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Existence and Multiplicity of Solutions for Anisotropic Discrete Boundary Value Problems Involving the $(p_1(t), p_2(t))$-Laplacian

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Abstract For an anisotropic discrete nonlinear problem with a variable exponent, we demonstrate both the existence and multiplicity of nontrivial solutions in this study. Our technique is based on variational methods.

Keywords Discrete nonlinear boundary value problems, nontrivial solution, $p(t)$-Laplacian, critical point theory

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1. Introduction

In this paper, we study the existence and multiplicity of nontrivial solutions for the following discrete anisotropic problem

$$(P)\quad \begin{cases} -\sum\limits_{i=1}^{2} \Delta(|\Delta y(t-1)|^{p_i(t-1)-2}\Delta y(t-1)) = h(t, y(t)),\ t \in [1, N]_{\mathbb{Z}}, \\ y(0) = y(N+1) = 0, \end{cases}$$

where $N \geq 2$ is an integer, $[1, N]_{\mathbb{Z}}$ is the discrete interval given by $\{1, 2, 3..., N\}$, $\Delta y(t) = y(t+1) - y(t)$ is the forward difference operator, $h : [1, N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function in the second variable and $p_1, p_2 : [0, N]_{\mathbb{Z}} \longrightarrow [2, \infty[$.

As usual, a solution of (P) is a function $y : [0, N+1]_{\mathbb{Z}} \longrightarrow \mathbb{R}$ which satisfies both equations of (P).

We would like to point out that issue (P) is a discrete equivalent of the variable exponent anisotropic problem

$$\begin{cases} -\sum\limits_{j=1}^{N}\sum\limits_{i=1}^{2} \dfrac{\partial}{\partial x_j}\left(|\dfrac{\partial y}{\partial x_j}|^{p_{i,j}(x)-2}\dfrac{\partial y}{\partial x_j}\right) = h(x, y), \quad x \in \Omega, \\ y(x) = 0, \quad x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded domain with a smooth boundary, $h \in C\left(\overline{\Omega} \times \mathbb{R}, \mathbb{R}\right)$ is a given function that satisfies certain properties, and $p_{i,j}(x)$ are continuous functions on $\overline{\Omega}$, with $p_{i,j}(x) \geq 2$ for $(i, j, x) \in [1, 2]_{\mathbb{Z}} \times [1, N]_{\mathbb{Z}} \times \Omega$.

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The theory of nonlinear difference equations has been widely used to study discrete models appearing in many fields such as computer science, economics, neural networks, ecology, cybernetics, mechanical engineering, statistics, optimal control, electrical circuit analysis, population dynamics, biology and other fields; (see, for example [1, 21, 22, 26]). The existence and multiplicity of solutions to boundary value issues for difference equations with the $p(.)$-Laplacian operator have recently attracted more attention. Fixed point theorems in cones are typically used to get these results on this issue (see [3, 19, 23, 24] and references therein). It is widely recognized that, variational methods, critical point theory and also monotonicity methods are powerful tools to investigate the existence and multiplicity of solutions of various problems, (see the monographs [4, 6–12, 14–18, 20, 27–30]).

In this paper, we shall study the existence and multiplicity of nontrivial solutions of (P), via min-max methods and Mountain Pass Theorem.

To state our main results, we use the following notation:

$$p_i^+ = \max_{t\in[0,N]_{\mathbb{Z}}} p_i(t), \;\; p_i^- = \min_{t\in[0,N]_{\mathbb{Z}}} p_i(t), \quad \text{for } i = 1, 2;$$

$$p_{\max}^+ = \max\{p_1^+, p_2^+\}, \;\; p_{\max}^- = \max\{p_1^-, p_2^-\}, \;\; p_{\min}^- = \min\{p_1^-, p_2^-\}.$$

The following theorems are the key findings of this paper:

Theorem 1.1. *Assume that*

(H_1) there exists $\delta > 2^{p_{\max}^+ + 1}(N+1)^{\frac{p_{\max}^+}{2}}$ such that

$$\lim\inf_{|x|\to\infty} \frac{p_{\min}^- H(t,x)}{|x|^{p_{\max}^+}} \geq \delta, \;\forall t \in [1,N]_{\mathbb{Z}},$$

where

$$H(t,x) = \int_0^x h(t,s)ds \quad \textit{for} \quad (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}.$$

Then the problem (P) has at least one solution.

Example 1.1. Let us consider a continuous function $h : [1,N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ given by the formula

$$h(t,x) = 2^{p_{\max}^+ + 2t} \frac{p_{\max}^+}{p_{\min}^-} (N+1)^{\frac{p_{\max}^+}{2}} |x|^{p_{\max}^+ - 2} x.$$

Clearly

$$H(t,x) = \frac{2^{p_{\max}^+ + 2t}}{p_{\min}^-} (N+1)^{\frac{p_{\max}^+}{2}} |x|^{p_{\max}^+}.$$

It is easy to see that

$$\lim\inf_{|x|\to\infty} \frac{p_{\min}^- H(t,x)}{|x|^{p_{\max}^+}} = 2^{p_{\max}^+ + 2t} (N+1)^{\frac{p_{\max}^+}{2}} \geq 2^{p_{\max}^+ + 2} (N+1)^{\frac{p_{\max}^+}{2}}.$$

Then $H(t,x)$ satisfies the condition (H_1) with $\delta = 2^{p_{\max}^+ + 2} (N+1)^{\frac{p_{\max}^+}{2}}$.

Theorem 1.2. *Suppose that* (H_1), (H_2) *and* $h(t,0)=0$ *for any* $t \in [1,N]_{\mathbb{Z}}$, *where*

(H_2) $\lim\limits_{|x|\to 0} \dfrac{H(t,x)}{|x|^{p^+_{\max}}} = 0, \ \forall t \in [1,N]_{\mathbb{Z}}$.

Then the problem (P) *has at least two nontrivial solutions, one of which is non-negative and the other is non-positive.*

Example 1.2. Put $h : [1,N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ by the formula

$$h(t,x) = \begin{cases} 2^{p^+_{\max}+2}\dfrac{p^+_{\max}}{p^-_{\min}}(N+1)^{\frac{p^+_{\max}}{2}}|x|^{p^+_{\max}-2}xe^t, & |x|>1,\ t\in[1,N]_{\mathbb{Z}},\\ 2^{p^+_{\max}+2}\dfrac{p^+_{\max}}{p^-_{\min}}(N+1)^{\frac{p^+_{\max}}{2}}|x|^{p^+_{\max}}xe^t, & |x|\le 1,\ t\in[1,N]_{\mathbb{Z}}. \end{cases}$$

By the expression of h, we have for all $t \in [1,N]_{\mathbb{Z}}$

$$H(t,x) = \begin{cases} \dfrac{2^{p^+_{\max}+2}}{p^-_{\min}}(N+1)^{\frac{p^+_{max}}{2}}|x|^{p^+_{\max}}e^t - \dfrac{2^{p^+_{max}+3}}{p^-_{\min}(p^+_{\max}+2)}(N+1)^{\frac{p^+_{\max}}{2}}e^t, |x|>1,\\ 2^{p^+_{\max}+2}\dfrac{p^+_{\max}}{p^-_{\min}(p^+_{\max}+2)}(N+1)^{\frac{p^+_{\max}}{2}}|x|^{p^+_{\max}+2}e^t,\ |x|\le 1. \end{cases}$$

Direct calculations yield

$$\lim\inf_{|x|\to\infty} \frac{p^-_{\min}H(t,x)}{|x|^{p^+_{\max}}} = 2^{p^+_{\max}+2}(N+1)^{\frac{p^+_{\max}}{2}}e^t \ge 2^{p^+_{\max}+2}(N+1)^{\frac{p^+_{\max}}{2}},$$

and

$$\lim_{|x|\to 0}\frac{H(t,x)}{|x|^{p^+_{\max}}} = 0.$$

Thus $H(t,x)$ satisfies the conditions (H_1) with $\delta = 2^{p^+_{\max}+2}(N+1)^{\frac{p^+_{\max}}{2}}$ and (H_2).

Theorem 1.3. *Suppose that* (H_2) *and* (H_3) *hold, where*

(H_3) $\lim\limits_{|x|\to\infty}\left(H(t,x) - \dfrac{2p^-_{\max}}{(p^-_{\min})^2}\lambda^+_N|x|^{p^+_{\max}}\right) = +\infty$, *for any* $t\in[1,N]_{\mathbb{Z}}$ *where* $\lambda^+_N = \max\left\{\lambda^{(1)}_N, \lambda^{(2)}_N\right\}$, *with*

$$\lambda^{(i)}_N = \sup\left\{\frac{\sum\limits_{t=1}^{N+1}|\Delta y(t-1)|^{p_i(t-1)}}{\sum\limits_{t=1}^{N}|y(t)|^{p^+_i}} \mid y\in E^N : \|y\|\ge 1\right\},\ \textit{for } i=1,2; \quad (1.1)$$

$$E^N = \{y : [0,N+1]_{\mathbb{Z}} \longrightarrow \mathbb{R} \mid y(0)=y(N+1)=0\}, \quad (1.2)$$

and

$$\|y\| = \left(\sum_{t=1}^{N+1}|\Delta y(t-1)|^2\right)^{\frac{1}{2}}.$$

Then the problem (P) *has at least two nontrivial solutions.*

Remark 1.1. It is easy to see that $\lambda_N^+ > 0$ and we will see later that λ_N^+ is finite.

Example 1.3. Let us consider a continuous function $h : [1, N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ given by the formula

$$h(t,x) = \begin{cases} \ln(t+1)(1 + p^+_{\max} \ln|x|)|x|^{p^+_{\max}-2}x, & |x| > 1, t \in [1,N]_{\mathbb{Z}}, \\ \ln(t+1)|x|^{p^+_{\max}-1}x, & |x| \le 1, t \in [1,N]_{\mathbb{Z}}. \end{cases}$$

Clearly, we have

$$H(t,x) = \begin{cases} \ln(t+1)(|x|^{p^+_{\max}} \ln|x| + \frac{1}{p^+_{\max}+1}), & |x| > 1, t \in [1,N]_{\mathbb{Z}}, \\ \frac{\ln(t+1)}{p^+ + 1}|x|^{p^+_{\max}+1}, & |x| \le 1, t \in [1,N]_{\mathbb{Z}}. \end{cases}$$

After a simple calculation, we get

$$\lim_{|x|\to\infty} \left(H(t,x) - \frac{p^-_{\max}}{(p^-_{\min})^2} \lambda_N^+ |x|^{p^+_{\max}} \right) = +\infty \text{ and } \lim_{|x|\to 0} \frac{H(t,x)}{|x|^{p^+_{\max}}} = 0, \text{for any } t \in [1,N]_{\mathbb{Z}}.$$

Then $H(t,x)$ satisfies the conditions (H_2) and (H_3).

The structure of this paper is as follows: Section 2 contains some preliminary lemmas. The main results will be proved in Section 3.

2. Preliminary lemmas

The vector space E^N defined in (1.2) is an N-dimensional Hilbert space with the inner product

$$\langle y, z \rangle = \sum_{t=1}^{N} \Delta y(t-1)\Delta z(t-1), \quad \forall y, z \in E^N,$$

while the corresponding norm is given by

$$\|y\| = \left(\sum_{t=1}^{N+1} |\Delta y(t-1)|^2 \right)^{\frac{1}{2}}.$$

We list also some inequalities that will be used later.

Lemma 2.1. *(see [13]) Let* $p : [0, N]_{\mathbb{Z}} \longrightarrow [2, \infty[$. *Then put*

$$p^+ = \max_{t\in[0,N]_{\mathbb{Z}}} p(t) \text{ and } p^- = \min_{t\in[0,N]_{\mathbb{Z}}} p(t).$$

For every $y \in E^N$, *we have*

(A_1) $\quad \sum_{t=1}^{N+1} |\Delta y(t-1)|^{p(t-1)} \ge N^{\frac{p^+-2}{2}} \|y\|^{p^+}$, *with* $\|y\| \le 1$.

(A_2) $\quad \sum_{t=1}^{N+1} |\Delta y(t-1)|^{p(t-1)} \ge N^{\frac{2-p^-}{2}} \|y\|^{p^-} - (N+1)$, *with* $\|y\| > 1$.

(A_3) $\quad \sum_{t=1}^{N} |y(t)|^m \le N(N+1)^{m-1} \sum_{t=1}^{N+1} |\Delta y(t-1)|^m, \ \forall m > 1.$

(A_4) $\max_{t\in[1,N]_{\mathbb{Z}}} |y(t)| < (N+1)^{\frac{1}{q}} \left(\sum_{t=1}^{N+1} |\Delta y(t-1)|^p\right)^{\frac{1}{p}}$, $\forall p, q > 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$.

(A_5) $\sum_{t=1}^{N+1} |\Delta y(t-1)|^m \leq 2^m \sum_{t=1}^{N} |y(t)|^m$, $\forall m \geq 2$.

(A_6) $\sum_{t=1}^{N+1} |\Delta y(t-1)|^{p(t-1)} \leq (N+1)\|y\|^{p^+} + (N+1)$.

(A_7) $\sum_{t=1}^{N+1} |\Delta y(t-1)|^m \leq (N+1)\|y\|^m$, $\forall m \geq 1$.

(A_8) $\sum_{t=1}^{N+1} |\Delta y(t-1)|^m \geq (N+1)^{\frac{2-m}{2}}\|y\|^m$, $\forall m \geq 2$.

Remark 2.1. From (A_6), it is easy to see that λ_N^+ defined in Theorem 1.5 is finite.

The functional associated to problem (P) is defined by $\Phi : E^N \longrightarrow \mathbb{R}$,

$$\Phi(y) = \sum_{t=1}^{N+1}\sum_{i=1}^{2} \frac{1}{p_i(t-1)}|\Delta y(t-1)|^{p_i(t-1)} - \sum_{t=1}^{N} H(t, y(t)). \tag{2.1}$$

Since $h : [1, N]_{\mathbb{Z}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, then Φ is well defined, of classe $C^1(E^N, \mathbb{R})$ and its Gâteaux derivative is given by

$$\Phi'(y).z = \sum_{t=1}^{N+1}\sum_{i=1}^{2} |\Delta y(t-1)|^{p_i(t-1)-2}\Delta y(t-1)\Delta z(t-1) - \sum_{t=1}^{N} h(t, y(t))z(t), \tag{2.2}$$

for any $z \in E^N$.

By the summation by parts formula, Φ' can be written as

$$\Phi'(y).z = \sum_{t=1}^{N}\left[-\sum_{i=1}^{2} \Delta(|\Delta y(t-1)|^{p_i(t-1)-2}\Delta y(t-1)) - h(t, y(t))\right] z(t),$$

for any $z \in E^N$.

Finding the solutions to the problem (P) is equivalent to getting the critical point of the functional Φ.

Now, we consider the truncated problem

$$(P_\pm) \quad \begin{cases} -\sum_{i=1}^{2} \Delta(|\Delta y(t-1)|^{p_i(t-1)-2}\Delta y(t-1)) = h_\pm(t, y(t)), \quad t \in [1, N]_{\mathbb{Z}}, \\ y(0) = y(N+1) = 0, \end{cases}$$

where

$$h_\pm(t, x) = \begin{cases} h(t, x), & \text{if} \pm x \geq 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.3}$$

For $y \in E^N$, we denote by $y^+ = \max(y, 0)$ and $y^- = \max(-y, 0)$ the positive and negative parts of y.

It is clear to see that $y^+ \geq 0$, $y^- \geq 0$, $y = y^+ - y^-$, $y^+.y^- = 0$, $y^\pm = \frac{1}{2}(|y| \pm y)$ and $y^\pm \leq |y|$.

Lemma 2.2. *All solutions of* (P_+) *(resp.* (P_-)*) are non-negative (resp. non positive) solutions of* (P).

Proof.
Define
$$\Phi_\pm : E^N \longrightarrow \mathbb{R},$$
$$\begin{aligned}\Phi_\pm(y) &= \sum_{t=1}^{N+1}\sum_{i=1}^{2}\frac{1}{p_i(t-1)}|\Delta y(t-1)|^{p_i(t-1)} - \sum_{t=1}^{N} H_\pm(t,y(t))\\ &= \sum_{t=1}^{N+1}\sum_{i=1}^{2}\frac{1}{p_i(t-1)}|\Delta y(t-1)|^{p_i(t-1)} - \sum_{t=1}^{N} H(t,y^\pm(t)),\end{aligned}$$
where $H_\pm(t,x) = \int_0^x h_\pm(t,s)\,\mathrm{d}s$.

It is easy to see that
$$\Delta y^+(t-1)\Delta y^-(t-1) \le 0, \quad \forall t \in [1,N+1]_{\mathbb{Z}}.$$
Now, we show that
$$|\Delta y^-(t-1)| \le |\Delta y(t-1)|, \quad \forall t \in [1,N+1]_{\mathbb{Z}}.$$
Indeed,
$$\begin{aligned}|\Delta y^-(t-1)| &= |y^-(t) - y^-(t-1)|\\ &= |\frac{1}{2}(|y(t)| - y(t)) - \frac{1}{2}(|y(t-1)| - y(t-1))|\\ &\le \frac{1}{2}\left[|y(t) - y(t-1)|| + |y(t) - y(t-1)|\right]\\ &\le |\Delta y(t-1)|.\end{aligned}$$
Let y be a solution of (P_+), or equivalently y be a critical point of Φ_+. Taking $z = y^-$ in
$\langle \Phi'_+(y), z\rangle = \sum_{t=1}^{N+1}\sum_{i=1}^{2} |\Delta y(t-1)|^{p_i(t-1)-2}\Delta y(t-1)\Delta z(t-1) - \sum_{t=1}^{N} h_+(t,y(t))z(t),$
we have
$$\begin{aligned}\langle \Phi'_+(y), y^-\rangle &= \sum_{t=1}^{N+1}\sum_{i=1}^{2} |\Delta y(t-1)|^{p_i(t-1)-2}\Delta y(t-1)\Delta y^-(t-1)\\ &= \sum_{t=1}^{N+1}\sum_{i=1}^{2} |\Delta y(t-1)|^{p_i(t-1)-2}\Delta (y^+(t-1) - y^-(t-1))\Delta y^-(t-1)\\ &= \sum_{t=1}^{N+1}\sum_{t=1}^{2} |\Delta y(t-1)|^{p_i(t-1)-2}\Delta y^+(t-1)\Delta y^-(t-1)\\ &\quad - |\Delta y(t-1)|^{p_i(t-1)-2}(\Delta y^-(t-1))^2.\end{aligned}$$
Therefore, we deduce that
$$\sum_{t=1}^{N+1}\sum_{i=1}^{2} |\Delta y(t-1)|^{p_i(t-1)-2}\left[-\Delta y^+(t-1)\Delta y^-(t-1)\right]$$

$$+\sum_{t=1}^{N+1}\sum_{i=1}^{2}|\Delta y(t-1)|^{p_i(t-1)-2}(\Delta y^-(t-1))^2=0.$$

Since,

$$-\Delta y^+(t-1)\Delta y^-(t-1)\geq 0,\ \ \forall t\in[1,N+1]_{\mathbb{Z}},$$

then, we get

$$|\Delta y(t-1)|^{p_i(t-1)-2}(\Delta y^-(t-1))^2=0,\ \ \forall (i,t)\in[1,2]_{\mathbb{Z}}\times[1,N+1]_{\mathbb{Z}}.$$

On the other hand,

$$\begin{aligned}|\Delta y^-(t-1)|^{p_i(t-1)} &= |\Delta y^-(t-1)|^{p_i(t-1)-2}\left(\Delta y^-(t-1)\right)^2\\ &\leq |\Delta y(t-1)|^{p_i(t-1)-2}\left(\Delta y^-(t-1)\right)^2=0,\end{aligned}$$

for any $(i,t)\in[1,2]_{\mathbb{Z}}\times[1,N+1]_{\mathbb{Z}}$.

So $y^-=0$ and $y=y^+$ is also a critical point of Φ with critical value $\Phi(y)=\Phi_+(y)$.

Similarly, nontrivial critical points of Φ_- are non-positive solutions of (P). The proof is complete. □

Definition 2.1. Let E be a real Banach space and $\Phi: E\longrightarrow\mathbb{R}$ be a C^1 functional. We say that a functional Φ satisfies the Palais-Smale (PS) condition, if every sequence $(y_n)\subset E$ such that $(\Phi(y_n))$ is bounded and $\Phi'(y_n)\to 0$ as $n\to\infty$, contains a convergent subsequence. The sequence (y_n) is called a (PS) sequence.

Lemma 2.3. *(see [25]) Let E be a reflexive Banach space. If a functional $\Phi\in C^1(E,\mathbb{R})$ is weakly lower semi continuous and anti-coercive, i.e. $\lim\limits_{\|y\|\longrightarrow\infty}\Phi(y)=-\infty$, then there exists $\overline{y}\in E$ such that $\Phi(\overline{y})=\sup\limits_{y\in E}\Phi(y)$ and u is also a critical point of Φ, i.e $\Phi'(\overline{y})=0$.*

Lemma 2.4. *(Mountain Pass Lemma [2])*
Let Φ be a C^1 functional on a Banach space E that satisfies the (PS) condition and $\Phi(0)=0$. Suppose that:

σ_1) there exist ρ, $\alpha>0$ such that $\Phi(y)\geq\alpha$ for all $y\in E$ with $\|y\|_H=\rho$,

σ_2) there exists $e\in E$, with $\|e\|_E>\rho$ such that $\Phi(e)\leq 0$.

Then,

$$c=\inf_{g\in\Gamma}\max_{s\in[0,1]}\Phi(g(s))\geq\alpha,$$

where

$$\Gamma=\{g\in C([0,1],E)\mid\ g(0)=0,\ g(1)=e\}$$

is a critical value of Φ.

3. Proofs of the main results

3.1. Proof of Theorem 1.1.

From (H_1), there exists $R>0$ such that

$$H(t,x)\geq\frac{1}{p^-_{\min}}(\delta-\varepsilon)|x|^{p^+_{\max}},\ \ \forall (t,|x|)\in[1,N]_{\mathbb{Z}}\times\,]R,+\infty[\,,$$

where

$$0 < \varepsilon < \delta - 2^{p^+_{\max}+1}(N+1)^{\frac{p^+_{\max}}{2}}. \tag{3.1}$$

On the other hand, by the continuity of $x \longrightarrow H(t,x) - \frac{1}{p^-_{\min}}(\delta-\varepsilon)|x|^{p^+_{\max}}$, there exists $d > 0$ such that

$$H(t,x) - \frac{1}{p^-_{\min}}(\delta-\varepsilon)|x|^{p^+_{\max}} \geq -d, \quad \forall (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,R]\,.$$

Thus, we deduce that

$$H(t,x) \geq \frac{1}{p^-_{\min}}(\delta-\varepsilon)|x|^{p^+_{\max}} - d, \quad \forall (t,x) \in [1,N]_{\mathbb{Z}} \times \mathbb{R}. \tag{3.2}$$

According to (A_5), (A_8) and (3.2), we obtain

$$\begin{aligned}\sum_{t=1}^{N} H(t,y(t)) &\geq \frac{(\delta-\varepsilon)}{p^-_{\min}} \sum_{t=1}^{N} |y(t)|^{p^+_{\max}} - dN \\ &\geq 2^{-p^+_{\max}}(N+1)^{\frac{2-p^+_{\max}}{2}} \frac{(\delta-\varepsilon)}{p^-_{\min}} \|y\|^{p^+_{\max}} - dN. \end{aligned} \tag{3.3}$$

Now, using the preceding inequality and (A_6), we have

$$\begin{aligned}\Phi(y) &\leq \frac{2(N+1)}{p^-_{\min}}\|y\|^{p^+_{\max}} - \frac{2^{-p^+_{\max}}}{p^-_{\min}}(\delta-\varepsilon)(N+1)^{\frac{2-p^+_{\max}}{2}}\|y\|^{p^+_{\max}} + \frac{2(N+1)}{p^-_{\min}} + dN \\ &\leq \frac{2^{-p^+_{\max}}}{p^-_{\min}}(N+1)^{\frac{2-p^+_{\max}}{2}}\left[2^{p^+_{\max}+1}(N+1)^{\frac{p^+_{\max}}{2}} - (\delta-\varepsilon)\right]\|y\|^{p^+_{\max}} \\ &\quad + \frac{2(N+1)}{p^-_{\min}} + dN.\end{aligned}$$

Then, in view of (3.1), $\Phi(y) \to -\infty$ as $\|y\| \to \infty$. Thus, Φ is anti-coercive and bounded from the above, hence there is a maximum point of Φ at some $y^* \in E^N$ i.e., $\Phi(y^*) = \sup_{y\in E^N} \Phi(y)$, which is a critical point of Φ. Hence y^* is a solution of (P). The proof of Theorem 1.3 is complete. □

3.2. Proof of Theorem 1.3.

To apply the Mountain Pass Theorem, we will do separate studies of the (PS) condition compactness of $\Phi_\pm$ and its geometry.

Lemma 3.1. *Assume that (H_1) holds. Then the functional Φ_+ satisfies the (PS) condition.*

Proof. Let $(y_n) \subset E^N$ be a (PS) sequence for the functional Φ_+, i.e.,

$$|\Phi_+(y_n)| \leq C \text{ and } \Phi'_+(y_n) \longrightarrow 0 \text{ as } n \to \infty,$$

where C is a constant. Let us show that (y_n) is bounded in E^N. Since $y_n = y_n^+ - y_n^-$, we will prove that (y_n^+) and (y_n^-) are bounded.

Suppose that (y_n^-) is unbounded. Then there exists an integer $n_0 > 0$ such that

$$\|y_n^-\| \geq 2(N+1) \quad \text{for } n \geq n_0. \tag{3.4}$$

Since $\Delta y_n^+(t-1)\Delta y_n^-(t-1) \leq 0$ and $|\Delta y_n^-(t-1)| \leq |\Delta y_n(t-1)|$, $\forall t \in [1, N+1]_{\mathbb{Z}}$, then, we have

$$\begin{aligned}
\langle \Phi_+'(y_n), y_n^- \rangle &= \sum_{t=1}^{N+1}\sum_{i=1}^{2} |\Delta y_n(t-1)|^{p_i(t-1)-2}\Delta y_n(t-1)\Delta y_n^-(t-1) \\
&- \sum_{t=1}^{N} h_+(t, y_n(t)) y_n^-(t) \\
&= \sum_{t=1}^{N+1}\sum_{i=1}^{2} |\Delta y_n(t-1)|^{p_i(t-1)-2}\Delta y_n^+(t-1)\Delta y_n^-(t-1) \\
&- |\Delta y_n(t-1)|^{p_i(t-1)-2}(\Delta y_n^-(t-1))^2 \\
&\leq -\sum_{t=1}^{N+1}\sum_{i=1}^{2} |\Delta y_n^-(t-1)|^{p_i(t-1)}.
\end{aligned}$$

Using the above inequality and (A_2), we obtain for any $n \geq n_0$

$$\langle \Phi_+'(y_n), y_n^- \rangle \leq -N^{\frac{2-p_1^-}{2}}\|y_n^-\|^{p_1^-} - N^{\frac{2-p_2^-}{2}}\|y_n^-\|^{p_2^-} + 2(N+1).$$

This implies that

$$N^{\frac{2-p_1^-}{2}}\|y_n^-\|^{p_1^-} + N^{\frac{2-p_2^-}{2}}\|y_n^-\|^{p_2^-} - 2(N+1) \leq \langle \Phi_+'(y_n), -y_n^- \rangle \leq \|\Phi_+'(y_n)\|\|y_n^-\|.$$

Therefore,

$$N^{\frac{2-p_1^-}{2}}\|y_n^-\|^{p_1^-} \leq \|\Phi_+'(y_n)\|\|y_n^-\| + 2(N+1),$$

and

$$N^{\frac{2-p_1^-}{2}}\|y_n^-\|^{p_1^- -1} \leq \|\Phi_+'(y_n)\| + 1. \tag{3.5}$$

Since $\Phi_+'(y_n) \longrightarrow 0$ as $n \to \infty$, then for any $\varepsilon > 0$, there exists an integer n_1 with $n_1 \geq n_0$ such that

$$\|\Phi_+'(y_n)\| < \varepsilon, \ \forall n \geq n_1.$$

Combining the preceding inequality and (3.5), we get

$$\|y_n^-\|^{p_1^- -1} \leq (\varepsilon+1)\, N^{\frac{p_1^- -2}{2}} \quad \text{for any } n \geq n_1,$$

which means that (y_n^-) is bounded. Thus we obtain a contradiction.

Now, we will prove that (y_n^+) is bounded. We argue by contradiction. Suppose that $\|y_n^+\| \longrightarrow \infty$ as $n \to \infty$.

From the proof of Theorem 1.1,

$$\sum_{t=1}^{N} H(t, y_n^+(t)) \geq 2^{-p_{\max}^+}(N+1)^{\frac{2-p_{\max}^+}{2}} \frac{(\delta-\varepsilon)}{p_{\min}^-}\|y_n^+\|^{p_{\max}^+} - dN, \tag{3.6}$$

where

$$0 < \varepsilon < \delta - 2^{p^+_{\max}}(N+1)^{\frac{p^+_{\max}}{2}}. \tag{3.7}$$

By (A_6) and (3.6), we have

$$\begin{aligned}
&\Phi_+(y_n)\\
&= \sum_{t=1}^{N+1}\sum_{i=1}^{2}\frac{1}{p_i(t-1)}|\Delta y_n(t-1)|^{p_i(t-1)} - \sum_{t=1}^{N} H(t, y_n^+(t))\\
&\leq \sum_{i=1}^{2}\frac{N+1}{p_i^-}\left[\|y_n^+ - y_n^-\|^{p_i^+} + 1\right] - 2^{-p^+_{\max}}(N+1)^{\frac{2-p^+_{\max}}{2}}\frac{(\delta-\varepsilon)}{p^-_{\min}}\|y_n^+\|^{p^+_{\max}} + dN\\
&\leq \frac{2(N+1)}{p^-_{\min}}\left[\left(\|y_n^+\| + \|y_n^-\|\right)^{p^+_{\max}} + 1\right] - 2^{-p^+_{\max}}(N+1)^{\frac{2-p^+_{\max}}{2}}\frac{(\delta-\varepsilon)}{p^-_{\min}}\|y_n^+\|^{p^+_{\max}}\\
&\quad + dN\\
&\leq \frac{2^{-p^+_{\max}}(N+1)^{\frac{2-p^+_{\max}}{2}}}{p^-_{\min}}\left[2^{p^+_{\max}+1}(N+1)^{\frac{p^+_{\max}}{2}}\left(1 + \frac{\|y_n^-\|}{\|y_n^+\|}\right)^{p^+_{\max}} - (\delta-\varepsilon)\right]\\
&\quad \times \|y_n^+\|^{p^+_{\max}} + \frac{2(N+1)}{p^-_{\min}} + dN.
\end{aligned}$$

So, we deduce that

$$\begin{aligned}
-C \leq & \frac{2^{-p^+_{\max}}(N+1)^{\frac{2-p^+_{\max}}{2}}}{p^-_{\min}}\left[2^{p^+_{\max}+1}(N+1)^{\frac{p^+_{\max}}{2}}\left(1 + \frac{\|y_n^-\|}{\|y_n^+\|}\right)^{p^+_{\max}} - (\delta-\varepsilon)\right]\\
& \times \|y_n^+\|^{p^+_{\max}} + \frac{2(N+1)}{p^-_{\min}} + dN.
\end{aligned}$$

Then in view of (3.7) and the fact that (y_n^-) is bounded, we get

$$\begin{aligned}
&\frac{2^{-p^+_{\max}}(N+1)^{\frac{2-p^+_{\max}}{2}}}{p^-_{\min}}\left[2^{p^+_{\max}+1}(N+1)^{\frac{p^+_{\max}}{2}}\left(1 + \frac{\|y_n^-\|}{\|y_n^+\|}\right)^{p^+_{\max}} - (\delta-\varepsilon)\right]\|y_n^+\|^{p^+_{\max}}\\
&\quad + \frac{2(N+1)}{p^-_{\min}} + dN \longrightarrow -\infty,
\end{aligned}$$

as $n \to \infty$.

This is a contradiction, hence (y_n^+) is bounded. It follows that (y_n) is bounded. The proof is complete. □

Lemma 3.2. *Assume that (H_2) holds, then there exist $r > 0$ and $\alpha > 0$ such that $\Phi_+(y) \geq \alpha$, for all $y \in \partial B_r \cap E^N$, where B_r denote the open ball in E^N about 0 of radius r and ∂B_r denote its boundary.*

Proof. Using the condition (H_2), for any $\varepsilon > 0$ there exists $R > 0$ such that

$$|H(t,x)| \leq \varepsilon |x|^{p^+_{\max}}, \quad \forall (t, |x|) \in [1,N]_{\mathbb{Z}} \times [0,R]. \tag{3.8}$$

Let $y \in E^N$ such that $\|y\| \leq r$ with $r = \min\left\{\frac{R}{\sqrt{N+1}}, 1\right\}$. From (A_4) it follows

$$|y^+(t)| \leq |y(t)| \leq \max_{t\in[1,N]_{\mathbb{Z}}} |y(t)| \leq R, \quad \forall t \in [1,N]_{\mathbb{Z}}.$$

Therefore, we deduce that

$$|H(t, y^+(t))| \leq \varepsilon |y^+(t)|^{p^+_{\max}} \leq \varepsilon |y(t)|^{p^+_{\max}}, \quad \forall t \in [1, N]_{\mathbb{Z}}.$$

Using the preceding inequality and (A_1), (A_3), (A_7), we obtain

$$\begin{aligned} \Phi_+(y) &= \sum_{t=1}^{N+1} \sum_{i=1}^{2} \frac{1}{p_i(t-1)} |\Delta u(t-1)|^{p_i(t-1)} - \sum_{t=1}^{N} H(t, y^+(t)) \\ &\geq \left[\frac{N^{\frac{p_1^+-2}{2}}}{p_1^+} + \frac{N^{\frac{p_2^+-2}{2}}}{p_2^+} - \varepsilon N (N+1)^{p^+_{\max}} \right] \|y\|^{p^+_{\max}}. \end{aligned}$$

Let us choose $\varepsilon > 0$ such that $\varepsilon < \frac{1}{N(N+1)^{p^+_{\max}}} \left[\frac{N^{\frac{p_1^+-2}{2}}}{p_1^+} + \frac{N^{\frac{p_2^+-2}{2}}}{p_2^+} \right]$. It follows that there exist $r > 0$ and $\alpha > 0$ such that

$$\Phi_+(y) \geq \alpha, \ \forall y \in \partial B_r \cap E^N.$$

The proof of Lemma 3.2 is complete. □

Proof of Theorem 1.3. In order to apply the Mountain Pass Theorem, we must prove that

$$\Phi_+(sy) \longrightarrow -\infty \ \text{ as } \ s \to \infty, \text{ for certain } y \in E^N.$$

Let $y \in E^N$, $y > 0$, $\|y\| > 1$ and $s > 1$. From (A_6) and (3.6), we have

$$\begin{aligned} &\Phi_+(sy) \\ \leq &\sum_{i=1}^{2} \frac{N+1}{p_i^-} \left[s^{p_i^+} \|y\|^{p_i^+} + 1 \right] - 2^{-p^+_{\max}} (N+1)^{\frac{2-p^+_{\max}}{2}} \frac{(\delta - \varepsilon)}{p^-_{\min}} s^{p^+_{\max}} \|y\|^{p^+_{\max}} + dN \\ \leq &\frac{2(N+1)}{p^-_{\min}} \left[s^{p^+_{\max}} \|y\|^{p^+_{\max}} + 1 \right] - 2^{-p^+_{\max}} (N+1)^{\frac{2-p^+_{\max}}{2}} \frac{(\delta - \varepsilon)}{p^-_{\min}} s^{p^+_{\max}} \|y\|^{p^+_{\max}} + dN \\ \leq &\frac{2^{-p^+_{\max}}}{p^-_{\min}} (N+1)^{\frac{2-p^+_{\max}}{2}} s^{p^+_{\max}} \left(2^{p^+_{\max}+1} (N+1)^{\frac{p^+_{\max}}{2}} - (\delta - \varepsilon) \right) \|y\|^{p^+_{\max}} \\ &+ \frac{2(N+1)}{p^-_{\min}} + dN, \end{aligned}$$

where $0 < \varepsilon < \delta - 2^{p^+_{\max}} (N+1)^{\frac{p^+_{\max}}{2}}$. Therefore

$$\Phi_+(sy) \longrightarrow -\infty \ \text{ as } \ s \to \infty.$$

It follows that there exists $y^{**} \in E^N$ such that $\|y^{**}\| > r$ and $\Phi_+(y^{**}) < 0$.

According to the Mountain Pass Theorem, Φ_+ admits a critical value $c \geq \alpha$ which is characterized by

$$c = \inf_{g \in \Gamma} \max_{s \in [0,1]} \Phi_+(g(s)),$$

where

$$\Gamma = \{ g \in C([0,1], E^N) / \ g(0) = 0, \ g(1) = y^{**} \}.$$

Then, the functional Φ_+ has a critical point y_+ with $\Phi_+(y_+) \geq \alpha$.

But, $\Phi_+(0) = 0$, that is $y_+ \neq 0$. Therefore, the problem (P_+) has a nontrivial solution which by Lemma 2.2, is non-negative solution of the problem (P).

Similarly, using Φ_-, we show that there exists furthermore a non-positive solution. The proof of Theorem 1.2 is now complete. $\square$

Proof of Theorem 1.4. From the condition (H_2), for

$$\varepsilon = \frac{1}{2N(N+1)^{p^+_{\max}}} \left[\frac{N^{\frac{p_1^+-2}{2}}}{p_1^+} + \frac{N^{\frac{p_2^+-2}{2}}}{p_2^+} \right]$$

there exists $R > 0$ such that

$$|H(t,x)| \leq \varepsilon |x|^{p^+_{\max}}, \;\; \forall (t,|x|) \in [1,N]_{\mathbb{Z}} \times [0,R]\,. \tag{3.9}$$

Let $y \in E^N$, $\|y\| \leq \rho$ with $\rho = \min\left\{ \frac{R}{\sqrt{N+1}}, 1 \right\}$. By (A_4) it follows

$$|y(t)| \leq \max_{t\in[1,N]_{\mathbb{Z}}} |u(t)| \leq R, \;\; \forall t \in [1,N]_{\mathbb{Z}}.$$

So, we deduce that

$$|H(t,y(t))| \leq \varepsilon |y(t)|^{p^+_{\max}}, \;\; \forall t \in [1,N]_{\mathbb{Z}}.$$

By (A_1), (A_3) and (A_7), we have

$$\begin{aligned} \Phi(y) &\geq \frac{N^{\frac{p_1^+-2}{2}}}{p_1^+} \|y\|^{p_1^+} + \frac{N^{\frac{p_2^+-2}{2}}}{p_2^+} \|y\|^{p_2^+} - \varepsilon N(N+1)^{p^+_{\max}} \|y\|^{p^+_{\max}} \\ &\geq \left[\frac{N^{\frac{p_1^+-2}{2}}}{p_1^+} + \frac{N^{\frac{p_2^+-2}{2}}}{p_2^+} - \varepsilon N(N+1)^{p^+_{\max}} \right] \|y\|^{p^+_{\max}} \\ &\geq \frac{1}{2} \left[\frac{N^{\frac{p_1^+-2}{2}}}{p_1^+} + \frac{N^{\frac{p_2^+-2}{2}}}{p_2^+} \right] \|y\|^{p^+_{\max}}. \end{aligned}$$

Take $\alpha = \frac{1}{2} \left[\frac{N^{\frac{p_1^+-2}{2}}}{p_1^+} + \frac{N^{\frac{p_2^+-2}{2}}}{p_2^+} \right] \rho^{p^+_{\max}} > 0$. Then,

$$\Phi(y) \geq \alpha > 0, \;\; \forall y \in E^N \;\text{ with } \|y\| = \rho. \tag{3.10}$$

Now, by contradiction we prove that Φ is anti-coercive. Let $K \in \mathbb{R}$ and $(y_n) \subset E^N$ such that

$$\|y_n\| \longrightarrow \infty \;\text{ and }\; \Phi(y_n) \geq K.$$

Putting $z_n = \frac{y_n}{\|y_n\|}$, one has $\|z_n\| = 1$. Since $\dim E^N < \infty$, there exists $z \in E^N$ such that

$$\|z_n - z\| \longrightarrow 0, \text{ as } n \to \infty \text{ and } \|z\| = 1.$$

In particular $z \neq 0$, we pose $\Theta = \{t \in [1,N]_{\mathbb{Z}} / \; z(t) \neq 0\}$.

For $t \in \Theta$, $|y_n(t)| \longrightarrow \infty$. Using (1.1), we have

$$\begin{aligned}
K \leq & \frac{1}{p_1^-}\lambda_N^{(1)}\sum_{t=1}^{N}|y_n(t)|^{p_1^+} + \frac{1}{p_2^-}\lambda_N^{(2)}\sum_{t=1}^{N}|y_n(t)|^{p_2^+} \\
& - \sum_{t=1}^{N}\left[H(t,y_n(t)) - \frac{2p_{\max}^-}{(p_{\min}^-)^2}\lambda_N^+|y_n(t)|^{p_{\max}^+}\right] - \frac{2p_{\max}^-}{(p_{\min}^-)^2}\lambda_N^+\sum_{t=1}^{N}|y_n(t)|^{p_{\max}^+} \\
\leq & \frac{2}{p_{\min}^-}\lambda_N^+\left[1 - \frac{p_{\max}^-}{p_{\min}^-}\right]\sum_{t=1}^{N}|y_n(t)|^{p_{\max}^+} - \sum_{t=1}^{N}\left[H(t,y_n(t)) - \frac{2p_{\max}^-}{(p_{\min}^-)^2}\lambda_N^+|y_n(t)|^{p_{\max}^+}\right] \\
\leq & -\sum_{t\in\Theta}\left[H(t,y_n(t)) - \frac{2p_{\max}^-}{(p_{\min}^-)^2}\lambda_N^+|y_n(t)|^{p_{\max}^+}\right] \\
& - \sum_{t\in[1,N]_{\mathbb{Z}}\setminus\Theta}\left[H(t,y_n(t)) - \frac{2p_{\max}^-}{(p_{\min}^-)^2}\lambda_N^+|y_n(t)|^{p_{\max}^+}\right].
\end{aligned}$$

From the condition (H_3), we deduce that

$$-\sum_{t\in\Theta}\left[H(t,y_n(t)) - \frac{2p_{\max}^-}{(p_{\min}^-)^2}\lambda_N^+|y_n(t)|^{p_{\max}^+}\right] \longrightarrow -\infty, \quad \text{as} \quad n \to \infty.$$

The sequence $(y_n(t))$ is bounded for any $t \in [1,N]_{\mathbb{Z}}\setminus\Theta$ and H is continuous, then there exists a constant $M \in \mathbb{R}$ such that

$$-\sum_{t\in[1,N]_{\mathbb{Z}}\setminus\Theta}\left[H(t,y_n(t)) - \frac{2p_{\max}^-}{(p_{\min}^-)^2}\lambda_N^+|y_n(t)|^{p_{\max}^+}\right] \leq M.$$

Therefore, we get

$$K \leq -\sum_{t\in\Theta}\left[H(t,y_n(t)) - \frac{2p_{\max}^-}{(p_{\min}^-)^2}\lambda_N^+|y_n(t)|^{p_{\max}^+}\right] + M \longrightarrow -\infty, \quad \text{as} \quad n \to \infty.$$

This a contradiction. Hence Φ is anti-coercive on E^N. So, we can choose e large enough to ensure that $\Phi(e) < 0$, and that any (PS) sequence (y_n) is bounded. In view of the fact that the dimension of E^N is finite, we see that Φ satisfies the (PS) condition. Since $\Phi(0) = 0$, then all the conditions of Lemma 2.4 are satisfied. Thus Φ possesses a critical value

$$c \geq \alpha = \frac{1}{2}\left[\frac{N^{\frac{p_1^+-2}{2}}}{p_1^+} + \frac{N^{\frac{p_2^+-2}{2}}}{p_2^+}\right]\rho^{p_{\max}^+} > 0,$$

where

$$c = \inf_{g\in\Gamma}\max_{s\in[0,1]}\Phi(g(s)),$$

and

$$\Gamma = \{g \in C([0,1], E^N)/\ g(0) = 0,\ g(1) = e\}.$$

Let $y_1 \in E^N$ such that $\Phi(y_1) = c$. Clearly, y_1 is a nontrivial solution of the problem (P).

On the other hand, since Φ is bounded from the above and anti-coercive, then there is a maximum point of Φ at some $y_2 \in E^N$ i.e., $\Phi(y_2) = \sup_{y \in E^N} \Phi(y)$.

Using (3.10), we obtain

$$\Phi(y_2) = \sup_{y \in E^N} \Phi(y) \geq \sup_{y \in \partial B_\rho} \Phi(y) > 0.$$

Hence y_2 is a nontrivial solution of the problem (P).

If $y_1 \neq y_2$, then we have two nontrivial solutions y_1 and y_2.

Otherwise, similar to the proof of Theorem 1.3 in [5], we get two different critical points of Φ on E^N.

Consequently, the problem (P) has at least two nontrivial solutions. The proof of Theorem 1.3 is complete. □

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References

[1] R.P. Agarwal, *Difference Equations and Inequalities, Theory, Methods, and Applications*, 2nd edition, Marcel Dekker, New York, 2000.

[2] A. Ambrosetti and P.H. Rabinowitz, *Dual variational methods in critical points theory and applications*, J. Funct. Anal., 1973, 14, 349-381.

[3] R. Avery and J. Henderson, *Existence of three positive pseudo-symmetric solutions for a one dimensional discrete p-Laplacian*, J. Differ. Equ. Appl., 2004, 10(6), 529-539.

[4] A.R. El Amrouss and O. Hammouti, *Spectrum of discrete 2n-th order difference operator with periodic boundary conditions and its applications*, Opuscula Math., 2021, 41(4), 489-507.

[5] A.R. El Amrouss and O. Hammouti, *Multiplicity of solutions for the discrete boundary value problem involving the p-Laplacian*, Arab J. Math. Sci., 2023, 29(1), 73-82.

[6] A.R. El Amrouss and O. Hammouti, *Existence of multiple solutions to a discrete 2n-th order periodic boundary value problem via variational method*, Sci. Bulletin-Series A, 2021, 83(3), 159-170.

[7] A.R. El Amrouss and F. Kissi, *Multiplicity of solutions for a general $p(x)$-Laplacian Dirichlet problem*, Arab J. Math. Sci., 2013, 19(2) , 205-216.

[8] M. El Ouaarabi, C. Allalou, S. Melliani, *Existence Result for a Double Phase Problem Involving the $(p(x), q(x))$-Laplacian Operator*, Math. Slovaca, 2023, 73(4), 969-982.

[9] M. El Ouaarabi, C. Allalou, S. Melliani, *Weak Solutions for Double Phase Problem Driven by the $(p(x), q(x))$-Laplacian Operator under Dirichlet Boundary Conditions*, Bol. Soc. Parana. Mat., 2023, 41, 1-14.

[10] M. El Ouaarabi, C. Allalou, S. Melliani, *On a class of p(x)-Laplacian-like Dirichlet problem depending on three real parameters*, Arab J. Math. Sci., 2022, 11, 227-239.

[11] M. El Ouaarabi, C. Allalou, S. Melliani, *Existence of weak solutions for $p(x)$-Laplacian-like problem with $p(x)$-Laplacian operator under Neumann boundary condition*, Sao Paulo J. Math. Sci., 2022, 17(2), 1057-1075.

[12] M. El Ouaarabi, C. Allalou, S. Melliani, *p(x)-Laplacian-Like Neumann Problems in Variable-Exponent Sobolev Spaces via Topological Degree Methods*, Filomat, 2022, 36(17), 5973-5984.

[13] M. Galewski and R. Wieteska, *Existence and multiplicity of positive solutions for discrete anisotropic equations*, Turk. J. Math., 2014, 38, 297-310.

[14] O. Hammouti, *Existence of Three Solutions For a Discrete p-Laplacian Boundary Value Problem*, Palest. J. Math., 2024, 13(2), 144-155.

[15] O. Hammouti, *Existence and multiplicity of solutions for nonlinear 2n-th order difference boundary value problems.* J. Elli. Par. Equa., 2022, 8(2), 1-17.

[16] O. Hammouti, A.R. El Amrouss, *Existence of Solutions to a Discrete Problems for Fourth Order Nonlinear p-Laplacian Via Variational Method*, Bol. Soc. Parana. Mat., 2023, 41, 1-9.

[17] O. Hammouti, S. Taarabti, R. P. Agarwal, *Anisotropic discrete boundary value problems*, Appl. Anal. Discret. Math. 2023, 17, 232-248.

[18] O. Hammouti, N. Makran, S. Taarabti, *Multiplicity of solutions for discrete 2n-th order periodic boundary value problem*, J. Elliptic Parabol. Equ. 2023, 9, 1195-1209.

[19] Z. He, *On the existence of positive solutions of p-Laplacian difference equations*, J. Comput. Appl. Math., 2003, 16, 1, 193-201.

[20] M. K. Moghadam, Y. Khalili, R. Wieteska, *Existence of two solutions for a fourth-order difference problem with $p(k)$ exponent*, Afr. Mat., 2020, 31 959-970.

[21] W. G. Kelley and A. C. Peterson, *Difference Equations, an Introduction with Applications*, 2nd edition. Academic Press, New York, 2001.

[22] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher order with Applications*, Kluwer Academic Publishers, 1993, 256.

[23] Y. Li and L. Lu, *Existence of positive solutions of p-Laplacian difference equations*, Appl. Math. Lett., 2006, 19(10), 1019-1023.

[24] Y. Liu and W. Ge, *Twin positive solutions of boundary value problems for finite difference equations with p-Laplacian operator*, J. Math. Anal. Appl. 2003, (27)8, 551-561.

[25] J. Mawhin, *Problèmes de Dirichlet variationnels non linéaires.* Les Presses de l'Université de Montréal, 1987, 1.

[26] D. Motreanu and V. Rădulescu, *Variational and non-variational methods in nonlinear analysis and boundary value problems*, Nonconvex Optim. Appl., Kluwer, 2003, 67.

[27] N. Moujane, M. El Ouaarabi, C. Allalou, *Study of some elliptic system of $(p(x),q(x))$-Kirchhoff type with convection*, J. Elli. Par. Equa., 2023, 9(2), 687-704.

[28] N. S. Papageorgiou, F. Vetro, P. Winker, *Sign changing solutions for critical double phase problems with variable exponent*, Z. Anal. Anwend. 2023, 42(1), 235-251.

[29] Y. Soukaina, M. El Ouaarabi, A. Chakir, H. Khalid, *Existence result for double phase problem involving the* $(p(x), q(x))$*-Laplacian-like operators*, Int. J. Anal. Appl., 2023, 14, 3201-3210.

[30] J. Zuo, O. Hammouti, S. Taarabti, *Multiplicity of Solutions for Discrete 2n-TH Order Periodic Boundary Value Problem with* φ_p*-Laplacian*, Axioms, 2024, 13(3), 1-15.