A(u_0) < 0$, and let T be the existence time of $u(t)$. First from $B(u_0) = e$, $A(u_0) < 0$ and Lemma 2.2 we can deduce $A_\delta(u_0) < 0$ and $B(u_0) < h(\delta)$, i.e. $u_0(x) \in U_\delta$ for $\delta_1 < \delta < \delta_2$. Next we prove $u(t) \in U_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 < t < T$. If it is false, let $t_0 \in (0, T)$ be the first time such that $u(t) \in U_\delta$ for $0 \leqslant t < t_0$ and $u(t_0) \in \partial U_\delta$, i.e. $A_\delta(u(t_0)) = 0$ or $B(u(t_0)) = h(\delta)$ for some $\delta \in (\delta_1, \delta_2)$. So (2.1) implies $B(u(t_0)) = h(\delta)$ is impossible. If $A_\delta(u(t_0)) = 0$, then $A_\delta(u(t)) < 0$ for $0 < t < t_0$ and Lemma 3.1 yields $\|u(t)\|_{1,2} > z(\delta)$ and $\|u(t_0)\|_{1,2} \geqslant z(\delta)$. Therefore, by the definition of $h(\delta)$ we have $B(u(t_0)) \geqslant h(\delta)$, which contradicts (2.1). □

Now, we present the second main result of this paper.

Theorem 3.1. *Let $u_0(x) \in W^{1,2}(\Omega)$ and $g(u)$ satisfy* (C). *Assume that $B(u_0) < h$ and $A(u_0) < 0$. Then the solution of problem* (1.1) *must blow up in finite time i.e. there exists a $T > 0$ such that*

$$\lim_{t \to T} \int_0^t \|u\|_2^2 \, \mathrm{d}\tau = +\infty. \tag{3.2}$$

Proof of Theorem 3.1 Let $u(t)$ be any solution of problem (1.1) with $B(u_0) < h$ and $A(u_0) < 0$.

We consider the auxiliary function

$$\varphi_1(t) = \int_0^t \|u\|_2^2 \, \mathrm{d}\tau.$$

A direct calculation gives

$$\dot{\varphi}_1(t) = \|u\|_2^2,$$

and

$$\ddot{\varphi}_1(t) = 2\langle u_t, u\rangle = 2\left(\langle g(u), u\rangle - \|u\|_{1,2}^2\right) = -2A(u). \tag{3.3}$$

By (3.3), (2.2) and

$$\int_{\partial\Omega} u g(u) \mathrm{d}\rho \geqslant (p+1) \int_{\partial\Omega} G(u) \mathrm{d}\rho,$$

we can deduce

$$\begin{aligned} \ddot{\varphi}_1(t) &\geqslant 2(p+1) \int_0^t \|u_t\|_2^2 \, \mathrm{d}\tau + (p-1)\|u\|_{1,2}^2 - 2(p+1)B(u_0) \\ &\geqslant 2(p+1) \int_0^t \|u_t\|_2^2 \, \mathrm{d}\tau + (p-1)\dot{\varphi}_1(t) - 2(p+1)B(u_0), \end{aligned}$$

and

$$\begin{aligned} \varphi_1\ddot{\varphi}_1 - \frac{p+1}{2}(\dot{\varphi}_1)^2 \geqslant & 2(p+1)\left[\int_0^t \|u\|_2^2 \, \mathrm{d}\tau \int_0^t \|u_t\|_2^2 \, \mathrm{d}\tau - \left(\int_0^t \langle u, u_t\rangle \mathrm{d}\tau\right)^2\right] \\ & + (p-1)\varphi_1\dot{\varphi}_1 - (p+1)\|u_0\|_2^2 \dot{\varphi}_1 \\ & - 2(p+1)B(u_0)\varphi_1 + \frac{p+1}{2}\|u_0\|_2^2. \end{aligned}$$

According to the Hölder inequality, we deduce that

$$\begin{aligned}\varphi_1\ddot{\varphi}_1-\frac{p+1}{2}\left(\dot{\varphi}_1\right)^2 \geqslant &(p-1)\varphi_1\dot{\varphi}_1-(p+1)\left\|u_0\right\|_2^2\dot{\varphi}_1\\ &-2(p+1)B\left(u_0\right)\varphi_1+\frac{p+1}{2}\left\|u_0\right\|_2^2.\end{aligned} \tag{3.4}$$

1. If $B\left(u_0\right)\leqslant 0$, then

$$\varphi_1\ddot{\varphi}_1-\frac{p+1}{2}\left(\dot{\varphi}_1\right)^2 \geqslant (p-1)\varphi_1\dot{\varphi}_1-(p+1)\left\|u_0\right\|_2^2\dot{\varphi}_1.$$

The following task is to claim that $A(u)<0$ for $t>0$. Arguing by contradiction, we assume that there exists a $t_0>0$ such that $A\left(u\left(t_0\right)\right)=0$. Let $t_0>0$ be the first time such that $A(u(t))=0$. Then $A(u(t))<0$ for $0\leqslant t<t_0$. From Lemma 3.1 we obtain $\|u\|_{1,2}>z(1)$ for $0<t<t_0$. Consequently, we get $\left\|u\left(t_0\right)\right\|_{1,2}\geqslant z(1)$ and $B\left(u\left(t_0\right)\right)\geqslant h$ which contradicts (2.2). Then, from (3.3) we have $\ddot{\varphi}_1(t)>0$ for $t>0$. By this and $\dot{\varphi}_1(0)=\left\|u_0\right\|_2^2\geqslant 0$ there exists a $t_0\geqslant 0$ such as $\dot{\varphi}_1\left(t_0\right)>0$ and

$$\varphi_1(t)\geqslant\dot{\varphi}_1\left(t_0\right)\left(t-t_0\right)+\varphi_1\left(t_0\right)\geqslant\dot{\varphi}_1\left(t_0\right)\left(t-t_0\right),\quad t\geqslant t_0.$$

Then for sufficiently large t we can deduce $(p-1)\varphi_1>(p+1)\left\|u_0\right\|_2^2$ and

$$\varphi_1(t)\,\ddot{\varphi}_1(t)-\frac{p+1}{2}\left(\dot{\varphi}_1(t)\right)^2>0. \tag{3.5}$$

Since, for $t>0$

$$\left(\varphi_1^{-\beta}(t)\right)''=-\frac{\beta}{\varphi_1^{\beta+2}(t)}\left(\varphi_1(t)\,\ddot{\varphi}_1(t)-(\beta+1)\dot{\varphi}_1(t)^2\right),$$

we see that for $\beta=\frac{p-1}{2}$ we have $\left(\varphi_1^{-\beta}(t)\right)''<0$. Therefore $\varphi_1^{-\beta}(t)$ is concave for sufficiently large t, and there exists a finite time T such that $\varphi_1^{-\beta}(t)\rightarrow 0$. In other words,

$$\lim_{t\rightarrow T^-}\varphi_1(t)=+\infty.$$

2. If $0<B\left(u_0\right)<h$, then by Lemma 3.2, we have $u(t)\in U_\delta$ for $1<\delta<\delta_2$ and $t>0$, where δ_2 is the larger root of equation $h(\delta)=B(u_0)$. Therefore $A_\delta(u)<0$ and (From Lemma 3.1) $\|u\|_{1,2}>z(\delta)$ for $\delta\in(1,\delta_2)$ and $t>0$. Then, we have $A_{\delta_2}(u)\leqslant 0$ and $\|u\|_{1,2}\geqslant z\left(\delta_2\right)$ for $t>0$.Thus (3.3) gives

$$\begin{aligned}&\ddot{\varphi}_1(t)=-2A(u)=2\left(\delta_2-1\right)\|u\|_{1,2}^2-2A_{\delta_2}(u)\geqslant 2\left(\delta_2-1\right)z^2\left(\delta_2\right)>0,\\ &0\leq t<\infty,\\ &\dot{\varphi}_1(t)\geqslant 2\left(\delta_2-1\right)z^2\left(\delta_2\right)t+\dot{\varphi}_1(0)\geqslant 2\left(\delta_2-1\right)z^2\left(\delta_2\right)t,\quad 0\leq t<\infty,\\ &\varphi_1(t)\geqslant\left(\delta_2-1\right)z^2\left(\delta_2\right)t^2+\varphi_1(0)=\left(\delta_2-1\right)z^2\left(\delta_2\right)t^2,\quad 0\leq t<\infty.\end{aligned}$$

Therefore for sufficiently large t we get

$$\begin{aligned}&\frac{1}{2}(p-1)\varphi_1(t)>(p+1)\left\|u_0\right\|_2^2,\\ &\frac{1}{2}(p-1)\dot{\varphi}_1(t)>2(p+1)B\left(u_0\right).\end{aligned}$$

Hence from (3.4) we again obtain (3.5) for sufficiently large t. The remainder of the proof is similar to that in the proof of (i).

4. Asymptotic behavior

In this final section, we prove the asymptotic behavior of weak solutions. The main result is given in the following theorem.

Theorem 4.1. *Let $u_0(x) \in W^{1,2}(\Omega)$ and $g(u)$ satisfy* (C). *Suppose also that $B(u_0) < h$ and $A(u_0) > 0$. Then, for the weak global solution u of problem* (1.1), *there exists a constant $\omega > 0$ such that*

$$\|u\|_2^2 \le \|u_0\|_2^2 \,\mathrm{e}^{-\omega t}, \quad 0 \le t < \infty. \tag{4.1}$$

Proof of Theorem 4.1 By Theorem 2.1, we know that there exists a global weak solution $u(t) \in L^\infty\left(0,\infty; W^{1,2}(\Omega)\right) \cap C\left([0,T]; L^2(\Omega) \times L^2(\partial\Omega,\rho)\right)$ to problem (1.1). Let $u(t)$ be any global weak solution of problem (1.1) with $B(u_0) < h$ and $A(u_0) > 0$. Consequently,

$$\langle u_t, v\rangle + \langle u, v\rangle + \langle \nabla u, \nabla v\rangle = \langle g(u), v\rangle_0, \quad \forall v \in W^{1,2}(\Omega), t \in (0,T). \tag{4.2}$$

Multiplying (4.2) by any $h(t) \in C[0,\infty)$, we have

$$\langle u_t, h(t)v\rangle + \langle u, h(t)v\rangle + \langle \nabla u, \nabla(h(t)v)\rangle = \langle g(u), h(t)v\rangle_0, \quad \forall v \in W^{1,2}(\Omega), t \in (0,T)$$

and

$$\langle u_t, \varphi\rangle + \langle u, \varphi\rangle + \langle \nabla u, \nabla \varphi\rangle = \langle g(u), \varphi\rangle_0, \quad \forall \varphi \in L^\infty\left(0,\infty; W^{1,2}(\Omega)\right),\ t \in (0,T). \tag{4.3}$$

Setting $\varphi = u$, (4.3) implies

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + A(u) = 0, \quad 0 \le t < \infty. \tag{4.4}$$

By $0 < B(u_0) < h, A(u_0) > 0$ and Lemma (2.3), we get $u(t) \in S_\delta$ for $\delta_1 < \delta < \delta_2$ and $0 \le t < \infty$, where $\delta_1 < \delta_2$ are the two roots of equation $h(\delta) = B(u_0)$. Consequently, we obtain $A_\delta(u) \ge 0$ for $\delta_1 < \delta < \delta_2$ and $A_{\delta_1}(u) \ge 0$ for $0 \le t < \infty$. Then, (4.4) leads to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + (1-\delta_1)\|u\|_{1,2}^2 + A_{\delta_1}(u) = 0, \quad 0 \le t < \infty,$$

accordingly

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_2^2 + (1-\delta_1)\|u\|_2^2 \le 0, \quad 0 \le t < \infty.$$

Finally, Gronwall's inequality leads to

$$\|u\|_2^2 \le \|u_0\|_2^2 \,\mathrm{e}^{-2(1-\delta_1)t}, \quad 0 \le t < \infty.$$

This completes the proof of the Theorem.

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References

[1] A. Audrito and T. Kukuljan, *Regularity theory for fully nonlinear parabolic obstacle problems*, Journal of Functional Analysis, 2023, 285(10), 110-116.

[2] I. Babuska, and J.E. Osborn, *Eigenvalue problems. Handbook of Numerical Analysis, Finite Element Method (Part I)*, North-Holland, Amsterdam, 1991, 2, 641-787.

[3] I. Bejenaru, J.I. Diaz and I. Vrabie, *An abstract approximate controllability result and applications to elliptic and parabolic system with dynamics boundary conditions*, Electron Journal Differential Equations, 2001, 50, 1-19.

[4] V.J. Below, M. Cuesta and G.P. Mailly, *Qualitative results for parabolic equations involving the p-Laplacian under dynamical boundary conditions*, North-Western European Journal of Mathematics, 2018, 4, 59-97.

[5] S. Bergman and M. Schiffer, *Kernel Functions and Elliptic Differential Equations in Mathematical Physics*, Academic Press, New York, 1953.

[6] A. Bermudez, R. Rodriguez and D. Santamarina, *A fnite element solution of an added mass formulation for coupled fluid-solid vibrations*, Numer. Math, 2000, 87, 201-227.

[7] A.P. Calderon, *On a inverse boundary value problem. In: Seminar in numerical analysis and its applications to continuum physics*, Soc. Brasileira de Matemaatica, Rio de Janeiro,1980, 65-73.

[8] C. Conca, J. Planchard and M. Vanninathanm, *Fluid and Periodic Structures*, John Wiley and Sons, New York, 1995.

[9] J.I. Diaz, G.Hetzer and L. Tello, *An energy balance climate model with hysteresis*, Nonlinear Analysis, 2006, 64(9), 2053-2074.

[10] J.I.Diaz and L. Tello, *On a climate model with a dynamic nonlinear diffusive boundary condition*, Discrete and Continuous Dynamical Systems Series S, 2008, 1(2), 253-262.

[11] C. Enache, *Blow-up phenomena for a class of quasilinear parabolic problems under Robin boundary condition*, Appl. Math. Lett, 2011, 24(3), 288-292.

[12] A.H. Erhardt, *The Stability of Parabolic Problems with Nonstandard $p(x,t)$-Growth*, Mathematics, 2017, 5(4), 50.

[13] J.F. Escobar, *The geometry of the first non-zero Steklov eigenvalue*, J. Funct Anal, 1997, 150(2), 544-56.

[14] A. Lamaizi, A. Zerouali, O. Chakrone and B. Karim, *Global existence and blow-up of solutions for parabolic equations involving the Laplacian under nonlinear boundary conditions*, Turkish Journal of Mathematics, 2021, 45(6), 2103-2111.

[15] J.L. Lions, *Quelques methods de resolution des problemes aux limites nonlineairs*, Dunod, Paris, 1969.

[16] W. Liu, Y. Chen, J. Zhou and Q. Liang, *Unconditional error analysis of linearized BDF2 mixed virtual element method for semilinear parabolic problems on polygonal meshes*, Journal of Computational and Applied Mathematics, 2024, 115-864.

[17] L.E.Payne, and D.H. Sattinger, *Saddle points and instability of nonlinear hyperbolic equations*, Israel. J. Math, 1975, 22(3-4), 273-303.

[18] L.E. Payne and P.W. Schaefer, *Bounds for blow-up time for the heat equation under nonlinear boundary conditions*, Proceedings of the Royal Society of Edinburgh, 2009, 139(6), 1289-1296.

[19] L., Yachenga, and Z. Junshengc, *Nonlinear parabolic equations with critical initial conditions* $J(u_0) = d$ *OR* $I(u_0) = 0$, Nonlinear Analysis, 2004, 58(7-8), 873-883.

[20] L. Yacheng, X. Runzhang and Y. Tao, *Global existence, nonexistence and asymptotic behavior of solutions for the Cauchy problem of semilinear heat equations*, Nonlinear Analysis, 2008, 68(11), 3332-3348.

[21] H.Yi, Y.Chen, Y. Wang and Y. Huang, *A two-grid immersed finite element method with the Crank-Nicolson time scheme for semilinear parabolic interface problems*, Applied Numerical Mathematics, 2023, 189, 1-22.

[22] Q. Zhao and Y. Cao, *Initial boundary value problem of pseudo-parabolic Kirchhoff equations with logarithmic nonlinearity*, Mathematical Methods in the Applied Sciences, 2024, 47(2), 799-816.