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Three Weak Solutions for $(p(x),q(x))$-Biharmonic Problem with Hardy Weight with Two Parameters

Ibrahim Chamlal[1,†], Mohamed Talbi[2], Najib Tsouli[1] and Mohammed Filali[1]

Abstract The focus of this study is on the existence of three solutions to $(p(\cdot), q(\cdot))$-biharmonic operator with an $s(x)$-Hardy term under no-flux boundary conditions. Our method is based on the variational method and critical point results.

Keywords Variational method, singular problem, $p(x)$-biharmonic operator

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1. Introduction

Singular elliptic problems have been intensively studied in the last decades, and have drawn attention in many types of contexts and applications, including heat conduction theory, boundary layer phenomena, biological pattern formation, morphogenesis and chemical heterogeneous catalysts (see [8, 32, 35, 36, 38, 41]).

In 2023 A. Khaleghi and A. Razani [26] studied the following $(p(x), q(x))$-biharmonic problem containing a singular term with exponent constant

$$\begin{cases} \Delta^2_{p(x)} u + \Delta^2_{q(x)} u + \theta(x) \frac{|u|^{s-2}u}{|x|^{2s}} = \lambda f(x,u) & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N (N > 2)$ is a bounded domain with boundary of class C^1; $p, q \in C_+(\bar{\Omega})$, $\theta \in L^\infty(\Omega)$ is a real positive function, $1 < s < N/2$, λ is a positive parameter, and f is a Carathéodory function. They proved the existence and multiplicity of weak solutions of this problem, through the use of variational approaches and critical point results.

Also in [3], A. Ayoujil et al, examined a class of $(p_1(\cdot), p_2(\cdot))$-biharmonic of the form

$$\begin{cases} \Delta\left(|\Delta u|^{p_1(x)-2}\Delta u\right) + \Delta\left(|\Delta u|^{p_2(x)-2}\Delta u\right) = f(x,u) \text{ in } \Omega, \\ u = \Delta u = 0 \text{ on } \partial\Omega. \end{cases}$$

Several studies have focused on different equations in the $p-q-$laplacien oprator (see [9, 24, 28, 33, 43]). However in [40], Honghui Yin and Zuodong Yang, studied

†the corresponding author.
Email address: ibra4.chamlal@gmail.com (I. Chamlal), talbijdi@gmail.com (M. Talbi), n.tsouli@ump.ac.ma (N. Tsouli), filali1959@yahoo.fr (M. Filali)
[1]Department of Mathematics, Mohammed First University, Oujda, 60000, Morocco
[2]Department of Mathematics, Regional Centre of Trades Education and Training, Oujda, 60000, Morocco

the following problem.

$$\begin{cases} -\Delta_p u - \Delta_q u = \theta V(x)|u|^{r-2}u + |u|^{p^*-2}u + \lambda f(x,u), \quad x \in \Omega, \\ u = 0 \quad x \in \partial\Omega, \end{cases}$$

They showed that the problem has an infinite weak solution. Additionally, they obtained some results for the case $1 < q < p < r < p^*$, by using variational methods.

Weihua Wang [39] obtained multiple solutions for $\triangle_p^2 u = \frac{\mu|u|^{r-2}u}{|x|^s} + f(x,u)$ with Dirichlet boundary conditions, and the same problem with Navier boundary conditions, where $2 < 2p < N, p \leq r < p^*(s) = \frac{(N-s)p}{N-2p} \leq p^*(0) := p^*, \mu \geq 0$.

Over the past few years, there has been a lot of interest in the $p(x)$-biharmonic problem involving the $s(x)-$Hardy weight. It is the reason why this paper is a significant step in that direction. In this article, we focus on a particular class of singular fourth-order elliptic problems with no-flux boundary conditions.

$$(P_\lambda) \begin{cases} \Delta^2_{p(x)} u + \Delta^2_{q(x)} u + a(x)|u|^{p(x)-2}u = \mu m(x) \dfrac{|u|^{s(x)-2}u}{|x|^{s(x)}} + \lambda f(x,u) \text{ in } \Omega, \\ u = constant, \ \Delta u = 0, \qquad \text{on } \partial\Omega, \\ \int_{\partial\Omega} \frac{\partial}{\partial n}\Big(|\Delta u|^{p(x)-2}\Delta u\Big) + \frac{\partial}{\partial n}\Big(|\Delta u|^{q(x)-2}\Delta u\Big) ds = 0, \end{cases}$$

where $\Delta^2_{p(x)}u$ is the $p(x)$-biharmonic operator, $\Omega \subset \mathbb{R}^N (N > 2)$ is a smooth bounded domain and $0 \in \Omega$, λ is a real positive parameter, and the fonctions $p(x), q(x), r(x) \in C(\overline{\Omega})$.

We start by giving the assumptions that we will consider for our problem (PV).
H(r,q,p) $\quad 1 < q^- < q^+ < p^- < p^+ < \frac{N}{2}$,
where $\quad h^- := \min_{x\in\bar{\Omega}} h(x), \quad h^+ := \max_{x\in\bar{\Omega}} h(x)$.

(s) $p^+ < s^- \leq s(x) < p_2^\star(x)$ for all $x \in \overline{\Omega}$, and $s^+ - \frac{1}{2} < s^-$.

(a) $a \in L^\infty(\Omega)$ and there exists $a_0 > 0$ such that $a(x) \geq a_0$ for all $x \in \Omega$.

(f_1)
$$|f(x,t)| \leq a_1 + a_2|t|^{z(x)-1} \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$
where $a_1, \ a_2 > 0$ and $1 < z(x) < p^-$, $\forall x \in \overline{\Omega}$.

(m) $m \in L^{\gamma(x)}(\Omega)$ is a changing sign function, where $\gamma \in C_+(\overline{\Omega})$ and $\frac{1}{p_2^\star(x)} + \frac{1}{\gamma(x)} < \frac{1}{s(x)}$ for all $x \in \overline{\Omega}$.

This paper has the following structure. In Section 2, we list a few standard definitions, fundamental properties, and background information on generalized Lebesgue-Sobolev spaces. In Section 3 under the case of the variable exponent, we prove the Sobolev-Hardy type compact embedding theorem and provide some preliminary results.

We get the existence of one weak solution nontrivial for the problem (P_λ) in Section 4. After that in Section 5, we show that the problem (P_λ) has two and three solutions.

2. Preliminaries

First, we review some fundamental knowledge on the variable exponent Lebesgue-Sobolev. In this paper, we assume that,

(H) p verifies $1 < p^- \le p^+ < \infty$ and is log-Hölder continuous function in $\overline{\Omega}$.

$M(\Omega)$ is the set of all real functions that are measurable and defined on Ω.

We introduce the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{ u \in M(\Omega) : \int_\Omega |u|^{p(x)} \, \mathrm{d}x < \infty \right\},$$

with the norm

$$|u|_{p(x)} = \inf \left\{ \mu > 0 : \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \le 1 \right\}.$$

Now, we get the inequalities of Hölder

$$\left| \int_\Omega u(x)v(x)\, dx \right| < 2|u|_{p(x)}|v|_{p'(x)}, \tag{2.1}$$

for all $u \in L^{p'(x)}(\Omega)$ and $v \in L^{p(x)}(\Omega)$ and $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$. Additionally, we have that for any $u \in L^{p(x)}(\Omega), v \in L^{q(x)}(\Omega)$ and $w \in L^{r(x)}(\Omega)$, if $\frac{1}{p(x)} + \frac{1}{q(x)} + \frac{1}{r(x)} = 1$,

$$\int_\Omega |uvw| \mathrm{d}x < 3|u|_{p(x)}|v|_{q(x)}|w|_{r(x)}, \tag{2.2}$$

(see [19, Proposition 2.4 and Proposition 2.5]). There will be a need for the proposition that follows.

Proposition 2.1. *(see [13, Lemma 2.1]). For $p_1, p_2 \in M(\Omega)$, such that $p_1 \in L^\infty(\Omega)$ and $1 < p_1(x)p_2(x) \le \infty$, for a.e. $x \in \Omega$. Let $u \in L^{p_2(x)}(\Omega)$ such that $u \ne 0$. Then*

(i) $|u|_{p_1(\cdot)p_2(\cdot)} \le 1$, *then* $|u|^{p_1^+}_{p_1(\cdot)p_2(\cdot)} \le \left| |u|^{p_1(\cdot)} \right|_{p_2(\cdot)} \le |u|^{p_1^-}_{p_1(\cdot)p_2(\cdot)}$,

(ii) $|u|_{p_1(\cdot)p_2(\cdot)} \ge 1$, *then* $|u|^{p_1^-}_{p_1(\cdot)p_2(\cdot)} \le \left| |u|^{p_1(\cdot)} \right|_{p_2(\cdot)} \le |u|^{p_1^+}_{p_1(\cdot)p_2(\cdot)}$.

Moving on to the definition of $W^{k,p(x)}(\Omega)$ (the Sobolev space with variable exponent)

$$W^{k,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : D^\alpha u \in L^{p(x)}(\Omega), |\alpha| \le k \right\},$$

endowed with the norm

$$\|u\|_{W^{k,p(x)}(\Omega)} = \sum_{|\alpha| \le k} |D^\alpha u|_{p(x)},$$

is a reflexive and separable Banach space. (See [27]).

We have $C^\infty(\overline{\Omega})$ is dense in $W^{k,p(x)}(\Omega)$, and denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C^\infty(\overline{\Omega})$ in $W^{1,p(x)}(\Omega)$, (see [10, Section 6.5.3] and [12, Theorem 3.7]).

Thereafter, we consider the weighted variable exponent of Lebesgue space, which is defined as follows:

Let $b \in \mathbf{M}(\Omega)$ and $b(x) > 0$ for all $x \in \Omega$. We define

$$L^{p(x)}_{b(x)}(\Omega) = \left\{ u \in \mathbf{M}(\Omega) : \int_\Omega b(x)|u(x)|^{p(x)} dx < \infty \right\}$$

with the norm,

$$|u|_{L^{p(x)}_{b(x)}(\Omega)} = |u|_{(p(x),b(x))} = \inf \left\{ \gamma > 0 : \int_\Omega b(x) \left| \frac{u(x)}{\gamma} \right|^{p(x)} dx \leqslant 1 \right\}.$$

$\left(L^{p(x)}_{b(x)}(\Omega), |\cdot|_{L^{p(x)}_{b(x)}} \right)$ is a Banach space. For properties of this norm, (see [18]).

We will need the following theorem.

Theorem 2.1. *(see [10, Section 6] and [18, Theorem 2.3]). Let $q \in C(\overline{\Omega})$ with $1 < q(x) \leq p_2^*(x)$ for each $x \in \overline{\Omega}$, ($p_2^*(x) = \frac{Np(x)}{N-2p(x)}$, if $p(x) < \dfrac{N}{2}$). Then there is a continuous embedding compact $W^{2,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$.*

In this work, we opted for the norm:

$$\|u\|_a = \inf \left\{ \mu > 0 : \int_\Omega \left(\left| \frac{\Delta u(x)}{\mu} \right|^{p(x)} + a(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} \right) dx \leq 1 \right\},$$

this norm is equivalent to the usual norm $\|\cdot\|_{W^{2,p(x)}(\Omega)}$. (See [16, Remark 2.1]). We present the following subspace of $W^{2,p(\cdot)}(\Omega)$.

$$\begin{aligned} Y &= \left\{ u \in W^{2,p(x)}(\Omega) \ : u|_{\partial\Omega} \equiv constant \right\} \\ &= \left\{ u + c \ : u \in W^{2,p(x)}(\Omega) \cap W_0^{1,p(x)}(\Omega), \ c \in \mathbb{R} \right\}. \end{aligned}$$

$\left(Y, \|\cdot\|_{W^{2,p(x)}(\Omega)}\right)$ is a reflexive and separable Banach space (see [6, Theorem 4]). In the space Y, we will search for the weak solution of our problem. As a result, we take into account the functional $\Theta : Y \to \mathbb{R}$ such that

$$\Theta(u) = \int_\Omega \left[|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right] dx.$$

The following inequalities have a significant connection to the norm $\|.\|_a$, (See for example [6, Proposition 1]). For $u \in W^{2,p(x)}(\Omega)$ we have:

$$\|u\|_a \geq 1 \Rightarrow \|u\|_a^{p^-} \leq \Theta(u) \leq \|u\|_a^{p^+}. \tag{2.3}$$

$$\|u\|_a \geq 1 \Rightarrow \|u\|_a^{p^-} \leq \Theta(u) \leq \|u\|_a^{p^+}. \tag{2.4}$$

Proposition 2.2. *(see [18, Theorem 2.2]) Assume that $p_1(x), p_2(x) \in C_+(\overline{\Omega})$. If $p_1(x) \leq p_2(x)$, then $W^{2,p_1(x)}(\Omega)$ can be imbedded into $W^{2,p_2(x)}(\Omega)$ continuously.*

Proposition 2.3. *([15, Proposition 2.5] and [1, Proposition 1.6]) $\Lambda_p : Y \to \mathbb{R}$ is a functional defined by*

$$\Lambda_p(u) = \int_\Omega \frac{1}{p(x)} \left[|\Delta u|^{p(x)} + a(x)|u|^{p(x)} \right] dx + \int_\Omega \frac{1}{q(x)} |\Delta u|^{q(x)} dx,$$

which verifies the following assumptions.

(i) Λ *is of class* C^1, *with the following Gâteaux derivative*

$$\langle \Lambda'(u), \varphi \rangle = \int_\Omega |\Delta u|^{p(x)-2} \Delta u \Delta \varphi dx + \int_\Omega |\Delta u|^{q(x)-2} \Delta u \Delta \varphi dx + \int_\Omega a(x)|u|^{p(x)-2} u \varphi dx.$$

(ii) *For any* $u \in Y$ *and any subsequence* $(u_n)_n \subset Y$ *such that* $u_n \rightharpoonup u$ *in* Y, *there holds*

$$\Lambda(u) \leq \lim_{n\to\infty} inf \Lambda(u_n).$$

(iii) *The mapping* $\Lambda' : Y \to Y'$ *is of type* (S_+), *that is* $u_n \rightharpoonup u$ *and*

$$\limsup_{n\to\infty} \langle \Lambda'(u_n), (u_n - u) \rangle \leq 0 \text{ implies that } u_n \longrightarrow u.$$

3. Proof of main result

In the present paper, the lemma that follows is important.

Lemma 3.1. *Let us consider* $r(x)$, $\gamma(x) \in C_+(\overline{\Omega})$, *and*

$$r(x) < \frac{N - \gamma(x)}{N} p_2^\star(x) = p_2^\star(\gamma) \qquad \text{for all } x \in \overline{\Omega}. \tag{3.1}$$

Then an embedding $W^{2,p(x)}(\Omega) \hookrightarrow L^{r(x)}_{|x|^{-\gamma(x)}}(\Omega)$ *is compact.*

Proof. We notice $|x|^{-\gamma(x)} \in L^{\frac{N-\epsilon}{\gamma(x)}}(\Omega)$ such that ϵ is a positive constant small enough. Let $u \in W^{2,p(x)}(\Omega)$. Set $h(x) = \left(\frac{N-\epsilon}{\gamma(x)}\right)' r(x) = \frac{(N-\epsilon)r(x)}{N-\epsilon-\gamma(x)}$. Then (3.1) implies $h(x) < p_2^*(x)$ and by Theorem 2.3 (in [18]) there is a compact embedding $W^{2,p(x)}(\Omega) \hookrightarrow L^{h(x)}(\Omega)$, then for $u \in W^{2,p(x)}(\Omega)$ we have $|u(x)|^{r(x)} \in L^{\frac{N-\epsilon}{N-\gamma(x)-\epsilon}}(\Omega)$ and, by inequality (2.1), we deduce

$$\int_\Omega \frac{|u|^{r(x)}}{|x|^{\gamma(x)}} dx \leqslant 2 \Big| |x|^{-\gamma(x)} \Big|_{\frac{N-\epsilon}{\gamma(x)}} \Big| |u|^{r(x)} \Big|_{\frac{(N-\epsilon)}{N-\gamma(x)-\epsilon}} < \infty.$$

This proves $W^{2,p(x)}(\Omega) \subset L^{r(x)}_{|x|^{-\gamma(x)}}(\Omega)$. Now let $(u_n) \subset W^{2,p(x)}(\Omega)$ and $u_n \rightharpoonup 0$ in $W^{2,p(x)}(\Omega)$. Using Theorem 2.1, we conclude that $u_n \longrightarrow 0$ in $L^{h(x)}(\Omega)$ and from this we get $\Big| |u_n|^{r(x)} \Big|_{\frac{N-\epsilon}{N-\gamma(x)-\epsilon}} \longrightarrow 0$. Hence, we have

$$\int_\Omega \frac{|u|^{r(x)}}{|x|^{\gamma(x)}} dx \leqslant 2 \Big| |x|^{-\gamma(x)} \Big|_{\frac{N-\epsilon}{\gamma(x)}} \Big| |u|^{r(x)} \Big|_{\frac{N-\epsilon}{N-\gamma(x)-\epsilon}} \longrightarrow 0.$$

That implies $\Big| u_n \Big|_{(r(x), |x|^{-\gamma(x)})} \longrightarrow 0$. Therefore, the embedding $W^{2,p(x)}(\Omega) \hookrightarrow L^{r(x)}_{|x|^{-\gamma(x)}}(\Omega)$ is compact. The proof is complete. □

In addition to the density result and the boundary conditions, we apply Green's formula and the fact that Y is a closed subspace of $\left(W^{2,p(x)}(\Omega), \|.\|_{W^{2,p(x)}(\Omega)}\right)$. Here is the definition that we provide:

Definition 3.1. $u \in Y$ is a weak solution of the problem (P_λ) if

$$\int_\Omega |\Delta u|^{p(x)-2}\Delta u \Delta \upsilon dx + \int_\Omega a(x)|u|^{p(x)-2}u\upsilon dx + \int_\Omega |\Delta u|^{q(x)-2}\Delta u\Delta \upsilon dx$$
$$- \mu \int_\Omega m(x)\frac{|u|^{s(x)-2}}{|x|^{s(x)}}u\upsilon dx - \lambda \int_\Omega f(x,u)udx = 0$$

for all $\upsilon \in Y$.

We apply the critical point theory to problem (P_λ) in order to find a weak solution. As a result, we associate the functional

$$T_\lambda(u) = J_1(u) - \lambda I_1(u),$$

where

$$J_1(u) = \Lambda_p(u) + \int_\Omega \frac{1}{q(x)}|\Delta u|^{q(x)}dx - \mu \int_\Omega \frac{m(x)}{s(x)}\frac{|u|^{s(x)}}{|x|^{s(x)}}dx,$$

and

$$I_1(u) = \int_\Omega F(x,u)dx.$$

From (i)-(ii) in Proposition 2.3 and Lemma 3.1, we can conclude that the energy functional $T_\lambda \in C^1(Y,\mathbb{R})$ and is weakly lower semicontinuity. Its Gâteaux derivative is defined as follows:

$$\begin{aligned}\langle T_\lambda'(u), \upsilon\rangle = &\int_\Omega |\Delta u|^{p(x)-2}\Delta u\Delta \upsilon dx + \int_\Omega a(x)|u|^{p(x)-2}u\upsilon dx \\ &+ \int_\Omega |\Delta u|^{q(x)-2}\Delta u\Delta\upsilon dx - \mu\int_\Omega m(x)\frac{|u|^{s(x)-2}}{|x|^{s(x)}}u\upsilon dx \\ &- \lambda\int_\Omega f(x,u)udx,\end{aligned}$$

for all $\upsilon \in Y$.

4. The existence result

The following theorem provides a base for our first solution of the problem (P_λ).

Theorem 4.1. *[37] Given a reflexive real Banach space X, consider two Gâteaux differentiable functionals $\Phi, \Psi : X \to \mathbb{R}$, such that Φ is coercive, sequentially weakly lower semicontinuous, and strongly continuous. Further, suppose that Ψ is an upper semicontinuous sequentially weak function. Put*

$$\varphi(r) := \inf_{u\in\Phi^{-1}(]-\infty,r[)} \frac{\left(\sup_{\upsilon\in\Phi^{-1}(1-\infty,r)}\Psi(\upsilon)\right) - \Psi(u)}{r - \Phi(u)}.$$

Then, for every $r > \inf_X \Phi$ and every $\lambda \in]0, 1/\varphi(r)[$,the restriction of $J_\lambda := \Phi - \lambda\Psi$ to $\Phi^{-1}(]-\infty, r[)$ admits a global minimum, which is a critical point (local minimum) of J_λ in X.

Theorem 4.2. *Let $f:\Omega\times\mathbb{R}\to\mathbb{R}$ is a carathéodory and that (f_1) is satisfied. Then there are $\mu^\star$ and $\lambda^\star$ two positive constants, such that the problem (P_λ) admits at least one non-trivial weak solution $u_\lambda\in Y$ for each $\mu\in]0,\mu^\star[$ and $\lambda\in]0,\lambda^\star[$.*

Proof. First we need to prove that J_1 is coercive for every $\lambda>0$ and $\mu>0$. Since $s^+-\frac{1}{2}<s^-$, then there exists θ such that $s^+-\frac{1}{2}<\theta<s^-$, wiche implies $2(s^--\theta)<2(s^+-\theta)<1$. Let us take σ any measurable function satisfying

$$\max\{\frac{\gamma(x)}{1+\theta\gamma(x)},\frac{p_2^\star(s(x))}{p_2^\star(s(x))+\theta-s(x)}\}<\sigma(x)<\min\{\frac{\gamma(x)p_2^\star(s(x))}{p_2^\star(s(x))+\theta\gamma(x)},\frac{1}{1+\theta-s(x)}\},$$

for all $x\in\overline{\Omega}$ and

$$\theta(\frac{\sigma^+}{\sigma^-}+1)\leq s^-.$$

This implies that $\sigma\in L^\infty(\Omega)$ and $1<\sigma(x)<\gamma(x)$, for any $x\in\overline{\Omega}$. Then, we have

$$1<\frac{\theta\sigma(x)\gamma(x)}{\gamma(x)-\sigma(x)}<p_2^\star(s(x)),\qquad 1<\frac{(s(x)-\theta)\sigma(x)}{\sigma(x)-1}<p_2^\star(s(x)),\quad \forall x\in\overline{\Omega}.$$

So there exist constants C_1 and C_2 such that

$$\Big|\frac{u}{|x|}\Big|_{\frac{\theta\sigma(x)\gamma(x)}{\gamma(x)-\sigma(x)}}\leq C_1\|u\|_a,$$

$$\Big|\frac{u}{|x|}\Big|_{\frac{(s(x)-\theta)\sigma}{\sigma(x)-1}}\leq C_2\|u\|_a,$$

for all $x\in\overline{\Omega}$. So we have the following

$$\begin{aligned}\Big|\int_\Omega\frac{m(x)}{s(x)}\frac{|u|^{s(x)}}{|x|^{s(x)}}dx\Big| &\leq\frac{1}{s^-}\int_\Omega|m(x)\frac{|u|^\theta}{|x|^\theta}|\frac{|u|^{s(x)-\theta}}{|x|^{s(x)-\theta}}dx\\ &\leq\frac{2}{s^-}\Big|m(x)\frac{|u|^\theta}{|x|^\theta}\Big|_{\sigma(x)}\Big|\frac{|u|^{s(x)-\theta}}{|x|^{s(x)-\theta}}\Big|_{\frac{\sigma(x)}{\sigma(x)-1}}\\ &\leq\frac{2}{s^-}\Big||m(x)|^{\sigma(x)}\Big|^{\frac{1}{\sigma^-}}_{\frac{\gamma(x)}{\sigma(x)}}\Big|\frac{|u|^{\theta\sigma(x)}}{|x|^{\theta\sigma(x)}}\Big|^{\frac{1}{\sigma^-}}_{\frac{\gamma(x)}{\gamma(x)-\sigma(x)}}\Big|\frac{|u|^{s(x)-\theta}}{|x|^{s(x)-\theta}}\Big|_{\frac{\sigma(x)}{\sigma(x)-1}},\end{aligned}$$

for all $u\in Y$, with $\Big|m(x)\frac{|u|^\theta}{|x|^\theta}\Big|_{\sigma(x)}>1$. Thus, we have

$$\begin{aligned}\Big|m(x)\frac{|u|^\theta}{|x|^\theta}\Big|_{\sigma(x)} &\leq 1+\frac{2}{s^-}\Big||m(x)|^{\sigma(x)}\Big|^{\frac{1}{\sigma^-}}_{\frac{\gamma(x)}{\sigma(x)}}\Big|\frac{|u|^{\theta\sigma(x)}}{|x|^{\theta\sigma(x)}}\Big|^{\frac{1}{\sigma^-}}_{\frac{\gamma(x)}{\gamma(x)-\sigma(x)}}\\ &\leq 1+\frac{2}{s^-}(1+|m|^{\frac{\sigma^+}{\sigma^-}}_{\sigma(x)})(1+\Big|\frac{|u|}{|x|}\Big|^{\frac{\theta\sigma^+}{\sigma^-}}_{\frac{\theta\sigma(x)\gamma(x)}{\gamma(x)-\sigma(x)}})\leq C_3(1+\|u\|^{\frac{\theta\sigma^+}{\sigma^-}})\end{aligned}$$

for any $u\in Y$. Similarly

$$\Big|\frac{|u|^{s(x)-\theta}}{|x|^{s(x)-\theta}}\Big|_{\frac{\sigma(x)}{\sigma(x)-1}}\leq 1+\Big|\frac{u}{|x|}\Big|^{s^+-\theta}_{\frac{\sigma(s(x)-\theta)}{\sigma(x)-1}}\leq 1+C_4\|u\|^{s^+-\theta},\qquad\forall u\in Y.$$

By the above information, we deduce that

$$\Big|\int_\Omega \frac{m(x)}{s(x)}\frac{|u|^{s(x)}}{|x|^{s(x)}}dx\Big| \le C_3(1+\|u\|^{\frac{\theta\sigma^+}{\sigma^-}})(1+C_4\|u\|^{s^+-\theta})$$

for all $u \in Y$.

Since $\int_\Omega \frac{1}{q(x)}|\Delta u|^{q(x)}dx \ge 0$, with $\|u\| > 1$ we have

$$\begin{aligned}|J_1(u)| &\ge \frac{1}{p^+}\|u\|^{p^-} - \mu C_3(1+\|u\|^{\frac{\theta\sigma^+}{\sigma^-}})(1+C_4\|u\|^{s^+-\theta})\\ &\ge \frac{1}{p^+}\|u\|^{p^-} - C_5\mu(1+\|u\|^{\frac{2\theta\sigma^+}{\sigma^-}}+\|u\|^{2(s^+-\theta)}).\end{aligned} \tag{4.1}$$

Since $p^- > 1 > 2(s^+-\theta) > \frac{2\theta\sigma^+}{\sigma^-}$, then $J_1(u) \to +\infty$ as $\|u\| \to +\infty$. Thus J_1 is coercive. Clearly we have $\inf_{u\in Y} J_1(u) \le 0$. Now we can set $r \in]0,+\infty[$, and consider the function

$$\chi(r) = \frac{\sup_{u\in J_1^{-1}(]-\infty,r[)} I_1(u)}{r}.$$

By the above inequality, for $0 < \mu < \dfrac{1}{3p^+C_5} = \mu^\star$, and $\|u\| \ge 1$ we obtain

$$\begin{aligned}|J_1(u)| &\ge \frac{1}{p^+}\|u\|^{p^-} - \mu C_5(1+2\|u\|^{2(s^+-\theta)}),\\ &\ge (\frac{1}{p^+}-3\mu C_5)\|u\|^{2(s^+-\theta)}.\end{aligned}$$

Then for $J_1(u) < r$, we get

$$\|u\| \le (\frac{r}{\frac{1}{p^+}-\mu 3C_5})^{\frac{1}{2(s^+-\theta)}} \qquad \forall u \in Y. \tag{4.2}$$

Taking into account hypothesis (f_1), there exists C and C_z, such that

$$\begin{aligned}I_1(u) &= \int_\Omega F(x,u)dx \le a_1\|u\|_{L^1(\Omega)} + \frac{a_2}{z^-}\|u\|^{z^\pm}_{L^{z(x)}(\Omega)}\\ &\le Ca_1\|u\| + C_z\frac{a_2}{z^-}\|u\|^{z^\pm}.\end{aligned} \tag{4.3}$$

Now, we have

$$I_1(u) < a_1C(\frac{r}{\frac{1}{p^+}-3\mu C_5})^{\frac{1}{2(s^+-\theta)}} + \frac{a_2C_z}{z^-}(\frac{r}{\frac{2}{p^+}-3\mu C_5})^{\frac{z^\pm}{2(s^+-\theta)}}$$

for every $u \in Y$, such that $J_1(u) < r$ and $\|u\| \ge 1$. Hence

$$I_1(u) < C_6 + a_1C(\frac{r}{\frac{1}{p^+}-3\mu C_5})^{\frac{1}{2(s^+-\theta)}} + \frac{a_2C_z}{z^-}(\frac{r}{\frac{2}{p^+}-3\mu C_5})^{\frac{z^\pm}{2(s^+-\theta)}}$$

with $C_6 = a_1 + \frac{a_2C_z}{z^-}$.

$$\sup_{u\in(]-\infty,r[)} I_1(u) \le C_6 + a_1Cr^{\frac{1}{2(s^+-\theta)}}(\frac{1}{p^+}-3\mu C_5)^{\frac{1}{2(\theta-s^+)}}$$

$$+\frac{a_2C_z}{z^-}r^{\frac{z^\pm}{2(s^+-\theta)}}(\frac{1}{p^+}-3\mu C_5)^{\frac{z^\pm}{2(\theta-s^+)}}.$$

Then

$$\chi(r)\leq\frac{C_6}{r}+a_1Cr^{\frac{1}{2(s^+-\theta)}-1}(\frac{1}{p^+}-3\mu C_5)^{\frac{1}{2(\theta-s^+)}}\\
+\frac{a_2C_z}{z^-}r^{\frac{z^\pm}{2(s^+-\theta)}-1}(\frac{1}{p^+}-3\mu C_5)^{\frac{z^\pm}{2(\theta-s^+)}},$$

for every $r>0$. In particular

$$\chi(\xi^{2(s^+-\theta)})\leq\frac{C_6}{\xi^{2(s^+-\theta)}}+a_1C(\frac{1}{p^+}-3\mu C_5)^{\frac{1}{2(\theta-s^+)}}\xi^{1-2(s^+-\theta)}\\
+\frac{a_2C_z}{z^-}(\frac{1}{p^+}-3\mu C_5)^{\frac{z^\pm}{2(\theta-s^+)}}\xi^{z^\pm-2(s^+-\theta)}. \tag{4.4}$$

Now, observe that

$$\varphi(\xi^{2(s^+-\theta)})=\inf_{u\in J_1^{-1}(]-\infty,\xi^{2(s^+-\theta)}[)}\frac{\sup_{v\in J_1^{-1}(]-\infty,r[)}I_1(v)-I_1(u)}{r-J_1(u)}\leq\chi(\xi^{2(s^+-\theta)}), \tag{4.5}$$

because there exists $u_0\in J_1^{-1}(]-\infty,\xi^{2(s^+-\theta}[)$ such that $J_1(u_0)=I_1(u_0)=0$. In conclusion, the inequalities (4.4) and (4.5), give us

$$\varphi(\xi^{2(s^+-\theta)})\leq\chi(\xi^{2(s^+-\theta)})\leq\frac{C_6}{\xi^{2(s^+-\theta)}}+a_1C(\frac{1}{p^+}-3\mu C_5)^{\frac{1}{2(\theta-s^+)}}\xi^{1-2(s^+-\theta)}\\
+\frac{a_2C_z}{z^-}(\frac{1}{p^+}-3\mu C_5)^{\frac{z^\pm}{2(\theta-s^+)}}\xi^{z^\pm-2(s^+-\theta)}=\frac{1}{\lambda^\star}<\frac{1}{\lambda}.$$

Otherwise,

$$\lambda\in\Big]0,\frac{z^-\xi^{1-2(s^+-\theta)}}{z^-C_6\xi+z^-a_1C(\frac{1}{p^+}-3\mu C_5)^{\frac{1}{2(\theta-s^+)}}+a_2C_z(\frac{1}{p^+}-3\mu C_5)^{\frac{z^\pm}{2(\theta-s^+)}}\xi^{z^\pm-1}}\Big[\\
\subseteq]0,\frac{1}{\varphi(\xi^{2(s^+-\theta)})}[.$$

Then by Theorem 4.1, there exists a function $u_\lambda\in J_1^{-1}(]-\infty,\xi^{2(s^+-\theta)}[)$ such that

$$T'_\lambda(u_\lambda)=J'_1(u_\lambda)-\lambda I'_1(u_\lambda)=0,$$

and, specifically, the restriction of T_λ to $J_1^{-1}(]-\infty,\xi^{2(s^+-\theta)}[)$ has a global minimum represented by u_λ.

It remains to show that u_λ is non-trivial. We have two cases. If $f(x,0)=0$, then there exists $\varphi>0$ such that $t\varphi\in J_1^{-1}(]-\infty,\xi^{2(s^+-\theta)}[)$ for $t\in(0,1)$ small enough, we have

$$T_\lambda(t\varphi)\leq\frac{t^{p^-}}{p^-}\|\varphi\|^{p^\pm}+\frac{C_qt^{q^-}}{q^-}\|\varphi\|^{q^\pm}-\frac{t^{s^+}}{s^+}\int_\Omega m(x)\frac{|\varphi|^{s(x)}}{|x|^{s(x)}}dx-a_1t\int_\Omega|\varphi|dx$$

$$- a_2\frac{t^{z^+}}{z^+}\int_\Omega |\varphi|^{z(x)}dx$$

Since $z^+ < q^- < p^- < s^+$ and for t small enough, we have $J_\lambda(t\varphi) < 0$. Thus $J_\lambda(u_\lambda) < J_\lambda(t\varphi) < 0 = J_\lambda(0)$.

Else, if $f(x,0) \neq 0$ in Ω, the function u_λ cannot be trivial. Hence for $\mu \in [0,\mu^\star[$ and for every $\lambda \in [0,\lambda^\star[$, the problem (P_λ) admits a non-trivial solution faible $u_\lambda \in Y$. □

5. The multiplicity result

Theorem 5.1. *(see [4, Theorem 3.2]) Assume that X is a real Banach space and that there are two continuously Gâteaux differentiable functionals $\Phi, \Psi : X \to \mathbb{R}$ such that Φ is bounded from below and $\Phi(0) = \Psi(0) = 0$. Consider that $r > 0$ and assume that for every*

$$\lambda \in]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}[,$$

the functional $I_\lambda := \Phi - \lambda\Psi$ is unbounded from below and satisfies the Palais-Smale condition. Then, for every

$$\lambda \in]0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r[)}\Psi(u)}[,$$

the functional I_λ admits two different critical points.

Theorem 5.2. *Let $f : \Omega\times\mathbb{R} \to \mathbb{R}$ be a carathéodory with $f(x,0) = 0$ in Ω, and satisfy (f_1). There exist two positive constants, $\mu^\star$ and $\lambda^\star$, such that the problem (P_λ) has at least two distinct nontrivial weak solutions in Y, for every $\mu \in]0,\mu^\star[$ and $\lambda \in]0,\lambda^\star[$.*

Lemma 5.1. *The functional T_λ verifies the Palais-Smale condition in Y.*

Proof. Let us assume that there is a convergent subsequence $\{u_n\} \subseteq Y$. Utilizing (f_1) and the Holder inequality, we obtain

$$\begin{aligned} J_1(u) = \int_\Omega F(x,u(x))dx &\le a_1\int_\Omega |u|dx + \frac{a_2}{z^-}\int_\Omega |u|^{z(x)}dx \\ &\le a_1C\|u\| + \frac{a_2C_z}{z^-}\|u\|^{z^\pm}. \end{aligned}$$

From (4.1), one has

$$\begin{aligned} < T_\lambda(u_n), u_n > =&< I_1(u_n), u_n > -\lambda < J_1(u_n), u_n > \\ &\ge \frac{1}{p^+}\|u_n\|^{p^\pm} - C_5\mu(1 + \|u_n\|^{\frac{2\theta\sigma^+}{\sigma^-}} + \|u_n\|^{2(s^+-\theta)}) \\ &- \lambda\Big(a_1C\|u_n\| + \frac{a_2C_z}{z^-}\|u_n\|^{z^\pm}\Big). \end{aligned}$$

Then, by applying the second property of (P.S), we have

$$\|u\|^{p^\pm} \le C_5\mu(1 + \|u_n\|^{\frac{2\theta\sigma^+}{\sigma^-}} + \|u_n\|^{2(s^+-\theta)}) + \lambda\Big(a_1C\|u_n\| + \frac{a_2C_z}{z^-}\|u_n\|^{z^\pm}\Big),$$

and since $\frac{2\theta\sigma^+}{\sigma^-} < 2(s^+ - \theta) < 1 < z^\pm < p^\pm$, it follows that $\{u_n\}$ is bounded in Y.

Consequently, there is a subsequence still indicated by $\{u_n\}$, and $u \in Y$ such that

$$u_n \rightharpoonup u \quad \text{in } Y \quad \text{as } n \to \infty, \tag{5.1}$$

then

$$\lim_{n\to\infty} < T'_\lambda(u_n), u_n - u >= 0.$$

Specifically, we have

$$\begin{aligned} 0 = \lim_{n\to\infty} \Big(& \int_\Omega |\Delta u_n|^{p(x)-2}\Delta u_n \Delta(u_n - u)dx + \int_\Omega |\Delta u_n|^{q(x)-2}\Delta u_n \Delta(u_n - u)dx \\ & + \int_\Omega a(x)\frac{|u_n|^{p(x)-2}}{|x|^{p(x)}}u_n(u_n - u)dx - \mu \int_\Omega m(x)\frac{|u_n|^{s(x)-2}}{|x|^{\alpha(x)}}u_n(u_n - u)dx \\ & - \lambda \int_\Omega f(x, u_n)(u_n - u)dx\Big). \end{aligned}$$

By Hölder inequality (2.2), we get

$$\begin{aligned} \Big|\int_\Omega m(x)\frac{|u_n|^{s(x)-2}}{|x|^{s(x)}}u_n(u_n - u)dx\Big| &\le 2\Big|m(x)\frac{|u_n|^{s(x)-1}}{|x|^{s(x)-1}}\Big|_{s'(x)}\Big|\frac{u_n - u}{|x|}\Big|_{s(x)} \\ &\le 3|m|_{\gamma(x)}\Big|\frac{|u_n|^{s(x)-1}}{|x|^{s(x)-1}}\Big|_{v(x)}\Big|\frac{u_n - u}{|x|}\Big|_{s(x)}, \end{aligned}$$

where $v(x) = \dfrac{s(x)}{\gamma(x)(s(x) - 1)}$, then by using Proposition 2.1 and Lemma 3.1 we obtain

$$\Big|\frac{|u_n|^{s(x)-1}}{|x|^{s(x)-1}}\Big|_{v(x)} \le \Big|\frac{u_n}{|x|}\Big|^{s^\pm-1}_{(s(x)-1)v(x)} = \Big|\frac{u_n}{|x|}\Big|^{s^\pm-1}_{\frac{s(x)}{\gamma(x)}} \le \Big(C_{\frac{s}{\gamma}}\|u_n\|^{(\frac{s}{\gamma})^\pm}\Big)^{(s^\pm-1)}.$$

Since (u_n) is bounded in Y, there exists a constant C' such that,

$$\Big|\frac{|u_n|^{s(x)-1}}{|x|^{s(x)-1}}\Big|_{v(x)} \le C'.$$

Thus to show that $\lim\limits_{n\longrightarrow\infty}\Big|\int_\Omega m(x)\frac{|u_n|^{s(x)-2}}{|x|^{s(x)}}u_n(u_n - u)dx\Big| = 0$, we prove that $\lim\limits_{n\longrightarrow\infty}\Big|\frac{u_n - u}{|x|}\Big|_{s(x)} = 0$.

For every $\epsilon > 0$ we have

$$\begin{aligned} \int_\Omega \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}}dx &= \int_{B(0,\epsilon)} \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}}dx + \int_{\Omega\setminus B(0,\epsilon)} \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}}dx \\ &\le \int_{B(0,\epsilon)} \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}}dx + \Big(\frac{1}{\epsilon^{s^+}} + \frac{1}{\epsilon^{s^-}}\Big)\int_\Omega |u_n - u|^{s(x)}dx \end{aligned}$$

and due to embedding $Y \hookrightarrow L^{s(x)}(\Omega)$, we get $\left(\frac{1}{\epsilon^{s^+}} + \frac{1}{\epsilon^{s^-}}\right)|u_n - u|_{s(x)}^{s(x)} \to 0$ as $n \to +\infty$.

However,

$$\int_{B(0,\epsilon)} \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}} dx = \int_\Omega \chi_{B(0,\epsilon)} \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}} dx,$$

we have $s^+ < p_2^\star$ then there exists α such that $s^+ < \alpha < p_2^\star$.

Lemma 3.1, Proposition 2.1, and Hölder inequality allow us to deduce

$$\begin{aligned}\int_\Omega \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}} dx &\leq 2|B(0,\epsilon)|^{\frac{\alpha}{\alpha - s(x)}} \left|\frac{|u_n - u|^{s(x)}}{|x|^{s(x)}}\right|_{\frac{\alpha}{s(x)}} + o(n)\\ &\leq 2|(B(0,\epsilon)|^{\frac{\alpha}{\alpha - s^-}} \left(\left|\frac{|u_n - u|}{|x|}\right|_\alpha^{s^+} + \left|\frac{|u_n - u|}{|x|}\right|_\alpha^{s^-}\right) + o(n)\\ &\leq C\epsilon^{\frac{N\alpha}{\alpha - s^-}} (\|u_n - u\|_a^{s^+} + \|u_n - u\|_a^{s^-}) + o(n).\end{aligned}$$

Since (u_n) is bounded in Y,then $\|u_n - u\|_a \leq M$ for all $n \in \mathbb{N}$, which $M > 0$.

Consequently,

$$\int_\Omega \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}} dx \leq C\epsilon^{\frac{N\alpha}{\alpha - s^-}} (M^{s^+} + M^{s^-}) + o(n),$$

thus, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \int_\Omega \frac{|u_n - u|^{s(x)}}{|x|^{s(x)}} dx \leq C\epsilon^{\frac{N\alpha}{\alpha - s^-}} (M^{s^+} + M^{s^-}),$$

which implies that, for $\epsilon \to 0$

$$\lim_{n\longrightarrow\infty} \left|\frac{u_n - u}{|x|}\right|_{s(x)} = 0.$$

Then

$$\lim_{n\longrightarrow\infty} \left|\int_\Omega m(x) \frac{|u_n|^{s(x)-2}}{|x|^{s(x)}} u_n(u_n - u) dx\right| = 0.$$

From (f_1), Hölder inequality and Proposition 2.1 we have

$$\begin{aligned}\left|\int_\Omega f(x,u_n)(u_n - u)dx\right| &\leq C\|u_n - u\|_{L^1} + 2C\left||u_n|^{z(x)-1}\right|_{z'(x)} |u_n - u|_{z(x)}\\ &\leq C\|u_n - u\|_{L^1} + 2C\left(|u_n|_{z(x)}^{z^+-1} + |u_n|_{z(x)}^{z^--1}\right)|u_n - u|_{z(x)}.\end{aligned}$$

Using the compact embedding $Y \hookrightarrow L^1(\Omega)$ and $Y \hookrightarrow L^{r(x)}(\Omega)$, we obtain

$$\lim_{n\longrightarrow\infty} \int_\Omega f(x,u_n)(u_n - u)dx = 0.$$

We infer from (a) and (2.1) that

$$\left|\int_\Omega a(x)|u_n|^{p(x)-2} u_n(u_n - u)dx\right| \leq 2\|a\|_\infty \left||u_n|^{p(x)-1}\right|_{p'(x)} |u_n - u|_{p(x)}.$$

Using the compact embedding $Y \hookrightarrow L^{p(x)}(\Omega)$, we find

$$\lim_{n\longrightarrow\infty}\int_\Omega a(x)|u_n|^{p(x)-2}u_n(u_n-u)dx = 0.$$

Therefore, we arrive at

$$\lim_{n\longrightarrow\infty}\int_\Omega (|\Delta u_n|^{p(x)-2}+|\Delta u_n|^{p(x)-2})\Delta u_n\Delta(u_n-u)dx = 0.$$

Consequently, Proposition 2.3 and the weak convergence (5.1) suggest that $u_n \longrightarrow u$ in Y as $n \longrightarrow +\infty$.

This leads us to say that T_λ satisfies the palais-smale condition. □

Proof. [Proof of theorem 5.2.] All theorem 5.1 hypothesis has been proved. Then, T_λ admits at least two different critical points, which are the weak solutions of problem (P_λ), for every $\lambda \in]0,\lambda^\star[$. □

Theorem 5.3. *(see [5, Theorem 2.1]) X is a reflexive real Banach space. Let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable whose Gâteaux derivative is compact. Let $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on X^* such that*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Let $r > 0$ and $\bar{x} \in X$, such that $r < \Phi(\bar{x})$, and

i) $\frac{\sup_{\Phi(x)<r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$.

ii) for each $\lambda \in \Lambda_r :=]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)<r}\Psi(x)}[$, the functional $I_\lambda := \Phi - \lambda\Psi$ is coercive.

Then, for every $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ has at least three different critical points in X.

We shall use the following assumption:

(f_2) There exists $\nu(x)$ such that $p^+ < \nu^- < \nu(x) < p_2^\star(x)$, and

$$\limsup_{s\to 0}\sup_{x\in\Omega}\frac{F(x,s)}{|s|^{\nu(x)}} < +\infty.$$

Theorem 5.4. *Assume that $H(p,q,s)$ (a), (m), (f_1) and (f_2) hold. For $f(x,0) = 0$ in Ω there exist several positive constants $\lambda^\star$, $\overline{\lambda}$ and $\mu^\star$, for every $\mu \in]0,\mu^\star[$ and $\lambda \in]\overline{\lambda},\lambda^\star[$, the problem (P_λ) has at least three different solutions.*

Proof. Fix $\mu \in]0,\mu^\star[$. By the condition (f_2), there exists $\zeta \in [0,1]$, and C_7 such that

$$F(x,s) < C_7|s|^{\nu(x)} \qquad \text{for all } s \in [-\zeta,\zeta] \qquad \text{a.e} \qquad x \in \Omega,$$

and by (f_1), then there exists $M > 0$, such that

$$F(x,s) < M|s|^{\nu^-} \qquad \text{for all } s \in \mathbb{R}.$$

On the other hand, we have

$$\int_\Omega F(x,u)dx < M\int_\Omega |u|^{\nu^-}dx \le C_7\|u\|^{\nu^-} \le C_8 r^{\frac{\nu^-}{2(s^+-\theta)}},$$

when $\|u\| \le (\frac{r}{\frac{1}{p^+}-\mu 3C_6})^{\frac{1}{2(s^+-\theta)}}$. Since $\nu^- > 1 > 2(s^+-\theta)$, we obtain

$$\lim_{r\to 0^+}\frac{1}{r}\sup_{\|u\|\le(\frac{r}{\xi})^{\frac{1}{2(s^+-\theta)}}}\{J_1(u)\} = 0,$$

where $\xi = \frac{1}{p^+} - \mu 3C_6$. Next, let $u_1 \in C(\overline{\Omega})$ be a positive fonction in Ω with $I_1(u_1) > 0$, and $F(x,u_1) > 0$ for a.e $x \in \Omega$. We get

$$J_1(u_1) = \int_\Omega F(x,u_1(x))dx > 0.$$

As a result, we can get $r \in \Big(0, \min\{I_1(u_1),\xi\}\Big)$, such that

$$\sup_{\|u\|\le(\frac{r}{\xi})^{\frac{1}{2(s^+-\theta)}}}\{J_1(u)\} < r\frac{J_1(u_1)}{I_1(u_1)}.$$

Now, let $u \in I_1^{-1}(-\infty,r]$. Then by (4.2), we have

$$\|u\| \le \left(\frac{r}{\xi}\right)^{\frac{1}{2(s^+-\theta)}},$$

then we can infer that $I_1^{-1}(-\infty,r] \subset \{u\in Y : \xi\|u\|^{2(s^+-\theta)} \le r\}$. Hence

$$\sup_{u\in I_1^{-1}(-\infty,r]}\{J_1(u)\} < r\frac{J_1(u_1)}{I_1(u_1)} = r\overline{\lambda}.$$

As a result, the assumption (i) of Theorem 5.3 is fulfilled. Therefore, we have to show that the functional T_λ is coercive. From (4.1) and by (f_1) we have

$$T_\lambda(u) \ge \frac{1}{p^+}\|u\|^{p^-} - C_6\mu(1+\|u\|^{\frac{2\theta\sigma^+}{\sigma^-}} + \|u\|^{2(s^+-\theta)}) - \lambda(a_1C\|u\| + \frac{a_2C_z}{z^-}\|u\|^{z^\pm}),$$

$\frac{2\theta\sigma^+}{\sigma^-} < 2(s^+-\theta) < 1 < z^\pm < p^\pm$, proving that the functional T_λ is coercive. Consequently, (ii) is satisfied. Therefore, every assumption in Theorem 5.3 is verified. Following that, the functional T_λ permits at least three distinct critical points, which are the weak solutions of the problem (P_λ), for each $\lambda \in]\overline{\lambda},\lambda^\star[$. □

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Declarations

Conflict of interest. The authors affirm that they do not have any competing interests.

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