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Orbital Stability of the Sum of N Peakons for the CH-mCH Equation*

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Abstract This paper is concerned with a generalization of the modified Camassa-Holm equation with both cubic and quadratic nonlinearities (also known as the CH-mCH equation). We mainly prove the orbital stability of the train of peakons for the CH-mCH equation in energy space, using energy arguments and combining the method of orbital stability of a single peakon with the monotonicity of the local energy norm.

Keywords Camassa-Holm equation, CH-mCH equation, peakons, multi-peakons, orbital stability

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1. Introduction

In this paper, we consider the multi-peakon solutions of the following CH-mCH equation [15]

$$m_t + k_1((u^2 - u_x^2)m)_x + k_2(2u_x m + u m_x) = 0, t > 0, x \in \mathbb{R}, \tag{1.1}$$

where $m = u - u_{xx}$, k_1 and k_2 are two arbitrary constants, Eq. (1.1) is completely integrable and admits the Lax pair and bi-Hamiltonian structure [38]. The Cauchy problem and well-posedness were considered in [28].

Notice that when $k_1 = 0, k_2 = 1$, Eq. (1.1) reduces to the Camassa-Holm (CH) equation

$$m_t + 2u_x m + u m_x = 0, \quad m = u - u_{xx}, \tag{1.2}$$

which was derived as a model for shallow water waves [3], where $u(t,x)$ denotes the free surface above the flat bottom. Eq. (1.2) has many interesting properties: the existence of peaked solutions and multi-peakons [1, 3], wave-breaking phenomena [7–9] and geometric formulations [6]. Fuchssteiner and Fokas [16] first noted that Eq. (1.2) has a bi-Hamiltonian structure and hence infinitely many conservation laws. Camassa and Holm [3] obtained the single peakons of Eq. (1.1), which takes the form [30],

$$u(t,x) = c\varphi(x - ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}, \tag{1.3}$$

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and the multi-peakons

$$u(t,x)=\sum_{i=1}^{N}p_i(t)e^{-|x-q_i(t)|}, \tag{1.4}$$

where $p_i(t)$ and $q_i(t)$ satisfy the Hamiltonian system

$$\begin{cases} \dot{p}_i=\sum_{j\neq i}p_ip_j\text{sign}(q_i-q_j)e^{-|q_i-q_j|}=-\frac{\partial H}{\partial q_i}, \\ \\ \dot{q}_i=\sum_j p_je^{-|q_i-q_j|}=\frac{\partial H}{\partial p_i}, \end{cases} \tag{1.5}$$

with the Hamiltonian

$$H=\frac{1}{2}\sum_{i,j=1}^{N}p_ip_je^{-|q_i-q_j|}. \tag{1.6}$$

Constantin and Strauss [11] proved orbital stability using energy as a Lyapunov function and basing on the conservation law of the CH equation. A variational approach for proving the orbital stability of the peakons was introduced by Constantin and Molinet [10]. The variational approach was extended to prove the orbital stability of the peakons for the other nonlinear wave equations [4, 17, 22, 25, 29, 33, 41]. Orbital stability of multi-peakon solutions was discussed by Dika and Molinet in [14].

When $k_1=1, k_2=0$, Eq. (1.1) reduces to the mCH\FORQ equation

$$m_t+\big((u^2-u_x^2)m\big)_x=0, \quad m=u-u_{xx}. \tag{1.7}$$

The orbital stability of the single peakons and the train of peakons for (1.7) was proved in [24] and [35], respectively. After that, Li [19] established the orbital stability of the peakons under $H^1\cap W^{1,4}$ norm.

We also introduce the gmCH equation proposed in [2]:

$$m_t+\big((u^2-u_x^2)^nm\big)_x=0, \quad m=u-u_{xx}, \tag{1.8}$$

where $n\geqslant 1$ is a positive integer. Eq. (1.8) becomes the fifth-order CH-type equation when $n=2$. The orbital stability of periodic peakons was examined by [32]. When $n=3$, Liu [26, 27] investigated the orbital stability of a higher-order nonlinear modified Camassa-Holm equation with peakons and multi-peakons. The local well-posedness and blow-up mechanism of Eq. (1.8) have been discussed in [39]. The orbital stability of peakons for Eq. (1.8) has been demonstrated by Guo et al. in [18]. Deng and Chen [13] have also proved the orbital stability of the sum of N peakons. Recently, a variety of CH-type equations have been explored, including the mCH-Novikov equation [31], the generalized cubic-quintic Camassa-Holm type equation [37], the b-family of FORQ/MCH equations [40], etc. Orbital stability of the single peakons and multi-peakons for the mCH-Novikov equation and the generalized cubic-quintic Camassa-Holm type equation has been proved by [5, 12, 36, 37]. For the Camassa-Holm-type equations, different wave profiles of φ for different types of phase orbits were classified using dynamical system theory in [20, 21].

More generally, Eq. (1.1) also has single peakons, periodic peakons and multi-peakons. Its orbital stability has been proved by Liu et al. in [23]. In this paper, we prove that the multi-peakons of Eq. (1.1) are orbitally stable in energy space.

For the convenience of narration, we introduce the relevant definition of Sobolev space. Let $\Omega \subset \mathbb{R}^n$ be an open set. For positive integer n and $1 \leq p < \infty$, we denote $D^n u = \{D^\beta u : |\beta| = n\}$,

$$|D^n u| = \left(\sum_{|\beta|=n} \left|D^\beta u\right|^2 \right)^{\frac{1}{2}}, \quad \|D^n u\|_{p,\Omega} = \left(\sum_{|\beta|=n} \int_\Omega \left|D^\beta u\right|^p dx \right)^{\frac{1}{p}}.$$

Definition 1.1. *Assume that k is a positive integer, define $W_p^k(\Omega) = \{u : \ D^\beta u \in L^p(\Omega), |\beta| \leq k\}$, then the norm*

$$\|u\|_{W_p^k(\Omega)} = \begin{cases} \left(\sum\limits_{n \leq k} \|D^n u\|_{p,\Omega}^p \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \sum\limits_{n \leq k} \operatorname{ess\,sup}_\Omega |D^n u|, & p = \infty. \end{cases}$$

From the above definition, the space W_p^k that gives the norm $\|\cdot\|_{W_p^k(\Omega)}$ is a Banach space. When $p = 2$, it is denoted as $W_2^k(\Omega) = H^k(\Omega)$, then $H^k(\mathbb{R}^n)$ is the integral exponential Sobolev space. Let $s \in \mathbb{R}$, define the real exponential Sobolev space $H^s(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : (1 + |y|^2)^{\frac{s}{2}} \hat{u} \in L^2(\mathbb{R}^n)\}$, where $\hat{u}$ is the Fourier transform of u and $\mathcal{S}'(\mathbb{R}^n)$ is the dual space of the rapidly decreasing function space $\mathcal{S}(\mathbb{R}^n)$.

According to the above explanation, the main result of this paper is described as the following theorem.

Theorem 1.1. *Let $0 < c_1 < \cdots < c_N$ be given. There exist A, ε_0, $L_0 > 0$ such that for any $u_0 \in H^s(\mathbb{R})$ with some $s > 5/2$ which satisfies $0 \leqslant (1 - \partial_x^2) u_0(x) \not\equiv 0$, any $0 < \varepsilon < \varepsilon_0$ and $L > L_0$, if*

$$\left\| u_0 - \sum_{i=1}^N \varphi_{c_i}\left(\cdot - z_i^0\right) \right\|_{H^1(\mathbb{R})} \leqslant \varepsilon^2 \tag{1.9}$$

for some z_i^0 satisfying

$$z_i^0 - z_{i-1}^0 > L \quad (i = 2, \ldots, N), \tag{1.10}$$

then the corresponding solution $u(t,x) \in C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$ with initial data $u(0,x) = u_0(x)$ and maximal existence time $T > 0$ exists, and there exist $x_1(t), \ldots, x_N(t)$ defined on $[0,T)$, such that

$$\sup_{t \in [0,T)} \left\| u(t,\cdot) - \sum_{i=1}^N \varphi_{c_i}\left(\cdot - x_i(t)\right) \right\|_{H^1(\mathbb{R})} \leqslant A\left(\sqrt{\varepsilon} + L^{-\frac{1}{8}}\right) \tag{1.11}$$

and for $i = 2, \ldots, N$,

$$x_i(t) - x_{i-1}(t) > \frac{L}{2}, \quad \forall t \in [0,T). \tag{1.12}$$

The remainder of this paper is organized as follows. In Section 2, we provide a set of definitions and lemmas that need to be used below. In Section 3, we go through four subsections to finish the proof portion of Theorem 1.1. In Subsection

3.1, we will show the crucial Lemma 3.1, which requires a large number of summation formulas and a large number of estimates for its proof. In Subsection 3.2, we demonstrate the monotonicity of functionals. In Subsection 3.3, we obtain a global identity and local estimator for conserved quantities. In Subsection 3.4, we summarize the proof of Theorem 1.1.

2. Preliminaries

In this section, we present some definitions and lemmas used in the subsequent proofs. We first review the local well-posedness results for the Cauchy problem associated with Eq. (1.1), some properties for strong solutions, and two basic invariants, which will be frequently used in the rest of the paper. We are concerned with the Cauchy problem for the CH-mCH equation on both the line and the unit circle:

$$\begin{cases} m_t + k_1((u^2 - u_x^2)m)_x + k_2(2u_x m + u m_x) = 0, t > 0, x \in \mathbb{R}, \\ m = u - u_{xx}, \\ u(0,x) = u_0(x), x \in \mathbb{R}. \end{cases} \tag{2.1}$$

We first give the definition of a strong solution as follows.

Definition 2.1 ([23]). If $u \in C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$, with $s > 5/2$ and some $T > 0$, satisfies (2.1), then u is called a strong solution on $[0,T)$. If u is a strong solution on $[0,T)$ for every $T > 0$, then it is called a global strong solution.

Lemma 2.1 ([23]). *Let $u_0 \in H^s(\mathbb{R})$, with $s > 5/2$. Then there exists a time $T > 0$ such that the initial value problem (2.1) has a unique strong solution $u \in C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$ and the map $u_0 \mapsto u$ is continuous from a neighborhood of u_0 in $H^s(\mathbb{R})$ into $C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$.*

Since $m = u - u_{xx}$, Eq. (1.1) can be rewritten as the following nonlinear partial differential equation:

$$\begin{aligned} & u_t + k_1\left(u^2 - \frac{1}{3}u_x^2\right)u_x + k_1(1-\partial_x^2)^{-1}\partial_x\left(\frac{2}{3}u^3 + uu_x^2\right) + \frac{k_1}{3}(1-\partial_x^2)^{-1}(u_x^3) \\ & + k_2 u u_x + k_2(1-\partial_x^2)^{-1}\partial_x\left(u^2 + \frac{1}{2}{u_x}^2\right) = 0. \end{aligned} \tag{2.2}$$

Notice that $\left(1-\partial_x^2\right)^{-1} f = G * f$ for all $f \in L^2$, where $G(x) \triangleq e^{-|x|}/2$. In fact, from this formulation, one can define weak solutions of (2.1) as follows.

Definition 2.2 ([23]). Given initial data $u_0 \in W^{1,3}(\mathbb{R})$, a function $u \in L^\infty_{loc}([0,T), W^{1,3}_{loc}(\mathbb{R}))$ is said to be a weak solution to the initial value problem (2.1) if it satisfies the following identity:

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}} \left[u\partial_t\phi + \frac{k_1}{3}u^3\partial_x\phi + \frac{k_1}{3}u_x^3\phi + k_1(1-\partial_x^2)^{-1}\left(\frac{2}{3}u^3 + uu_x^2\right)\partial_x\phi \right. \\ & \left. - \frac{k_1}{3}\left(1-\partial_x^2\right)^{-1}\left(u_x^3\right)\phi + \frac{k_2}{2}u^2\partial_x\phi + k_2\left(1-\partial_x^2\right)^{-1}\left(u^2 + \frac{1}{2}u_x^2\right)\partial_x\phi \right] dxdt \end{aligned}$$

$$+\int_{\mathbb{R}} u_0(x)\phi(x,0)dx = 0, \tag{2.3}$$

for any smooth test function $\phi(t,x) \in C_c^\infty([0,T)\times\mathbb{R})$. If u is a weak solution on $[0,T)$ for every $T>0$, then it is called a global weak solution.

Lemma 2.2 ([23]). *If the initial data $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$, then the following two functions*

$$E(u) = \int_{\mathbb{R}} \left(u^2 + u_x^2\right) dx \tag{2.4}$$

and

$$F(u) = \int_{\mathbb{R}} k_1(u^4 + 2u^2u_x^2 - \frac{1}{3}u_x^4) + 2k_2(u^3 + uu_x^2)dx \tag{2.5}$$

are invariants for Eq. (1.1)*. Furthermore, if $m_0 = (1-\partial_x^2)u_0$ does not change sign, then $m(t,x)$ will not change sign for any $t \in [0,T)$. It turns out that if $m_0 \geqslant 0$, then the corresponding solution $u(t,x)$ is positive and satisfies [34]*

$$|u_x(t,x)| \leqslant u(t,x), \quad \forall (t,x) \in [0,T)\times\mathbb{R}. \tag{2.6}$$

Remark 2.1. Notice that $E(u)$ and $F(u)$ represent conservation of energy, and $E(u)$ and $\|u\|_{H^1(\mathbb{R})}$ have a special relationship, namely $E(u) = \|u\|^2_{H^1(\mathbb{R})}$. Therefore, space $H^1(\mathbb{R})$ is also called the energy space.

Lemma 2.3 ([23]). *For any $c > 0$, the peaked function of the form*

$$u(t,x) = \varphi_c(x-ct) = ae^{-|x-ct|}, \tag{2.7}$$

where

$$a = \tfrac{3}{4}\tfrac{-k_2 \pm \sqrt{k_2^2 + \frac{8}{3}k_1 c}}{k_1}, k_1 \neq 0, k_2^2 + \tfrac{8}{3}k_1 c \geqslant 0,$$

is a global weak solution to the Eq. (1.1) *in the sense of Definition 2.2.*

Lemma 2.4 ([23]). *Let $k_1 > 0$ and $k_2 \leqslant 0$. Let φ_c be the peaked soliton defined in* (2.7)*, with wave speed satisfying $c > 2k_2^2/3k_1$. Assume that $u_0 \in H^s(\mathbb{R})$, $s > 5/2$, satisfies $0 \not\equiv m_0(x) = (1-\partial_x^2)u_0(x) \geqslant 0$. Then there exists $\delta_0 > 0$, depending on k_1, k_2, c and $\|u_0\|_{H^s(\mathbb{R})}$, such that if*

$$\|u_0 - \varphi_c\|_{H^1(\mathbb{R})} < \delta, \quad 0 < \delta < \delta_0,$$

then the corresponding positive solution $u(t,x)$ of the Cauchy problem for the CH-mCH equation (2.1) *with initial data $u(0,x) = u_0(x)$ satisfies*

$$\sup_{t\in[0,T)} \|u(t,\cdot) - \varphi_c(\cdot - \xi(t))\|_{H^1(\mathbb{R})} < A\,\delta^{1/4},$$

where $T > 0$ is the maximal existence time, $\xi(t) \in \mathbb{R}$ is the point at which the solution $u(t,\cdot)$ achieves its maximum, and the constant $A > 0$ depends on k_1, k_2, the wave speed c and the $\|u_0\|_{H^1(\mathbb{R})}$.

3. Proof of Theorem 1.1

In this section, the proof of Theorem 1.1 is divided into four parts. The following H^1 neighborhood is defined for $\alpha > 0$ and $L > 0$ for all the sums of N peakons of fixed speeds $c_1, \ldots, c_N$, with spatial shifts z_i that satisfied $z_i - z_{i-1} \geqslant L$,

$$U(\alpha, L) = \left\{ u \in H^1(\mathbb{R}); \inf_{z_i - z_{i-1} \geqslant L} \left\| u - \sum_{i=1}^{N} \varphi_{c_i}(\cdot - z_i) \right\|_{H^1(\mathbb{R})} < \alpha \right\}. \tag{3.1}$$

By a standard continuity argument, as $u(t,x)$ is continuous in $H^s(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$, with $s > 5/2$, to prove Theorem 1.1, it suffices to show that there exist $A > 0, L_0 > 0$, and $\varepsilon_0 > 0$ such that for all $L > L_0$ and $0 < \varepsilon < \varepsilon_0$, if u_0 satisfies $m_0 \geqslant 0$, (1.9) and (1.10), and if for some $0 < t^* < T$,

$$u(t) \in U\left(A\left(\sqrt{\varepsilon} + L^{-\frac{1}{8}}\right), \frac{L}{2} \right), \quad \forall t \in [0, t^*], \tag{3.2}$$

then

$$u(t^*) \in U\left(\frac{A}{2}\left(\sqrt{\varepsilon} + L^{-\frac{1}{8}}\right), \frac{2L}{3} \right). \tag{3.3}$$

Therefore, we only have to verify (3.3) for some $L > L_0$ and $0 < \varepsilon < \varepsilon_0$ under the hypothesis of (3.2), with A, L_0, and ε_0 to be specified later.

3.1. Modulation

In this subsection, we will be proving that if the solution $u(t)$ is still close to a manifold of the train of N peakons for $t \in [0, t^*]$, we can decompose $u(t)$ into the sum of N modulated peakons plus a function $v(t)$ that stays small in $H^1(\mathbb{R})$: $u(t,x) = \sum_{i=1}^{N} \varphi_{c_i}\big(x - \tilde{x}_i(t)\big) + v(t,x)$. Moreover, it will be shown that the different bumps of u that are individually close to a peakon get away from each other as time evolves.

Lemma 3.1. *Let the initial data u_0 satisfy the assumptions given in Theorem 1.1. There exist $\alpha_0 \ll 1$ and $L_0 \gg 1$ depending only on $(c_i)_{i=1}^N$ such that if for $0 < \alpha < \alpha_0$ and $L > L_0$, the corresponding solution $u(t)$ satisfies for some $0 < t^* < T$,*

$$u(t) \in U\left(\alpha, \frac{L}{2}\right), \quad \forall t \in [0, t^*], \tag{3.4}$$

then there exist unique C^1 functions

$$\tilde{x}_i : [0, t^*] \to \mathbb{R}, \quad i = 1, \ldots, N, \tag{3.5}$$

such that if we define $v(t,x)$ by

$$v(t) = u(t) - \sum_{i=1}^{N} R_i(t), \quad \text{where } R_i(t) = \varphi_{c_i}(\cdot - \tilde{x}_i(t)), \tag{3.6}$$

then the following properties hold for all $j \in \{1, 2, \ldots, N\}$ and $t \in [0, t^*]$:

$$\int_{\mathbb{R}} v(t) \partial_x R_j dx = 0, \tag{3.7}$$

$$\|v(t)\|_{H^1(\mathbb{R})} \leqslant O(\sqrt{\alpha}), \tag{3.8}$$

$$|\dot{\tilde{x}}_j(t) - c_j| \leqslant O(\sqrt{\alpha}) + O(L^{-1}), \tag{3.9}$$

$$|\tilde{x}_j(t) - \tilde{x}_{j-1}(t)| \geqslant \frac{3L}{4} + \frac{(c_j - c_{j-1})t}{2}. \tag{3.10}$$

Furthermore, define $\mathcal{J}_j(t) = [y_j(t), y_{j+1}(t)]$, *with*

$$y_1 = -\infty,\ y_{N+1} = +\infty \text{ and } y_j(t) = \frac{\tilde{x}_{j-1}(t) + \tilde{x}_j(t)}{2},\ j = 2, \ldots, N, \tag{3.11}$$

it holds

$$|\xi_j(t) - \tilde{x}_j(t)| < \frac{L}{12}, \tag{3.12}$$

where $\xi_1(t), \ldots, \xi_N(t)$ *are any points such that*

$$u(t, \xi_j(t)) = \max_{x \in \mathcal{J}_j(t)} u(t, x), \quad t \in [0, t^*],\ j = 1, \ldots, N. \tag{3.13}$$

Proof. According to the proof method in [14, 24], we can create N C^1-functions $\tilde{x}_1(t), \ldots, \tilde{x}_i(t)$ on $[0, t^*]$ meeting an appropriate orthogonality condition by using the implicit function theorem. We only need to prove (3.9) here, the rest of the proof is similar to [14, 24]. Now, we prove that the speed of $\tilde{x}_j$ stays close to c_j on $[0, t^*]$. Notice that

$$\partial_x^2 R_j(t) = -2a_j \delta(\tilde{x}_j(t)) + R_j(t). \tag{3.14}$$

Differentiating (3.7) with respect to t, we get

$$\begin{aligned}\int_{\mathbb{R}} v_t(t)\partial_x R_j(t)dx &= \dot{\tilde{x}}_j(t)\langle \partial_x^2 R_j(t), v(t)\rangle_{H^{-1},H^1} \\ &= \dot{\tilde{x}}_j(t)\left(\int_{\mathbb{R}} R_j(t)v(t)dx - 2a_j v(t, \tilde{x}_j(t))\right),\end{aligned}$$

therefore,

$$\int_{\mathbb{R}} v_t(t)\partial_x R_j(t)dx \leqslant |\dot{\tilde{x}}_j| O(\|v\|_{H^1}) \leqslant O(\|v\|_{H^1})|\dot{\tilde{x}}_j - c_j| + O(\|v\|_{H^1}). \tag{3.15}$$

On the other hand, substituting $u(t,x) = \sum_{i=1}^{N} R_i(t) + v(t,x)$ into (2.2) and using the following equation of $R_i(t)$:

$$\begin{aligned}&\partial_t R_i + (\dot{\tilde{x}}_i - c_i)\partial_x R_i + k_1\left(R_i^2 - \frac{1}{3}(\partial_x R_i)^2\right)\partial_x R_i \\ &\quad + k_1(1-\partial_x^2)^{-1}\partial_x\left(\frac{2}{3}R_i^3 + R_i(\partial_x R_i)^2\right) + \frac{k_1}{3}(1-\partial_x^2)^{-1}\left(\partial_x R_i\right)^3 \\ &\quad + k_2 R_i \partial_x R_i + k_2(1-\partial_x^2)^{-1}\partial_x\left(R_i^2 + \frac{1}{2}(\partial_x R_i)^2\right) = 0.\end{aligned}$$

We find that $v(t,x)$ satisfies on $[0,t^*]$:

$$
\begin{aligned}
&v_t-\sum_{i=1}^N(\dot{\tilde{x}}_i-c_i)\partial_x R_i\\
&\quad=-\frac{k_1}{3}\partial_x\left(\left(v+\sum_{i=1}^N R_i\right)^3-\sum_{i=1}^N R_i^3\right)-\frac{k_2}{2}\partial_x\left(\left(v+\sum_{i=1}^N R_i\right)^2-\sum_{i=1}^N R_i^2\right)\\
&\qquad+\frac{k_1}{3}\left(\left(v_x+\sum_{i=1}^N \partial_x R_i\right)^3-\sum_{i=1}^N (\partial_x R_i)^3\right)\\
&\qquad-\frac{k_1}{3}(1-\partial_x^2)^{-1}\left(\left(v_x+\sum_{i=1}^N \partial_x R_i\right)^3-\sum_{i=1}^N (\partial_x R_i)^3\right)\\
&\qquad-k_1\left(1-\partial_x^2\right)^{-1}\partial_x\left(\frac{2}{3}\left(v+\sum_{i=1}^N R_i\right)^3+\left(v+\sum_{i=1}^N R_i\right)\left(v_x+\sum_{i=1}^N \partial_x R_i\right)^2\right.\\
&\qquad\left.-\sum_{i=1}^N R_i\left(\frac{2}{3}R_i^2+(\partial_x R_i)^2\right)\right)\\
&\qquad-k_2(1-\partial_x^2)^{-1}\partial_x\left(\left(v+\sum_{i=1}^N R_i\right)^2+\frac{1}{2}\left(v_x+\sum_{i=1}^N \partial_x R_i\right)^2\right.\\
&\qquad\left.-\sum_{i=1}^N R_i^2-\frac{1}{2}\sum_{i=1}^N(\partial_x R_i)^2\right).
\end{aligned}
\tag{3.16}
$$

Using the $L^2(\mathbb{R})$-scalar product with $\partial_x R_j$ and integrating by parts, we get for $t\in[0,t^*]$,

$$
\begin{aligned}
&-(\dot{\tilde{x}}_j-c_j)\int_{\mathbb{R}}(\partial_x R_j)^2dx\\
&\quad=-\int_{\mathbb{R}}v_t\partial_x R_j dx+\sum_{i\neq j}(\dot{\tilde{x}}_i-c_i)\int_{\mathbb{R}}(\partial_x R_i)(\partial_x R_j)dx\\
&\qquad+\frac{k_1}{3}\int_{\mathbb{R}}\left(\left(v+\sum_{i=1}^N R_i\right)^3-\sum_{i=1}^N R_i^3\right)\partial_x^2 R_j dx\\
&\qquad+\frac{k_1}{3}\int_{\mathbb{R}}\left(\left(v_x+\sum_{i=1}^N \partial_x R_i\right)^3-\sum_{i=1}^N (\partial_x R_i)^3\right)\partial_x R_j dx+B(t)\\
&\quad:=-\int_{\mathbb{R}}v_t\partial_x R_j dx+\sum_{i\neq j}(\dot{\tilde{x}}_i-c_i)\int_{\mathbb{R}}(\partial_x R_i)(\partial_x R_j)dx+T_1+T_2+B(t),
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
B(t)=&\frac{k_2}{2}\int_{\mathbb{R}}\left(\left(v+\sum_{i=1}^N R_i\right)^2-\sum_{i=1}^N R_i^2\right)\partial_x^2 R_j dx\\
&-\frac{k_1}{3}\int_{\mathbb{R}}(1-\partial_x^2)^{-1}\left(\left(v_x+\sum_{i=1}^N \partial_x R_i\right)^3-\sum_{i=1}^N (\partial_x R_i)^3\right)\partial_x R_j dx
\end{aligned}
$$

$$+ k_1 \int_{\mathbb{R}} (1-\partial_x^2)^{-1} \left(\frac{2}{3} \left(v + \sum_{i=1}^N R_i \right)^3 + \left(v + \sum_{i=1}^N R_i \right) \left(v_x + \sum_{i=1}^N \partial_x R_i \right)^2 \right.$$
$$\left. - \frac{2}{3} \sum_{i=1}^N R_i^3 - \sum_{i=1}^N R_i (\partial_x R_i)^2 \right) \partial_x^2 R_j dx$$
$$+ k_2 \int_{\mathbb{R}} (1-\partial_x^2)^{-1} \left(\left(v + \sum_{i=1}^N R_i \right)^2 + \frac{1}{2} \left(v_x + \sum_{i=1}^N \partial_x R_i \right)^2 \right.$$
$$\left. - \sum_{i=1}^N R_i^2 - \frac{1}{2} \sum_{i=1}^N (\partial_x R_i)^2 \right) \partial_x^2 R_j dx.$$

To estimate T_1, we denote

$$\begin{aligned} V &= \left(v + \sum_{i=1}^N R_i \right)^3 - \sum_{i=1}^N R_i^3 \\ &= v^3 + 3v^2 \sum_{i=1}^N R_i + 3v \left(\sum_{i=1}^N R_i \right)^2 + \left(\sum_{i=1}^N R_i \right)^3 - \sum_{i=1}^N R_i^3, \end{aligned}$$

it follows from (3.14) that

$$\frac{3}{k_1} T_1 = -2a_j V(t, \tilde{x}_j(t)) + \int_{\mathbb{R}} V R_j dx.$$

Since $\|v\|_{L^\infty(\mathbb{R})} \leqslant \frac{\sqrt{2}}{2} \|v\|_{H^1(\mathbb{R})} \leqslant O(\sqrt{\alpha})$, using the exponential decay of R_i, we derive for all $x \in \mathbb{R}$ that

$$|V(t,x)| \leqslant (O(\sqrt{\alpha}) + O(1)) O(\sqrt{\alpha}) + O\left(e^{-\frac{L}{8}} \right) \tag{3.18}$$

and

$$\int_{\mathbb{R}} V R_j dx \leqslant (O(\sqrt{\alpha}) + O(1)) O(\sqrt{\alpha}) + O\left(e^{-\frac{L}{8}} \right). \tag{3.19}$$

Together with (3.18) and (3.19), we conclude that

$$T_1 \leqslant O(\sqrt{\alpha}) + O\left(e^{-\frac{L}{8}} \right). \tag{3.20}$$

Next, estimating T_2 above, we directly compute to get

$$\begin{aligned} \frac{3}{k_1} T_2 = &\int_{\mathbb{R}} v_x^3 \partial_x R_j dx + 3 \int_{\mathbb{R}} v_x^2 \partial_x R_j \sum_{i=1}^N \partial_x R_i dx \\ &+ 3 \int_{\mathbb{R}} v_x \partial_x R_j \left(\sum_{i=1}^N \partial_x R_i \right)^2 dx \\ &+ \int_{\mathbb{R}} \left(\left(\sum_{i=1}^N \partial_x R_i \right)^3 - \sum_{i=1}^N (\partial_x R_i)^3 \right) \partial_x R_j dx. \end{aligned}$$

Since $\left(1-\partial_x^2\right)u_0(x) \geqslant 0$ for all $x \in \mathbb{R}$, it follows from (2.6) that

$$\begin{aligned}
\|v_x\|_{L^\infty(\mathbb{R})} &\leqslant \|u_x\|_{L^\infty(\mathbb{R})} + \left\|\sum_{i=1}^N \partial_x R_i\right\|_{L^\infty(\mathbb{R})} \\
&\leqslant \|u\|_{L^\infty(\mathbb{R})} + \sum_{i=1}^N \|\partial_x R_i\|_{L^\infty(\mathbb{R})} \\
&\leqslant \frac{\sqrt{2}}{2}\left\|v+\sum_{i=1}^N R_i\right\|_{H^1(\mathbb{R})} + \sum_{i=1}^N a_i^2 \\
&\leqslant O(\sqrt{\alpha}) + O(1).
\end{aligned}$$

Hence

$$\int_{\mathbb{R}} v_x^3 \partial_x R_j dx \leqslant a_i \|v_x\|_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} v_x^2 dx \leqslant \big(O(\sqrt{\alpha})+O(1)\big)\|v\|_{H^1(\mathbb{R})}^2.$$

By Hölder's inequality, we conclude that

$$\begin{aligned}
&3\int_{\mathbb{R}} v_x^2 \partial_x R_j \sum_{i=1}^N \partial_x R_i dx + 3\int_{\mathbb{R}} v_x \partial_x R_j \left(\sum_{i=1}^N \partial_x R_i\right)^2 dx \\
\leqslant & C\int_{\mathbb{R}} v_x^2 dx + C\|v\|_{H^1(\mathbb{R})} \leqslant C\left(\|v\|_{H^1(\mathbb{R})}+1\right)\|v\|_{H^1(\mathbb{R})} \\
\leqslant & \big(O(\sqrt{\alpha})+O(1)\big)O(\sqrt{\alpha}).
\end{aligned}$$

Applying the exponential decay of $\partial_x R_i$, we have

$$T_2 \leqslant O(\sqrt{\alpha}) + O\left(e^{-\frac{L}{8}}\right). \tag{3.21}$$

Using the same method as above to estimate $B(t)$, we obtain

$$B(t) \leqslant O(\sqrt{\alpha}) + O\left(e^{-\frac{L}{8}}\right). \tag{3.22}$$

It can be concluded that the results involved in (3.20)–(3.22) depend only on $(c_i)_{i=1}^N$. As a result, using (3.17) and the decay of $\partial_x R_i$, we present

$$\begin{aligned}
&a_j^2\left|\dot{\tilde{x}}_j - c_j\right| \\
&\leqslant \left|\int_{\mathbb{R}} v_t \partial_x R_j dx\right| + \sum_{i\neq j}\left(\left|\dot{\tilde{x}}_j\right| + c_i\right)\left|\int_{\mathbb{R}} (\partial_x R_i)(\partial_x R_j)\, dx\right| + O(\sqrt{\alpha}) + O\left(e^{-\frac{L}{8}}\right) \\
&\leqslant O(\sqrt{\alpha})\left|\dot{\tilde{x}}_j - c_j\right| + O(\sqrt{\alpha}) + O\left(e^{-\frac{L}{8}}\right),
\end{aligned}$$

which proves (3.9). This completes the proof of Lemma 3.1. □

3.2. Monotonicity property

In this subsection, we will illustrate the monotonicity of functionals. First, we need to present the following fundamental identity, which is based on weighted energy.

Lemma 3.2. *Let $u_0(x) \in H^s(\mathbb{R})$, $s > 5/2$ and $T > 0$ be the maximal time of existence of the corresponding strong solution $u(t,x)$ with initial data $u_0(x)$. Then for any smooth function $\omega(x)$, the following identity holds:*

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2)\omega dx\\
&\quad= -\frac{k_1}{2}\int_{\mathbb{R}}(u^2-u_x^2)^2\omega' dx + \frac{2k_1}{3}\int_{\mathbb{R}}u^4\omega' dx\\
&\qquad + k_2\int_{\mathbb{R}}uu_x^2\omega' dx + \frac{4k_1}{3}\int_{\mathbb{R}}u(1-\partial_x^2)^{-1}(u^3+3uu_x^2)\omega' dx\\
&\qquad - 2k_1\int_{\mathbb{R}}u(1-\partial_x^2)^{-1}(u_x^2 m)\omega' dx + k_2\int_{\mathbb{R}}u(1-\partial_x^2)^{-1}(2u^2+u_x^2)\omega' dx, \quad (3.23)
\end{aligned}$$

for all $t \in [0,T)$.

Proof. Since $u \in C([0,T), H^s(\mathbb{R})) \cap C^1([0,T), H^{s-1}(\mathbb{R}))$ with $s > 5/2$, we assume that $u(t,x)$ is smooth. Taking the derivative of (2.2) with respect to x gives

$$\begin{aligned}
u_{tx} =& -k_1\left(uu_x^2 + u^2u_{xx} - u_x^2u_{xx} - \frac{2}{3}u^3\right)\\
&- k_2\left(\frac{1}{2}u_x^2 + uu_{xx} - u^2\right) - k_1(1-\partial_x^2)^{-1}\left(\frac{2}{3}u^3 + uu_x^2\right)\\
&- k_2(1-\partial_x^2)^{-1}\left(u^2 + \frac{1}{2}u_x^2\right) - \frac{1}{3}k_1(1-\partial_x^2)^{-1}\partial_x\left(u_x^3\right). \quad (3.24)
\end{aligned}$$

Integrating by parts yields

$$\frac{d}{dt}\int_{\mathbb{R}}(u^2+u_x^2)\omega dx = 2\int_{\mathbb{R}}um_t\omega dx - 2\int_{\mathbb{R}}uu_{tx}\omega' dx := I_1 + I_2. \quad (3.25)$$

Using the equations for (1.1) and m, it is obtained by direct calculation that

$$\begin{aligned}
I_1 =& -2\int_{\mathbb{R}}u\left[k_1((u^2-u_x^2)m)_x + k_2\left(2u_xm + um_x\right)\right]\omega dx\\
=&2\int_{\mathbb{R}}u_x\left[k_1\left(u^2-u_x^2\right)m + k_2\left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx}\right)\right]\omega dx\\
&+ 2\int_{\mathbb{R}}u\left[k_1\left(u^2-u_x^2\right)m + k_2\left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx}\right)\right]\omega' dx. \quad (3.26)
\end{aligned}$$

Since

$$2k_1\int_{\mathbb{R}}u_x\left(u^2-u_x^2\right)m\omega dx = -\frac{k_1}{2}\int_{\mathbb{R}}(u^2-u_x^2)^2\omega' dx \quad (3.27)$$

and

$$2k_2\int_{\mathbb{R}}u_x\left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - uu_{xx}\right)\omega dx = -k_2\int_{\mathbb{R}}(u^3 - uu_x^2)\omega' dx, \quad (3.28)$$

then

$$I_1 = -\frac{k_1}{2}\int_{\mathbb{R}}(u^2-u_x^2)^2\omega' dx + 2k_2\int_{\mathbb{R}}(u^3 - u^2u_{xx})\omega' dx$$

$$+ 2k_1 \int_{\mathbb{R}} (u^4 - u^3 u_{xx} - u^2 u_x^2 + u u_x^2 u_{xx}) \omega' dx. \tag{3.29}$$

On the other hand, it follows from (3.24) that

$$\begin{aligned}
I_2 =& 2k_1 \int_{\mathbb{R}} u \left(u u_x^2 + u^2 u_{xx} - u_x^2 u_{xx} - \frac{2}{3} u^3 \right) \omega' dx \\
&+ 2k_1 \int_{\mathbb{R}} u \left(1 - \partial_x^2\right)^{-1} \left(\frac{2}{3} u^3 + u u_x^2 \right) \omega' dx \\
&+ \frac{2}{3} k_1 \int_{\mathbb{R}} u \left(1 - \partial_x^2\right)^{-1} \partial x \left(u_x^3\right) \omega' dx \\
&+ 2k_2 \int_{\mathbb{R}} u \left(\frac{1}{2} u_x^2 + u u_{xx} - u^2 \right) \omega' dx \\
&+ 2k_2 \int_{\mathbb{R}} u (1 - \partial_x^2)^{-1} \left(u^2 + \frac{1}{2} u_x^2 \right) \omega' dx \\
:=& A_1 + A_2 + A_3 + A_4 + A_5.
\end{aligned} \tag{3.30}$$

It is not difficult to estimate that

$$I_1 + A_1 + A_4 = -\frac{k_1}{2} \int_{\mathbb{R}} \left(u^2 - u_x^2\right)^2 \omega' dx + \frac{2}{3} k_1 \int_{\mathbb{R}} u^4 \omega' dx + k_2 \int_{\mathbb{R}} u u_x^2 \omega' dx. \tag{3.31}$$

For the term A_3, the calculation gives

$$A_3 = -2k_1 \int_{\mathbb{R}} u \left(\left(1 - \partial_x^2\right)^{-1} \left(u_x^2 m\right) \right) \omega' dx + 2k_1 \int_{\mathbb{R}} u \left(\left(1 - \partial_x^2\right)^{-1} \left(u u_x^2\right) \right) \omega' dx. \tag{3.32}$$

Thus, with the above calculations, we introduce from (3.25) that

$$\begin{aligned}
&\frac{d}{dt} \int_{\mathbb{R}} \left(u^2 + u_x^2\right) \omega dx \\
&= -\frac{k_1}{2} \int_{\mathbb{R}} \left(u^2 - u_x^2\right)^2 \omega' dx + \frac{2k_1}{3} \int_{\mathbb{R}} u^4 \omega' dx + k_2 \int u u_x^2 \omega' dx \\
&\quad + \frac{4k_1}{3} \int_{\mathbb{R}} u \left(\left(1 - \partial_x^2\right)^{-1} \left(u^3 + 3u u_x^2\right) \right) \omega' dx \\
&\quad - 2k_1 \int_{\mathbb{R}} u \left(\left(1 - \partial_x^2\right)^{-1} \left(u_x^2 m\right) \right) \omega' dx \\
&\quad + k_2 \int_{\mathbb{R}} u \left(1 - \partial_x^2\right)^{-1} \left(2u^2 + u_x^2\right) \omega' dx,
\end{aligned} \tag{3.33}$$

which proves this lemma. □

Next, we will prove the almost monotonicity of functions that are very close to the energy at the right of the $(i-1)$th bump of u, $i = 2, \ldots, N$. Considering the C^∞ function Ψ defined on $\mathbb{R}$ satisfying

$$\begin{cases} 0 < \Psi(x) < 1, \ \Psi'(x) > 0, & x \in \mathbb{R}, \\ |\Psi'''(x)| \leqslant 10 \Psi'(x), & x \in [-1, 1], \end{cases}$$

and

$$\Psi(x)=\begin{cases} e^{-|x|}, & x<-1,\\ 1-e^{-|x|}, & x>1.\end{cases}$$

Let $\Psi_K=\Psi(\frac{\cdot}{K})$, $K>0$, we define the weight function $\Phi_i=\Phi_i(t,x)$ by

$$\Phi_1=1-\Psi_{2,K},\quad \Phi_N=\Psi_{N,K},\quad \Phi_i=\Psi_{i,K}-\Psi_{i+1,K},\quad i=2,\dots,N-1,$$

where for $i=2,\dots,N$,

$$\Psi_{i,K}(t,x)=\Psi_K(x-y_i(t)) \ \text{ with } y_i(t) \text{ defined in (3.11).} \tag{3.34}$$

We find that $\sum\limits_{i=1}^{N}\Phi_i(t,x)\equiv 1,\ x\in\mathbb{R},\ t\in[0,t^*]$. Taking $L>0$ and $L/K>0$ large enough from the progressive nature of the exponential of Φ_i, it is inferred that

$$|1-\Phi_i|\leqslant 4e^{-\frac{L}{4K}} \ \text{ on } \left[\tilde{x}_i-\frac{L}{4},\tilde{x}_i+\frac{L}{4}\right] \tag{3.35}$$

and

$$|\Phi_i|\leqslant 4e^{-\frac{L}{4K}} \ \text{ on } \left[\tilde{x}_j-\frac{L}{4},\tilde{x}_j+\frac{L}{4}\right],\quad \text{for } j\neq i. \tag{3.36}$$

Define the following localized conserved version of E and F as

$$E_i(t)=E_i(u(t))=\int_{\mathbb{R}}(u^2+u_x^2)\Phi_i(t)dx, \tag{3.37}$$

$$\begin{aligned} F_i(t)&=F_i(u(t))\\ &=\int_{\mathbb{R}}\left(k_1\big(u^4+2u^2u_x^2-\frac{1}{3}u_x^4\big)+2k_2\big(u^3+uu_x^2\big)\right)\Phi_i(t)dx. \end{aligned} \tag{3.38}$$

Considering the weight function $\Psi_{j,K}(t,x)$ defined in (3.34), we introduce for $j=2,\dots,N$,

$$\mathcal{I}_{j,K}(t)=\int_{\mathbb{R}}\left(u^2(t,x)+u_x^2(t,x)\right)\Psi_{j,K}(t,x)dx. \tag{3.39}$$

In the following Lemma, we indicate that for a solution u of Eq. (1.1) in Lemma 3.1, the function $\mathcal{I}_{j,K}(t)$ is almost decreasing with time. Assuming $0<c_1<c_2<\cdots<c_N$, we set

$$\sigma_0=\frac{1}{4}\min\{c_1,c_2-c_1,\dots,c_N-c_{N-1}\}. \tag{3.40}$$

Lemma 3.3. *Let $u(t,x)$ be the strong solution of the Eq.* (1.1) *satisfying* (3.4) *on $[0,t^*]$ with initial data $u(0,x)=u_0(x)$. Assume $u_0(x)$ satisfies the assumptions given in Theorem 1.1. There exist $\alpha_0>0$ and $L_0>0$ depending only on $(c_i)_{i=1}^N$ such that if $0<\alpha<\alpha_0$ and $L>L_0$, then for $4\le K=O(L^{1/2})$,*

$$\mathcal{I}_{j,K}(t)-\mathcal{I}_{j,K}(0)\leqslant \frac{C}{\sigma_0}e^{-\frac{L}{8K}}, \tag{3.41}$$

for each $j\in\{2,...,N\}$ and any $t\in[0,t^*]$ with a positive constant C.

Proof. First, we fix $j \in \{2, ..., N\}$, establish with $\omega = \Psi_{j,K}$ in (3.23), and use $\frac{d}{dt}\Psi_{j,K}(t,x) = -\dot{y}_j(t)\partial_x\Psi_{j,K}(t,x)$, we obtain that for $t \in [0, t^*]$,

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}_{j,K}(t) =& \frac{d}{dt}\int_{\mathbb{R}} \left(u^2 + u_x^2\right) \Psi_{j,K}(t,x)dx \\
=& -\dot{y}_j(t)\int_{\mathbb{R}} \left(u^2 + u_x^2\right) \partial_x\Psi_{j,K}dx - \frac{k_1}{2}\int_{\mathbb{R}}(u^2 - u_x^2)^2\partial_x\Psi_{j,K}(t,x)dx \\
&+ \frac{2k_1}{3}\int_{\mathbb{R}} u^4\partial_x\Psi_{j,K}(t,x)dx + k_2\int_{\mathbb{R}} uu_x^2\partial_x\Psi_{j,K}(t,x)dx \\
&+ \frac{4k_1}{3}\int_{\mathbb{R}} u\left(1-\partial_x^2\right)^{-1}\left(u^3 + 3uu_x^2\right)\partial_x\Psi_{j,K}(t,x)dx \\
&- 2k_1\int_{\mathbb{R}} u\left(1-\partial_x^2\right)^{-1}\left(u_x^2 m\right)\partial_x\Psi_{j,K}(t,x)dx \\
&+ k_2\int_{\mathbb{R}} u\left(1-\partial_x^2\right)^{-1}\left(2u^2 + u_x^2\right)\partial_x\Psi_{j,K}(t,x)dx.
\end{aligned} \tag{3.42}$$

According to (3.9) in Lemma 3.1, for $0 < \alpha < \alpha_0$ and $L > L_0$, there are

$$\begin{aligned}
-\dot{y}_j(t) =& -\frac{\dot{\tilde{x}}_j(t) - c_j}{2} - \frac{\dot{\tilde{x}}_{j-1}(t) - c_{j-1}}{2} - \frac{c_{j-1} + c_j}{2} \\
\leqslant& -\frac{c_{j-1} + c_j}{2} + O(\sqrt{\alpha}) + O(L^{-1}) < -\frac{1}{2}c_1.
\end{aligned} \tag{3.43}$$

Next, aligned with (3.42), we only cover two scenarios in this discussion: (1) $k_1 > 0$, $k_2 > 0$; (2) $k_1 > 0, k_2 \leqslant 0$.

(1) $k_1 > 0, k_2 > 0$, using the inequality of (2.6) and $\partial_x\Psi_{j,K} = \frac{1}{K}\Psi'(\frac{x-y_j(t)}{K}) > 0$, we reduce (3.42) to

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}_{j,K}(t) \leqslant& -\frac{c_1}{2}\int_{\mathbb{R}} \left(u^2 + u_x^2\right) \partial_x\Psi_{j,K}dx \\
&+ \frac{2k_1}{3}\int_{\mathbb{R}} u^4\partial_x\Psi_{j,K}(t,x)dx + k_2\int_{\mathbb{R}} uu_x^2\partial_x\Psi_{j,K}(t,x)dx \\
&+ \frac{4k_1}{3}\int_{\mathbb{R}} u\left(1-\partial_x^2\right)^{-1}\left(u^3 + 3uu_x^2\right)\partial_x\Psi_{j,K}(t,x)dx \\
&+ k_2\int_{\mathbb{R}} u\left(1-\partial_x^2\right)^{-1}\left(2u^2 + u_x^2\right)\partial_x\Psi_{j,K}(t,x)dx.
\end{aligned} \tag{3.44}$$

For further estimates, we define the interval D_j by

$$D_j = \left[\tilde{x}_{j-1}(t) + \frac{L}{4}, \tilde{x}_j(t) - \frac{L}{4}\right],$$

and divide $\mathbb{R}$ as $\mathbb{R} = D_j \cup D_j^c$. Note that according to (3.10) and (3.40), for $x \in D_j^c$,

$$|x - y_j(t)| \geqslant \frac{\tilde{x}_j(t) - \tilde{x}_{j-1}(t)}{2} - \frac{L}{4} \geqslant \frac{c_j - c_{j-1}}{4}t + \frac{L}{8} \geqslant \sigma_0 t + \frac{L}{8}, \tag{3.45}$$

and then for $K = O(\sqrt{L})$ and a large enough L_0,

$$\frac{|x - y_j(t)|}{K} \geqslant \frac{\sigma_0 t + \frac{L}{8}}{K} > 1, \tag{3.46}$$

which implies by definition of Ψ that

$$\partial_x \Psi_{j,K}(t,x) = \frac{1}{K}\Psi'\left(\frac{x-y_j(t)}{K}\right) \leqslant \frac{1}{K}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}, \quad x \in D_j^c. \tag{3.47}$$

Thus, using the conservation of $E(u) = \|u\|_{H^1(\mathbb{R})}^2$, there exists a constant $C > 0$ such that

$$\frac{2}{3}k_1 \int_{D_j^c} u^4 \partial_x \Psi_{j,K} dx \leqslant \frac{C}{K}\|u_0\|_{H^1(\mathbb{R})}^4 e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})} \tag{3.48}$$

and

$$k_2 \int_{D_j^c} uu_x^2 \partial_x \Psi_{j,K} dx \leqslant \frac{C}{K}\|u_0\|_{H^1(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}. \tag{3.49}$$

Moreover, notice that $|x - \tilde{x}_i(t)| > L/4$ for any $x \in D_j$ and each $i \in \{1, ..., N\}$. Thus, from (3.8) and the exponential decay of $\varphi_{c_i}(x - \tilde{x}_i(t))$, we have

$$\begin{aligned}
&\|u(t,x)\|_{L^\infty(D_j)} \\
&\leqslant \left\| u(t,x) - \sum_{i=1}^N \varphi_{c_i}\big(x - \tilde{x}_i(t)\big) \right\|_{L^\infty(D_j)} + \left\| \sum_{i=1}^N \varphi_{c_i}\big(x - \tilde{x}_i(t)\big) \right\|_{L^\infty(D_j)} \\
&\leqslant \frac{\sqrt{2}}{2} \left\| u(t,x) - \sum_{i=1}^N \varphi_{c_i}\big(x - \tilde{x}_i(t)\big) \right\|_{H^1(\mathbb{R})} + \sum_{i=1}^N \left\| \varphi_{c_i}\big(x - \tilde{x}_i(t)\big) \right\|_{L^\infty(D_j)} \\
&= O(\sqrt{\alpha}) + O(e^{-\frac{L}{8}}).
\end{aligned} \tag{3.50}$$

Therefore, using (3.48), (3.49) and (3.50), we obtain

$$\begin{aligned}
&\frac{2}{3}k_1 \int_{\mathbb{R}} u^4 \partial_x \Psi_{j,K} dx \\
&\quad = \frac{2}{3}k_1 \int_{D_j} u^4 \partial_x \Psi_{j,K} dx + \frac{2}{3}k_1 \int_{D_j^c} u^4 \partial_x \Psi_{j,K} dx \\
&\quad \leqslant \frac{2}{3}k_1 \|u\|_{L^\infty(D_j)}^2 \int_{D_j} u^2 \partial_x \Psi_{j,K} dx + \frac{C}{K}\|u_0\|_{H^1(\mathbb{R})}^4 e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})} \\
&\quad \leqslant \frac{c_1}{16} \int_{\mathbb{R}} (u^2 + u_x^2) \partial_x \Psi_{j,K} dx + \frac{C}{K}\|u_0\|_{H^1(\mathbb{R})}^4 e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}
\end{aligned} \tag{3.51}$$

and

$$\begin{aligned}
&k_2 \int_{\mathbb{R}} u{u_x}^2 \partial_x \Psi_{j,K} dx \\
&\quad = k_2 \int_{D_j} uu_x^2 \partial_x \Psi_{j,K} dx + k_2 \int_{D_j^c} uu_x^2 \partial_x \Psi_{j,K} dx \\
&\quad \leqslant k_2 \|u(t)\|_{L^\infty(D_j)} \int_{D_j} u_x^2 \partial_x \Psi_{j,K} dx + \frac{C}{K}\|u_0\|_{H^1(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})} \\
&\quad \leqslant \frac{c_1}{16} \int_{\mathbb{R}} (u^2 + u_x^2) \partial_x \Psi_{j,K} dx + \frac{C}{K}\|u_0\|_{H^1(\mathbb{R})}^3 e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}.
\end{aligned} \tag{3.52}$$

By a similar method as above, we get

$$\begin{aligned}
&\frac{4}{3}k_1\int_{D_j^c} u(1-\partial_x^2)^{-1}(u^3+3uu_x^2)\partial_x\Psi_{j,K}dx\\
&\leqslant 4k_1\|u\|_{L^\infty(\mathbb{R})}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(t,x)|\int_{\mathbb{R}} G*(u^3+uu_x^2)dx\\
&\leqslant 4k_1\|u\|^2_{L^\infty(\mathbb{R})}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(t,x)|\int_{\mathbb{R}} G*(u^2+u_x^2)dx\\
&\leqslant 2k_1\|u\|^2_{L^\infty(\mathbb{R})}\sup_{x\in D_j^c}|\partial_x\Psi_{j,K}(t,x)|\int_{\mathbb{R}} e^{-|x|}dx\int_{\mathbb{R}}(u^2+u_x^2)dx\\
&\leqslant \frac{C}{K}\|u_0\|^4_{H^1(\mathbb{R})}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})},
\end{aligned}\tag{3.53}$$

where $G(x)=e^{-|x|}/2$ is the Green function of $(1-\partial_x^2)^{-1}$.

However, according to the definition of Ψ, $|\Psi'''(x)|\leqslant 10\Psi'(x), x\in\mathbb{R}$, we discover that

$$\begin{aligned}
(1-\partial_x^2)\partial_x\Psi_{j,K}(t,x)&=\partial_x\Psi_{j,K}(t,x)-\frac{1}{K^3}\Psi'''\left(\frac{x-y_j(t)}{K}\right)\\
&\geqslant\left(1-\frac{10}{K^2}\right)\partial_x\Psi_{j,K}(t,x),
\end{aligned}\tag{3.54}$$

when the parameters $K\geqslant 4$, we have

$$(1-\partial_x^2)^{-1}\partial_x\Psi_{j,K}(t,x)\leqslant\left(1-\frac{10}{K^2}\right)^{-1}\partial_x\Psi_{j,K}(t,x).\tag{3.55}$$

Taking $K\geqslant 4$ and noting that $m_0\not\equiv 0$ and $\partial_x\Psi_{j,K}(t,x)>0$, we conclude that

$$\begin{aligned}
&\frac{4}{3}k_1\int_{D_j} u(1-\partial_x^2)^{-1}(u^3+3uu_x^2)\partial_x\Psi_{j,K}dx\\
&\leqslant 4k_1\|u\|_{L^\infty(D_j)}\int_{\mathbb{R}}(u^3+uu_x^2)(1-\partial_x^2)^{-1}\partial_x\Psi_{j,K}dx\\
&\leqslant C\|u\|_{L^\infty(D_j)}\|u\|_{H^1(\mathbb{R})}\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx.
\end{aligned}\tag{3.56}$$

Using (3.8), for $t\in[0,t^*]$,

$$\begin{aligned}
\|u(t)\|_{H^1(\mathbb{R})}=\|u_0\|_{H^1(\mathbb{R})}&\leqslant\|v_0\|_{H^1(\mathbb{R})}+\left\|\sum_{i=1}^N\varphi_{c_i}(\cdot-\tilde{x}_i(0))\right\|_{H^1(\mathbb{R})}\\
&\leqslant O(\sqrt{\alpha})+\sum_{i=1}^N\|\varphi_{c_i}\|_{H^1(\mathbb{R})}=O(\sqrt{\alpha})+\sum_{i=1}^N\sqrt{2}a_i,
\end{aligned}\tag{3.57}$$

which along with (3.50) gives

$$\frac{4}{3}k_1\int_{D_j} u(1-\partial_x^2)^{-1}(u^3+3uu_x^2)\partial_x\Psi_{j,K}dx$$

$$\leqslant C\big(O(\sqrt{\alpha})+O(e^{-\frac{L}{8}})\big)\left(O(\sqrt{\alpha})+\sum_{i=1}^{N}\sqrt{2}a_i\right)\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx. \quad (3.58)$$

Therefore, there exist $\alpha_0 \ll 1$ and $L_0 \gg 1$ such that for $0<\alpha<\alpha_0$ and $L>L_0$, combining with (3.53), it gives rise to

$$\begin{aligned}
&\frac{4}{3}k_1\int_{\mathbb{R}}u(1-\partial_x^2)^{-1}(u^3+3uu_x^2)\partial_x\Psi_{j,K}dx\\
&=\frac{4}{3}k_1\int_{D_j}u(1-\partial_x^2)^{-1}(u^3+3uu_x^2)\partial_x\Psi_{j,K}dx\\
&\quad+\frac{4}{3}k_1\int_{D_j^c}u(1-\partial_x^2)^{-1}(u^3+3uu_x^2)\partial_x\Psi_{j,K}dx\\
&\leqslant\frac{c_1}{16}\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx+\frac{C}{K}\|u_0\|^4_{H^1(\mathbb{R})}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}. \qquad (3.59)
\end{aligned}$$

Similarly, we can calculate that

$$\begin{aligned}
&k_2\int_{\mathbb{R}}u(1-\partial_x^2)^{-1}(2u^2+u_x^2)\partial_x\Psi_{j,K}dx\\
&\quad\leqslant\frac{c_1}{16}\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx+\frac{C}{K}\|u_0\|^3_{H^1(\mathbb{R})}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}. \qquad (3.60)
\end{aligned}$$

Combining (3.51), (3.52), (3.59) and (3.60), we infer that

$$\frac{d}{dt}\mathcal{I}_{j,K}(t)\leqslant-\frac{c_1}{4}\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx+\frac{C}{K}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}. \qquad (3.61)$$

Using the Gronwall argument on $[0,t]$ with $t\leqslant t^*$, we find that for any $t\in[0,t^*]$,

$$\mathcal{I}_{j,K}(t)-\mathcal{I}_{j,K}(0)\leqslant\frac{C}{K}\int_0^t e^{-\frac{1}{K}(\sigma_0\tau+\frac{L}{8})}d\tau\leqslant\frac{C}{\sigma_0}e^{-\frac{L}{8K}}.$$

(2) $k_1>0, k_2\leqslant 0$, similarly, we deduce by (3.42) that

$$\begin{aligned}
\frac{d}{dt}\mathcal{I}_{j,K}(t)\leqslant&-\frac{c_1}{2}\int_{\mathbb{R}}\left(u^2+u_x^2\right)\partial_x\Psi_{j,K}dx+\frac{2k_1}{3}\int_{\mathbb{R}}u^4\partial_x\Psi_{j,K}(t,x)dx\\
&+\frac{4k_1}{3}\int_{\mathbb{R}}u\left(1-\partial_x^2\right)^{-1}\left(u^3+3uu_x^2\right)\partial_x\Psi_{j,K}(t,x)dx.
\end{aligned}$$

Similar to the discussion method in (1) above, we obtain the results that

$$\frac{d}{dt}\mathcal{I}_{j,K}(t)\leqslant-\frac{c_1}{4}\int_{\mathbb{R}}(u^2+u_x^2)\partial_x\Psi_{j,K}dx+\frac{C}{K}e^{-\frac{1}{K}(\sigma_0 t+\frac{L}{8})}.$$

Integrating for time $[0,t]$ with $t\leqslant t^*$, we summarize that for $t\in[0,t^*]$,

$$\mathcal{I}_{j,K}(t)-\mathcal{I}_{j,K}(0)\leqslant\frac{C}{K}\int_0^t e^{-\frac{1}{K}(\sigma_0\tau+\frac{L}{8})}d\tau\leqslant\frac{C}{\sigma_0}e^{-\frac{L}{8K}}.$$

This completes the proof of Lemma 3.3. □

3.3. Global identity and localized estimate

In this subsection, we establish the global identity and the localized estimate. For $Z = (z_1, \ldots, z_N)$, we denote

$$R_{z_i}(x) = \varphi_{c_i}(x - z_i) = a_i \varphi(x - z_i) = a_i e^{-|x-z_i|}. \tag{3.62}$$

It is obvious that $R_{z_i}(x)$ has the peak at $x = z_i$ and hence

$$\max_{x\in\mathbb{R}} R_{z_i}(x) = R_{z_i}(z_i) = a_i. \tag{3.63}$$

A direct calculation shows

$$E(R_{z_i}) = \int_{\mathbb{R}} \left(\varphi_{c_i}^2 + \partial_x \varphi_{c_i}^2\right) dx = 2a_i^2 \tag{3.64}$$

and

$$\begin{aligned} F(R_{z_i}) = \int_{\mathbb{R}} k_1 \left(\varphi_{c_i}^4 + 2\varphi_{c_i}^2 \varphi_x^2 - \frac{\varphi_x^4}{3}\right) \\ + 2k_2 \left(\varphi_{c_i}^3 + \varphi_{c_i} \varphi_x^2\right) dx = \frac{4}{3} a_i^3 \left(k_1 a_i + 2k_2\right). \end{aligned} \tag{3.65}$$

Using (3.64), we provide a global identity, which is the generalization of Lemma 2.3 in [23].

Lemma 3.4. *For any $(z_1, \ldots, z_N) \in \mathbb{R}^N$ such that $|z_i - z_{i-1}| > L/2$ with $L > 0, i = 2, ..., N$, and any $u \in H^1(\mathbb{R})$, we have*

$$\begin{aligned} &\left\| u(x) - \sum_{i=1}^{N} R_{z_i}(x) \right\|_{H^1(\mathbb{R})}^2 \\ &= E(u) - \sum_{i=1}^{N} E(\varphi_{c_i}) - 4\sum_{i=1}^{N} a_i \big(u(z_i) - a_i\big) + O(e^{-L/4}), \end{aligned} \tag{3.66}$$

where the constant involved in $O(e^{-L/4})$ depends only on $c_1, \ldots, c_N$.

Proof. This lemma has been proved in [14, 24], and we leave out the steps here. □

In the following lemma, we build a localized estimate that establishes a connection between E_i and F_i through polynomial inequalities, where the functions E_i and F_i are independent of time since we fix $\tilde{x}_1 < \cdots < \tilde{x}_N$. For convenience, we take $K = \sqrt{L}/8$ to derive the appropriate estimates.

Lemma 3.5. *Given N real numbers $\tilde{x}_1 < \cdots < \tilde{x}_N$ with $\tilde{x}_i - \tilde{x}_{i-1} \geqslant 3L/4$. Define the interval $\mathcal{J}_i$ as in (3.11). Suppose that, for any fixed positive function $u \in H^s(\mathbb{R})$ with $s > 5/2$, and each $i = 1, \ldots, N$, there exists $\xi_i \in \mathcal{J}_i$ such that*

$$u(\xi_i) = \max_{x\in\mathcal{J}_i} u(x) := M_i \quad \text{and} \quad |\xi_i - \tilde{x}_i| < \frac{L}{12}. \tag{3.67}$$

Then, for each $i = 1, \ldots, N$, we have

$$F_i(u) \leqslant \left(\frac{4}{3} k_1 M_i^2 + 2k_2 M_i\right) E_i(u) - \frac{4}{3} k_1 M_i^4 - \frac{4}{3} k_2 M_i^3 + O(L^{-\frac{1}{2}}). \tag{3.68}$$

Proof. Let $i=1,\dots,N$ be fixed and take $\xi_i \in \mathcal{J}_i$ satisfying (3.67). We set

$$g_i(x)=\begin{cases} u(x)-u_x(x), & x<\xi_i,\\ u(x)+u_x(x), & x>\xi_i. \end{cases}$$

Direct computation yields

$$\begin{aligned}\int_{\mathbb{R}} g_i^2(x)\Phi_i(x)dx &= \int_{-\infty}^{\xi_i}(u-u_x)^2\,\Phi_i dx+\int_{\xi_i}^{+\infty}(u+u_x)^2\,\Phi_i dx\\ &=\int_{\mathbb{R}}\left(u^2+u_x^2\right)\Phi_i dx-2\int_{-\infty}^{\xi_i}uu_x\Phi_i dx+2\int_{\xi_i}^{+\infty}uu_x\Phi_i dx\\ &=E_i(u)-2M_i^2\Phi_i(\xi_i)+\int_{-\infty}^{\xi_i}u^2\partial_x\Phi_i dx-\int_{\xi_i}^{+\infty}u^2\partial_x\Phi_i dx. \end{aligned} \tag{3.69}$$

Next, following [23], we define the functions as follows

$$h_1(x)=\begin{cases} u^2(x)-\frac{2}{3}u(x)u_x(x)-\frac{1}{3}u_x^2(x), & x<\xi_i,\\ \\ u^2(x)+\frac{2}{3}u(x)u_x(x)-\frac{1}{3}u_x^2(x), & x>\xi_i \end{cases} \tag{3.70}$$

and

$$h_2(x)=u(x). \tag{3.71}$$

Therefore, using (3.70) and (3.71), we denote

$$\begin{aligned}&\int_{\mathbb{R}} h\left(x\right)g_i^2\left(x\right)\Phi_i\left(x\right)dx\\ &\quad:=k_1\int_{\mathbb{R}}h_1\left(x\right)g_i^2\left(x\right)\Phi_i\left(x\right)dx+2k_2\int_{\mathbb{R}}h_2\left(x\right)g_i^2\left(x\right)\Phi_i\left(x\right)dx. \end{aligned} \tag{3.72}$$

A direct calculation indicates that

$$\begin{aligned}\int_{\mathbb{R}}h_1\left(x\right)g_i^2\left(x\right)\Phi_i\left(x\right)dx &=\int_{-\infty}^{\xi_i}\left(u^2-\frac{2}{3}uu_x-\frac{1}{3}u_x^2\right)\left(u-u_x\right)^2\Phi_i dx\\ &\quad+\int_{\xi_i}^{\infty}\left(u^2+\frac{2}{3}uu_x-\frac{1}{3}u_x^2\right)\left(u+u_x\right)^2\Phi_i dx\\ &=\int_{\mathbb{R}}\left(u^4+2u^2u_x^2-\frac{1}{3}u_x^4\right)\Phi_i dx-\frac{4}{3}M_i^4\Phi_i\left(\xi_i\right)\\ &\quad+\frac{2}{3}\int_{-\infty}^{\xi_i}u^4\partial_x\Phi_i dx-\frac{2}{3}\int_{\xi_i}^{+\infty}u^4\partial_x\Phi_i dx. \end{aligned} \tag{3.73}$$

Using a similar method as above, one derives

$$\begin{aligned}\int_{\mathbb{R}}h_2\left(x\right)g_i^2\left(x\right)\Phi_i\left(x\right)dx &=\int_{-\infty}^{\xi_i}u\left(u-u_x\right)^2\Phi_i dx+\int_{\xi_i}^{\infty}u\left(u+u_x\right)^2\Phi_i dx\\ &=\int_{\mathbb{R}}\left(u^3+uu_x^2\right)\Phi_i dx-\frac{4}{3}M_i^3\Phi_i\left(\xi_i\right) \end{aligned}$$

$$+\frac{2}{3}\int_{-\infty}^{\xi_i} u^3\partial_x\Phi_i dx-\frac{2}{3}\int_{\xi_i}^{+\infty} u^3\partial_x\Phi_i dx. \tag{3.74}$$

Combining (3.73) with (3.74), we obtain

$$\begin{aligned}&\int_{\mathbb{R}} h\left(x\right)g_i^2\left(x\right)\Phi_i\left(x\right)dx\\ =&F_i\left(u\right)-\frac{4}{3}k_1M_i^4\Phi_i\left(\xi_i\right)+\frac{2}{3}k_1\int_{-\infty}^{\xi_i}u^4\partial_x\Phi_i dx-\frac{2}{3}k_1\int_{\xi_i}^{\infty}u^4\partial_x\Phi_i dx\\ &-\frac{8}{3}k_2M_i^3\Phi_i\left(\xi_i\right)+\frac{4}{3}k_2\int_{-\infty}^{\xi_i}u^3\partial_x\Phi_i dx-\frac{4}{3}k_2\int_{\xi_i}^{\infty}u^3\partial_x\Phi_i dx.\end{aligned} \tag{3.75}$$

We know $h_1\left(x\right)\leqslant\frac{4}{3}u^2(x)\leqslant\frac{4}{3}M_i^2$ and $h_2\left(x\right)\leqslant M_i$, so from (3.69), we deduce that

$$\begin{aligned}&\int_{\mathbb{R}} h_1\left(x\right)g^2\left(x\right)\Phi_i\left(x\right)dx\\ \leqslant&\frac{4}{3}\int_{\mathbb{R}}u^2\left(x\right)g^2\left(x\right)\Phi_i\left(x\right)dx\\ =&\frac{4}{3}\int_{\mathcal{J}_i}u^2(x)g^2(x)\Phi_i(x)dx+\frac{4}{3}\sum_{k\neq i,k=1}^{N}\int_{\mathcal{J}_k}u^2(x)g^2(x)\Phi_i(x)dx\\ \leqslant&\frac{4}{3}M_i^2E_i\left(u\right)-\frac{8}{3}M_i^4\Phi_i\left(\xi_i\right)+\frac{4}{3}\sum_{k\neq i,k=1}^{N}\int_{\mathcal{J}_k}u^2(x)g^2(x)\Phi_i(x)dx\\ &+\frac{4}{3}M_i^2\int_{-\infty}^{\xi_i}u^2\partial_x\Phi_i dx-\frac{4}{3}M_i^2\int_{\xi_i}^{\infty}u^2\partial_x\Phi_i dx.\end{aligned} \tag{3.76}$$

In a similar manner, we get

$$\begin{aligned}&\int_{\mathbb{R}} h_2\left(x\right)g^2\left(x\right)\Phi_i\left(x\right)dx\\ \leqslant&M_iE_i\left(u\right)-2M_i^3\Phi_i\left(\xi_i\right)+\sum_{k\neq i,k=1}^{N}\int_{\mathcal{J}_k}u(x)g^2(x)\Phi_i(x)dx\\ &+M_i\int_{-\infty}^{\xi_i}u^2\partial_x\Phi_i dx-M_i\int_{\xi_i}^{\infty}u^2\partial_x\Phi_i dx.\end{aligned} \tag{3.77}$$

Due to the construction of Φ_i and the exponential decay of Ψ, taking $K=\sqrt{L}/8$, clearly there are constants $C>0$,

$$|\partial_x\Phi_i|=\frac{1}{K}\Psi'\leqslant\frac{C}{K}\leqslant O(L^{-\frac{1}{2}}). \tag{3.78}$$

Using (3.72), (3.75)-(3.78) and the Sobolev embedding $\|u\|_{L^\infty(\mathbb{R})}\leqslant\frac{\sqrt{2}}{2}\|u\|_{H^1(\mathbb{R})}$, estimates can be derived that

$$\begin{aligned}F_i\left(u\right)\leqslant&\frac{4}{3}k_1M_i^2E_i\left(u\right)-\frac{4}{3}k_1M_i^4+\frac{4}{3}k_1M_i^4\left(1-\Phi_i\left(\xi_i\right)\right)\\ &+2k_2M_iE_i\left(u\right)-\frac{4}{3}k_2M_i^3+\frac{4}{3}k_2M_i^3\left(1-\Phi_i\left(\xi_i\right)\right)+O(L^{-\frac{1}{2}}).\end{aligned} \tag{3.79}$$

On the other hand, since $|\xi_i - \tilde{x}_i| < L/12$, using (3.35), we find

$$|1 - \Phi_i(\xi_i)| \leqslant 4e^{-\frac{L}{4K}} \leqslant O(L^{-\frac{1}{2}}). \tag{3.80}$$

From (3.79) and (3.80), we deduce that

$$F_i(u) \leqslant \left(\frac{4}{3}k_1 M_i^2 + 2k_2 M_i\right) E_i(u) - \frac{4}{3}k_1 M_i^4 - \frac{4}{3}k_2 M_i^3 + O(L^{-\frac{1}{2}}),$$

which proves this lemma. □

In the next lemma, we use the method in [24] to estimate the differences between the local maximum of the solution $u(t,x)$ and the maximum of each single peakon.

Lemma 3.6. *Let $u(t,x)$ be the strong solution of (1.1) satisfying (3.4) on $[0,t^*]$ as in Lemma 3.1 with initial data $u(0,x) = u_0(x)$ satisfying the assumptions given in Theorem 1.1. Let us set for $i \in \{1,\ldots,N\}$,*

$$M_i(t) = \max_{x \in \mathcal{J}_i(t)} u(t,x) = u\big(t, \xi_i(t)\big), \quad \forall t \in [0,t^*], \tag{3.81}$$

where the interval $\mathcal{J}_i(t)$ is defined in (3.11). Then, we have the estimate

$$\sum_{i=1}^{N} \left(\frac{4}{3}k_1 a_i^2 + \frac{8}{3}k_2 a_i\right)^{\frac{1}{2}} |M_i(t) - a_i| \leqslant O(\varepsilon) + O(L^{-\frac{1}{4}}), \tag{3.82}$$

where the constants in $O(\cdot)$ depend on $(c_i)_{i=1}^N$ and $\|u_0\|_{H^s(\mathbb{R})}$.

Proof. By the construction of Φ_i, for any $u \in H^s(\mathbb{R})(s > 5/2)$, we have

$$E(u) = \sum_{i=1}^{N} E_i(u), \quad F(u) = \sum_{i=1}^{N} F_i(u), \tag{3.83}$$

where $E_i(u)$, $F_i(u)$ are defined by (3.37) and (3.38). Furthermore, since u_0 satisfies (1.9) and (1.10), there exist $z_1^0 < z_2^0 < \cdots < z_N^0$ satisfying $z_i^0 - z_{i-1}^0 > L/2$, meaning that if we denote

$$R_{Z^0} = \sum_{i=1}^{N} \varphi_{c_i}\left(\cdot - z_i^0\right),$$

then

$$\|u_0 - R_{Z^0}\|_{H^1(\mathbb{R})} \leqslant \epsilon^2. \tag{3.84}$$

Using (3.84) and applying Minkowski inequality, we have

$$\begin{aligned}
&|E(u_0) - E(R_{Z^0})| \\
&\leqslant \left(\|u_0\|_{H^1(\mathbb{R})} + \|R_{Z^0}\|_{H^1(\mathbb{R})}\right) \left|\|u_0\|_{H^1(\mathbb{R})} - \|R_{Z^0}\|_{H^1(\mathbb{R})}\right| \\
&\leqslant \left(\|u_0 - R_{Z^0}\|_{H^1(\mathbb{R})} + 2\|R_{Z^0}\|_{H^1(\mathbb{R})}\right) \|u_0 - R_{Z^0}\|_{H^1(\mathbb{R})} \\
&\leqslant \left(\varepsilon^2 + 2\sum_{i=1}^{N} \|\varphi_{c_i}\|_{H^1(\mathbb{R})}\right) \varepsilon^2
\end{aligned}$$

$$\leqslant O\left(\varepsilon^2\right), \tag{3.85}$$

which means that

$$\begin{aligned}|E(u_0)-\sum_{i=1}^{N}E(\varphi_{c_i})| &\leqslant |E(u_0)-E(R_{Z^0})|+|E(R_{Z^0})-\sum_{i=1}^{N}E(\varphi_{c_i})|\\ &\leqslant O(\varepsilon^2)+O(e^{-\frac{L}{4}}).\end{aligned} \tag{3.86}$$

We fix $t\in[0,t^*]$ and note that by (3.68) in Lemma 3.5, the following inequality holds

$$F_i(u)-\Big(\frac{4k_1}{3}M_i^2+2k_2M_i\Big)E_i(u)+\frac{4k_1}{3}M_i^4+\frac{4k_2}{3}M_i^3\leqslant O(L^{-\frac{1}{2}}). \tag{3.87}$$

Now, we define the polynomial $P^i(y)$ by

$$P^i(y)=\frac{4k_1}{3}y^4+\frac{4k_2}{3}y^3-\left(\frac{4k_1}{3}y^2+2k_2y\right)E_i(u)+F_i(u). \tag{3.88}$$

Associated with the peakon φ_{c_i}, using (3.64) and (3.65), $P^i(y)$ takes the form

$$\begin{aligned}P_0^i(y)&=\frac{4k_1}{3}y^4+\frac{4k_2}{3}y^3-\Big(\frac{4k_1}{3}y^2+2k_2y\Big)E(\varphi_{c_i})+F(\varphi_{c_i})\\ &=\frac{4k_1}{3}y^4+\frac{4k_2}{3}y^3-\Big(\frac{4k_1}{3}y^2+2k_2y\Big)\big(2a_i^2\big)+\frac{4}{3}a_i^3\big(k_1a_i+2k_2\big)\\ &=(y-a_i)^2\Big(\frac{4k_1}{3}y^2+\frac{8k_1}{3}a_iy+\frac{4k_2}{3}y+\frac{4k_1}{3}a_i^2+\frac{8k_2}{3}a_i\Big).\end{aligned} \tag{3.89}$$

According to (3.87), (3.88) and (3.89), we obtain

$$P_0^i(M_i)=P^i(M_i)+\Big(\frac{4k_1}{3}M_i^2+2k_2M_i\Big)\big(E_i(u)-E(\varphi_{c_i})\big)-\big(F_i(u)-F(\varphi_{c_i})\big), \tag{3.90}$$

which yields

$$\begin{aligned}\Big(\frac{4k_1}{3}a_i^2+\frac{8k_2}{3}a_i\Big)\big(M_i(t)-a_i\big)^2\leqslant&\Big(\frac{4k_1}{3}M_i^2+2k_2M_i\Big)\big(E_i(u)-E(\varphi_{c_i})\big)\\ &-\big(F_i(u)-F(\varphi_{c_i})\big)+O(L^{-\frac{1}{2}}),\end{aligned} \tag{3.91}$$

where the solution $u(t,x)$ is positive. Summing over i from (3.91), we get

$$\begin{aligned}&\sum_{i=1}^{N}\Big(\frac{4k_1}{3}a_i^2+\frac{8k_2}{3}a_i\Big)\big(M_i(t)-a_i\big)^2\\ &\quad\leqslant\sum_{i=1}^{N}Q_i\big(E_i(u)-E_i(u_0)\big)+\sum_{i=1}^{N}Q_i\big(E_i(u_0)-E(\varphi_{c_i})\big)\\ &\qquad-\sum_{i=1}^{N}\big(F_i(u)-F(\varphi_{c_i})\big)+O(L^{-\frac{1}{2}})\end{aligned}$$

$$:= P_1 + P_2 + P_3 + O(L^{-\frac{1}{2}}), \tag{3.92}$$

where $Q_i = \left(\frac{4k_1}{3}M_i^2 + 2k_2 M_i\right)$. We now derive three useful estimates. We first estimate the term P_2. Using (3.86), we infer that

$$\begin{aligned} M_i(t) &\leqslant \|u(t,x)\|_{L^\infty(\mathbb{R})} \leqslant \left(\frac{1}{2}E(u_0)\right)^{\frac{1}{2}} \\ &\leqslant \left(\frac{1}{2}\sum_{i=1}^N E(\varphi_{c_i})\right)^{\frac{1}{2}} + O(\varepsilon) + O(e^{-\frac{L}{8}}) \leqslant \left(2\sum_{i=1}^N a_i^2\right)^{\frac{1}{2}}, \end{aligned} \tag{3.93}$$

for $0 < \varepsilon < \varepsilon_0$, $L > L_0$ with $0 < \varepsilon_0 \ll 1$ and $L_0 \gg 1$ both depending only on $(c_i)_{i=1}^N$. Then, by (3.84), the exponential decay of φ_{c_i} and Φ_i, and the definition of E_i, it follows that

$$\begin{aligned} &\sum_{i=1}^N |E_i(u_0) - E(\varphi_{c_i})| \\ &\leqslant \sum_{i=1}^N \left| \|u_0\|^2_{H^1(\mathcal{J}_i(0))} - \|\varphi_{c_i}\|^2_{H^1(\mathcal{J}_i(0))} \right| + O(L^{-\frac{1}{2}}) \\ &\leqslant \sum_{i=1}^N \left(\|u_0 - R_{z^0}\|_{H^1(\mathcal{J}_i(0))} + \sum_{k=1,k\neq i}^N \|\varphi_{c_k}\|_{H^1(\mathcal{J}_i(0))} \right) \\ &\quad \cdot \left(\|u_0 - R_{z^0}\|_{H^1(\mathbb{R})} + 2\sqrt{2}\sum_{i=1}^N a_i \right) + O(L^{-\frac{1}{2}}) \\ &\leqslant O(\varepsilon^2) + O(L^{-\frac{1}{2}}), \end{aligned}$$

which together with (3.93) yields

$$P_2 \leqslant \sum_{i=1}^N \left(\frac{4k_1}{3}M_i^2 + 2k_2 M_i\right) |E_i(u_0) - E(\varphi_{c_i})| \leqslant O(\varepsilon^2) + O(L^{-\frac{1}{2}}). \tag{3.94}$$

Since u_0 satisfies (3.84), similar arguments were used in Lemma 3.5 in [18], we have

$$\left| F(u_0) - F\left(\sum_{i=1}^N \varphi_{c_i}(\,\cdot - z_i^0)\right) \right| \leqslant O(\varepsilon^2),$$

this means that from $z_i^0 - z_{i-1}^0 > L/2$ and (3.83) that

$$\begin{aligned} P_3 &\leqslant \left| F(u_0) - \sum_{i=1}^N F(\varphi_{c_i}) \right| \\ &\leqslant \left| F(u_0) - F\left(\sum_{i=1}^N \varphi_{c_i}(\,\cdot - z_i^0)\right) \right| + \left| F\left(\sum_{i=1}^N \varphi_{c_i}(\,\cdot - z_i^0)\right) - \sum_{i=1}^N F(\varphi_{c_i}) \right| \\ &\leqslant O(\varepsilon^2) + O(e^{-\frac{L}{4}}). \end{aligned} \tag{3.95}$$

Using (3.83) along with the definition of the weight function Φ_i and the Abel transform, we obtain

$$
\begin{aligned}
P_1 &= Q_N(t)\sum_{i=1}^{N}\big(E_i\big(u(t)\big)-E_i(u_0)\big)\\
&\quad -\sum_{j=1}^{N-1}\big(Q_{j+1}(t)-Q_j(t)\big)\sum_{i=1}^{j}\big(E_i\big(u(t)\big)-E_i(u_0)\big)\\
&= -\sum_{j=1}^{N-1}\big(Q_{j+1}(t)-Q_j(t)\big)\\
&\quad \cdot\left(\int_{\mathbb{R}}\big(u^2(t)+u_x^2(t)\big)\big(1-\Psi_{j+1,K}\big)dx-\int_{\mathbb{R}}\big(u_0^2+u_{0x}^2\big)\big(1-\Psi_{j+1,K}\big)dx\right)\\
&= \sum_{j=1}^{N-1}\big(Q_{j+1}(t)-Q_j(t)\big)\big(\mathcal{I}_{j+1,K}(t)-\mathcal{I}_{j+1,K}(0)\big).
\end{aligned}
$$

According to (3.12), one obtains

$$
\Big|\xi_i(t)-\tilde{x}_i(t)\Big| < \frac{L}{12}, \quad \forall t\in\Big[0,t^*\Big],
$$

which along with (3.8) gives rise to

$$
\begin{aligned}
&\left\|u(t,x)-\sum_{i=1}^{N}\varphi_{c_i}\big(x-\xi_i(t)\big)\right\|_{H^1(\mathbb{R})}\\
&\leqslant \left\|u(t,x)-\sum_{i=1}^{N}\varphi_{c_i}\big(x-\tilde{x}_i(t)\big)\right\|_{H^1(\mathbb{R})}\\
&\quad +\left\|\sum_{i=1}^{N}\varphi_{c_i}\big(x-\xi_i(t)\big)-\sum_{i=1}^{N}\varphi_{c_i}\big(x-\tilde{x}_i(t)\big)\right\|_{H^1(\mathbb{R})}\\
&\leqslant \|v(t,x)\|_{H^1(\mathbb{R})}+\sum_{i=1}^{N}\big\|\varphi_{c_i}\big(x-\xi_i(t)\big)-\varphi_{c_i}\big(x-\tilde{x}_i(t)\big)\big\|_{H^1(\mathbb{R})}\\
&\leqslant O(\sqrt{\alpha})+O(e^{-\frac{L}{4}}).
\end{aligned}
\tag{3.96}
$$

Therefore, employing (3.8), (3.10), (3.12) and the exponential decay of φ_{c_i}, it seems that

$$
\begin{aligned}
&\big|u\big(t,\xi_i(t)\big)-a_i\big|\\
&\leqslant \left|u\big(t,\xi_i(t)\big)-\sum_{j=1}^{N}\varphi_{c_j}\big(\xi_i(t)-\xi_j(t)\big)\right|+\sum_{j\neq i,j=1}^{N}\varphi_{c_j}\big(\xi_i(t)-\xi_j(t)\big)\\
&\leqslant \left\|u\big(t,x(t)\big)-\sum_{j=1}^{N}\varphi_{c_j}\big(x-\xi_j(t)\big)\right\|_{L^\infty(\mathbb{R})}+O(e^{-\frac{L}{4}})\\
&\leqslant O(\sqrt{\alpha})+O(e^{-\frac{L}{4}}),
\end{aligned}
$$

that is,

$$|M_i(t) - a_i| \leqslant O(\sqrt{\alpha}) + O(e^{-\frac{L}{4}}).$$

This implies that

$$M_N(t) > M_{N-1}(t) > \cdots > M_1(t) > 0, \tag{3.97}$$

for $\alpha \ll 1$ and $L \gg 1$, due to $0 < c_1 < \cdots < c_N$ and the positivity of the solution $u(t,x)$. Notice that $\alpha = O(\sqrt{\varepsilon} + L^{-1/8})$.

Using (3.97), $Q_{j+1} - Q_j > 0$, we infer that

$$P_1 \leqslant Ce^{-\frac{L}{8K}} = O\big(L^{-\frac{1}{2}}\big), \tag{3.98}$$

because of $K = O(\sqrt{L})$.

Substituting (3.94), (3.95) and (3.98) into (3.92) yields

$$\sum_{i=1}^{N}\left(\frac{4k_1}{3}a_i^2 + \frac{8k_2}{3}a_i\right)\big(M_i(t) - a_i\big)^2 \leqslant O(\varepsilon^2) + O\Big(L^{-\frac{1}{2}}\Big). \tag{3.99}$$

Hence, the desired result follows immediately from (3.99), that is

$$\sum_{i=1}^{N}\left(\frac{4k_1}{3}a_i^2 + \frac{8k_2}{3}a_i\right)^{\frac{1}{2}} |M_i(t) - a_i| \leqslant O(\varepsilon) + O\Big(L^{-\frac{1}{4}}\Big), \quad \forall t \in [0, t^*],$$

where the terms $O(\cdot)$ depend on $(c_i)_{i=1}^N$ and $\|u_0\|_{H^s(\mathbb{R})}$. □

3.4. End the proof of Theorem 1.1

To complete the proof of Theorem 1.1, in view of (3.3), it suffices to prove that there exists a constant $C > 0$ independent of A such that at time t^*, there exist $z_1 < z_2 < \cdots < z_N$ with $z_i - z_{i_1} > L/2 \geqslant L_0/2 \gg 1$ satisfying

$$\left\| u(t^*, x) - \sum_{i=1}^{N} \varphi_{c_i}(x - z_i) \right\|_{H^1(\mathbb{R})} \leqslant C(\sqrt{\varepsilon} + L^{\frac{1}{8}}).$$

To this end, we need to take in (3.66), $z_i = \xi_i(t^*) \in \mathcal{J}_i(t^*, x), i = 1, \ldots, N$, where $\xi_i(t^*)$, $1 \leqslant i \leqslant N$ are defined by (3.13), which implies that

$$M_i(t^*) = \max_{x \in \mathcal{J}_i(t^*)} u(t^*, x) = u(t^*, \xi_i(t^*)).$$

Using (3.10) and (3.12), it is observed that

$$\begin{aligned}\xi_i(t^*) - \xi_{i-1}(t^*) &\geqslant \tilde{x}_i(t^*) - \tilde{x}_{i-1}(t^*) - |\xi_i(t^*) - \tilde{x}_i(t^*)| - |\xi_{i-1}(t^*) - \tilde{x}_{i-1}(t^*)| \\ &\geqslant \frac{3L}{4} - \frac{L}{6} > \frac{L}{2}.\end{aligned}$$

From (3.66), (3.82) and (3.86), it follows that

$$\begin{aligned}
&\left\| u(t^*,x) - \sum_{i=1}^{N} \varphi_{c_i}\big(x-\xi_i(t^*)\big) \right\|_{H^1(\mathbb{R})}^2 \\
&= E(u(t^*)) - \sum_{i=1}^{N} E(\varphi_{c_i}) - 4\sum_{i=1}^{N} a_i\big(M_i(t^*) - a_i\big) + O(e^{-\frac{L}{4}}) \\
&\leqslant E(u_0) - \sum_{i=1}^{N} E(\varphi_{c_i}) + 4\sum_{i=1}^{N} a_i|M_i(t^*) - a_i| + O(e^{-\frac{L}{4}}) \\
&\leqslant O(\varepsilon) + O(L^{-\frac{1}{4}}).
\end{aligned}$$

Therefore, for $0<\varepsilon<\varepsilon_0$, $L>L_0$ with $0<\varepsilon_0\ll 1$ and $L_0\gg 1$ both depending only on $(c_i)_{i=1}^N$, we conclude that

$$\left\| u(t^*,x) - \sum_{i=1}^{N} \varphi_{c_i}\big(x-\xi_i(t^*)\big) \right\|_{H^1(\mathbb{R})} \leqslant C(\sqrt{\varepsilon} + L^{-\frac{1}{8}}),$$

where the positive constant C depends only on $(c_i)_{i=1}^N$ and $\|u_0\|_{H^1(\mathbb{R})}$, not on A. Therefore, by choosing $A=2C$, we obtain Theorem 1.1. □

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